NUMERICAL INVARIANTS FOR HYPER-KÄHLER MANIFOLDS

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ABSTRACT. We prove some elementary results on the various constants attached to a hyper-Khler manifold, in particular, in the presence of a class that is isotropic for the Beauville–Bogomolov–Fujiki form.

1. INTRODUCTION

We begin by recalling a few standard facts about hyper-Kähler manifolds.

1.1. **The Beauville–Bogomolov–Fujiki form.** The *Beauville–Bogomolov–Fujiki* form of a hyper-Kähler manifold X of dimension $2n$ is usually defined as a canonical integral nondivisible[1] quadratic form q_X on the free abelian group $H^2(X,\mathbf{Z})$ which satisfies

$$\forall\alpha\in H^2(X,\mathbf{Z})\qquad \int_X \alpha^{2n}=c_Xq_X(\alpha)^n,$$

where c_X (the *Fujiki constant*) is a positive rational number. Both q_X and c_X are invariant by deformation.

Assume $q_X(\alpha)=0$. Then one can show $\alpha^{n+1}=0$ and, by comparing the coefficients of t^n in the relation

$$\int_X (t\alpha+\beta)^{2n}=c_Xq_X(t\alpha+\beta)^n=c_X(2tb_X(\alpha,\beta)+q_X(\beta))^n,$$

we get

$$\forall\beta\in H^2(X,\mathbf{R})\qquad \frac{1}{2^n}\binom{2n}{n}\int_X\alpha^n\beta^n=c_Xb_X(\alpha,\beta)^n. \tag{1}$$

1.2. **The Hirzebruch–Riemann–Roch theorem.** The Hirzebruch–Riemann–Roch theorem takes the following form on hyper-Kähler manifolds.

Theorem 1.1. *Let X be a hyper-Kähler manifold of dimension $2n$. There exists a degree-n polynomial $P_{RR,X}\in\mathbf{Q}[T]$, called the* Huybrechts–Riemann–Roch *polynomial, such that, for every holomorphic line bundle L on X, one has*

$$\chi(X,L)=P_{RR,X}(q_X(c_1(L))).$$

The polynomial $P_{RR,X}$ has the following properties:

(a) its constant term is $\chi(X,\mathscr{O}_X)=n+1$;
(b) its leading term is $\frac{c_X}{(2n)!}T^n$;

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[1]By this, we mean that the associated bilinear form b_X is integral and nondivisible on $H^2(X,\mathbf{Z})$.

(c) it is invariant by deformation;
(d) its coefficients are all positive (Jiang).

Example 1.2. The polynomial $P_{RR,X}$ has been computed for all known examples:

- When X is of $\mathrm{K3}^{[n]}$ deformation type, we have
$$P_X(T) = \binom{\frac{1}{2}T + n + 1}{n}.$$
The formula is the same (with $n = 5$) for the OG10 deformation type (Ríos Ortiz).
- When X is of Kum_n deformation type, we have
$$P_X(T) = (n+1)\binom{\frac{1}{2}T + n}{n}.$$
The formula is the same (with $n = 3$) for the OG6 deformation type (Ríos Ortiz).

The next two lemmas are elementary.

Lemma 1.3. *Let X be a hyper-Kähler manifold. For every class $\alpha \in H^2(X, \mathbf{Z})$, one has $P_{RR,X}(q_X(\alpha)) \in \mathbf{Z}$.*

Proof. Since the period map is surjective (Huybrechts), the class α becomes the first Chern class of some holomorphic line bundle on a deformation X' of X, hence $P_{RR,X'}(q_{X'}(\alpha)) \in \mathbf{Z}$. Since the polynomial $P_{RR,X}$ and the form q_X are deformation invariant, the result follows. □

Lemma 1.4. *Let X be a hyper-Kähler manifold of dimension $2n$. Assume that there is a class $\mathsf{l} \in H^2(X, \mathbf{Z})$ such that $q_X(\mathsf{l}) = 0$. For every $\mathsf{m} \in H^2(X, \mathbf{Z})$, the number*
$$a(\mathsf{l}, \mathsf{m}) := \frac{1}{n!}\int_X \mathsf{l}^n \mathsf{m}^n$$
is an integer.

Proof. Set
$$\forall k \in \mathbf{Z} \qquad P(k) := P_{RR,X}(q_X(k\mathsf{l} + \mathsf{m})) = P_{RR,X}(2kb_X(\mathsf{l}, \mathsf{m}) + q_X(\mathsf{m})).$$
Then P is a polynomial of degree n whose leading coefficient is (use property (b) above and (1))
$$\frac{c_X}{(2n)!}(2b_X(\mathsf{l}, \mathsf{m}))^n = \frac{1}{(n!)^2}\int_X \mathsf{l}^n \mathsf{m}^n = \frac{a(\mathsf{l}, \mathsf{m})}{n!}. \tag{2}$$
By Lemma 1.3, the polynomial P takes integral values on integers, hence $a(\mathsf{l}, \mathsf{m})$ is an integer. □

2. Properties of the Huybrechts–Riemann–Roch polynomial

2.1. **Coefficients of the Huybrechts–Riemann–Roch polynomial.** For each positive integer n, we define the positive integer
$$C_n := \gcd_{r_0,\dots,r_n \in \mathbf{Z}} \prod_{0 \le j < k \le n} (r_j^2 - r_k^2).$$

One computes easily $C_1 = 1$, $C_2 = 12$, and, with a computer,

$$\begin{aligned} C_3 &= 2^5 \cdot 3^3 \cdot 5, \\ C_4 &= 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7, \\ C_5 &= 2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^2, \\ C_6 &= 2^{27} \cdot 3^{14} \cdot 5^6 \cdot 7^3 \cdot 11, \\ C_7 &= 2^{37} \cdot 3^{19} \cdot 5^8 \cdot 7^5 \cdot 11^2 \cdot 13. \end{aligned}$$

Let us write

$$P_{RR,X}(T) =: a_n T^n + \cdots + a_1 T + a_0$$

with $a_0 = n + 1$.

Proposition 2.1. *For each $i \in \{0, \ldots, n\}$, the coefficient a_i of the polynomial $P_{RR,X}(T)$ belongs to $\frac{1}{2^i C_n}\mathbf{Z}$ (and to $\frac{1}{C_n}\mathbf{Z}$ if the quadratic form q_X is not even). In particular, $c_X \in \frac{(2n)!}{2^n C_n}\mathbf{Z}$.*

Proof. Let q be an integer represented by the quadratic form q_X. For all $r_0, \ldots, r_n \in \mathbf{Z}$, the integers $r_0^2 q, \ldots, r_n^2 q$ are also represented by q_X, so that $P_{RR,X}(r_j^2 q) = \sum_{i=0}^n a_i r_j^{2i} q^i$ is an integer for all $j \in \{0, \ldots, n\}$. The corresponding linear system with unknowns $a_0 q^0, \ldots, a_n q^n$ has a Vandermonde matrix (r_j^{2i}), so we get

$$a_i q^i \prod_{0 \le j < k \le n} (r_j^2 - r_k^2) \in \mathbf{Z}$$

for all $i \in \{0, \ldots, n\}$, which implies $a_i q^i C_n \in \mathbf{Z}$. Since the integral bilinear form associated with q_X is not divisible, the gcd of all integers q represented by q_X is either 2 (if the form q_X is even) or 1 (if it is not) and the proposition follows. □

In particular, we get $c_X \in \frac{1}{2}\mathbf{Z}$ when $n = 2$, and $c_X \in \frac{1}{48}\mathbf{Z}$ when $n = 3$. For any n, Proposition 2.1 gives the lower bound $c_X \ge \frac{(2n)!}{2^n C_n}$, but what we are really interested in, in order to prove boundedness properties of hyper-Kähler manifolds, would be to find an upper bound on c_X.

Remark 2.2. Assume that q_X represents all large enough even numbers (this is the case for all known examples). Then $P_{RR,X}(T)$ takes integral values on all large enough even numbers and this implies that its leading coefficient is in $\frac{1}{n! 2^n}\mathbf{Z}$, hence $c_X \in (2n-1)!!\mathbf{Z}$.

2.2. **Nieper's normalization of the Beauville–Bogomolov form.** It is nice to work with an integral quadratic form such as q_X, but this normalization introduces the constant c_X about which little is known. Nieper-Wißkirchen introduced a different, more intrinsic, normalization by defining a rational quadratic form λ_X on $H^2(X, \mathbf{Q})$ by the formula

$$\forall \alpha \in H^2(X, \mathbf{Q}) \qquad \frac{1}{(2n)!} \int_X \alpha^{2n} = A_X \lambda_X(\alpha)^n, \qquad \text{where } A_X := \int_X \mathrm{td}^{1/2}(X).$$

The quadratic form λ_X is a positive rational multiple of the quadratic form q_X and we define a positive rational constant n_X by

$$2q_X = n_X \lambda_X,$$

so that $c_X n_X^n = 2^n (2n)! A_X$. One has (Jiang)

$$0 < A_X < 1.$$

Example 2.3. When X is of $\mathrm{K3}^{[n]}$ or OG10 deformation type, one has $c_X = (2n-1)!!$, $n_X = n+3$, and $A_X = \frac{(n+3)^n}{n!2^{2n}}$.

When X is of Kum_n or OG6 deformation type, one has $c_X = (n+1)(2n-1)!!$, $n_X = n+1$, and $A_X = \frac{(n+1)^{n+1}}{n!2^{2n}}$.

The Hirzebruch–Riemann–Roch theorem takes the form

$$\chi(X, L) = Q_{RR,X}(\lambda_X(c_1(L))),$$

where $Q_{RR,X}(T) = P_{RR,X}(\frac{1}{2}n_X T)$ is a polynomial with positive rational coefficients which was expressed by Nieper in terms of the Chern numbers of X. The formula is

$$Q_{RR,X}(T) = \int_X \exp\Big(-\sum_{k=1}^{+\infty} \frac{B_{2k}}{2k} \operatorname{ch}_{2k}(X) T_{2k}\Big(\sqrt{\tfrac{1}{4}T+1}\Big)\Big), \tag{3}$$

where

- the B_{2k} are the Bernoulli numbers;
- the $\operatorname{ch}_{2k} \in H^{2k,2k}(X)$ are the homogeneous components of the Chern character of X;
- the $T_{2k}(Y)$ are the (even) Chebyshev polynomials, defined by $T_{2k}(\cos\theta) = \cos 2k\theta$.

We prove curious symmetry relations for the Huybrechts–Riemann–Roch polynomials $P_{RR,X}(T)$ and $Q_{RR,X}(T)$ for which we have no geometric explanations.

Proposition 2.4. *The polynomial $Q_{RR,X}(T)$ satisfies the symmetry relation*

$$Q_{RR,X}(-T-4) = (-1)^n Q_{RR,X}(T).$$

Equivalently,

$$P_{RR,X}(-T-2n_X) = (-1)^n P_{RR,X}(T).$$

When n is odd, $-n_X$ is therefore a negative rational root of $P_{RR,X}(T)$ (in all known examples, it is actually an integer).

Proof. Let P_k be the degree-k polynomial such that $P_k(T) = T_{2k}\Big(\sqrt{\frac{1}{4}T+1}\Big)$. Set $\cos\theta := \sqrt{\frac{1}{4}T+1}$, so that $T = 4(\cos^2\theta - 1) = -4\sin^2\theta$. We compute

$$\begin{aligned} P_k(-T-4) &= T_{2k}\Big(\sqrt{-\tfrac{1}{4}T}\Big) = T_{2k}(\sin\theta) = T_{2k}(\cos(\theta - \tfrac{\pi}{2})) = \cos(2k(\theta - \tfrac{\pi}{2})) \\ &= (-1)^k \cos 2k\theta = (-1)^k T_{2k}(\cos\theta) = (-1)^k T_{2k}\Big(\sqrt{\tfrac{1}{4}T+1}\Big) = (-1)^k P_k(T). \end{aligned}$$

By (3), the polynomial $Q_{RR,X}(T)$ is a linear combination of polynomials of the type

$$P_{k_1}(T)\cdots P_{k_r}(T)\int_X \operatorname{ch}_{2k_1}(X)\cdots\operatorname{ch}_{2k_r}(X)$$

for $k_1 + \cdots + k_r = n$. The proposition therefore follows. ◻

Corollary 2.5. *The polynomial$P_{RR,X}(T)$ is a linear combination with rational coefficients of the polynomials $(T+n_X)^{n-2j}$, for $0 \le j \le n/2$. In particular, one has*

$$P_{RR,X}(T) = \frac{c_X}{(2n)!}(T+n_X)^n + O(T^{n-2}).$$

When $n=2$, we have

$$P_{RR,X}(T)=\frac{c_X}{24}T^2+\frac{c_Xn_X}{12}T+3,$$

and when $n=3$,

$$P_{RR,X}(T)=\frac{c_X}{720}(T+n_X)^3+\Big(\frac{4}{n_X}-\frac{c_X}{720}n_X^2\Big)(T+n_X).$$

Proof. The first statement follows from the proposition, and the first relation from the fact that the leading coefficient of $P_{RR,X}(T)$ is $\frac{c_X}{(2n)!}$. This immediately gives the case $n=2$. When $n=3$, we write

$$P_{RR,X}(T)=\frac{c_X}{720}(T+n_X)^3+b(T+n_X),$$

where $b\in\mathbf{Q}$ satisfies

$$\frac{c_X}{720}n_X^3+bn_X=4,$$

which gives

$$b=\frac{4}{n_X}-\frac{c_X}{720}n_X^2$$

and implies the result. □

3. The Huybrechts–Riemann–Roch polynomial in the presence of an isotropic class

3.1. **Lagrangian fibrations.** Let X be a hyper-Kähler manifold of dimension $2n$ with a fibration $f\colon X\to\mathbf{P}^n$. The class $\mathsf{l}:=c_1(f^*\mathscr{O}_{\mathbf{P}^n}(1))\in H^2(X,\mathbf{Z})$ satisfies $q_X(\mathsf{l})=0$. Let $\mathsf{m}\in H^2(X,\mathbf{Z})$ be another class, not necessarily of type $(1,1)$; we assume $q_X(\mathsf{l},\mathsf{m})>0$.

Any smooth fiber $X_b:=f^{-1}(b)$ is a Lagrangian complex torus of dimension n (Matsushita). One shows that the hyperplane $\mathsf{l}^\perp$ is contained in the kernel of the restriction map $r_b\colon H^2(X,\mathbf{C})\to H^2(X_b,\mathbf{C})$. Since the restriction of a Kähler class on X is a Kähler class on X_b, the map r_b has rank exactly 1 and its image is generated by a Kähler class. The rational class $r_b(\mathsf{m})$ is a positive multiple of a Kähler class, hence is an ample class on X_b. In particular, X_b is an abelian variety and the "degree" of the polarization $r_b(\mathsf{m})=\mathsf{m}|_{X_b}$ is the positive integer

$$\frac{1}{n!}\int_{X_b}(\mathsf{m}|_{X_b})^n=\frac{1}{n!}\int_X\mathsf{l}^n\mathsf{m}^n=a(\mathsf{l},\mathsf{m})$$

already considered above.

We will concentrate on the case $a(\mathsf{l},\mathsf{m})=1$, when some Kähler class on X induces a principal polarization on the smooth fibers. The main idea of the proof of the result below comes from Ríos Ortiz, who used it to compute the Huybrechts–Riemann–Roch polynomial for hyper-Kähler manifolds of $\mathrm{K3}^{[n]}$ or OG10 deformation type, since they have deformations with a Lagrangain fibration of this type.

Theorem 3.1 (Ríos Ortiz, Debarre–Huybrechts–Macrì–Voisin)**.** *Let X be a hyper-Kähler manifold of dimension $2n$ with a Lagrangian fibration $f\colon X\to\mathbf{P}^n$. Set $\mathsf{l}:=c_1(f^*\mathscr{O}_{\mathbf{P}^n}(1))\in H^2(X,\mathbf{Z})$ and assume that there exists $\mathsf{m}\in H^2(X,\mathbf{Z})$ such that $a(\mathsf{l},\mathsf{m})=1$. Then, $b_X(\mathsf{l},\mathsf{m})=\pm1$, the quadratic form q_X is even, $c_X=(2n-1)!!$,*

$$P_{RR,X}(T)=\binom{\frac{T}{2}+n+1}{n},$$

and the sublattice $\mathbf{Z}\mathsf{l} \oplus \mathbf{Z}\mathsf{m}$ *of* $(H^2(X,\mathbf{Z}), q_X)$ *is isomorphic to a hyperbolic plane.*

It is conjectured that a class $\mathsf{l} \in H^2(X,\mathbf{Z})$ with $q_X(\mathsf{l}) = 0$ (which always exists when $b_2(X) \geq 5$ by Myers' theorem) should, after deformation, come from a Lagrangian fibration with base $\mathbf{P}^n$ as above. It is then natural to make the following, much more restrictive, conjecture.

Conjecture 3.2 (Debarre–Huybrechts–Macrì–Voisin). Let X be a hyper-Kähler manifold of dimension $2n$ with classes $\mathsf{l}, \mathsf{m} \in H^2(X,\mathbf{Z})$ such that

$$\int_X \mathsf{l}^{2n} = 0 \quad \text{and} \quad \int_X \mathsf{l}^n \mathsf{m}^n = n!$$

or, equivalently $q_X(\mathsf{l}) = 0$ and $a(\mathsf{l},\mathsf{m}) = 1$. Then,

$$P_{RR,X}(T) = \binom{\frac{1}{2}T + n + 1}{n}.$$

Knowing that $P_{RR,X}$ has this form already implies the bound $b_2(X) \leq n + 17 + \frac{12}{n+1}$ on the second Betti number (Benedetti–Song).

We may even make the more ambitious following conjecture (verified for all known examples of hyper-Kähler manifolds and proved for $n = 2$ by Debarre–Huybrechts–Macrì–Voisin).

Conjecture 3.3. Let X be a hyper-Kähler manifold of dimension $2n$ with classes $\mathsf{l}, \mathsf{m} \in H^2(X,\mathbf{Z})$ such that

$$\int_X \mathsf{l}^{2n} = 0 \quad \text{and} \quad \int_X \mathsf{l}^n \mathsf{m}^n = an!\ , \quad \text{with} \ \ a \in \{1, \dots, n\}.$$

Then $a = 1$ and X is of $\mathrm{K3}^{[n]}$ or of OG10 deformation type.

Our main result is the following result which almost proves Conjecture 3.2 in dimension 6.

Proposition 3.4. *Let* X *be a hyper-Kähler manifold of dimension* 6 *with classes* $\mathsf{l}, \mathsf{m} \in H^2(X,\mathbf{Z})$ *such that* $q_X(\mathsf{l}) = 0$ *and* $a(\mathsf{l},\mathsf{m}) = 1$*. We have* $b_X(\mathsf{l},\mathsf{m}) = 1$*, the quadratic form* q_X *is even, the sublattice* $\mathbf{Z}\mathsf{l} \oplus \mathbf{Z}\mathsf{m}$ *of* $(H^2(X,\mathbf{Z}), q_X)$ *is isomorphic to a hyperbolic plane, the Fujiki constant is* $c_X = 15$*, and*

$$P_{RR,X}(T) = \binom{\frac{T}{2} + 4}{3} - \frac{6 - n_X}{16} T^2,$$

where $n_X \in \{2, 6\}$.

According to Conjecture 3.2, the case $n_X = 2$ should not occur. We will prove this proposition in the last section.

Again when $n = 3$, we get a much weaker result in the case $a(\mathsf{l},\mathsf{m}) = 2$, which by Conjecture 3.3 should not occur at all.

Proposition 3.5. *Let* X *be a hyper-Kähler manifold of dimension* 6 *with classes* $\mathsf{l}, \mathsf{m} \in H^2(X,\mathbf{Z})$ *such that* $q_X(\mathsf{l}) = 0$ *and* $a(\mathsf{l},\mathsf{m}) = 2$*. We have* $b_X(\mathsf{l},\mathsf{m}) = 1$*, the quadratic form* q_X *is even, the Fujiki constant is* $c_X = 30$*, and*

$$P_{RR,X}(T) = \frac{1}{24}T^3 + \frac{n_X}{8}T^2 + \left(\frac{4}{n_X} + \frac{n_X^2}{12}\right)T + 4,$$

where $n_X \in \{1, 2, 3, 4\}$.

3.2. **Divisibility properties.** Let X be as usual a hyper-Kähler manifold of dimension $2n$ and let $\mathsf{l},\mathsf{m}\in H^2(X,\mathbf{Z})$ such that $q_X(\mathsf{l})=0$ and $a(\mathsf{l},\mathsf{m})\in\mathbf{Z}_{>0}$. By (2), we have

$$c_X b_X(\mathsf{l},\mathsf{m})^n = a(\mathsf{l},\mathsf{m})\frac{(2n)!}{2^n n!}. \tag{4}$$

Using Jiang's bound, we obtain

$$n_X^n = \frac{2^n(2n)!A_X}{c_X} < \frac{2^n(2n)!}{c_X} = \frac{2^{2n}n!b_X(\mathsf{l},\mathsf{m})^n}{a(\mathsf{l},\mathsf{m})}$$

hence

$$n_X < 4b_X(\mathsf{l},\mathsf{m})\sqrt[n]{\frac{n!}{a(\mathsf{l},\mathsf{m})}}. \tag{5}$$

Lemma 3.6. *We have*

$$n!b_X(\mathsf{l},\mathsf{m})^n \mid a(\mathsf{l},\mathsf{m})C_n \qquad (\text{and } n!2^n b_X(\mathsf{l},\mathsf{m})^n \mid a(\mathsf{l},\mathsf{m})C_n \text{ if } q_X \text{ is not even}).$$

Proof. By Proposition 2.1,

$$\frac{c_X 2^n C_n}{(2n)!} = \frac{a(\mathsf{l},\mathsf{m})C_n}{n!b_X(\mathsf{l},\mathsf{m})^n}$$

is an integer (and a stronger property holds when q_X is not even). □

Lemma 3.7. *We have*

$$a(\mathsf{l},\mathsf{m})\Big(\frac{q_X(\mathsf{m})+n_X}{2b_X(\mathsf{l},\mathsf{m})} - \frac{n-1}{2}\Big)\in\mathbf{Z}.$$

In particular,

$$n_X \in \mathbf{Z} + \frac{2b_X(\mathsf{l},\mathsf{m})}{a(\mathsf{l},\mathsf{m})}\mathbf{Z}$$

so that n_X is an integer when $a(\mathsf{l},\mathsf{m})\in\{1,2\}$.

Proof. For every $t\in\mathbf{Z}$, the number

$$P(t) := P_{RR,X}(q_X(t\mathsf{l}+\mathsf{m})) = P_{RR,X}(2tb_X(\mathsf{l},\mathsf{m})+q_X(\mathsf{m}))$$

is an integer. By Corollary 2.5, we have

$$\begin{aligned} P(t) &= \frac{c_X}{(2n)!}(2tb_X(\mathsf{l},\mathsf{m})+q_X(\mathsf{m})+n_X)^n + O(t^{n-2}) \\ &= \frac{a(\mathsf{l},\mathsf{m})}{b_X(\mathsf{l},\mathsf{m})^n 2^n n!}\big(2^n b_X(\mathsf{l},\mathsf{m})^n t^n + n2^{n-1}b_X(\mathsf{l},\mathsf{m})^{n-1}(q_X(\mathsf{m})+n_X)t^{n-1}\big) + O(t^{n-2}) \\ &= \frac{a(\mathsf{l},\mathsf{m})}{n!}t^n + \frac{a(\mathsf{l},\mathsf{m})}{b_X(\mathsf{l},\mathsf{m})2(n-1)!}(q_X(\mathsf{m})+n_X)t^{n-1} + O(t^{n-2}). \end{aligned}$$

This is an integer for all $t\in\mathbf{Z}$, hence so is

$$\begin{aligned} P(t) - a(\mathsf{l},\mathsf{m})\binom{t+n-1}{n} &= P(t) - a(\mathsf{l},\mathsf{m})\,\frac{t^n + \frac{n(n-1)}{2}t^{n-1}}{n!} + O(t^{n-2}) \\ &= \Big(\frac{q_X(\mathsf{m})+n_X}{2b_X(\mathsf{l},\mathsf{m})} - \frac{n-1}{2}\Big)\frac{a(\mathsf{l},\mathsf{m})}{(n-1)!}t^{n-1} + O(t^{n-2}). \end{aligned}$$

This implies the lemma. □

3.3. **Proof of Proposition 3.4.** We restrict ourselves to the case $n = 3$ and prove Proposition 3.4 (curiously, our method gives worse results when $n = 2$). We assume $a(\mathsf{l}, \mathsf{m}) = 1$. Since $C_3 = 2^5 \cdot 3^3 \cdot 5$, we obtain from Lemma 3.6

$$b_X(\mathsf{l}, \mathsf{m})^3 \mid 2^4 \cdot 3^2 \cdot 5 \qquad \text{(and } b_X(\mathsf{l}, \mathsf{m})^3 \mid 2 \cdot 3^2 \cdot 5 \text{ if } q_X \text{ is not even)},$$

so that $b_X(\mathsf{l}, \mathsf{m}) \in \{1, 2\}$ (and $b_X(\mathsf{l}, \mathsf{m}) = 1$ if q_X is not even).

Assume $b_X(\mathsf{l}, \mathsf{m}) = 1$. We have $c_X = 15$ from (4), Lemma 3.7 gives $q_X(\mathsf{m}) + n_X \in 2\mathbf{Z}$, and (5) gives $n_X < 4\sqrt[3]{6} < 7.2$, so that $n_X \in \{1, 2, 3, 4, 5, 6, 7\}$. Furthermore, we have, by Proposition 2.5,

$$P_{RR,X}(T) = \frac{1}{48}(T + n_X)^3 + \Big(\frac{4}{n_X} - \frac{1}{48}n_X^2\Big)(T + n_X).$$

For all values q taken by q_X, this must be an integer when $T = q$, so that

$$48n_X \mid n_X(q + n_X)^3 + (192 - n_X^3)(q + n_X). \tag{6}$$

In particular, $n_X \mid 192q$. If $n_X \in \{5, 7\}$, this implies $n_X \mid q$, which is impossible because the gcd of all integers q represented by q_X is either 1 or 2. Otherwise, $n_X \mid 12$, hence $16n_X \mid 192$, hence we obtain

$$16 \mid (q + n_X)^3 - n_X^2(q + n_X) = q(q + n_X)(q + 2n_X). \tag{7}$$

- When $n_X = 1$, the relation (7) is equivalent to $q \equiv 0, 6, 8, 14, 15 \pmod{16}$. The case $q \equiv 15 \pmod{16}$ is impossible since $4q$ is also represented but not in this list, hence $q \equiv 0, 6, 8, 14 \pmod{16}$ and q_X is even. This contradicts the fact that $q_X(\mathsf{m}) + n_X$ is even.
- When $n_X = 2$, the relation (7) is equivalent to q even.
- When $n_X = 3$, the only possible odd value is $q \equiv 13 \pmod{16}$. This implies that $4q \equiv 4 \pmod{16}$ should also be represented, but 4 does not satisfy the relation (7). So q_X is even, which contradicts the fact that $q_X(\mathsf{m}) + n_X$ is even.
- When $n_X = 4$, the relation (7) is equivalent to $4 \mid q$, which is impossible because the gcd of all integers q represented by q_X is either 1 or 2.
- When $n_X = 6$, the relation (7) is equivalent to q even.

All in all, we get $n_X \in \{2, 6\}$ and q_X even.

Assume $b_X(\mathsf{l}, \mathsf{m}) = 2$. The quadratic form q_X is even, we have $c_X = \frac{15}{8}$ from (4), Lemma 3.7 gives $q_X(\mathsf{m}) + n_X \in 4\mathbf{Z}$, so that n_X is an even integer, and (5) gives $n_X < 8\sqrt[3]{6} < 14.6$, so that $m_X := \frac{1}{2}n_X \in \{1, 2, 3, 4, 5, 6, 7\}$. As above, we deduce from Proposition 2.5 that

$$\frac{1}{8 \cdot 48}(2q + 2m_X)^3 + \Big(\frac{2}{m_X} - \frac{1}{8 \cdot 48}4m_X^2\Big)(2q + 2m_X)$$

is an integer for all values $2q$ taken by q_X, so that

$$48m_X \mid m_X(q + m_X)^3 + (192 - m_X^3)(q + m_X).$$

This is "the same" relation as (6) and the discussion above allows us to conclude that q must be even, so that all values taken by q_X are divisible by 4. This is impossible because the gcd of all values taken by q_X is 2. So this case does not occur.

Remark 3.8. It follows from results of Benedetti–Song that when $n_X = 6$, we have $b_2(X) \leq 23$ and there is equality if and only if $c_2(X) \in SH^2(X, \mathbf{R}) \subset H^4(X, \mathbf{R})$. In the case $n_X = 2$, one has $c_2(X) \notin SH^2(X, \mathbf{R})$.

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