COMPLETE CURVES IN THE MODULI SPACE OF POLARIZED K3 SURFACES AND HYPER-KÄHLER MANIFOLDS

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1. Introduction and statement of main results

Let $\mathscr{F}^0_{2e}$ be the (19-dimensional irreducible quasi-projective) coarse moduli space of *polarized* K3 surfaces of degree $2e$. Via the period map, it can be realized as the complement in the period domain of (the trace of) a (possibly reducible) Heegner divisor D_{2e}.

In 1998, Borcherds, Katzarkov, Pantev, and Shepherd-Barron

- proved that $\mathscr{F}^0_2$ is quasi-affine,[1] hence contains no complete curves;
- constructed a complete curve in $\mathscr{F}^0_{28}$.

Their idea was to start from a nonisotrivial family of principally polarized abelian surfaces, which exists because the coarse moduli space $\mathscr{A}_2$ of principally polarized abelian surfaces has a small boundary in its Satake compactification, and to construct a suitable polarization on the associated family of (smooth) Kummer surfaces (which needs to be ample on *all* Kummer surfaces in the family). We extend their construction to prove our first result.

Theorem 1.1. *For each integer $e \geq 62$ or in the set*

$$\{14, 18, 26, 28, 29, 32, 34, 36, 38, 40, 42, 44, 45, 46, 47, 49, 50, 53, 54, 56, 57, 59, 60\},$$

there exists a nonisotrivial complete family of K3 surfaces with a relative polarization of degree $2e$ or, equivalently, a complete curve in the moduli space $\mathscr{F}^0_{2e}$.

It is plausible that $\mathscr{F}^0_{2e}$ contain complete curves for all $e \geq 2$.

Coincidentally, the range $e \geq 62$ is the same for which Gritsenko–Hulek–Sankaran proved that $\mathscr{F}^0_{2e}$ is of general type (and also for $e \in \{46, 50, 54, 57, 58, 60\}$).

Remarks 1.2. (1) Our construction also yields the existence[2] of projective *rational* curves in $\mathscr{F}^0_{2e}$ for all degrees $2e = 4a^2 - 8c^2$, where a, c are relatively prime integers such that $a > 2c > 0$. Since $\mathscr{F}^0_{2e}$ is only a coarse moduli space, this does not translate into the existence of nonisotrivial families of polarized K3 surfaces parametrized by $\mathbf{P}^1$. In fact, such families do not exist: if a compact complex manifold T is the base of a family of smooth projective varieties with

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[1] This means that it is isomorphic to an open subset of an affine scheme. This is done by producing an automorphic form whose (ample) divisor on the projective Baily–Borel compactification of the period domain is a linear combination with positive coefficients of the two components of the Heegner divisor D_2.

[2] This follows from the existence of a complete rational Shimura curve in $\mathscr{A}_2$ corresponding for example to principally polarized abelian surfaces whose endomorphism rings contain a maximal order of an indefinite quaternion algebra over $\mathbf{Q}$ of reduced discriminant 10.

trivial canonical bundle and the period map is immersive at some point, T is of general type (Zuo, Brunebarbe). The existence of (rational) twistor lines in moduli space of hyper-Kähler manifolds shows that the presence of the polarization is essential.

(2) The situation is very different for the moduli space $\mathscr{F}_{2e}$ of *quasi-polarized* K3 surfaces of degree $2e$:[3] because of the existence of its Baily–Borel projective compactification with one-dimensional boundary, $\mathscr{F}_{2e}$ contains complete subvarieties of dimension 17 (it was proved by van der Geer–Katsura that this is the maximal possible dimension).

(3) A long time ago, Mumford noted (using again the Satake compactification) that the coarse moduli space M_g of smooth projective curves of genus g contains complete curves for all $g \geq 3$ (the scheme M_2 is affine). Moreover, Oort proved in 1974 that there exists a number g such that M_g contains a complete *rational* curve (although, again, there are no nonisotrivial complete families of smooth curves over curves of genus ≤ 1). Using the same ideas, Bryan–Donagi–Stipsicz proved in 2001 that $M_{3h^3-h^2+1}$ contains a complete rational curve for all integers $h \geq 2$.

(4) Let S be a smooth projective surface and let H be an ample divisor on S. The moduli space of rank-2 H-stable *locally free* sheaves over S with fixed determinant and $c_2 \gg 0$ contains complete curves (Hirschowitz–Hulek when $S = \mathbf{P}^2$ (1990), Huybrechts (1993) and Ballico (1994) in general).

(5) There exist nonisotrivial complete families of Gushel–Mukai varieties of dimension 5, 4, or 3 (Debarre–Kuznetsov; in dimension 6, the moduli space is affine). In dimensions 5 or 4, there even exist nonisotrivial families parametrized by $\mathbf{P}^1$. This is not clear in dimension 3 (no nonisotrivial families of smooth Fano threefolds parametrized by $\mathbf{P}^1$ are known).

Building on Theorem 1.1, we also obtain an extension to families of hyper-Kähler manifolds of $\mathrm{K3}^{[n]}$- or $\mathrm{Kum}_{[n]}$-type for all $n \geq 2$.

Theorem 1.3. *Let $n \geq 2$.*

(1) *Given a complete curve in $\mathscr{F}^0_{2e}$ and relatively prime positive integers a, b such that $a > b\sqrt{(n-1)^2 + 4(n-1)}$, we can construct a complete nonisotrivial family of nth Hilbert powers of K3 surfaces with a polarization of Beauville–Bogomolov–Fujiki square $2ea^2 - 2b^2(n-1)$ and divisibility $\gcd(a, 2(n-1))$.*

(2) *Given a positive integer e and relatively prime positive integers a, b such that $a > b(n+1)$, we can construct a complete nonisotrivial family of generalized Kummer manifolds of dimension $2n$ with a polarization of Beauville–Bogomolov–Fujiki square $2ea^2 - 2b^2(n+1)$ and divisibility $\gcd(a, 2(n+1))$.*

When $n = 2$, the minimal square that one obtains in case (2) corresponds to $e = b = 1$ and $a = 4$: it is $2 \cdot 4^2 - 2 \cdot 3 = 26$. The divisibility is 2.

2. Proof of Theorem 1.1

We start from a complete curve $T' \to \mathscr{A}_2$[4] that we lift to a fine moduli space of abelian surfaces with full level-m structures (with $m \geq 2$) to obtain a family $\mathscr{A} \to T$ (where $T \to T'$ is a finite cover) of abelian surfaces with a relative principal polarization $\mathscr{H}_{\mathscr{A}}$ on $\mathscr{A}$; since m

[3] "Quasi-polarized" means that the line bundle is only nef and big, not necessarily ample. The moduli space $\mathscr{F}^0_{2e}$ is the complement in $\mathscr{F}_{2e}$ of a Heegner divisor.

[4] One could use more generally the moduli space $\mathscr{A}_{2,(1,d)}$ of abelian surfaces with a polarization of type $(1, d)$, for any $d > 0$, but this tends to produce higher degrees $2e$ and does not seem to add to our list.

is even, there are sixteen sections corresponding to the 2-torsion points in the fibers. We blow up the images of these sixteen sections and take the quotient by the lift of the involution given by multiplication by -1 on $\mathscr{A}$ to obtain a nonisotrivial family $\pi\colon \mathscr{K} \to T$ of (smooth) Kummer surfaces. The image $\mathscr{E} \subset \widehat{\mathscr{K}}$ of the exceptional divisor of the blowup splits as the sum of irreducible divisors $\mathscr{E}_1, \dots, \mathscr{E}_{16}$ and its class is divisible by 2 (it is the branch locus of the quotient map). The relative polarization $\mathscr{H}_{\mathscr{A}}$ induces a relative quasi-polarization $\mathscr{H}_{\mathscr{K}}$ on $\mathscr{K} \to T$ of degree 4.

We consider line bundles on $\mathscr{K}$ of the type

$$a\mathscr{H}_{\mathscr{K}} - \sum_{i=1}^{16} a_i \mathscr{E}_i$$

and we want them to be relatively ample, that is, ample on each fiber. The following proposition provides an ampleness criterion valid on all Kummer surfaces (we do not require the polarization on the abelian surface to be principal, although we use the result only in that case).

Proposition 2.1. *On any Kummer surface, any rational class*

$$aH - \sum_{i=1}^{16} a_i E_i,$$

where $a_1 \geq \cdots \geq a_{16} > 0$ *and such that* $a > a_1 + a_2 + a_3 + a_4$*, is ample.*

Proof. We use ideas of Garbagnati–Sarti, who proved similar results when the Picard number is minimal: it is easy to adapt their proof to the general case. ☐

When $H^2 = 4$, this gives relative polarizations on $\mathscr{K}$ in degrees

$$2e = 4a^2 - 2\sum_{i=1}^{16} a_i^2$$

and we need to examine which positive even integers $2e$ can be written in this way with $a_1 \geq \cdots \geq a_{16} > 0$ and $a > a_1 + a_2 + a_3 + a_4$. We also need to make sure that the class is primitive (warning: the classes $H, E_1, \dots, E_{16}$ *do not* generate the Néron–Severi group of a Kummer surface over $\mathbf{Z}$).

For large e, we use results such as: *any integer* $e \geq 163$ *can be written as*

$$e = 2a^2 - \sum_{i=1}^{15} a_i^2 - 1,$$

where $a, a_1, \dots, a_{15}$ *are integers,* $a_1 \geq \cdots \geq a_{15} \geq 1$*, and* $a > a_1 + a_2 + a_3 + a_4$ and check the other cases using a computer program. The original case of degree 28 was obtained using the (integral) ample class

$$3H - \frac{1}{2}\sum_{i=1}^{16} E_i.$$

The family $\mathscr{K} \to T$ of polarized K3 surfaces induces a nonconstant map $T \to \mathscr{F}^0_{2e}$. This maps descends to a nonconstant map $T' \to \mathscr{F}^0_{2e}$ when, for example, $a_1, \dots, a_{16}$ are all equal (and integral).

3. Proof of Theorem 1.3

We construct complete curves in the moduli spaces of polarized hyper-Kähler manifolds using moduli space techniques. For simplicity, we will restrict ourselves to the case of hyper-Kähler manifolds of Kum_n-type (with $n \geq 2$). The case of hyper-Kähler manifolds of $\mathrm{K3}^{[n]}$-type is more involved: it requires the machinery of twisted surfaces and Bridgeland stability conditions.

Let (A, H) be a polarized abelian surface. The generalized Kummer variety $\mathrm{Kum}_n(A)$ is a hyper-Kähler manifold of dimension $2n$ and H induces a nef and big divisor H_n on $\mathrm{Kum}_n(A)$. There is also a canonical class δ which is half of the class of the divisor in $\mathrm{Kum}_n(A)$ parametrizing nonreduced subschemes.

Fundamental work of Bayer–Macrì allows us to identify some ample classes in $\mathrm{Kum}_n(A)$ and implies the following extension of Proposition 2.1 (in the case of Kummer surfaces, the class δ corresponds to $\frac{1}{2}\sum_{i=1}^{16} E_i$).

Proposition 3.1. *On any generalized Kummer variety* $\mathrm{Kum}_n(A)$*, the class* $aH_n - \delta$ *is ample for all real numbers* $a > n+1$.

As in the case of surfaces, the statement is optimal: the class $(n+1)H_n - \delta$ is *not* ample on $\mathrm{Kum}_n(A)$ for certain polarized abelian surfaces (A, H).

This construction extends to families and gives the following result.

Theorem 3.2. *Let* $\mathscr{A} \to T$ *be a smooth family of abelian surfaces with a relatively ample divisor* $\mathscr{H}_{\mathscr{A}}$*. Assume that the class of the divisor on* $\mathrm{Kum}_n(\mathscr{A}/T)$ *parametrizing nonreduced subschemes is divisible by 2 and let* $\delta_{\mathscr{A}}$ *be a half. For all positive integers* a, b *such that* $a > b(n+1)$*, the divisor*

$$a\mathscr{H}_{\mathscr{A},n} - b\delta_{\mathscr{A}}$$

is relatively ample on $\mathrm{Kum}_n(\mathscr{A}/T) \to T$.

In particular, given a complete curve of primitively polarized abelian surfaces of degree $2d$ and positive integers a, b such that $\gcd(a, b) = 1$ and $a > b(n+1)$, we obtain, after a finite base change, a complete nonisotrivial family of generalized Kummer manifolds of dimension $2n$ with a polarization of square $2da^2 - 2b^2(n+1)$ and divisibility $\gcd(a, 2(n+1))$.

Question 3.3. What is the largest dimension $d(e)$ of a compact subvariety of $\mathscr{F}^0_{2e}$? The only result I know in this direction is that $d(e)$ is no larger that the largest dimension of a compact subvariety of $\mathscr{F}_{2e}$, which is 17 by Remark 1.2(2). So, for $e \geq 62$, the integer $d(e)$ is between 1 and 17.