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INTERMEDIATE JACOBIANS OF VERRA AND GUSHEL-MUKAI THREEFOLDS

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ABSTRACT. The Prym map, which takes a étale double cover of smooth projective connected curves to its Prym variety, is not injective in any dimension: Donagi's tetragonal construction (published in 1981) proves that it is 3-to-1 on the locus of double covers of tetragonal curves. Verra then noted in 1992 (and belatedly published in 2004) that the Prym map is also 2-to-1 on the locus of double covers of plane sextic curves. The reason for that is that a general Verra threefold T (a divisor of bidegree (2,2) in $\mathbf{P}^2 \times \mathbf{P}^2$) has two conic bundle structures, with discriminants plane sextic curves, and the intermediate Jacobian $J(T)$ of T is isomorphic to the two associated Prym varieties. We will explain the links between the 9-dimensional intermediate Jacobian $J(T)$, whose theta divisor was studied in detail by Verra, and the 10-dimensional intermediate Jacobians of Gushel–Mukai threefolds. The ultimate aim is to understand the theta divisors of the latter and prove a Torelli-type theorem for Gushel–Mukai threefolds.

1. Prym varieties

1.1. **Definition of Prym varieties.** Let $\pi\colon \widetilde{C} \to C$ be an étale double cover between smooth projective connected curves. Its Prym variety $\mathrm{Prym}(\widetilde{C}/C)$ is the connected component of the origin of the kernel of the norm map $\mathrm{Nm}_\pi\colon \mathrm{Pic}(\widetilde{C}) \to \mathrm{Pic}(C)$. The principal polarization on the Jacobian $J(\widetilde{C})$ induces twice a principal polarization on $\mathrm{Prym}(\widetilde{C}/C)$. If $g \coloneqq g(C)$, one has $g(\widetilde{C}) = 2g-1$ and $\dim(P) = g-1$.

It is also convenient to consider the translate

$$P \coloneqq \{\widetilde{L} \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}_\pi(\widetilde{L}) = \omega_C,\ h^0(\widetilde{C},\widetilde{L}) \text{ even}\}$$

of $\mathrm{Prym}(\widetilde{C}/C)$ in $\mathrm{Pic}^{2g-2}(\widetilde{C})$. The trace on P of the canonical theta divisor in $\mathrm{Pic}^{2g-2}(\widetilde{C})$ is twice the theta divisor defined set-theoretically by

$$\Xi \coloneqq \{[\widetilde{L}] \in P \mid h^0(\widetilde{C},\widetilde{L}) \geq 2\}.$$

There is a Riemann–Kempf type description of the singular locus of Ξ: it is the union of

$$\mathrm{Sing}_{\mathrm{ex}}(\Xi) = \{[\widetilde{L}] \in P \mid \widetilde{L} = (\pi^*M)(\widetilde{E}),\ h^0(C,M) \geq 2,\ \widetilde{E} \geq 0\} \quad \text{(exceptional singularities)},$$
$$\mathrm{Sing}_{\mathrm{st}}(\Xi) = \{[\widetilde{L}] \in P \mid h^0(\widetilde{C},\widetilde{L}) \geq 4\} \quad \text{(stable singularities)}.$$

Exceptional singularities, which are easier to describe, only occur for special curves C, while $\mathrm{Sing}_{\mathrm{st}}(\Xi)$, which is harder to describe, is, at least when $g \geq 7$, always nonempty of codimension everywhere ≤ 6 in P.

Why are Prym varieties important?

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- First, following Beauville, one can extend the above definition to certain (ramified) double covers (called *allowable covers*) of singular, possibly reducible, curves and these generalized Pryms include all Jacobians of curves (they are Pryms of Wirtinger coverings). So in some sense, Pryms are generalizations of Jacobians of curves for which one still has a description of the theta divisor and its singular set.
- Pryms also occur when one consider (smooth) Fano threefolds T with a conic bundle structure $T \to \mathbf{P}^2$. The discriminant curve $C \subset \mathbf{P}^2$, which parametrizes singular fibers, has a connected étale double cover $\widetilde{C} \to C$ (where $\widetilde{C}$ parametrizes components of singular fibers) whose Prym variety is isomorphic, as a principally polarized abelian variety, to the intermediate Jacobian $J(T) := H^{1,2}(T)/H^3(T, \mathbf{Z})$. So again, this gives us access to the geometry of the theta divisor of the intermediate Jacobians of certain Fano threefolds. In fact, I know of only one case (Voisin's analysis of the intermediate Jacobian of quartic double solids) where this geometry has been described when the intermediate Jacobian is *not* a Prym.
- In several instances (see below), Torelli-type theorems have been obtained as a consequence of the description of the singularities of the theta divisors.

1.2. **The Prym map.** The Prym construction can be done in families and induces a morphism

$$P_g \colon \mathscr{R}_g \longrightarrow \mathscr{A}_{g-1}$$

from the (connected) $(3g-3)$-dimensional moduli space $\mathscr{R}_g$ of connected étale double covers of smooth projective connected curves of genus g to the $g(g-1)/2$-dimensional moduli space $\mathscr{A}_{g-1}$ of principally polarized abelian varieties of dimension $g-1$. This map is not proper but can be compactified to $\overline{P_g} \colon \overline{\mathscr{R}_g} \longrightarrow \mathscr{A}_{g-1}$ by including Beauville's allowable covers (the image of $\overline{P_g}$ then contains the Jacobian locus). The dimension are as follows:

g	$\dim(\mathscr{R}_g)$	$\dim(\mathscr{A}_{g-1})$	P_g
2	3	1	dominant
3	6	3	dominant
4	9	6	dominant
5	12	10	dominant
6	15	15	dominant, generically 27-to-1 (Donagi–Smith)
7	18	21	neither injective nor dominant
≥ 8	$3g-3$	21	neither injective nor dominant

There are two "reasons" why the Prym map P_g is never injective.

1.2.1. *The tetragonal construction (Donagi, 1981).* When C is a tetragonal curve (that is, C has a morphism of degree 4 to $\mathbf{P}^1$), Donagi constructed étale double covers of two other tetragonal curves with the same Prym variety. In particular, P_g is never injective.

For $g \geq 6$, let $\mathscr{T}_g \subset \mathscr{R}_g$ be the (connected) $(2g+3)$-dimensional locus of connected double étale covers of tetragonal curves of genus g. I proved that for $g \geq 13$, over the locus $P_g(\mathscr{T}_g)$, the map P_g is generically 3-to-1.

1.2.2. *The Verra construction (Verra, 1992).* For any étale double cover of a general (genus-10) smooth plane sextic $C \subset \mathbf{P}^2$, Verra constructed an étale double cover of another such sextic curve with the same Prym variety. So there is another reason why P_{10} is not injective. More precisely, Verra proved that the restriction of P_{10} to the (19-dimensional) locus $\mathscr{S}_{10}$ of connected étale double covers of smooth plane sextics is generically 2-to-1 onto its image (he does not prove that P_{10} is generically 2-to-1 over $P_{10}(\mathscr{S}_{10})$). We explain Verra's construction in the next section.

2. Verra threefolds and quartic double solids

2.1. **Verra threefolds.** Let $\mathbf{P}$ and $\mathbf{P}'$ be projective planes and let $T \subset \mathbf{P} \times \mathbf{P}'$ be a general smooth hypersurface of bidegree $(2, 2)$ (called a *Verra threefold*). It has two conic bundle structures given by the projections $T \to \mathbf{P}$ and $T \to \mathbf{P}'$, with general (nonisomorphic) sextic discriminant curves $C \subset \mathbf{P}$ and $C' \subset \mathbf{P}'$ and étale double covers $\widetilde{C} \to C$ and $\widetilde{C}' \to C'$ whose Prym varieties are, as we saw above, isomorphic to the 9-dimensional principally polarized intermediate Jacobian $J(T)$.

As explained above, these isomorphisms can be used to describe the singularities of a theta divisor of $J(T)$. Actually, it is worth looking into both Donagi's and Verra's constructions: we will see how the noninjectivity of the Prym map can be "explained" by the structure of this singular set.

For double étale covers of a curve C with a g^1_4 or a g^2_6, that is, of a curve with Clifford index 2, the set of exceptional singularities has a component of codimension 6 (the same codimension as the set of stable singularities): just take $M = g^1_4$ or a $g^2_6(-x)$ in the definition of $\mathrm{Sing}_{\mathrm{ex}}(\Xi)$.

More precisely, when C is tetragonal, $g \geq 13$, the curve C is neither hyperelliptic, nor trigonal, nor bielliptic, and the g^1_4 has no double fibers, I proved that

$$\mathrm{Sing}(\Xi) = S_1 \cup S_2 \cup S_3,$$

where each S_i is irreducible of codimension 6 in P, with class $8[\Xi]^6/6!$, and consists of exceptional singularities for one of the three coverings related by the tetragonal construction. The point here is that there is no a priori way to distinguish between S_1, S_2, and S_3: the exceptional or stable nature of a singular point is not intrinsic to the Prym variety.

When C is a general plane sextic, Verra proved that

$$\mathrm{Sing}(\Xi) = S_1 \cup S_2 \cup S,$$

where each S_i is irreducible of codimension 6 in P, with class $12[\Xi]^6/6!$, and consists of exceptional singularities for one of the two coverings obtained via Verra's construction, and S has codimension 6 in P, class $4[\Xi]^6/6!$, and consists of stable singularities for both coverings appearing in Verra's construction.

Verra further proved that Ξ has quadratic singularities at general points of S and that the intersection in

$$\mathbf{P}(T_{J(T),0}) \xrightarrow{\sim} \mathbf{P}(H^0(T, \mathscr{O}_T(1))^\vee)$$

(where $\mathscr{O}_T(1)$ corresponds to the Segre embedding $\Phi\colon T \subset \mathbf{P} \times \mathbf{P}' \hookrightarrow \mathbf{P}^8$) of the corresponding (rank-6) quadrics is $\Phi(T)$. In particular, one can recover T from $(J(T), \Xi)$ (Torelli theorem), hence the two étale double covers $\widetilde{C} \to C$ and $\widetilde{C}' \to C'$ from their common Prym variety.

2.2. **Quartic double solids.** A quartic double solid is a double cover $W \to \mathbf{P}^3$ branched along a quartic surface $B \subset \mathbf{P}^3$. The threefold W is smooth whenever B is smooth and it is then a Fano threefold of index 2. Quartic double solids were used by Artin–Mumford in 1972 to construct unirational nonrational threefolds and were later studied in great detail by Clemens in 1983. The intermediate Jacobian $J(W)$ has dimension 10 and Voisin proved in 1988 that the singular locus of a theta divisor Θ has a component S of codimension 5 (I later proved that S is unique), that Θ has quadratic singularities at general points of S, and that the intersection in

$$\mathbf{P}(T_{J(W),0}) \xrightarrow{\sim} \mathbf{P}(H^0(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(2))^\vee)$$

of the corresponding (rank-5) quadrics is $\Phi(B)$, where $\Phi \colon \mathbf{P}^3 \hookrightarrow \mathbf{P}^9$ is the second Veronese embedding. In particular, one can recover W from $(J(W), \Theta)$ (Torelli theorem).

Question 2.1 (Prokhorov). *Is a (general) quartic double solid birationally isomorphic to a conic bundle?* If the answer is affirmative, its intermediate Jacobian should be a (generalized) Prym variety. What we know about the singular locus of a theta divisor does not contradict this.

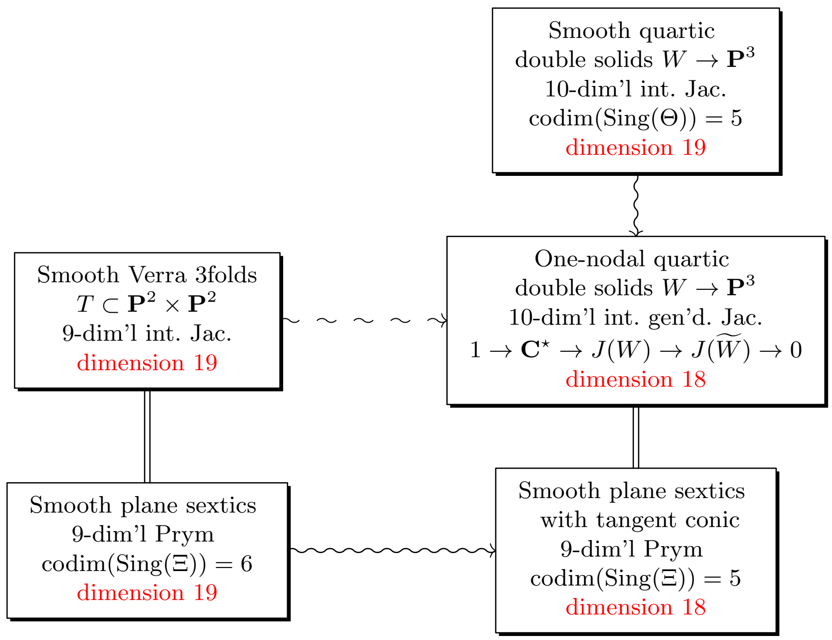
2.3. **Ramification.** Verra showed that the restriction $\mathscr{S}_{10} \to P_{10}(\mathscr{S}_{10})$ of the Prym map, which has generic degree 2, ramifies along the locus of double covers of plane sextic curves with an everywhere tangent conic. The Verra threefold is "replaced" by a nodal quartic double solid: if $W \to \mathbf{P}^3$ is a double cover branched along a quartic surface $B \subset \mathbf{P}^3$ with one node, projection from the node p of W defines a conic bundle $\widetilde{W} := \mathrm{Bl}_p W \to \mathbf{P}^2$ with discriminant curve a smooth sextic curve with an everywhere tangent conic (the image of the tangent cone to W at p). In terms of intermediate Jacobians, there is an extension

$$1 \to \mathbf{C}^\star \to J(W) \to J(\widetilde{W}) \to 0 \qquad (1)$$

with class $e_W \in J(\widetilde{W})/\pm 1$. The intermediate Jacobian $(J(\widetilde{W}), \Xi)$ is a 9-dimensional Prym (because $\widetilde{W}$ is a conic bundle) and the singular locus of Ξ has codimension 5 and a unique component of that dimension; its class is $2\xi^5/5!$ and it consists of stable singularities.

Remark 2.2. As to the generically 3-to-1 Prym map $\mathscr{T}_g \to P_g(\mathscr{T}_g)$, I am not sure what the ramification is. I studied the case where the g^1_4 has two double fibers (but this is a codimension-2 condition) and proved that indeed, in that case, two of the double covers are isomorphic (and the third tetragonal curve is the union of a trigonal curve and a $\mathbf{P}^1$). The singular locus of the theta divisor also becomes 5-codimensional.

2.4. **Summary.** We summarize the information gathered so far in a diagram (the numbers in red are the dimensions of the families).



Remark 2.3. The fact that the singular loci of all the theta divisors of intermediate Jacobians that appear here have codimension > 4 implies that all the (smooth) Fano threefolds involved (Verra threefolds, quartic double solids, desingularizations of one-nodal quartic double solids) are irrational.

Remark 2.4. I am not completely sure about the construction of the degeneration indicated by the broken wiggly arrow from Verra threefolds to one-nodal quartic double solids.

3. Gushel–Mukai threefolds

Gushel–Mukai threefolds are Fano threefolds of index 1 and degree 10. A general such Fano threefold is an ordinary Gushel–Mukai threefold, that is, a smooth complete intersection

$$X \coloneqq \mathsf{Gr}(2, V_5) \cap Q \cap \mathbf{P}^7 \subset \mathbf{P}(\textstyle\bigwedge^2 V_5) = \mathbf{P}^9, \tag{2}$$

where V_5 is a 5-dimensional complex vector space and Q is a quadric. The principally polarized intermediate Jacobian $J(X)$ has dimension 10. Gushel–Mukai threefolds form a 22-dimensional family $\mathscr{M}$ and it is known that the period map

$$\mathscr{M} \longrightarrow \mathscr{A}_{10}$$

has smooth 2-dimensional (nonconnected) fibers.

Almost nothing is known about the geometry of the theta divisor of $J(X)$. We will approach this question by looking at singular (nodal) degenerations of X.

3.1. **Factorial nodal Gushel–Mukai threefolds.** When the quadric Q in (2) is singular at a point o of the smooth fourfold $\mathsf{Gr}(2, V_5) \cap \mathbf{P}^7$, the ordinary Gushel–Mukai threefold X acquires a factorial node at o. The projection $p\colon X \dashrightarrow \mathbf{P}^2$ from the 4-dimensional linear space $\mathbf{T}_{X,o} \subset \mathbf{P}^7$ (classically called "double projection") is then a (birational) conic bundle. There is another such structure $p'\colon X \dashrightarrow \mathbf{P}^2$, more difficult to describe,[1] and one proves that the

[1] It comes from the fact that the image of the projection $X \dashrightarrow \mathbf{P}^6$ from o is a (singular) intersection of three quadrics.

rational map

$$(p,p')\colon X \dashrightarrow \mathbf{P}^2\times\mathbf{P}^2$$

is birational onto a Verra threefold $T\subset\mathbf{P}^2\times\mathbf{P}^2$.

Since factorial nodal Gushel–Mukai threefolds "depend on 21 parameters" and Verra threefolds depend on only 19, two extra parameters are involved: these two parameters correspond to the choice of a point in the union of two "special surfaces" (in the sense of Beauville) in $J(T)$. However, the generalized intermediate Jacobian $J(X)$ only depends on T: it is an extension

$$1\to\mathbf{C}^\star\to J(X)\to J(T)\to 0$$

with fixed (canonical) extension class $e_T\in J(T)/\pm 1$.

Remark 3.1 (Extension classes)*.* Let $\pi\colon\widetilde{C}\to C$ be a connected étale double cover of a general plane sextic $C\subset\mathbf{P}^2$. The canonical extension class $e_T\in\operatorname{Prym}(\widetilde{C}/C)/\pm 1$ only depends on π. When C acquires an everywhere tangent conic, the double cover π comes from a one-nodal quartic double solid $W\to\mathbf{P}^3$ and there is a canonical extension class $e_W\in\operatorname{Prym}(\widetilde{C}/C)/\pm 1$ which only depends on π (Section 2.3). One checks that this class e_W is the limit of the classes e_T.

3.2. **Nonfactorial nodal Gushel–Mukai threefolds.** Let $\rho\colon W\to\mathbf{P}^3$ be a quartic double solid, with associated involution $\tau\in\operatorname{Aut}(W)$ and anticanonical morphism

$$\varphi_W\coloneqq\varphi_{\rho^*\mathscr{O}_{\mathbf{P}^3}(2))}\colon\widetilde{W}\longrightarrow\mathbf{P}^{10}.$$

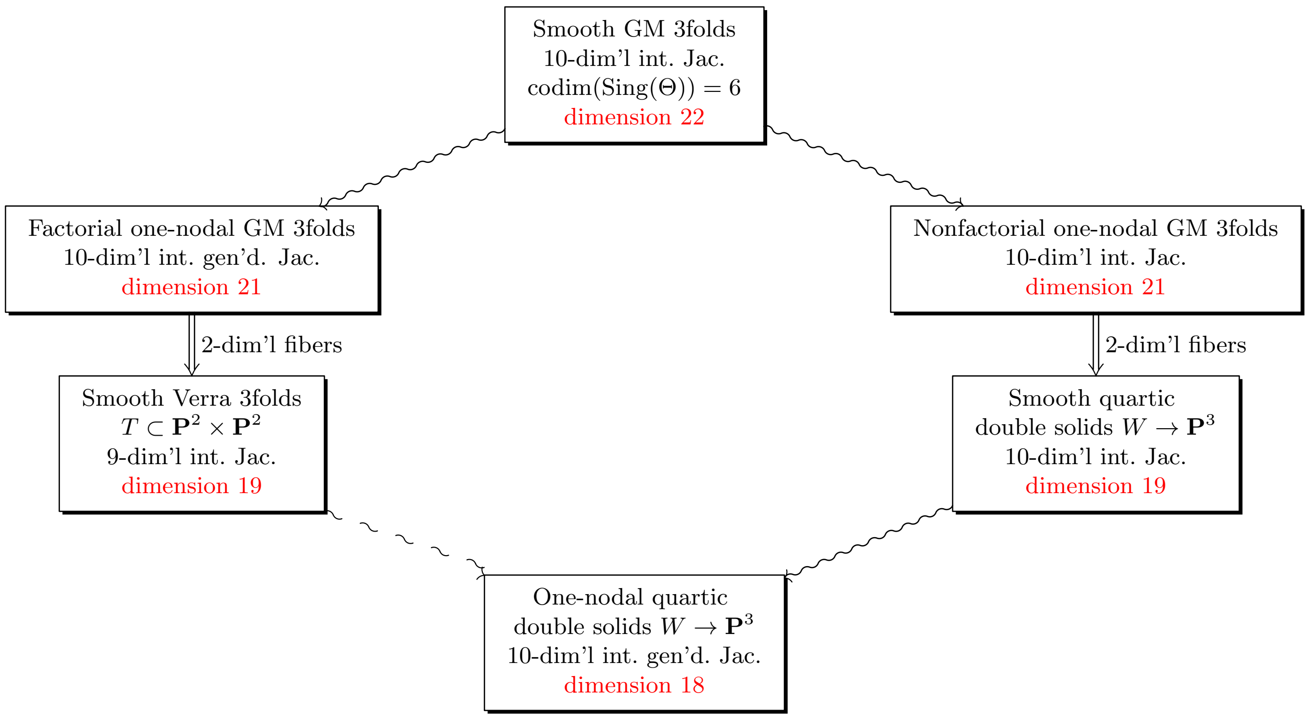
Choose a line $\ell\subset W$ (that is, a curve that maps isomorphically onto a line in $\mathbf{P}^3$; there is a 2-dimensional family of such curves) and let $\varepsilon\colon\operatorname{Bl}_\ell W\to W$ be the blowup of ℓ. The curve $\varphi_W(\ell)$ is a conic and the anticanonical morphism of $\operatorname{Bl}_\ell W$ is the composition

$$\operatorname{Bl}_\ell W\xrightarrow{\varepsilon}W\xrightarrow{\varphi_W}\mathbf{P}^{10}\overset{\pi_{\langle\varphi_W(\ell)\rangle}}{\dashrightarrow}\mathbf{P}^7.$$

This morphism is a small contraction: it only contracts the curve $\tau(L)$. Its image in $\mathbf{P}^7$ is a Gorenstein non-$\mathbf{Q}$-factorial Fano threefold of degree 10 with one node which is a deformation of (smooth) Gushel–Mukai threefolds (Brown–Corti–Zucconi (2002), Przyjalkowski–Cheltsov–Shramov (2005), Prokhorov (2019)).[2]

We can now complete our previous diagram as follows by adding smooth, factorial nodal, and nonfactorial nodal Gushel–Mukai threefolds.

[2]This singular Gushel–Mukai threefold is not an intersection of quadrics. The double projection to $\mathbf{P}^2$ has image a conic and is a rational fibration in cubic surfaces.



Question 3.2 (Prokhorov). *Is a (general) Gushel–Mukai threefold birationally isomorphic to a conic bundle?* If the answer is affirmative, its intermediate Jacobian should be a (generalized) Prym variety and so should the intermediate Jacobian of a general quartic double solid.

4. Andreotti–Mayer loci

What does all this tell us about the singularities of the theta divisor of the intermediate Jacobian of a (general) Gushel–Mukai threefold? To answer this question, it is convenient to introduce the closed Andreotti–Mayer loci

$$\mathscr{N}_p^k := \{(A, \Theta) \in \mathscr{A}_p \mid \dim(\mathrm{Sing}(\Theta)) \geq k\}.$$

Because the set of stable singularities has codimension ≤ 6, we see that $P_g(\mathscr{R}_g) \subset \mathscr{N}_{g-1}^{g-7}$.

Mumford studied the boundary of these loci for $\mathbf{C}^\star$-extensions. He proved that if a $\mathbf{C}^\star$-extension of a principally polarized abelian variety $(A, \Theta) \in \mathscr{A}_{p-1}$ with extension class $a \in A/\pm 1$ is in the closure of $\mathscr{N}_p^k$, then

- either $(A, \Theta) \in \mathscr{N}_{p-1}^k$;
- or $\dim(\mathrm{Sing}(\Theta) \cap \mathrm{Sing}(\Theta_a))) \geq k - 1$;
- or $\dim(\mathrm{Sing}(\Theta \cap \Theta_a)) \geq k$.

The second condition obviously implies $(A, \Theta) \in \mathscr{N}_{p-1}^{k-1}$ and, assuming that a generates A (for example, when A is simple and a is nontorsion), I proved that the third condition also implies $(A, \Theta) \in \mathscr{N}_{p-1}^{k-1}$.

Using the fact that

- $P_{10}(\mathscr{S}_{10})$ is not contained in $\mathscr{N}_9^4$ (Verra);
- a general element of $P_{10}(\mathscr{S}_{10})$ is simple;
- a very general extension class e_T is nontorsion;

we deduce that the intermediate Jacobian of a general Gushel–Mukai threefold is not in $\mathscr{N}_{10}^5$.

Proposition 4.1. *The singular locus of a theta divisor of the 10-dimensional intermediate Jacobian of a general Gushel–Mukai threefold has dimension at most 4. In particular, a general Gushel–Mukai threefold is not rational.*

The nonrationality was known before (by a simpler argument). Mongardi and I recently gave the first explicit example of a nonrational (smooth) Gushel–Mukai threefold. There is a conjecture of Iliev that gives a precise description of the singular locus of a theta divisor of the intermediate Jacobian of a (general) Gushel–Mukai threefold; in particular, its dimension should be exactly 4.

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