GUSHEL–MUKAI VARIETIES WITH MANY SYMMETRIES AND AN EXPLICIT IRRATIONAL GUSHEL–MUKAI THREEFOLD

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Abstract. We construct an explicit smooth Fano complex threefold with Picard number 1, index 1, and degree 10 (also known as a Gushel–Mukai threefold) and prove that it is not rational by showing that its intermediate Jacobian has a faithful $\mathrm{PSL}(2,\mathbf{F}_{11})$-action. Along the way, we construct Gushel–Mukai varieties of various dimensions with rather large (finite) automorphism groups. The starting point of all these constructions is an Eisenbud–Popescu–Walter sextic with a faithful $\mathrm{PSL}(2,\mathbf{F}_{11})$-action discovered by the second author in 2013.

To Fabrizio Catanese, on the occasion of his 70+1st birthday

1. Introduction

The problem of the rationality of unirational smooth Fano complex threefolds has now been solved in most cases but there are still some unanswered questions. For example, Beauville established in [B1, Theorem. 5.6(ii)], by a degeneration argument using the Clemens–Griffiths criterion, that a *general* Fano threefold with Picard number 1, index 1, and degree 10 (also known as a Gushel–Mukai, or GM, threefold) is irrational, but not a single smooth example was known, although it is expected that all of these Fano threefolds are irrational. One of the main results of this article is the construction of a complete 2-dimensional family of such examples (Corollary 5.3), including one such threefold defined (over $\mathbf{Q}$) by explicit equations (Section 3.3, Corollary 5.4).

Our starting point was a remarkable EPW (for Eisenbud–Popescu–Walter) sextic hypersurface $Y_{\mathbb{A}} \subset \mathbf{P}^5$, constructed in [Mo3], with a faithful action by the simple group $\mathbb{G} := \mathrm{PSL}(2,\mathbf{F}_{11})$ of order 660 (Section 3.2). We prove that the automorphism group of $Y_{\mathbb{A}}$ is exactly $\mathbb{G}$ (Proposition 4.1) and that it is the only quasi-smooth EPW sextic with an automorphism of order 11 (Theorem 4.2).

From this sextic, one can construct GM varieties of various dimensions with exotic properties. Using [DK2], we obtain for example families of GM varieties of dimensions 4 or 6 with middle-degree Hodge groups of maximal rank 22 (Section 4.4).

Another application is the construction of GM varieties with large (finite) automorphism groups. The foremost example is a GM fivefold $X^5_{\mathbb{A}}$ with automorphism group $\mathbb{G}$ (Corollary 4.4(2)) but we also construct GM varieties of various dimensions with automorphism groups $\mathbf{Z}/11\mathbf{Z}$, D_{12}, $\mathbf{Z}/6\mathbf{Z}$, $\mathbf{Z}/3\mathbf{Z}$, D_{10}, $\mathbf{Z}/5\mathbf{Z}$, $\mathfrak{A}_4$, $(\mathbf{Z}/2\mathbf{Z})^2$, or $\mathbf{Z}/2\mathbf{Z}$ (Table 2).

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By [DK5], the intermediate Jacobians of the GM varieties of dimension 3 or 5 obtained from the sextic $Y_{\mathbb{A}}$ are all isomorphic to a fixed principally polarized abelian variety $(\mathbb{J}, \theta)$ of dimension 10. This applies in particular to $X^5_{\mathbb{A}}$, and the $\mathbb{G}$-action on $X^5_{\mathbb{A}}$ induces a faithful $\mathbb{G}$-action on $(\mathbb{J}, \theta)$. We use this fact to prove that the GM threefolds that we construct from $Y_{\mathbb{A}}$ are not rational: by the Clemens–Griffiths criterion ([CG, Corollary 3.26]), it suffices to prove that their (common) intermediate Jacobian $(\mathbb{J}, \theta)$ is not a product of Jacobians of curves. For this, we follow [B2, B3] and use the fact that $(\mathbb{J}, \theta)$ has "too many automorphisms" (because of the $\mathbb{G}$-action). Note that the GM threefolds themselves may have no nontrivial automorphisms. This is how we produce a complete 2-dimensional family of irrational GM threefolds, all mutually birationally isomorphic.

The 10-dimensional principally polarized abelian variety $(\mathbb{J}, \theta)$ seems an interesting object of study. The 10-dimensional complex representation attached to the $\mathbb{G}$-action is irreducible and defined over $\mathbf{Q}$. This implies that $(\mathbb{J}, \theta)$ is indecomposable and isogeneous to the product of 10 copies of an elliptic curve (Propositions 5.1 and 5.8). We conjecture, but were unable to prove, that $(\mathbb{J}, \theta)$ is isomorphic to an explicit 10-dimensional principally polarized abelian variety that we construct in Proposition C.5.

The situation is reminiscent of that of the Klein cubic threefold $W \subset \mathbf{P}^4$: Klein proved in [K] that W has a faithful linear $\mathbb{G}$-action; one hundred years later, Adler proved in [A1] that the automorphism group of W is exactly $\mathbb{G}$ and Roulleau showed in [R] that W is the only smooth cubic threefold with an automorphism of order 11. The intermediate Jacobian of W is a principally polarized abelian variety of dimension 5 isomorphic to the product of 5 copies of an elliptic curve with complex multiplication and Adler proved in [A2] that it is the only abelian variety of dimension 5 with a faithful action of $\mathbb{G}$. This is the reason why we call our sextic $Y_{\mathbb{A}}$ the Klein EPW sextic. We also refer to [CKS] for the construction of a one-dimensional family of threefolds with $\mathfrak{S}_6$-actions whose intermediate Jacobians are isogeneous to the product of 5 copies of varying elliptic curves ([CKS, Remark 4.5]).

Our proofs heavily use the construction by O'Grady in [O2] of canonical double covers of quasi-smooth EPW sextics called double EPW sextics (see also [DK4]). They are smooth hyperkähler fourfolds whose automorphisms may, thanks to Verbitsky's Torelli Theorem, be determined using lattice theory. We also use the close relationship between EPW sextics and GM varieties developed in [IM, DK1, DK2, DK3, DK5] and surveyed in [D].

The article is organized as follows. In Section 2, we recall basic facts about EPW sextics and GM varieties. In Section 3, we describe explicitly the Klein Lagrangian $\mathbb{A}$ and the Klein EPW sextic $Y_{\mathbb{A}}$, and we prove that the EPW sextic $Y_{\mathbb{A}}$ is quasi-smooth. In Section 4, we prove that the automorphism group of $Y_{\mathbb{A}}$ is $\mathbb{G}$ and that $Y_{\mathbb{A}}$ is the only quasi-smooth EPW sextic with an automorphism of order 11. We also discuss the possible automorphism groups and some Hodge groups of the various GM varieties that can be constructed from the Lagrangian $\mathbb{A}$. In Section 5, we introduce the important surface $\widetilde{Y}^{\geq 2}_{A}$ (a double étale cover of the singular locus of $Y_{\mathbb{A}}$) and its Albanese variety $(\mathbb{J}, \theta)$. We prove our irrationality results for GM threefolds and discuss the structure of the 10-dimensional principally polarized abelian variety $(\mathbb{J}, \theta)$.

The rest of the article consists of appendices. In the long Appendix A, we gather old and new general results on automorphisms of double EPW sextics and of double EPW surfaces. Appendix B recalls a few classical facts about representations of the group $\mathbb{G}$. Appendix C discusses decomposition results for abelian varieties with automorphisms.

Notation. Let m be a positive integer; throughout this article, V_m denotes a complex vector space of dimension m and we set $\zeta_m := e^{\frac{2\pi i}{m}}$. As we did above, we denote by $\mathbb{G}$ the simple group $\mathrm{PSL}(2, \mathbf{F}_{11})$ of order 660.

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2. Eisenbud–Popescu–Walter sextics and Gushel–Mukai varieties

We recall in this section a few basic facts about Eisenbud–Popescu–Walter (or EPW for short) sextics and Gushel–Mukai (or GM for short) varieties.

2.1. **EPW sextics and their automorphisms.** Let V_6 be a 6-dimensional complex vector space. We endow $\bigwedge^3 V_6$ with the $\bigwedge^6 V_6$-valued symplectic form defined by wedge product. Given a Lagrangian subspace $A \subset \bigwedge^3 V_6$ and a nonnegative integer ℓ, one defines (see [O1, Section 2] or [DK1, Appendix B]) in $\mathbf{P}(V_6)$ the closed subschemes

$$Y_A^{\geq \ell} := \bigl\{[x] \in \mathbf{P}(V_6) \mid \dim\bigl(A \cap (x \wedge \textstyle\bigwedge^2 V_6)\bigr) \geq \ell\bigr\}$$

and the locally closed subschemes

$$Y_A^{\ell} := \bigl\{[x] \in \mathbf{P}(V_6) \mid \dim\bigl(A \cap (x \wedge \textstyle\bigwedge^2 V_6)\bigr) = \ell\bigr\} = Y_A^{\geq \ell} \smallsetminus Y_A^{\geq \ell+1}.$$

We henceforth assume that A contains no decomposable vectors (that is, no nonzero products $x \wedge y \wedge z$). The scheme $Y_A := Y_A^{\geq 1}$ is then an integral sextic hypersurface (called an *EPW sextic*) whose singular locus is the integral surface $Y_A^{\geq 2}$; the singular locus of that surface is the finite set $Y_A^{\geq 3}$ (see [DK1, Theorem B.2]) which is empty for A general.

One has moreover ([DK1, Proposition B.9])

$$\operatorname{Aut}(Y_A) = \{g \in \operatorname{PGL}(V_6) \mid (\textstyle\bigwedge^3 g)(A) = A\} \tag{1}$$

and this group is finite (see also [Be, Proposition 7.6]).

2.2. **GM varieties and their automorphisms.** A (smooth ordinary) GM variety of dimension $n \in \{3, 4, 5\}$ is the smooth complete intersection, in $\mathbf{P}(\bigwedge^2 V_5)$, of the Grassmannian $\mathsf{Gr}(2, V_5)$ in its Plücker embedding, a linear space $\mathbf{P}^{n+4}$, and a quadric. It is a Fano variety with Picard number 1, index $n - 2$, and degree 10.

There is a bijection between the set of isomorphism classes of (smooth ordinary) GM varieties X of dimension n and the set of isomorphism classes of triples (V_6, V_5, A), where $A \subset \bigwedge^3 V_6$ is a Lagrangian subspace with no decomposable vectors and $V_5 \subset V_6$ is a hyperplane such that

$$\dim(A \cap \textstyle\bigwedge^3 V_5) = 5 - n \tag{2}$$

(this bijection was first described in the proof of [IM, Proposition 2.1] when $n = 5$; for the general case, see [DK1, Theorem 3.10 and Proposition 3.13(c)] or [D, (2)]).

By [DK1, Lemma 2.29 and Corollary 3.11], we have

$$\operatorname{Aut}(X) \simeq \{g \in \operatorname{Aut}(Y_A) \mid g(V_5) = V_5\}. \tag{3}$$

3. The Klein Lagrangian

The following construction of an EPW sextic with a faithful $\mathbb{G}$-action first appeared in [Mo3, Example 4.5.2].

3.1. **The Klein Lagrangian $\mathbb{A}$ and the GM fivefold $X^5_{\mathbb{A}}$.** Let $\xi\colon \mathbb{G} \to \mathrm{GL}(V_\xi)$ be the irreducible representation of $\mathbb{G}$ of dimension 5 described in Appendix B. From the existence of a unique (up to multiplication by a nonzero scalar) $\mathbb{G}$-equivariant symmetric isomorphism

$$w\colon \bigwedge^2 V_\xi \xrightarrow{\sim} \bigwedge^2 V_\xi^\vee \tag{4}$$

as in (31), we infer that there is a unique $\mathbb{G}$-invariant quadric

$$\mathsf{Q} \subset \mathbf{P}(\bigwedge^2 V_\xi) \tag{5}$$

and that it is smooth. Since its equation does not lie in the image of the $\mathbb{G}$-equivariant morphism

$$V_\xi \simeq \bigwedge^4 V_\xi^\vee \hookrightarrow \mathrm{Sym}^2(\bigwedge^2 V_\xi^\vee),$$

which is the space of Plücker quadrics, the quadric Q does not contain the Grassmannian $\mathsf{Gr}(2, V_\xi)$. Therefore, it defines a GM fivefold

$$X^5_{\mathbb{A}} := \mathsf{Q} \cap \mathsf{Gr}(2, V_\xi) \tag{6}$$

with a faithful $\mathbb{G}$-action (we will show below that $X^5_{\mathbb{A}}$ is smooth).

The group $\mathbb{G}$ being simple nonabelian, the representation $\bigwedge^5 \xi$ is trivial. The isomorphism w from (4) therefore induces an isomorphism of representations

$$v\colon \bigwedge^2 V_\xi \xrightarrow{\sim} \bigwedge^2 V_\xi^\vee \otimes \bigwedge^5 V_\xi \xrightarrow{\sim} \bigwedge^3 V_\xi. \tag{7}$$

Since w is symmetric, v satisfies $v(x) \wedge y = x \wedge v(y)$ for all $x, y \in \bigwedge^2 V_\xi$.

Let $\chi_0\colon \mathbb{G} \to V_{\chi_0}$ be the trivial representation and consider the $\mathbb{G}$-representation

$$V_6 := V_{\chi_0} \oplus V_\xi.$$

The decomposition of $\bigwedge^3 V_6$ into irreducible $\mathbb{G}$-representations is

$$\bigwedge^3 V_6 = (V_{\chi_0} \wedge \bigwedge^2 V_\xi) \oplus \bigwedge^3 V_\xi \tag{8}$$

and, if e_0 is a generator of V_{χ_0}, the Lagrangian subspace $\mathbb{A} \subset \bigwedge^3 V_6$ associated with the GM fivefold $X^5_{\mathbb{A}}$ according to the general procedure outlined in Section 2.2 is the graph

$$\mathbb{A} := \{e_0 \wedge x + v(x) \mid x \in \bigwedge^2 V_\xi\}$$

of v. Conversely, $X^5_{\mathbb{A}}$ is the GM fivefold associated with the Lagrangian $\mathbb{A}$ and the hyperplane $V_\xi \subset V_6$ (referring to (2), note that $\mathbb{A} \cap \bigwedge^3 V_\xi = \{0\}$).

We will use the following notation. Let c and a be the elements of $\mathbb{G}$ defined in Appendix B and let $(e_1, \dots, e_5)$ be a basis of V_ξ in which $\xi(c)$ and $\xi(a)$ have matrices as in (30). Let $(e_1^\vee, \dots, e_5^\vee)$ be the dual basis of $V_\xi^\vee$. We also set $e_{i_1 \cdots i_r} = e_{i_1} \wedge \cdots \wedge e_{i_r} \in \bigwedge^r V_6$.

Proposition 3.1. *The GM fivefold $X^5_{\mathbb{A}}$ is smooth and the Lagrangian subspace $\mathbb{A}$ contains no decomposable vectors.*

Proof. The basis $(e_{ij})_{1 \le i < j \le 5}$ of $\bigwedge^2 V_\xi$ consists of eigenvectors of $\bigwedge^2 \xi(c)$, with eigenvalues all the primitive 11^{th} roots of 1, and similarly for the dual basis $(e_{ij}^\vee)_{1 \le i < j \le 5}$ of $\bigwedge^2 V_\xi^\vee$. Looking at the corresponding eigenvalues, we see that we may normalize the isomorphism w in (4) so that it satisfies $w(e_{12}) = -e_{13}^\vee$ (both are eigenvectors of $\bigwedge^2 \xi(c)$ with eigenvalue ζ_{11}^5). Applying $\bigwedge^2 \xi(a)$, we find

$$w(e_{12}) = -e_{13}^\vee,\ w(e_{23}) = -e_{24}^\vee,\ w(e_{34}) = -e_{35}^\vee,\ w(e_{45}) = e_{14}^\vee,\ w(e_{15}) = -e_{25}^\vee.$$

Since w is symmetric, we also have

$$w(e_{13}) = -e_{12}^\vee,\ w(e_{24}) = -e_{23}^\vee,\ w(e_{35}) = -e_{34}^\vee,\ w(e_{14}) = e_{45}^\vee,\ w(e_{25}) = -e_{15}^\vee.$$

The quadric Q from (5) is therefore defined by

$$x_{12}x_{13}+x_{23}x_{24}+x_{34}x_{35}-x_{45}x_{14}+x_{15}x_{25}=0. \tag{9}$$

A computer check with [M2] now ensures that the GM fivefold $X^5_{\mathbb{A}}$ defined by (6) is smooth. It follows from [DK1, Theorem 3.16] that $\mathbb{A}$ contains no decomposable vectors. ☐

The group $\mathbb{G}$ acts faithfully on the GM fivefold $X^5_{\mathbb{A}}$. Using the isomorphism (3), we see that it also acts faithfully on the EPW sextic $Y_{\mathbb{A}}$ by linear automorphisms that fix the hyperplane V_ξ. More precisely, the representation $\chi_0\oplus\xi\colon \mathbb{G}\hookrightarrow \mathrm{GL}(V_6)$ induces an embedding $\mathbb{G}\hookrightarrow \mathrm{Aut}(Y_{\mathbb{A}})\subset \mathrm{PGL}(V_6)$. We will prove in Proposition 4.1 that the embedding $\mathbb{G}\hookrightarrow \mathrm{Aut}(Y_{\mathbb{A}})$ is in fact an isomorphism.

3.2. **Explicit equations.** As we saw in the proof of Proposition 3.1, and with the notation of that proof, the isomorphism $v\colon \bigwedge^2 V_\xi \xrightarrow{\sim} \bigwedge^3 V_\xi$ from (7) may be defined by

$$\begin{aligned} v(e_{12})=e_{245},\ v(e_{23})=e_{135},\ v(e_{34})=e_{124},\ v(e_{45})=e_{235},\ v(e_{15})=-e_{134},\\ v(e_{13})=-e_{345},\ v(e_{24})=-e_{145},\ v(e_{35})=-e_{125},\ v(e_{14})=e_{123},\ v(e_{25})=e_{234}.\end{aligned} \tag{10}$$

This gives

$$\begin{aligned}\mathbb{A}=\langle e_{012}+e_{245},e_{013}-e_{345},e_{014}+e_{123},e_{015}-e_{134},e_{023}+e_{135},\\ e_{024}-e_{145},e_{025}+e_{234},e_{034}+e_{124},e_{035}-e_{125},e_{045}+e_{235}\rangle.\end{aligned} \tag{11}$$

One can readily see from this that the isomorphism $V_6\xrightarrow{\sim} V_6^\vee$ that sends e_0 to $-e_0^\vee$ and e_j to $e_j^\vee$ for $j\in\{1,\dots,5\}$ maps $\mathbb{A}$ onto its orthogonal $\mathbb{A}^\perp$, a Lagrangian subspace of $\bigwedge^3 V_6^\vee$; we say that $\mathbb{A}$ is *self-dual.* Also, if one starts from the dual representation $\xi^\vee$, one obtains the same Lagrangian $\mathbb{A}$.

Proposition 3.2. *The EPW sextic $Y_{\mathbb{A}}$ is defined by the equation*

$$\begin{aligned} &x_0^6+2x_0^3(x_1x_3^2+x_2x_4^2+x_3x_5^2+x_4x_1^2+x_5x_2^2)-4x_0(x_1^3x_2^2+x_2^3x_3^2+x_3^3x_4^2+x_4^3x_5^2+x_5^3x_1^2)\\ &+4x_0(x_1x_3x_4^3+x_2x_4x_5^3+x_3x_5x_1^3+x_4x_1x_2^3+x_5x_2x_3^3)-12x_0x_1x_2x_3x_4x_5\\ &+x_1^2x_3^4+x_2^2x_4^4+x_3^2x_5^4+x_4^2x_1^4+x_5^2x_2^4-4(x_1x_4x_5^4+x_2x_5x_1^4+x_3x_1x_2^4+x_4x_2x_3^4+x_5x_3x_4^4)\\ &-2(x_1x_3^3x_5^2+x_2x_4^3x_1^2+x_3x_5^3x_2^2+x_4x_1^3x_3^2+x_5x_2^3x_4^2)\\ &+6(x_1x_2x_3^2x_4^2+x_2x_3x_4^2x_5^2+x_3x_4x_5^2x_1^2+x_4x_5x_1^2x_2^2+x_5x_1x_2^2x_3^2)=0\end{aligned} \tag{12}$$

in $\mathbf{P}(V_6)$. The scheme $Y^{\ge 2}_{\mathbb{A}}$ is a smooth irreducible surface, so that the scheme $Y^{\ge 3}_{\mathbb{A}}$ is empty.

Proof. The scheme $Y_{\mathbb{A}}$ is the locus in $\mathbf{P}(V_6)$ where the map

$$x\wedge \textstyle\bigwedge^2 V_6 \longrightarrow \bigwedge^3 V_6/\mathbb{A}$$

drops rank. In the decomposition (8), the second summand is transverse to $\mathbb{A}$ and we can identify $\bigwedge^3 V_6/A$ with $\bigwedge^3 V_\xi$. Moreover, in the affine open subset U_0 of $\mathbf{P}(V_6)$ defined by $x_0\neq 0$, one has $x\wedge\bigwedge^2 V_6 = x\wedge \bigwedge^2 V_\xi$. In U_0, the scheme $Y_{\mathbb{A}}$ is therefore the locus where the map

$$x\wedge \textstyle\bigwedge^2 V_\xi \longrightarrow \bigwedge^3 V_\xi \xrightarrow{\ v^{-1}\ } \bigwedge^2 V_\xi$$

drops rank. Concretely, if $x=e_0+x_1e_1+\cdots+x_5e_5$, we see, using (11) and (10), that it maps e_{12} to

$$\begin{aligned} x\wedge e_{12}=e_{012}+x_3e_{123}+x_4e_{124}+x_5e_{125} &\xmapsto{\text{mod }\mathbb{A}} -e_{245}+x_3e_{123}+x_4e_{124}+x_5e_{125}\\ &\xmapsto{\ v^{-1}\ } -e_{12}+x_3e_{14}+x_4e_{34}-x_5e_{35}.\end{aligned}$$

All in all, using the basis $(e_{12}, e_{13}, e_{14}, e_{15}, e_{23}, e_{24}, e_{25}, e_{34}, e_{35}, e_{45})$ of $\bigwedge^2 V_\xi$, one sees that $Y_{\mathbb{A}} \cap U_0$ is defined as the determinant of the 10×10 matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & x_5 & -x_4 & 0 & 0 & x_2 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -x_5 & x_4 & -x_3 \\ x_3 & -x_2 & -1 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_4 & x_3 & -1 & 0 & 0 & 0 & -x_1 & 0 & 0 \\ 0 & x_5 & 0 & -x_3 & -1 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & -x_5 & x_4 & 0 & -1 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & x_4 & -x_3 & -1 & x_2 & 0 & 0 \\ x_4 & 0 & -x_2 & 0 & 0 & x_1 & 0 & -1 & 0 & 0 \\ -x_5 & 0 & 0 & x_2 & 0 & 0 & -x_1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & x_5 & 0 & -x_3 & 0 & x_2 & -1 \end{pmatrix}.$$

We obtain the equation (12) by homogenizing this determinant, computed with Macaulay2 ([M2]). We then check with Macaulay2 that $\mathrm{Sing}(Y_{\mathbb{A}})$ is a smooth surface (this reproves that $\mathbb{A}$ contains no decomposable vectors and proves in addition that $Y_{\mathbb{A}}^{\geq 3}$ is empty). ☐

3.3. The GM threefold $X^3_{\mathbb{A}}$.

We keep the notation above. By Proposition 3.2, $Y_{\mathbb{A}}^{\geq 3}$ is empty and, since $\mathbb{A}$ is self-dual, so is $Y_{\mathbb{A}^\perp}^{\geq 3}$. For all hyperplanes $V_5 \subset V_6$, we thus have

$$\dim(\mathbb{A} \cap \textstyle\bigwedge^3 V_5) \leq 2. \tag{13}$$

Consider the hyperplane $V_5 \subset V_6$ spanned by $e_0, \dots, e_4$. From the description (11), one sees that there is an inclusion

$$\langle e_{014} + e_{123}, e_{034} + e_{124} \rangle \subset \mathbb{A} \cap \textstyle\bigwedge^3 V_5$$

of vector spaces which, because of the inequality (13), is an equality. The associated GM variety is therefore smooth of dimension 3 (see Section 2.2); we denote it by $X^3_{\mathbb{A}}$. Using the automorphism $\xi(a)$ of V_6 that permutes the vectors $e_1, \dots, e_5$, we see that we get isomorphic GM threefolds if we start from hyperplanes spanned by e_0 and any four vectors among $e_1, \dots, e_5$.

Going through the procedure mentioned in Section 2.2, A. Kuznetsov found that $X^3_{\mathbb{A}}$ is the intersection, in $\mathbf{P}(\bigwedge^2 V_5)$, of the Grassmannian $\mathsf{Gr}(2, V_5)$, the linear space $\mathbf{P}^7$ with equations

$$x_{03} + x_{12} = x_{04} - x_{23} = 0,$$

and the quadric with equation

$$x_{01}x_{02} - x_{13}x_{14} - x_{24}x_{34} = 0.$$

4. EPW sextics and GM varieties with many automorphisms

As in Section 2.1, let V_6 be a 6-dimensional complex vector space and let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors. It defines an integral EPW sextic $Y_A \subset \mathbf{P}(V_6)$. As explained in more detail in Appendix A.1, there is a canonical double covering $\pi_A \colon \widetilde{Y}_A \to Y_A$ and, when $Y_A^{\geq 3} = \varnothing$, the fourfold $\widetilde{Y}_A$ is a smooth hyperkähler variety of $\mathrm{K3}^{[2]}$-type.

4.1. Automorphisms of the EPW sextic $Y_{\mathbb{A}}$.

We constructed at the end of Section 3.1 an injection $\mathbb{G} \hookrightarrow \mathrm{Aut}(Y_{\mathbb{A}})$. By Proposition 3.2, the double EPW sextic $\widetilde{Y}_{\mathbb{A}}$ is smooth and, by Proposition A.2, the group $\mathrm{Aut}(Y_{\mathbb{A}})$ is isomorphic to the group $\mathrm{Aut}^s_H(\widetilde{Y}_{\mathbb{A}})$ of symplectic isomorphisms of $\widetilde{Y}_{\mathbb{A}}$ that preserve the polarization H.

Proposition 4.1. *The automorphism group of the Klein EPW sextic $Y_{\mathbb{A}}$ is isomorphic to $\mathbb{G}$.*

Proof. It is enough to prove that $\mathrm{Aut}^s_H(\widetilde{Y}_{\mathbb{A}})$ is isomorphic to $\mathbb{G}$. Let $g \in \mathrm{Aut}^s_H(\widetilde{Y}_{\mathbb{A}})$. It acts on the orthogonal of H in $\mathrm{Pic}(\widetilde{Y}_{\mathbb{A}})$ which, by Corollary A.4, is the rank-20 lattice S discussed in Section A.3 and the action is faithful. Let us prove that g acts trivially on the discriminant group $\mathrm{Disc}(\mathsf{S})$.

By Corollary A.4, the lattice $H^\perp \simeq (-2)^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \subset H^2(\widetilde{Y}_{\mathbb{A}}, \mathbf{Z})$ (see (24)) primitively contains the lattices $\mathrm{Tr}(\widetilde{Y}_{\mathbb{A}}) \simeq (22)^{\oplus 2}$ and S and it is a finite extension of their direct sum. This extension is obtained by adding to $\mathrm{Tr}(\widetilde{Y}_{\mathbb{A}}) \oplus \mathsf{S}$ two elements $\frac{a_1+b_1}{11}$ and $\frac{a_2+b_2}{11}$, where a_1 and a_2 are orthogonal generators of $\mathrm{Tr}(\widetilde{Y}_{\mathbb{A}})$ of square 22, and b_1 and b_2 are classes in S of divisibility 11. Since g preserves $H^\perp$ and $\mathrm{Tr}(\widetilde{Y}_{\mathbb{A}})$, it follows readily that $g(b_i) = b_i + 11c_i$ for some $c_i \in \mathsf{S}$, which implies that g acts trivially on $\mathrm{Disc}(\mathsf{S})$, as claimed.

The proposition follows since, by [HM, Table 1, line 120], the group of isometries of S that act trivially on $\mathrm{Disc}(\mathsf{S})$ coincides with $\mathbb{G}$. $\square$

4.2. **GM varieties with many symmetries.** Proposition 4.1 can be used to determine the automorphism groups of the GM varieties constructed from the Lagrangian $\mathbb{A}$, and in particular the varieties $X^5_{\mathbb{A}}$ and $X^3_{\mathbb{A}}$ defined in Sections 3.1 and 3.3. By (3), all we have to do is determine the stabilizers of hyperplanes in V_6 under the $\mathbb{G}$-action. Since this action is conjugate to its dual, we might as well determine the stabilizers of lines in $V_6 = \mathbf{C}e_0 \oplus V_\xi$. We proceed in three steps:

- determine the various fixed-point sets of all subgroups of $\mathbb{G}$, listed up to conjugacy in [BCV, Figure 1];
- compute the stabilizers of these fixed points;
- find in which stratum $Y^0_{\mathbb{A}}$, $Y^1_{\mathbb{A}}$, or $Y^2_{\mathbb{A}}$ they lie (recall that $Y^3_{\mathbb{A}}$ is empty; in particular, the surface $Y^2_{\mathbb{A}}$ is projective).

A first useful remark is the following: *if $g \in \mathbb{G}$ is a nontrivial element of odd order, the fixed-point set of g in $Y_{\mathbb{A}}$ is finite.* Indeed, we will see below by a case-by-case analysis that the fixed-point set $\mathrm{Fix}(g)$ of g in $\mathbf{P}(V_6)$ is a union of lines and isolated points. Assume that a line $\Delta \subset \mathrm{Fix}(g)$ is contained in $Y_{\mathbb{A}}$. By Proposition A.2, g lifts to a symplectic automorphism $\tilde{g}$ of $\widetilde{Y}_{\mathbb{A}}$ which commutes with its covering involution ι. For any x in the curve $\pi_{\mathbb{A}}^{-1}(\Delta) \subset \widetilde{Y}_{\mathbb{A}}$, one has either $\tilde{g}(x) = x$ or $\tilde{g}(x) = \iota(x)$, hence $\tilde{g}^2(x) = x$. The curve $\pi_{\mathbb{A}}^{-1}(\Delta) \subset \widetilde{Y}_A$ is therefore contained in the fixed-point set of the nontrivial symplectic automorphism $\tilde{g}^2$. But this fixed-point set is, on the one hand, a disjoint union of surfaces and isolated points and, on the other hand, contained in $\pi_{\mathbb{A}}^{-1}(\mathrm{Fix}(g^2))$, whose dimension is at most 1 (because g^2 is again nontrivial of odd order), so we reach a contradiction. Moreover, 1 is not an eigenvalue for the action of g on the tangent space at a fixed point, hence any line in $\mathrm{Fix}(g)$ meets $Y^1_{\mathbb{A}}$ and $Y^2_{\mathbb{A}}$ transversely.

Furthermore, since g itself can be written as a square, we see that the fixed-point set of its symplectic lift $\tilde{g}$ (which has the same order) is the inverse image in $\widetilde{Y}_{\mathbb{A}}$ of $\mathrm{Fix}(g)$.

Our second tool will be the Lefschetz topological fixed point theorem for an automorphism g *with finite fixed-point set* on the regular projective surface $Y^2_{\mathbb{A}}$. This theorem reads

$$\#(\mathrm{Fix}(g) \cap Y^2_{\mathbb{A}}) = \sum_{i=0}^{4} (-1)^i \,\mathrm{Tr}(g^*|_{H^i(Y^2_{\mathbb{A}},\mathbf{Q})}) = 2 + \mathrm{Tr}(g^*|_{H^2(Y^2_{\mathbb{A}},\mathbf{Q})}).$$

The group $\mathbb{G}$ acts on $\mathbb{A}$ (via the representation $\bigwedge^2 V_\xi$) and $Y^2_{\mathbb{A}}$ and, by Proposition A.7, the isomorphism $H^2(Y^2_{\mathbb{A}}, \mathbf{C}) \simeq \bigwedge^2(\mathbb{A} \oplus \bar{\mathbb{A}})$ from (28) is equivariant for these actions. Using the fact that the representation $\bigwedge^2 V_\xi$ is self-dual and the formula

$$\chi_{\bigwedge^2(\bigwedge^2 V_\xi \oplus \bigwedge^2 V_\xi)}(g) = 2\chi_{\bigwedge^2(\bigwedge^2 V_\xi)}(g) + \chi_{\bigwedge^2 V_\xi \otimes \bigwedge^2 V_\xi}(g) = 2\chi_{\bigwedge^2 V_\xi}(g)^2 - \chi_{\bigwedge^2 V_\xi}(g^2),$$

one can then compute the numbers of fixed points of g in $Y^2_{\mathbb{A}}$ given in Table 1.

The Lefschetz theorem was also used to the same effect in [Mo3, Section 6.2] on hyperkähler varieties of K3$^{[2]}$-type. It gives, for symplectic automorphisms of $\widetilde{Y}_{\mathbb{A}}$ of odd prime order, the number (when finite) of fixed points on $\widetilde{Y}_{\mathbb{A}}$. By the remark made above, this is the number of fixed points on $Y^2_{\mathbb{A}}$ (which we get from Table 1) plus twice the number of fixed points on $Y^1_{\mathbb{A}}$. So we get from [Mo3, Section 6.2] the following numbers (except for the information between parentheses (when g has order 2 or 6), which will be a consequence of the discussion below—where it will not be used).

order of g	11	5	6	3	2
$\#(\mathrm{Fix}(g)\cap Y^2_{\mathbb{A}})$	5	2	3	3	(dim 1)
$\#(\mathrm{Fix}(g)\cap Y_{\mathbb{A}})$	5	8	(7)	15	(dim 2)

Table 1. Number (when finite) of fixed points on the surface $Y^2_{\mathbb{A}}$ and the fourfold $Y_{\mathbb{A}}$

We will see in the discussion below that these sets are in fact always finite, except when g has order 2. We can now go through the list of all subgroups of $\mathbb{G}$ from [BCV, Figure 1] and determine their various fixed-point sets. We will use the notation and results of Appendix B.

4.2.1. *The subgroups* $\mathbb{G}$ *and* $\mathbf{Z}/11\mathbf{Z}\rtimes\mathbf{Z}/5\mathbf{Z}$. The subgroups $\mathbf{Z}/11\mathbf{Z}\rtimes\mathbf{Z}/5\mathbf{Z}$ of $\mathbb{G}$ are all conjugate to the subgroup generated by the elements a and c of $\mathbb{G}$. We see from (30) that their only fixed point is $[e_0]$. It is on $Y^0_{\mathbb{A}}$ hence defines a GM fivefold, $X^5_{\mathbb{A}}$, already defined in Section 3.1, with automorphism group $\mathbb{G}$.

4.2.2. *The subgroups* $\mathbf{Z}/11\mathbf{Z}$. The subgroups $\mathbf{Z}/11\mathbf{Z}$ of $\mathbb{G}$ are all conjugate to the subgroup generated by the element c of $\mathbb{G}$. We see from (30) that there are 6 fixed points: the point $[e_0]$ (on $Y^0_{\mathbb{A}}$) and 5 other points. For these 5 points, which are all in the same $\mathbb{G}$-orbit, the stabilizers are exactly $\mathbf{Z}/11\mathbf{Z}$ (because the only nontrivial oversubgroups are $\mathbf{Z}/11\mathbf{Z}\rtimes\mathbf{Z}/5\mathbf{Z}$ and $\mathbb{G}$). Furthermore, using Table 1, one sees that they are in $Y^2_{\mathbb{A}}$ (this was already observed in Section 3.3). So we get isomorphic GM threefolds, $X^3_{\mathbb{A}}$, already defined in Section 3.3, with automorphism groups $\mathbf{Z}/11\mathbf{Z}$.

4.2.3. *The subgroups* $\mathbf{Z}/3\mathbf{Z}$*,* $\mathbf{Z}/6\mathbf{Z}$*, and* D_{12}. The elements of order 6 of $\mathbb{G}$ are all conjugate to the element b of $\mathbb{G}$. Since its character in the representation ξ is 1, it acts on V_ξ with eigenvalues $1,\zeta_6,\zeta_6^2,\zeta_6^4,\zeta_6^5$, for which we choose eigenvectors w_0,w_1,w_2,w_4,w_5. The fixed-point set of b consists of the line $\Delta_6=\langle[e_0],[w_0]\rangle$ and the 4 isolated points $[w_1]$, $[w_2]$, $[w_4]$, $[w_5]$. Any involution τ in $\mathbb{G}$ that, together with b, generates a dihedral group D_{12}, exchanges the eigenspaces corresponding to conjugate eigenvalues. Looking at the subgroup pattern of $\mathbb{G}$, one sees that the stabilizers of the 4 isolated points are $\mathbf{Z}/6\mathbf{Z}$, whereas those of points of $\Delta_6\smallsetminus\{[e_0]\}$ are D_{12} (a maximal proper subgroup).

The fixed-point set of an element of $\mathbb{G}$ of order 3 (such as b^2; they are all conjugate) is the union of Δ_6 and two other disjoint lines, $\Delta_3=\langle[w_1],[w_4]\rangle$ and $\Delta'_3=\tau(\Delta_3)=\langle[w_2],[w_5]\rangle$. The fixed-point set of the subgroup $D_6=\langle b^2,\tau\rangle$ is therefore the line Δ_6.

Consider now the isomorphism of representations $v\colon \bigwedge^2 V_\xi\xrightarrow{\sim}\bigwedge^3 V_\xi$ from (7). It maps the ζ_6^2-eigenspace for the action of $\bigwedge^2\xi(b)$ (spanned by $w_0\wedge w_2$) to the ζ_6^2-eigenspace for the action of $\bigwedge^3\xi(b)$ (spanned by $w_1\wedge w_2\wedge w_5$), so one can write

$$v(w_0\wedge w_2)=\alpha w_1\wedge w_2\wedge w_5$$

for some $\alpha \in \mathbf{C}$. By definition of $\mathbb{A}$, this implies $w_2 \wedge (e_0 \wedge w_0 - \alpha w_1 \wedge w_5) \in \mathbb{A}$. Similarly, upon considering ζ_6-eigenspaces, one sees that

$$v(w_2 \wedge w_5) = \beta w_1 \wedge w_2 \wedge w_4 + \gamma w_0 \wedge w_2 \wedge w_5,$$

for some $\beta, \gamma \in \mathbf{C}$, so that $w_2 \wedge (e_0 \wedge w_5 + \beta w_1 \wedge w_4 + \gamma w_0 \wedge w_5) \in \mathbb{A}$. This proves that $[w_2]$ is in $Y^2_{\mathbb{A}}$, and so is $[w_4] = \tau([w_2])$.

Consider the length-18 scheme $\mathrm{Fix}(b^2) \cap Y_{\mathbb{A}} = Y_{\mathbb{A}} \cap (\Delta_6 \cup \Delta_3 \cup \Delta'_3)$. We see from Table 1 that it has 15 points, 3 of them in $Y^2_{\mathbb{A}}$ (hence nonreduced) and fixed by b, therefore 12 of them in $Y^1_{\mathbb{A}}$ (reduced as noted above), none fixed by b. Since the set $\mathrm{Fix}(b^2) \cap Y^2_{\mathbb{A}}$ is τ-invariant and contains $[w_2]$ and $[w_4]$, and b acts as an involution with no fixed points on the set $\mathrm{Fix}(b^2) \cap Y^1_{\mathbb{A}} \cap \Delta_3$, whose cardinality is thus even, we see that each line Δ_6, Δ_3, Δ'_3 contains a single point of $Y^2_{\mathbb{A}}$ and 4 points of $Y^1_{\mathbb{A}}$; the points $[w_1]$ and $[w_5]$ are in $Y^0_{\mathbb{A}}$. In particular, the set $\mathrm{Fix}(b) \cap Y_{\mathbb{A}}$ has 7 points, as claimed in Table 1.

So altogether, we get GM varieties of dimensions 3, 4, or 5, with automorphism groups $\mathbf{Z}/3\mathbf{Z}$, of dimensions 3 or 5 with automorphism groups $\mathbf{Z}/6\mathbf{Z}$, and of dimensions 3 or 4 with automorphism groups D_{12}, and we see that no GM varieties $X_{\mathbb{A},V_5}$ have automorphism groups the dihedral group D_6 or the alternating group $\mathfrak{A}_5$.

4.2.4. *The subgroups* $\mathbf{Z}/5\mathbf{Z}$ *and* D_{10}. The subgroups $\mathbf{Z}/5\mathbf{Z}$ of $\mathbb{G}$ are all conjugate to the subgroup generated by the element a of $\mathbb{G}$. Since its character is 0, it acts on V_ξ with eigenvalues $1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4$. Its fixed-point set in $\mathbf{P}(V_6)$ therefore consists of a line Δ_5 passing through $[e_0]$ and 4 isolated points. Any involution τ in $\mathbb{G}$ that, together with a, generates a dihedral group D_{10}, exchanges the eigenspaces corresponding to conjugate eigenvalues. Looking at the subgroup pattern of $\mathbb{G}$, one sees that the stabilizers of the 4 isolated points are $\mathbf{Z}/5\mathbf{Z}$, whereas those of points of Δ_5 contain D_{10}. Since we saw above that $\mathfrak{A}_5$-stabilizers are not possible, the stabilizers are therefore D_{10} for all points of $\Delta_5 \smallsetminus \{[e_0]\}$.

Since $\#(\mathrm{Fix}(g) \cap Y_{\mathbb{A}}) = 8$ (Table 1), one sees that the line Δ_5 meets $Y_{\mathbb{A}}$ in only 4 points. Since $Y^1_{\mathbb{A}} \cap \Delta_5$ is reduced, at least one of them must be in $Y^2_{\mathbb{A}}$. Among the 4 isolated fixed points, the involution τ acts with no fixed points on the set of those that are in $Y^2_{\mathbb{A}}$, hence its cardinality is even. Since $\#(\mathrm{Fix}(g) \cap Y^2_{\mathbb{A}}) = 2$ (Table 1), the only possibility is that Δ_5 contains 2 points in $Y^1_{\mathbb{A}}$ and 2 points in $Y^2_{\mathbb{A}}$, and the 4 isolated points are in $Y^1_{\mathbb{A}}$. So altogether, we get GM fourfolds with automorphism groups $\mathbf{Z}/5\mathbf{Z}$ and GM varieties of dimensions 3, 4, or 5 with automorphism groups D_{10}.

4.2.5. *The subgroups* $\mathbf{Z}/2\mathbf{Z}$*,* $(\mathbf{Z}/2\mathbf{Z})^2$*, and* $\mathfrak{A}_4$. Since its character is 1, any order-2 element g of $\mathbb{G}$ acts on V_ξ with eigenvalues $1, 1, 1, -1, -1$. Its fixed-point set in $\mathbf{P}(V_6)$ therefore consists of the disjoint union of a 3-space $\mathbf{P}(V_4)$ passing through $[e_0]$ and a line Δ_2. Double EPW sextics with a symplectic involution were studied in [C, Theorem 5] and [Mo3, Theorem 6.2.3]: they prove that the fixed-point set is always the union of a smooth K3 surface and 28 isolated points. By [C, Proposition 17] (which holds under some generality assumptions which are satisfied by $\mathbb{A}$ because it contains no decomposable vectors), we obtain:

- $\mathrm{Fix}(g) \cap Y_{\mathbb{A}}$ is the union of a smooth quadric Q and a Kummer quartic S, both contained in $\mathbf{P}(V_4)$, and the 6 distinct points of $Y_{\mathbb{A}} \cap \Delta_2$;
- $\mathrm{Fix}(g) \cap Y^2_{\mathbb{A}}$ is contained in $\mathbf{P}(V_4)$ and is the disjoint union of the smooth curve $Q \cap S$ and the 16 singular points of S.

The fixed K3 surface in $\widetilde{Y}_{\mathbb{A}}$ mentioned above is a double cover of Q branched along $Q \cap S$. The images in $Y_{\mathbb{A}}$ of the 28 fixed points are the 6 points of $Y_{\mathbb{A}} \cap \Delta_2$ (which are in $Y^1_{\mathbb{A}}$) and the 16 singular points of S (which are in $Y^2_{\mathbb{A}}$).

The fixed-point set of any subgroup $(\mathbf{Z}/2\mathbf{Z})^2$ of $\mathbb{G}$ is a plane Π_4 passing through $[e_0]$ and 3 isolated points. This plane is contained in $\mathbf{P}(V_4)$ and contains the line Δ_6 fixed by any D_{12} containing $(\mathbf{Z}/2\mathbf{Z})^2$. For points in $\Pi_4 \smallsetminus \Delta_6$ and the 3 isolated points, the stabilizers are either $(\mathbf{Z}/2\mathbf{Z})^2$ or $\mathfrak{A}_4$.

As an $\mathfrak{A}_4$-representation, V_6 splits as the direct sum of the 3 characters (which span the plane Π_4) and the one irreducible representation of dimension 3. It follows that the fixed-point set of any $\mathfrak{A}_4$ containing $(\mathbf{Z}/2\mathbf{Z})^2$ has 3 points (corresponding to the 3 characters), all in Π_4. One of them is $[e_0]$ and the stabilizer of the other two is indeed $\mathfrak{A}_4$.

The plane Π_4 meets $Y_{\mathbb{A}}$ along the union of the conic $\Pi_4 \cap Q$ and the quartic curve $\Pi_4 \cap S$. Since the 1-dimensional part of $\mathrm{Fix}(g) \cap Y_{\mathbb{A}}^2$ is the smooth octic curve $Q \cap S$, its intersection with the plane Π_4 is finite nonempty. So we get points in $\Pi_4 \smallsetminus \Delta_6$ (with stabilizers $(\mathbf{Z}/2\mathbf{Z})^2$) in each of the strata.

Finally, the two fixed points of $\mathfrak{A}_4$ are not in $Y_{\mathbb{A}}$: if they were, we would obtain a point of $\widetilde{Y}_{\mathbb{A}}$ fixed by a symplectic action of $\mathfrak{A}_4$; however there are no representations of $\mathfrak{A}_4$ in $\mathrm{Sp}(\mathbf{C}^4)$ without trivial summands, so there are no points in $\widetilde{Y}_{\mathbb{A}}$ fixed by $\mathfrak{A}_4$. Therefore, we only get GM varieties of dimension 5 with automorphism groups $\mathfrak{A}_4$.

We sum up our results in a table:

aut. groups	$\mathbb{G}$	$\mathbf{Z}/11\mathbf{Z}$	D_{12}	$\mathbf{Z}/6\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$	D_{10}	$\mathbf{Z}/5\mathbf{Z}$	$\mathfrak{A}_4$	$(\mathbf{Z}/2\mathbf{Z})^2$	$\mathbf{Z}/2\mathbf{Z}$	$\{1\}$
$\dim(X_{\mathbb{A},V_5})$	5	3	3, 4	3, 5	3, 4, 5	3, 4, 5	4	5	3, 4, 5	3, 4, 5	3, 4, 5

TABLE 2. Possible automorphisms groups of (ordinary) GM varieties associated with the Lagrangian $\mathbb{A}$

4.3. **EPW sextics with an automorphism of order** 11. We use the injectivity of the period map (25) to characterize quasi-smooth EPW sextics with an automorphism of prime order at least 11.

Theorem 4.2. *The only quasi-smooth EPW sextic with an automorphism of prime order $p \geq 11$ is the EPW sextic $Y_{\mathbb{A}}$, and $p = 11$.*

Proof. Let Y_A be a quasi-smooth EPW sextic with an automorphism g of prime order $p \geq 11$. By Proposition A.2, g lifts to a symplectic automorphism of the same order of the smooth double EPW sextic $\widetilde{Y}_A$ which fixes the polarization H. By Corollary A.4, the transcendental lattice $\mathrm{Tr}(\widetilde{Y}_A)$ is isomorphic to the lattice $T := (22)^{\oplus 2}$ and is primitively embedded in the lattice $H^\perp$, with orthogonal complement isomorphic to the lattice S defined in Section A.3.

Lemma 4.3. *Any two primitive embeddings of T into the lattice $h^\perp$ with orthogonal complements isomorphic to S differ by an isometry in $\widetilde{O}(h^\perp)$.*

Proof. According to [N, Proposition 1.5.1] (see also [BCS, Proposition 2.7]), to primitively embed the lattice T into the lattice $h^\perp$, one needs subgroups $K_T \subset \mathrm{Disc}(T) \simeq (\mathbf{Z}/22\mathbf{Z})^2$ and $K_{h^\perp} \subset \mathrm{Disc}(h^\perp) \simeq (\mathbf{Z}/2\mathbf{Z})^2$ and an isometry $u\colon K_T \xrightarrow{\sim} K_{h^\perp}$ for the canonical $\mathbf{Q}/2\mathbf{Z}$-valued quadratic forms on these groups. The discriminant of the orthogonal complement is then $22^2 \cdot 2^2/\operatorname{Card}(K_T)^2$.

In our case, we want this orthogonal complement to be S, with discriminant group $(\mathbf{Z}/11\mathbf{Z})^2$. The only choice is therefore to take K_T to be the 2-torsion part of $\mathrm{Disc}(T)$ and $K_{h^\perp} = \mathrm{Disc}(h^\perp)$.

There are only two choices for u and they correspond to switching the two factors of $(\mathbf{Z}/2\mathbf{Z})^2$. Any two such embeddings $T \hookrightarrow h^\perp$ therefore differ by an isometry of $h^\perp$ and, upon composing with the involution of $h^\perp$ that switches the two (-2)-factors, we may assume that this isometry is in $\widetilde{O}(h^\perp)$. ☐

If we fix any embedding $T \hookrightarrow h^\perp$ as in the lemma, the period of $\widetilde{Y}_A$ therefore belongs to the (uniquely defined) image in the quotient $\widetilde{O}(h^\perp)\backslash\Omega_h$ of the set $\mathbf{P}(T \otimes \mathbf{C}) \cap \Omega_h$. This set consists of two conjugate points, one on each component of Ω_h, hence they are mapped to the same point in the period domain $\widetilde{O}(h^\perp)\backslash\Omega_h$. The theorem now follows from the injectivity of the polarized period map, which implies that $\widetilde{Y}_A$ and $\widetilde{Y}_{\mathbb{A}}$ are isomorphic by an isomorphism that respects the polarizations. Since these polarizations define the double covers π_A and $\pi_{\mathbb{A}}$, this isomorphism descends to an isomorphism between Y_A and $Y_{\mathbb{A}}$. ☐

Corollary 4.4. (1) *The only smooth double EPW sextic with a symplectic automorphism of prime order $p \geq 11$ fixing the polarization H is the Klein double sextic $\widetilde{Y}_{\mathbb{A}}$, and $p = 11$.*

(2) *The only (smooth ordinary) GM varieties with an automorphism of prime order $p \geq 11$ are the GM varieties $X^3_{\mathbb{A}}$ and $X^5_{\mathbb{A}}$, and $p = 11$.*

Proof. Part (1) is only a rephrasing of Theorem 4.2, using the isomorphism $\mathrm{Aut}^s_H(\widetilde{Y}_A) \xrightarrow{\sim} \mathrm{Aut}(Y_A)$ from Proposition A.2. For part (2), let X be a (smooth ordinary) GM variety with an automorphism of prime order $p \geq 11$ and let A be an associated Lagrangian. By (3), the quasi-smooth EPW sextic Y_A also has an automorphism of order p. It follows from Theorem 4.2 that we can take $A = \mathbb{A}$ and that $p = 11$. The result now follows from Sections 4.2.1 and 4.2.2. ☐

4.4. **GM varieties of dimensions 4 and 6 with many Hodge classes.** GM sixfolds do not appear in the definition given in Section 2.2. This is because they are *special* (as opposed to *ordinary*): they are double covers $\gamma\colon X \to \mathsf{Gr}(2, V_5)$ branched along the smooth intersection of $\mathsf{Gr}(2, V_5)$ with a quadric (a GM fivefold!). As mentioned in Section 2.2, to this GM fivefold correspond a Lagrangian $A \subset \bigwedge^3 V_6$ and a hyperplane $V_5 \subset V_6$ such that $A \cap \bigwedge^3 V_5 = \{0\}$, and conversely, given a Lagrangian A and a hyperplane $V_5 \subset V_6$ satisfying this property, one can construct a GM fivefold in $\mathsf{Gr}(2, V_5)$ and a GM sixfold $\gamma\colon X \to \mathsf{Gr}(2, V_5)$ branched along this GM fivefold.

When X is a GM fourfold, we let $\gamma\colon X \to \mathsf{Gr}(2, V_5)$ be the inclusion (in both cases, γ is called the Gushel map in [DK1]).

In both cases (that is, for GM varieties X of even dimensions $2m \in \{4, 6\}$ and associated Lagrangian A), there are, by [DK2, Theorem 5.1], isomorphisms

$$(H^{2m}(X, \mathbf{Z})_{00}, \smile) \simeq (H^2(\widetilde{Y}_A, \mathbf{Z})_0, (-1)^{m-1} q_{BB}) \tag{14}$$

of polarized Hodge structures, where

$$H^{2m}(X, \mathbf{Z})_{00} := \gamma^* H^{2m}(\mathsf{Gr}(2, V_5), \mathbf{Z})^\perp \subset H^{2m}(X, \mathbf{Z})$$

and $H^2(\widetilde{Y}_A, \mathbf{Z})_0$ is, in our previous notation, $H^\perp \subset H^2(\widetilde{Y}_A, \mathbf{Z})$.

We use these results to construct explicit families of GM varieties X of even dimensions $2m \in \{4, 6\}$ with groups $\mathrm{Hdg}^m(X) := H^{m,m}(X) \cap H^{2m}(X, \mathbf{Z})$ of Hodge classes of maximal rank $h^{m,m}(X) = 22$ ([DK2, Proposition 3.1]). We proceed as follows: start from the Lagrangian $\mathbb{A}$ and any hyperplane $V_5 \subset V_6$ that satisfies the condition $\dim(A \cap \bigwedge^3 V_5) = 3 - m$. We obtain,

- when $m = 2$, a family (parametrized by the fourfold $Y^1_{\mathbb{A}}$) of GM fourfolds,

- when $m = 3$, as explained above, a family (parametrized by the fivefold $\mathbf{P}(V_6) \smallsetminus Y_{\mathbb{A}}$) of GM sixfolds.

By (14), these GM $2m$-folds X all satisfy

$$\begin{aligned}\mathrm{Hdg}^m(X)((-1)^{m-1}) &\supset \gamma^* H^{2m}(\mathsf{Gr}(2,V_5),\mathbf{Z})((-1)^{m-1}) \oplus \mathsf{S}\\ &\simeq (2)^{\oplus 2} \oplus \mathsf{S}\\ &\simeq (2)^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} -2 & -1 \\ -1 & -6 \end{pmatrix}^{\oplus 2},\end{aligned}$$

a lattice of rank 22, the maximal possible value (the last isomorphism follows from Section A.3). The inclusion on the first line is in fact an isomorphism, because the lattice on the last line has no overlattices (its discriminant group has no nontrivial isotropic elements; see Section A.4).

Remark 4.5. Take $m = 2$. The integral Hodge conjecture in degree 2 for GM fourfolds was recently proved in [P, Corollary 1.2]. Therefore, we get a family (parametrized by the fourfold $Y^1_{\mathbb{A}}$) of GM fourfolds X such that all classes in $\mathrm{Hdg}^2(X)$ are classes of algebraic cycles.

Example 4.6. Take $m = 3$ and $V_5 = V_\xi$. We get a GM sixfold $X^6_{\mathbb{A}}$ which can be defined inside $\mathbf{P}(\mathbf{C}e_{00} \oplus \bigwedge^2 V_\xi)$ by the quadratic equation

$$x_{00}^2 = x_{12}x_{13} + x_{23}x_{24} + x_{34}x_{35} - x_{45}x_{14} + x_{15}x_{25}$$

(the right side is the equation (9) of the $\mathbb{G}$-invariant quadric $\mathsf{Q} \subset \mathbf{P}(\bigwedge^2 V_\xi)$) and the Plücker quadrics in the $(x_{ij})_{1\le i<j\le 5}$ that define $\mathsf{Gr}(2,V_\xi)$ in $\mathbf{P}(\bigwedge^2 V_\xi)$. Since the equation of Q is $\mathbb{G}$-invariant, we see that $\mathbf{Z}/2\mathbf{Z} \times \mathbb{G}$ acts on $\mathbf{C}e_{00} \oplus \bigwedge^2 V_\xi$ component-wise, and this group is $\mathrm{Aut}(X^6_{\mathbb{A}})$.

The integral Hodge conjecture in degree 3 is not known in general for GM sixfolds X, but it was proved in [P, Corollary 8.4] that the cokernel $V^3(X)$ (the *Voisin group*) of the cycle map

$$\mathrm{CH}^3(X) \longrightarrow \mathrm{Hdg}^3(X)$$

is 2-torsion. When $X = X^6_{\mathbb{A}}$, since the cycle map is surjective for $\mathsf{Gr}(2,V_\xi)$, the image of the cycle map, modulo $\gamma^* H^{2m}(\mathsf{Gr}(2,V_5),\mathbf{Z})$, is a $\mathbb{G}$-invariant, not necessarily saturated, sublattice of S of index a power of 2.

5. Irrational GM threefolds

5.1. **Double EPW surfaces and their automorphisms.** Let $Y_A \subset \mathbf{P}(V_6)$ be a quasi-smooth EPW sextic, where $A \subset \bigwedge^3 V_6$ is a Lagrangian subspace with no decomposable vectors. Its singular locus is the smooth surface $Y_A^{\ge 2}$ and, as explained in Appendix A.5, there is a canonical connected étale double covering $\widetilde{Y}_A^{\ge 2} \to Y_A^{\ge 2}$.

Let X be any (smooth) GM variety of dimension 3 or 5 associated with A and let $\mathrm{Jac}(X)$ be its intermediate Jacobian. It is a 10-dimensional abelian variety endowed with a canonical principal polarization θ_X. By [DK5, Theorem 1.1], there is a canonical principal polarization θ on $\mathrm{Alb}(\widetilde{Y}_A^{\ge 2})$ and a canonical isomorphism

$$(\mathrm{Jac}(X),\theta_X) \xrightarrow{\sim} (\mathrm{Alb}(\widetilde{Y}_A^{\ge 2}),\theta) \tag{15}$$

between 10-dimensional principally polarized abelian varieties. By (28), the tangent spaces at the origin of these abelian varieties are isomorphic to A.

The subgroup $\mathrm{Aut}(X)$ of $\mathrm{Aut}(Y_A)$ (see (3)) acts faithfully on both $\mathrm{Jac}(X)$ and $\mathrm{Alb}(\widetilde{Y}_A^{\ge 2})$ and, by Proposition A.8, the isomorphism above is $\mathrm{Aut}(X)$-equivariant.

5.2. **Explicit irrational GM threefolds.** Consider the Klein Lagrangian $\mathbb{A}$. By Proposition 4.1, we have $\mathrm{Aut}(Y_{\mathbb{A}}) \simeq \mathbb{G}$ and the analytic representation of the action of that group on $\mathrm{Alb}(\widetilde{Y}_A^{\geq 2})$ is, by Proposition A.7, the representation of $\mathbb{G}$ on $\mathbb{A}$, that is, the irreducible representation $\bigwedge^2 \xi$ of $\mathbb{G}$ (Section 3.1). In particular, $\mathbb{G}$ acts faithfully on the 10-dimensional principally polarized abelian variety

$$(\mathbb{J}, \theta) := (\mathrm{Alb}(\widetilde{Y}_{\mathbb{A}}^{\geq 2}), \theta) \tag{16}$$

by automorphisms that preserve the principal polarization θ. By Lemma C.2, any $\mathbb{G}$-invariant polarization on $\mathbb{J}$ is proportional to θ.

Proposition 5.1. *The principally polarized abelian variety $(\mathbb{J}, \theta)$ is indecomposable.*

Proof. If $(\mathbb{J}, \theta)$ is isomorphic to a product of $m \geq 2$ nonzero indecomposable principally polarized abelian varieties, such a decomposition is unique up to the order of the factors hence induces a morphism $u \colon \mathbb{G} \to \mathfrak{S}_m$ (the group $\mathbb{G}$ permutes the factors). Since the analytic representation is irreducible, the image of u is nontrivial and, the group $\mathbb{G}$ being simple, u is injective; but this is impossible because $\mathbb{G}$ contains elements of order 11 but not $\mathfrak{S}_m$, because $m \leq 10$. □

We can now prove our main result.

Theorem 5.2. *Any smooth GM threefold associated with the Lagrangian $\mathbb{A}$ is irrational.*

Proof. Let X be such a threefold. By Proposition A.7, the isomorphism $(\mathrm{Jac}(X), \theta_X) \xrightarrow{\sim} (\mathbb{J}, \theta)$ in (15) is $\mathbb{G}$-equivariant. We follow [B2, B3]: to prove that X is not rational, we apply the Clemens–Griffiths criterion ([CG, Corollary 3.26]); in view of Proposition 5.1, it suffices to prove that $(\mathbb{J}, \theta)$ is not the Jacobian of a smooth projective curve.

Suppose $(\mathbb{J}, \theta) \simeq (\mathrm{Jac}(C), \theta_C)$ for some smooth projective curve C of genus 10. The group $\mathbb{G}$ then embeds into the group of automorphisms of $(\mathrm{Jac}(C), \theta_C)$; by the Torelli theorem, this group is isomorphic to $\mathrm{Aut}(C)$ if C is hyperelliptic and to $\mathrm{Aut}(C) \times \mathbf{Z}/2\mathbf{Z}$ otherwise. Since any morphism from $\mathbb{G}$ to $\mathbf{Z}/2\mathbf{Z}$ is trivial, we see that $\mathbb{G}$ is a subgroup of $\mathrm{Aut}(C)$. This contradicts the fact that the automorphism group of a curve of genus 10 has order at most 432 ([LMFD]). □

Corollary 5.3. *There exists a complete family, with finite moduli morphism, parametrized by the smooth projective surface $Y_{\mathbb{A}}^{\geq 2}$, of irrational smooth ordinary GM threefolds.*

Proof. This follows from the theorem and [DK3, Example 6.8]. □

The theorem applies in particular to the GM threefold $X_{\mathbb{A}}^3$ defined in Section 3.3.

Corollary 5.4. *The GM threefold $X_{\mathbb{A}}^3$ is irrational.*

Remark 5.5. It is a general fact that all smooth GM varieties of the same dimension constructed from the same Lagrangian are birationally isomorphic ([DK1, Corollary 4.16]); in particular, all threefolds in the family of Corollary 5.3 are mutually birationally isomorphic.

Remark 5.6. The Clemens–Griffiths component of a principally polarized abelian variety is the product of its indecomposable factors that are not isomorphic to Jacobians of smooth projective curves, and the Clemens–Griffiths component of a Fano threefold is the Clemens–Griffiths component of its intermediate Jacobian; it follows from the Clemens–Griffiths method that the Clemens–Griffiths component of a Fano threefold is a birational invariant. By Proposition 5.1, the Clemens–Griffiths component of the GM threefolds constructed from the Lagrangian $\mathbb{A}$ is $(\mathbb{J}, \theta)$; in particular, these threefolds are not birationally isomorphic to any smooth cubic threefold (because their Clemens–Griffiths components all have dimension 5).

Remark 5.7. All GM fivefolds are rational (over $\mathbf{C}$); for that, one can invoke the general result [DK1, Proposition 4.2], but in the case of $X^5_{\mathbb{A}}$, one sees directly from its explicit equation (9) (together with the Plücker quadric equations) in $\mathbf{P}(\bigwedge^2 V_\xi) = \mathbf{P}^9$ that the projection $\mathbf{P}^9 \dashrightarrow \mathbf{P}^5$ from $\mathbf{P}(\langle e_{13}, e_{14}, e_{15}, e_{34}\rangle) = \mathbf{P}^3$ induces a birational isomorphism $X^5_{\mathbb{A}} \overset{\sim}{\dashrightarrow} \mathbf{P}^5$, proving that $X^5_{\mathbb{A}}$ is rational over $\mathbf{Q}$.

We do not know whether the smooth GM fourfolds associated with the Lagrangian $\mathbb{A}$ are rational (folklore conjectures say that they should be irrational, because they have no associated K3 surfaces; see Proposition A.5).

Let us go back to the 10-dimensional principally polarized abelian variety $(\mathbb{J}, \theta)$ defined by (16). It is acted on faithfully by the group $\mathbb{G}$, and the associated analytic representation $\mathbb{G} \to \mathrm{GL}(T_{\mathbb{J},0})$ is the irreducible representation $\bigwedge^2 \xi$ of $\mathbb{G}$ (Sections 5.1 and 5.2).

Proposition 5.8. *The abelian variety $\mathbb{J}$ is isogeneous to E^{10}, for some elliptic curve E.*

Proof. Since the analytic representation is irreducible and defined over $\mathbf{Q}$ (Appendix B), the proposition follows from Proposition C.1. □

Unfortunately, we were not able to say more about the elliptic curve E in the proposition: as explained in Remark C.4, the mere existence of a $\mathbb{G}$-action on E^{10} with prescribed analytic representation and of a $\mathbb{G}$-invariant polarization does not put any restriction on E.

We suspect that this curve E is isomorphic to the elliptic curve $E_\lambda := \mathbf{C}/\mathbf{Z}[\lambda]$, which has complex multiplication by $\mathbf{Z}[\lambda]$, where $\lambda := \frac{1}{2}(-1+\sqrt{-11})$. More precisely, we conjecture that $(\mathbb{J}, \theta)$ is isomorphic to the principally polarized abelian variety constructed in Proposition C.5.

Appendix A. Automorphisms of double EPW sextics

A.1. **Double EPW sextics and their automorphisms.** As in Section 2.1, let V_6 be a 6-dimensional complex vector space and let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors, with associated EPW sextic $Y_A \subset \mathbf{P}(V_6)$. There is a canonical double covering

$$\pi_A \colon \widetilde{Y}_A \longrightarrow Y_A \tag{17}$$

branched along the integral surface $Y_A^{\geq 2}$. The fourfold $\widetilde{Y}_A$ is called a *double EPW sextic* and its singular locus is the finite set $\pi_A^{-1}(Y_A^{\geq 3})$ ([O2, Section 1.2] or [DK1, Theorem B.7]). It carries the canonical polarization $H := \pi_A^* \mathscr{O}_{Y_A}(1)$ and the image of the associated morphism $\widetilde{Y}_A \to \mathbf{P}(H^0(\widetilde{Y}_A, H)^\vee)$ is isomorphic to Y_A. When $Y_A^{\geq 3} = \varnothing$, we say that Y_A is *quasi-smooth* and $\widetilde{Y}_A$ is a smooth hyperkähler variety of $\mathrm{K3}^{[2]}$-type.

Every automorphism of Y_A induces an automorphism of $\widetilde{Y}_A$ (see the proof of [DK1, Proposition B.8(b)]) that fixes the class H. Conversely, let $\mathrm{Aut}_H(\widetilde{Y}_A)$ be the group of automorphisms of $\widetilde{Y}_A$ that fix the class H. It contains the covering involution ι of π_A. Any element of $\mathrm{Aut}_H(\widetilde{Y}_A)$ induces an automorphism of $\mathbf{P}(H^0(\widetilde{Y}_A, H)^\vee) \simeq \mathbf{P}(V_6)$ hence descends to an automorphism of Y_A. This gives a central extension

$$0 \to \langle \iota \rangle \to \mathrm{Aut}_H(\widetilde{Y}_A) \to \mathrm{Aut}(Y_A) \to 1. \tag{18}$$

As we will check in (22), the space $H^2(\widetilde{Y}_A, \mathscr{O}_{\widetilde{Y}_A})$ has dimension 1. It is acted on by the group of automorphisms of $\widetilde{Y}_A$ and this defines another extension

$$1 \to \operatorname{Aut}^s_H(\widetilde{Y}_A) \to \operatorname{Aut}_H(\widetilde{Y}_A) \to \boldsymbol{\mu}_r \to 1. \tag{19}$$

The image of ι in $\boldsymbol{\mu}_r$ is -1 and $\operatorname{Aut}^s_H(\widetilde{Y}_A)$ is the subgroup of elements of $\operatorname{Aut}_H(\widetilde{Y}_A)$ that act trivially on $H^2(\widetilde{Y}_A, \mathscr{O}_{\widetilde{Y}_A})$ (when $Y_A^{\geq 3} = \varnothing$, these are exactly, by Hodge theory, the symplectic automorphisms—those that leave any symplectic 2-form on $\widetilde{Y}_A$ invariant).

We will show in the next proposition (which was kindly provided by A. Kuznetsov) that these extensions are both trivial. For that, we construct an extension

$$1 \to \boldsymbol{\mu}_2 \to \widetilde{\operatorname{Aut}}(Y_A) \to \operatorname{Aut}(Y_A) \to 1 \tag{20}$$

as follows. Recall from (1) that there is an embedding $\operatorname{Aut}(Y_A) \hookrightarrow \operatorname{PGL}(V_6)$. Let G be the inverse image of $\operatorname{Aut}(Y_A)$ via the canonical map $\operatorname{SL}(V_6) \to \operatorname{PGL}(V_6)$. It is an extension of $\operatorname{Aut}(Y_A)$ by $\boldsymbol{\mu}_6$ and we set $\widetilde{\operatorname{Aut}}(Y_A) := G/\boldsymbol{\mu}_3$.

The action of G on V_6 induces an action on $\bigwedge^3 V_6$ such that $\boldsymbol{\mu}_6$ acts through its cube, hence the latter action factors through an action of $\widetilde{\operatorname{Aut}}(Y_A)$. The subspace $A \subset \bigwedge^3 V_6$ is preserved by this action, hence we have a morphism of central extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \widetilde{\operatorname{Aut}}(Y_A) & \longrightarrow & \operatorname{Aut}(Y_A) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{C}^\times & \longrightarrow & \operatorname{GL}(A) & \longrightarrow & \operatorname{PGL}(A) & \longrightarrow & 1. \end{array} \tag{21}$$

Lemma A.1. *The vertical morphisms in* (21) *are injective.*

Proof. Let $g \in G \subset \operatorname{SL}(V_6)$. Assume that g acts trivially on A. Then it also acts trivially on $A^\vee$. There is a G-equivariant exact sequence $0 \to A \to \bigwedge^3 V_6 \to A^\vee \to 0$ which splits G-equivariantly because G is finite. It follows that G also acts trivially on $\bigwedge^3 V_6$. The natural morphism $\operatorname{PGL}(V_6) \to \operatorname{PGL}(\bigwedge^3 V_6)$ being injective, g is in $\boldsymbol{\mu}_6$. Finally, $\boldsymbol{\mu}_6/\boldsymbol{\mu}_3$ acts nontrivially on A, hence g is in $\boldsymbol{\mu}_3$ and its image in $\widetilde{\operatorname{Aut}}(Y_A)$ is 1. This proves that the middle vertical map in (21) is injective.

Assume now that g acts as $\lambda \operatorname{Id}_A$ on A. Its eigenvalues on $\bigwedge^3 V_6$ are then λ and λ^{-1}, both with multiplicity 10. Let $\lambda_1, \dots, \lambda_6$ be its eigenvalues on V_6. For all $1 \leq i < j < k \leq 6$, one then has $\lambda_i\lambda_j\lambda_k = \lambda$ or λ^{-1}. It follows that if i, j, k, l, m are all distinct, $\lambda_i\lambda_j\lambda_k, \lambda_i\lambda_j\lambda_l, \lambda_i\lambda_j\lambda_m$ can only take 2 values, hence $\lambda_k, \lambda_l, \lambda_m$ can only take 2 values. So, there are at most 2 distinct eigenvalues and one of the eigenspaces, say E_{λ_1}, has dimension at least 3. If $\lambda \neq \lambda^{-1}$, the eigenspace in $\bigwedge^3 V_6$ for the eigenvalue λ_1^3, which is either A or $A^\vee$, contains $\bigwedge^3 E_{\lambda_1}$. This contradicts the fact that A and $A^\vee$ contain no decomposable vectors. Therefore, $\lambda = \lambda^{-1}$ and g acts as $\pm \operatorname{Id}_A$, and the first part of the proof implies that the image of $\pm g$ in $\widetilde{\operatorname{Aut}}(Y_A)$ is 1. This proves that the rightmost vertical map in (21) is injective. ☐

The following proposition strengthened some results of Beri ([Be, Proposition 4.1]).

Proposition A.2 (Kuznetsov)**.** *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors. The extensions* (18) *and* (19) *are trivial and $r = 2$; more precisely, there is an isomorphism*

$$\operatorname{Aut}_H(\widetilde{Y}_A) \simeq \operatorname{Aut}(Y_A) \times \langle \iota \rangle$$

that splits (18) *and the factor* $\operatorname{Aut}(Y_A)$ *corresponds to the subgroup* $\operatorname{Aut}^s_H(\widetilde{Y}_A)$ *of* $\operatorname{Aut}_H(\widetilde{Y}_A)$.

Proof. We briefly recall from [O2, Section 1.2] (see also [DK4]) the construction of the double cover $\pi_A\colon \widetilde{Y}_A \to Y_A$. In the terminology of the latter article, one considers the Lagrangian subbundles $\mathscr{A}_1 := A\otimes \mathscr{O}_{\mathbf{P}(V_6)}$ and $\mathscr{A}_2 := \bigwedge^2 T_{\mathbf{P}(V_6)}(-3)$ of the trivial vector bundle $\bigwedge^3 V_6\otimes \mathscr{O}_{\mathbf{P}(V_6)}$, and the first Lagrangian cointersection sheaf $\mathscr{R}_1 := \mathrm{Coker}(\mathscr{A}_2 \hookrightarrow \mathscr{A}_1^\vee)$, a rank-1 sheaf with support Y_A. One sets ([DK4, Theorem 5.2(1)])

$$\widetilde{Y}_A = \mathrm{Spec}(\mathscr{O}_{Y_A} \oplus \mathscr{R}_1(-3)).$$

In particular, one has

$$H^2(\widetilde{Y}_A, \mathscr{O}_{\widetilde{Y}_A}) \simeq H^2(Y_A, \mathscr{R}_1(-3)) \simeq H^3(\mathbf{P}(V_6), \mathscr{A}_2(-3)) = H^3(\mathbf{P}(V_6), \textstyle\bigwedge^2 T_{\mathbf{P}(V_6)}(-6)) \simeq \mathbf{C}. \tag{22}$$

The subbundles $\mathscr{A}_1$ and $\mathscr{A}_2$ are invariant for the action of $\widetilde{\mathrm{Aut}}(Y_A)$ on $\bigwedge^3 V_6$, hence the sheaf $\mathscr{R}_1$ is $\widetilde{\mathrm{Aut}}(Y_A)$-equivariant. Finally, the line bundle $\mathscr{O}_{\mathbf{P}(V_6)}(-1)$ has a G-linearization (the subgroup $G \subset \mathrm{SL}(V_6)$ was defined right before Lemma A.1). It follows that $\mathscr{O}_{\mathbf{P}(V_6)}(-3)$ has an $\widetilde{\mathrm{Aut}}(Y_A)$-linearization, hence the same is true for the sheaf $\mathscr{R}_1(-3)$. Therefore, the group $\widetilde{\mathrm{Aut}}(Y_A)$ acts on $\widetilde{Y}_A$ and fixes the polarization H.

Observe now that since the nontrivial element of $\boldsymbol{\mu}_2 \subset \widetilde{\mathrm{Aut}}(Y_A)$ acts by -1 on A, hence also on $\mathscr{R}_1$, and since it acts by -1 on $\mathscr{O}(-1)$, hence also on $\mathscr{O}(-3)$, the group $\boldsymbol{\mu}_2$ acts trivially on $\mathscr{R}_1(-3)$, hence also on $\widetilde{Y}_A$. Therefore, the morphism $\widetilde{\mathrm{Aut}}(Y_A) \to \mathrm{Aut}_H(\widetilde{Y}_A)$ factors through the quotient $\widetilde{\mathrm{Aut}}(Y_A)/\boldsymbol{\mu}_2 = \mathrm{Aut}(Y_A)$. In other words, the surjection $\mathrm{Aut}_H(\widetilde{Y}_A) \to \mathrm{Aut}(Y_A)$ in (18) has a section and this central extension is trivial.

The action of the group $\mathrm{Aut}(\widetilde{Y}_A)$ on the 1-dimensional vector space $H^2(\widetilde{Y}_A, \mathscr{O}_{\widetilde{Y}_A})$ defines a morphism $\mathrm{Aut}(\widetilde{Y}_A) \to \mathbf{C}^\star$ that maps ι to -1. The lift $\widetilde{\mathrm{Aut}}(Y_A) \to \mathrm{Aut}(Y_A) \hookrightarrow \mathrm{Aut}_H(\widetilde{Y}_A)$ acts trivially on $H^2(\widetilde{Y}_A, \mathscr{O}_{\widetilde{Y}_A})$ because its action is induced by the action of $\mathrm{PGL}(V_6)$, which has no nontrivial characters. This gives a surjection $\mathrm{Aut}_H(\widetilde{Y}_A) \to \langle\iota\rangle$ which is trivial on the image of the section $\mathrm{Aut}(Y_A) \hookrightarrow \mathrm{Aut}_H(\widetilde{Y}_A)$. This implies that the extension (19) is also trivial and $r = 2$. The theorem is therefore proved. □

A.2. **Moduli space and period map of (double) EPW sextics.** Quasi-smooth EPW sextics admit an affine coarse moduli space $\mathbf{M}^{\mathrm{EPW},0}$, constructed in [O3] as a GIT quotient by $\mathrm{PGL}(V_6)$ of an affine open dense subset of the space of Lagrangian subspaces in $\bigwedge^3 V_6$.

Let $\widetilde{Y}$ be a hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type (such as a double EPW sextic). The lattice $H^2(\widetilde{Y}, \mathbf{Z})$ (endowed with the Beauville–Bogomolov quadratic form q_{BB}) is isomorphic to the lattice

$$L := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus (-2), \tag{23}$$

where U is the hyperbolic plane $\left(\mathbf{Z}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$, $E_8(-1)$ is the negative definite even unimodular rank-8 lattice, and (m) is the rank-1 lattice with generator of square m.

Fix a class $h \in L$ with $h^2 = 2$. These classes are all in the same $O(L)$-orbit and

$$h^\perp \simeq U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus (-2)^{\oplus 2}. \tag{24}$$

The space

$$\begin{aligned}\Omega_h &:= \{[x] \in \mathbf{P}(L \otimes \mathbf{C}) \mid x\cdot h = 0,\ x\cdot x = 0,\ x\cdot \bar{x} > 0\}\\ &= \{[x] \in \mathbf{P}(h^\perp \otimes \mathbf{C}) \mid x\cdot x = 0,\ x\cdot \bar{x} > 0\}\end{aligned}$$

has two connected components, interchanged by complex conjugation, which are Hermitian symmetric domains. It is acted on by the group

$$\{g \in O(L) \mid g(h) = h\},$$

also with two connected components, which is also the index-2 subgroup $\widetilde{O}(h^\perp)$ of $O(h^\perp)$ that consists of isometries that act trivially on the discriminant group $\mathrm{Disc}(h^\perp) \simeq (\mathbf{Z}/2\mathbf{Z})^2$. The quotient is an irreducible quasi-projective variety ([BB]) and the *period map*

$$\wp\colon \mathbf{M}^{\mathrm{EPW},0} \longrightarrow \widetilde{O}(h^\perp)\backslash\Omega_h, \quad [\widetilde{Y}] \longmapsto [H^{2,0}(\widetilde{Y})] \tag{25}$$

is algebraic ([Bo]). It is an open embedding by Verbitsky's Torelli theorem ([V, Ma, H]).

If $A \subset \bigwedge^3 V_6$ is a Lagrangian such that $\widetilde{Y}_A$ is smooth with period $[x] \in \mathbf{P}(L \otimes \mathbf{C})$ (well defined only up to the action of $\widetilde{O}(h^\perp)$), the Picard group $\mathrm{Pic}(\widetilde{Y}_A)$ is, by Hodge theory, isomorphic to $x^\perp \cap L$.

A.3. **The rank lattice S.** Since it plays a central role in this article, we introduce here the rank-20 lattice S, first defined in [Mo2, Example 2.9] (see also [Mo3, Example 2.5.9]) where it is denoted by S_{11}. It is defined by the 20×20 Gram matrix

$$\begin{pmatrix}
-4 & 1 & -2 & -2 & -1 & 1 & -1 & 1 & -1 & -1 & 2 & 1 & -1 & 2 & -1 & -2 & -2 & 2 & 1 & -1 \\
1 & -4 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 2 & -1 & -2 & 2 & 0 & -1 & 0 & 0 & -1 & -2 & 1 \\
-2 & -1 & -4 & -2 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 2 & -2 & -1 & -1 & 0 & 0 & 1 \\
-2 & -1 & -2 & -4 & 0 & 0 & -2 & 0 & -1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & -1 \\
-1 & -1 & -1 & 0 & -4 & 1 & -1 & 2 & -2 & -1 & 1 & 0 & -1 & 0 & -2 & -2 & 0 & 1 & 1 & -1 \\
1 & -1 & -1 & 0 & 1 & -4 & 0 & -1 & 0 & 1 & -2 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 1 \\
-1 & -1 & 0 & -2 & -1 & 0 & -4 & 1 & -2 & 1 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & 2 & 0 & -2 \\
1 & 1 & 1 & 0 & 2 & -1 & 1 & -4 & 0 & 0 & -1 & 1 & 1 & 0 & 2 & 1 & 0 & -1 & 1 & 0 \\
-1 & -1 & 0 & -1 & -2 & 0 & -2 & 0 & -4 & 0 & 0 & 1 & 1 & 0 & -1 & -2 & 0 & 2 & 0 & -2 \\
-1 & 2 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & -4 & 1 & 1 & -2 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
2 & -1 & 1 & 2 & 1 & -2 & 1 & -1 & 0 & 1 & -4 & -2 & 2 & -1 & 0 & 0 & 0 & -1 & -2 & 1 \\
1 & -2 & 0 & 1 & 0 & -1 & 1 & 1 & 1 & 1 & -2 & -4 & 1 & 0 & -1 & 0 & -1 & -1 & -2 & 2 \\
-1 & 2 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & -2 & 2 & 1 & -4 & 0 & -1 & 0 & 0 & 1 & 2 & 0 \\
2 & 0 & 2 & 1 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & -4 & 1 & 1 & 1 & 0 & 0 & -1 \\
-1 & -1 & -2 & 0 & -2 & -1 & 0 & 2 & -1 & 0 & 0 & -1 & -1 & 1 & -4 & -2 & -1 & 1 & 0 & 0 \\
-2 & 0 & -1 & 0 & -2 & 0 & -1 & 1 & -2 & 0 & 0 & 0 & 0 & 1 & -2 & -4 & -2 & 2 & 0 & -1 \\
-2 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & -2 & -4 & 1 & 0 & 0 \\
2 & -1 & 0 & 1 & 1 & 0 & 2 & -1 & 2 & 1 & -1 & -1 & 1 & 0 & 1 & 2 & 1 & -4 & 0 & 2 \\
1 & -2 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 1 & -2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & -4 & 1 \\
-1 & 1 & 1 & -1 & -1 & 1 & -2 & 0 & -2 & 0 & 1 & 2 & 0 & -1 & 0 & -1 & 0 & 2 & 1 & -4
\end{pmatrix}$$

It is negative definite, even, and contains no (-2)-classes. Its discriminant group is $(\mathbf{Z}/11\mathbf{Z})^2$ and its discriminant form is $\left(\begin{smallmatrix} -2/11 & 0 \\ 0 & -2/11 \end{smallmatrix}\right)$. It is in the same genus[1] as the lattice

$$S := E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} -2 & -1 \\ -1 & -6 \end{pmatrix}^{\oplus 2}.$$

However, the lattices S and S are not isomorphic (because S does not represent -2) but the indefinite lattices $(2) \oplus \mathsf{S}$ and $(2) \oplus S$ are by [N, Corollary 1.13.3].

A direct computation shows that the lattice S contains the rank-5 lattice with diagonal quadratic form $(-4,-4,-4,-6,-8)$. By [Bh, Section 6(iii)], the quadratic form on the last four variables represents every even negative integer with the exception of -2, and the first variable can be used to ensure that all these integers can be primitively represented in S.

[1] By Nikulin's celebrated result [N, Corollary 1.9.4], this means that they have same ranks, same signatures, and that their discriminant forms coincide.

A.4. **Automorphisms of prime order.** Let $\widetilde{Y}$ be a hyperkähler fourfold of K3$^{[2]}$-type. In the lattice $(H^2(\widetilde{Y},\mathbf{Z}),q_{BB})$ mentioned in Appendix A.2, we consider the *transcendental lattice*

$$\mathrm{Tr}(\widetilde{Y}) := \mathrm{Pic}(\widetilde{Y})^\perp \subset H^2(\widetilde{Y},\mathbf{Z}).$$

The automorphism group $\mathrm{Aut}(\widetilde{Y})$ acts faithfully by isometries on the lattice $(H^2(\widetilde{Y},\mathbf{Z}),q_{BB})$ and preserves the sublattices $\mathrm{Pic}(\widetilde{Y})$ and $\mathrm{Tr}(\widetilde{Y})$. If G is a subset of $\mathrm{Aut}(\widetilde{Y})$, we denote by $T_G(\widetilde{Y})$ the invariant lattice (of elements of $H^2(\widetilde{Y},\mathbf{Z})$ that are invariant by all elements of G) and by $S_G(\widetilde{Y}) := T_G(\widetilde{Y})^\perp$ its orthogonal in $H^2(\widetilde{Y},\mathbf{Z})$.

Many results are known about automorphisms of prime order p of hyperkähler fourfolds. We restrict ourselves to the case $p \geq 11$.

Theorem A.3. *Let $\widetilde{Y}$ be a projective hyperkähler fourfold of K3$^{[2]}$-type and let g be a symplectic automorphism of $\widetilde{Y}$ of prime order $p \geq 11$. Then $p = 11$ and there are inclusions $\mathrm{Tr}(\widetilde{Y}) \subset T_g(\widetilde{Y})$ and $S_g(\widetilde{Y}) \subset \mathrm{Pic}(\widetilde{Y})$. The lattice $S_g(\widetilde{Y})$ is isomorphic to the lattice S and $\rho(\widetilde{Y}) = 21$. The possible lattices $T_g(\widetilde{Y})$ are*

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 22 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 6 & 2 & 2 \\ 2 & 8 & -3 \\ 2 & -3 & 8 \end{pmatrix}.$$

Proof. The proof is a compilation of previously known results on symplectic automorphisms. The bound $p \leq 11$ is [Mo2, Corollary 2.13]. The inclusions and the properties of the lattice $S_g(\widetilde{Y})$ are in [Mo1, Lemma 3.5], the equality $\rho(\widetilde{Y}) = 21$ is in [Mo2, Proposition 1.2], the lattice $S_g(\widetilde{Y})$ is determined in [Mo3, Theorem 7.2.7], and the possible lattices $T_g(\widetilde{Y})$ in [BNS, Section 5.5.2]. □

This theorem applies in particular to (smooth) double EPW sextics $\widetilde{Y}_A$. We are interested in automorphisms that preserve the canonical degree-2 polarization H. By Proposition A.2, the group of these automorphisms, modulo the covering involution ι, is isomorphic to the group of automorphisms of the EPW sextic Y_A.

Corollary A.4. *Let $\widetilde{Y}_A$ be a smooth double EPW sextic and let g be an automorphism of $\widetilde{Y}_A$ of prime order $p \geq 11$ that fixes the polarization H. Then $p = 11$ and*[2]

$$S_g(X) \simeq \mathsf{S}, \qquad T_g(\widetilde{Y}_A) \simeq \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix} \oplus (22), \qquad \mathrm{Tr}(\widetilde{Y}_A) \simeq (22)^{\oplus 2},$$

$$\mathrm{Pic}(\widetilde{Y}_A) = \mathbf{Z}H \oplus \mathsf{S} \simeq (2) \oplus E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} -2 & -1 \\ -1 & -6 \end{pmatrix}^{\oplus 2}.$$

In particular, the fourfold $\widetilde{Y}_A$ has maximal Picard number 21.

Proof. By Proposition A.2, the automorphism g is symplectic (all nonsymplectic automorphisms have even order). Since $H \in T_g(\widetilde{Y}_A)$ and $q_{BB}(H) = 2$, and the second lattice in Theorem A.3 contains no classes of square 2, there is only one possibility for $T_g(\widetilde{Y}_A)$ (see also [Mo3, Section 7.4.4]). There are only two (opposite) classes of square 2 in that lattice, so we find $\mathrm{Tr}(\widetilde{Y}_A)$ as their orthogonal.

[2]In the given decomposition of the lattice $\mathrm{Pic}(\widetilde{Y}_A)$, the summand (2) is *not* generated by the polarization H, because S contains no (-2)-classes.

We know that $\mathrm{Pic}(\widetilde{Y}_A)$ is an overlattice of $\mathbf{Z}H \oplus S_g(\widetilde{Y}_A)$. Since the latter has no nontrivial overlattices (its discriminant group has no nontrivial isotropic elements), they are equal. Finally, the last isomorphism in the statement follows from the discussion at the end of Section A.3. □

We prove in Theorem 4.2 that the double EPW sextic $\widetilde{Y}_{\mathbb{A}}$ is the only smooth double EPW sextic with an automorphism of order 11 that fixes the polarization H.

In Hassett's terminology (recalled in [DM, Section 4]), a (smooth) double EPW sextic $\widetilde{Y}_A$ is *special of discriminant* d if there exists a primitive rank-2 lattice $K \subset \mathrm{Pic}(\widetilde{Y}_A)$ containing the polarization H such that $\mathrm{disc}(K^\perp) = -d$ (the orthogonal complement is taken in $(H^2(\widetilde{Y}_A, \mathbf{Z}), q_{BB})$); this may only happen when $d \equiv 0, 2, 4 \pmod 8$ and $d > 8$ ([DM, Proposition 4.1 and Remark 6.3]). The fourfold $\widetilde{Y}_A$ has an *associated K3 surface* if moreover the lattice $K^\perp$ is isomorphic to the opposite of the primitive cohomology lattice of a pseudo-polarized K3 surface (necessarily of degree d); a necessary condition for this to happen is $d \equiv 2, 4 \pmod 8$ (this was proved in [DIM, Proposition 6.6] for GM fourfolds but the computation is the same).

Proposition A.5. *The double EPW sextic $\widetilde{Y}_{\mathbb{A}}$ is special of discriminant d if and only if d is a multiple of* 8 *greater than* 8. *In particular, it has no associated K3 surfaces.*

Proof. Assume that $\widetilde{Y}_{\mathbb{A}}$ is special of discriminant d. Since $\mathrm{Pic}(\widetilde{Y}_{\mathbb{A}}) \simeq \mathbf{Z}H \oplus \mathsf{S}$, the required lattice K as above is of the form $\langle H, \kappa\rangle$, where $\kappa \in \mathsf{S}$ is primitive. Since $\mathrm{Disc}(\mathsf{S}) \simeq (\mathbf{Z}/11\mathbf{Z})^2$, the divisibility $\mathrm{div}_{\mathsf{S}}(\kappa)$ divides 11 and, since $\mathrm{Disc}(H^\perp) \simeq (\mathbf{Z}/2\mathbf{Z})^2$ (see [DM, (1)]), the divisibility $\mathrm{div}_{H^\perp}(\kappa)$ divides 2, but also divides $\mathrm{div}_{\mathsf{S}}(\kappa)$ (because $\mathsf{S} \subset H^\perp$). It follows that $\mathrm{div}_{H^\perp}(\kappa) = 1$. The lattice $\langle H, \kappa\rangle^\perp$ therefore has discriminant $4\kappa^2$ by the formula [DM, (4)].

It follows that $\widetilde{Y}_{\mathbb{A}}$ is special of discriminant d if and only if $d \equiv 0 \pmod 8$ and S primitively represents $-d/4$. The proposition then follows from the discussion at the end of Section A.3. □

A.5. **Double EPW surfaces and their automorphisms.** Let $Y_A \subset \mathbf{P}(V_6)$ be an EPW sextic, where $A \subset \bigwedge^3 V_6$ is a Lagrangian subspace with no decomposable vectors. By [DK4, Theorem 5.2(2)], there is a canonical connected double covering

$$\widetilde{Y}_A^{\geq 2} \longrightarrow Y_A^{\geq 2} \tag{26}$$

between integral surfaces, with covering involution τ, branched over the finite set $Y_A^{\geq 3}$.

We compare automorphisms of Y_A with those of $\widetilde{Y}_A^{\geq 2}$. Any automorphism of Y_A induces an automorphism of its singular locus $Y_A^{\geq 2}$. This defines a morphism $\mathrm{Aut}(Y_A) \to \mathrm{Aut}(Y_A^{\geq 2})$. Since $\mathrm{Aut}(Y_A)$ is a subgroup of $\mathrm{PGL}(V_6)$ and the surface $Y_A^{\geq 2}$ is not contained in a hyperplane, this morphism is injective.

Proposition A.6 (Kuznetsov)**.** *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors. Any element of* $\mathrm{Aut}(Y_A)$ *lifts to an automorphism of $\widetilde{Y}_A^{\geq 2}$. These lifts form a subgroup of* $\mathrm{Aut}(\widetilde{Y}_A^{\geq 2})$ *which is isomorphic to the group* $\widetilde{\mathrm{Aut}}(Y_A)$ *in the extension* (20) *via an isomorphism that takes $\langle\tau\rangle$ to $\boldsymbol{\mu}_2$.*

Proof. The proof follows the exact same steps as the proof of Proposition A.2, whose notation we keep. By [DK4, Theorem 5.2(2)], the surface $\widetilde{Y}_A^{\geq 2}$ is defined as

$$\widetilde{Y}_A^{\geq 2} = \mathrm{Spec}(\mathscr{O}_{Y_A^{\geq 2}} \oplus \mathscr{R}_2(-3)), \tag{27}$$

where $\mathscr{R}_2 = (\bigwedge^2 \mathscr{R}_1|_{Y_A^{\geq 2}})^{\vee\vee}$. As in the proof of Proposition A.2, the group $\widetilde{\mathrm{Aut}}(Y_A)$ acts on $\widetilde{Y}_A^{\geq 2}$ and the nontrivial element of $\boldsymbol{\mu}_2$ acts by -1 on both $\mathscr{R}_1$ and $\mathscr{O}(-3)$. It follows that it acts by 1 on $\mathscr{R}_2$ and by -1 on $\mathscr{R}_2(-3)$, hence as the involution τ on $\widetilde{Y}_A^{\geq 2}$. This proves the proposition. ☐

It is possible to deform the double cover (26) to the canonical double étale covering associated with the (smooth) variety of lines on a quartic double solid (see the proof of [DK5, Proposition 2.5]), so we can use Welters' calculations in [W, Theorem (3.57) and Proposition (3.60)]. In particular, the abelian group $H_1(\widetilde{Y}_A^{\geq 2}, \mathbf{Z})$ is free of rank 20 (and τ acts as $-\mathrm{Id}$) and there are canonical isomorphisms ([DK5, Proposition 2.5])

$$\begin{aligned} T_{\mathrm{Alb}(\widetilde{Y}_A^{\geq 2}),0} &\simeq H^1(\widetilde{Y}_A^{\geq 2}, \mathscr{O}_{\widetilde{Y}_A^{\geq 2}}) \simeq A, \\ H^2(Y_A^{\geq 2}, \mathbf{C}) &\simeq \textstyle\bigwedge^2 H^1(\widetilde{Y}_A^{\geq 2}, \mathbf{C}) \simeq \bigwedge^2 (A \oplus \bar{A}). \end{aligned} \tag{28}$$

The Albanese variety $\mathrm{Alb}(\widetilde{Y}_A^{\geq 2})$ is thus an abelian variety of dimension 10 and one can consider the analytic representation (see Section C.1)

$$\rho_a \colon \mathrm{Aut}(\widetilde{Y}_A^{\geq 2}) \longrightarrow \mathrm{GL}(T_{\mathrm{Alb}(\widetilde{Y}_A^{\geq 2}),0}) \simeq \mathrm{GL}(A).$$

Recall from Proposition A.6 that there is an injective morphism $\widetilde{\mathrm{Aut}}(Y_A) \hookrightarrow \mathrm{Aut}(\widetilde{Y}_A^{\geq 2})$.

Proposition A.7. *Let Y_A be a quasi-smooth EPW sextic. The restriction of the analytic representation ρ_a to the subgroup $\widetilde{\mathrm{Aut}}(Y_A)$ of $\mathrm{Aut}(\widetilde{Y}_A^{\geq 2})$ is the injective middle vertical map in the diagram* (21).

Proof. The morphism ρ_a is the representation of the group $\mathrm{Aut}(\widetilde{Y}_A^{\geq 2})$ on the vector space

$$T_{\mathrm{Alb}(\widetilde{Y}_A^{\geq 2}),0} \simeq H^1(\widetilde{Y}_A^{\geq 2}, \mathscr{O}_{\widetilde{Y}_A^{\geq 2}}).$$

As in the proof of [DK5, Proposition 2.5]), there are canonical isomorphisms

$$H^1(\widetilde{Y}_A^{\geq 2}, \mathscr{O}_{\widetilde{Y}_A^{\geq 2}}) \simeq H^1(Y_A^{\geq 2}, \mathscr{R}_2(-3)) \simeq H^1(Y_A^{\geq 2}, \mathscr{O}_{Y_A^{\geq 2}}(3))^\vee,$$

where the first isomorphism comes from (27) and the second one from Serre duality (because $\mathscr{R}_2$ is the canonical sheaf of $Y_A^{\geq 2}$).

As in the proof of Proposition A.6, the sheaf $\mathscr{O}_{Y_A^{\geq 2}}(3)$ has an $\widetilde{\mathrm{Aut}}(Y_A)$-linearization, where $\mathrm{Aut}(Y_A)$ acts on $Y_A^{\geq 2}$ by restriction and the nontrivial element of $\boldsymbol{\mu}_2$ acts by -1 on $\mathscr{O}_{Y_A^{\geq 2}}(3)$.

By construction, the resolution

$$0 \to (\textstyle\bigwedge^2 \mathscr{A}_2)(-6) \to (\mathscr{A}_1^\vee \otimes \mathscr{A}_2)(-6) \to (\mathrm{Sym}^2 \mathscr{A}_1)(-6) \oplus \mathscr{O}_{\mathbf{P}(V_6)}(-6) \to \mathscr{O}_{\mathbf{P}(V_6)} \to \mathscr{O}_{Y_A^{\geq 2}} \to 0$$

given in [DK2, (33)] is $\widetilde{\mathrm{Aut}}(Y_A)$-equivariant, hence induces an $\widetilde{\mathrm{Aut}}(Y_A)$-equivariant isomorphism

$$H^1(Y_A^{\geq 2}, \mathscr{O}_{Y_A^{\geq 2}}(3)) \simeq H^3(\mathbf{P}(V_6), (\mathscr{A}_1^\vee \otimes \mathscr{A}_2)(-3)) = A^\vee \otimes H^3(\mathbf{P}(V_6), \mathscr{A}_2(-3)).$$

As already noted during the proof of Proposition A.2, $\widetilde{\mathrm{Aut}}(Y_A)$ acts trivially on the 1-dimensional vector space $H^3(\mathbf{P}(V_6), \mathscr{A}_2(-3)) = H^3(\mathbf{P}(V_6), \bigwedge^2 T_{\mathbf{P}(V_6)}(-6))$. All this proves that the action of $\mathrm{Aut}(\widetilde{Y}_A^{\geq 2})$ on $T_{\mathrm{Alb}(\widetilde{Y}_A^{\geq 2}),0}$ is indeed given by the desired morphism. ☐

A.6. **Automorphisms of GM varieties.** Let as before V_6 be a 6-dimensional vector space and let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors. Let $V_5 \subset V_6$ be a hyperplane and let X be the associated (smooth ordinary) GM variety (Section 2.2). One has (see (3))

$$\mathrm{Aut}(X) \simeq \{g \in \mathrm{PGL}(V_6) \mid \textstyle\bigwedge^3 g(A) = A,\ g(V_5) = V_5\}.$$

Since the extension (20) splits (Proposition A.2), there is a lift

$$\mathrm{Aut}(X) \longrightarrow \mathrm{GL}(A) \tag{29}$$

(see (21)) which is injective by Lemma A.1.

When the dimension of X is either 3 or 5, its intermediate Jacobian $\mathrm{Jac}(X)$ is a 10-dimensional abelian variety. By [DK5, Theorem 1.1], it is canonically isomorphic to $\mathrm{Alb}(\widetilde{Y}_A^{\ge 2})$ (see (15)). Therefore, there is an isomorphism

$$T_{\mathrm{Jac}(X),0} \xrightarrow{\sim} T_{\mathrm{Alb}(\widetilde{Y}_A^{\ge 2}),0}.$$

Together with the isomorphism (28), this gives an analytic representation

$$\rho_{a,X} \colon \mathrm{Aut}(X) \longrightarrow \mathrm{GL}(T_{\mathrm{Jac}(X),0}) \xrightarrow{\sim} \mathrm{GL}(A).$$

Proposition A.8. *The analytic representation $\rho_{a,X}$ coincides with the injective morphism* (29)*. Equivalently, the isomorphism* (15) *is* $\mathrm{Aut}(X)$*-equivariant.*

Proof. Assume $\dim(X) = 3$ and choose a line $L_0 \subset X$. The isomorphism $\mathrm{Alb}(\widetilde{Y}_A^{\ge 2}) \xrightarrow{\sim} \mathrm{Jac}(X)$ was then constructed in [DK5, Theorem 4.4] from the Abel–Jacobi map

$$\mathsf{AJ}_{Z_{L_0}} \colon H_1(\widetilde{Y}_A^{\ge 2}, \mathbf{Z}) \longrightarrow H_3(X, \mathbf{Z})$$

associated with a family $Z_{L_0} \subset X \times \widetilde{Y}_A^{\ge 2}$ of curves on X parametrized by $\widetilde{Y}_A^{\ge 2}$. Although the family Z_{L_0} does depend on the choice of L_0, the map $\mathsf{AJ}_{Z_{L_0}}$ does not.

Let $g \in \mathrm{Aut}(X)$ (also considered as an automorphism of $\widetilde{Y}_A^{\ge 2}$). By the functoriality properties of the Abel–Jacobi map ([DK5, Lemma 3.1]), we obtain

$$\mathsf{AJ}_{Z_{L_0}} \circ g_* = \mathsf{AJ}_{(\mathrm{Id}_X \times g)^*(Z_{L_0})} = \mathsf{AJ}_{(g \times \mathrm{Id}_{\widetilde{Y}_A^{\ge 2}})_*(Z_{g^{-1}(L_0)})} = g_* \circ \mathsf{AJ}_{Z_{g^{-1}(L_0)}},$$

which proves the proposition. When $\dim(X) = 5$, the proof is similar, except that Z_{Π_0} is now a family of surfaces in X that depends on a plane $\Pi_0 \subset X$. □

Appendix B. Representations of the group $\mathbb{G}$

The group $\mathbb{G} := \mathrm{PSL}(2, \mathbf{F}_{11})$ is the only simple group of order $660 = 2^2 \cdot 3 \cdot 5 \cdot 11$. It is generated by the classes

$$a = \begin{pmatrix} 5 & 0 \\ 0 & 9 \end{pmatrix}, \quad b = \begin{pmatrix} 3 & 5 \\ -5 & 3 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

of respective orders 5, 6, and 11. We let I_2 be the identity matrix.

The group $\mathbb{G}$ has 8 irreducible $\mathbf{C}$-representations, of dimensions 1, 5, 5, 10, 10, 11, 12, and 12. Here is a character table for 4 of these irreducible representations.

As before, we have set (where $\zeta_{11} = e^{\frac{2i\pi}{11}}$)

$$\lambda := \zeta_{11}^{1^2} + \zeta_{11}^{2^2} + \zeta_{11}^{3^2} + \zeta_{11}^{4^2} + \zeta_{11}^{5^2} = \zeta_{11} + \zeta_{11}^3 + \zeta_{11}^4 + \zeta_{11}^5 + \zeta_{11}^9 = \tfrac{1}{2}(-1 + \sqrt{-11}).$$

Conjugation class	$[I_2]$	$[c]$	$[c^2]$	$[a]=[a^4]$	$[a^2]=[a^3]$	$[b]=[b^5]$	$[b^2]=[b^4]$	$[b^3]$
Cardinality	1	60	60	132	132	110	110	55
Order	1	11	11	5	5	6	3	2
χ_0	1	1	1	1	1	1	1	1
ξ	5	λ	$\bar\lambda$	0	0	1	-1	1
$\xi^\vee$	5	$\bar\lambda$	λ	0	0	1	-1	1
$\bigwedge^2\xi$	10	-1	-1	0	0	1	1	-2

Table 3. Partial character table for $\mathbb{G}$

The representation ξ (which appears in Section 3.1) has a realization in the matrix ring $\mathscr{M}_5(\mathbf{C})$ for which

$$\xi(a)=\begin{pmatrix}0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix},\quad \xi(c)=\begin{pmatrix}\zeta_{11}&0&0&0&0\\0&\zeta_{11}^4&0&0&0\\0&0&\zeta_{11}^5&0&0\\0&0&0&\zeta_{11}^9&0\\0&0&0&0&\zeta_{11}^3\end{pmatrix}. \tag{30}$$

Every irreducible character of $\mathbb{G}$ has Schur index 1 ([Se, § 12.2], [F, Theorem 6.1]). In particular, the representation $\bigwedge^2\xi$, having an integral character, can be defined over $\mathbf{Q}$ and even, by a theorem of Burnside ([Bu]), over $\mathbf{Z}$, that is, by a morphism $\mathbb{G}\to\mathrm{GL}(10,\mathbf{Z})$. The representation $\bigwedge^2\xi$ is self-dual, so there is a $\mathbb{G}$-equivariant isomorphism

$$w\colon \bigwedge^2 V_\xi \xrightarrow{\ \sim\ } \bigwedge^2 V_\xi^\vee, \tag{31}$$

unique up to multiplication by a nonzero scalar, and it is symmetric ([Se, prop. 38]).

Appendix C. Decomposition of abelian varieties with automorphisms

We gather here a few very standard notation and facts about abelian varieties. Let X be a complex abelian variety. We denote by $\mathrm{Pic}(X)$ the group of isomorphism classes of line bundles on X, by $\mathrm{Pic}^0(X)\subset\mathrm{Pic}(X)$ the subgroup of classes of line bundles that are algebraically equivalent to 0, and by $\mathrm{NS}(X)$ the Néron–Severi group $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$, a free abelian group of finite rank. The group $\mathrm{Pic}^0(X)$ has a canonical structure of an abelian variety; it is called the dual abelian variety. Any endomorphism u of X induces an endomorphism $\widehat{u}$ of $\mathrm{Pic}^0(X)$.

Given the class $\theta\in\mathrm{NS}(X)$ of a line bundle L on X, we let φ_θ be the morphism

$$\begin{aligned} X &\longrightarrow \mathrm{Pic}^0(X)\\ x &\longmapsto \tau_x^*L\otimes L^{-1}\end{aligned}$$

of abelian varieties, where τ_x is the translation by x (it is independent of the choice of the representative L of θ). When θ is a polarization, that is, when L is ample, φ_θ is an isogeny.

We say that θ is a principal polarization when φ_θ is an isomorphism. If $n:=\dim(X)$, this is equivalent to saying that the self-intersection number θ^n is $n!$. The associated *Rosati* *involution* on $\operatorname{End}(X)$ is then defined by $u \mapsto u' := \varphi_\theta^{-1} \circ \widehat{u} \circ \varphi_\theta$. The map

$$\begin{aligned} \iota_\theta \colon \operatorname{NS}(X) &\hookrightarrow \operatorname{End}(X) \\ \theta' &\longmapsto \varphi_\theta^{-1} \circ \varphi_{\theta'} \end{aligned}$$

is an injective morphism of free abelian groups whose image is the group $\operatorname{End}^s(X)$ of symmetric elements for the Rosati involution ([BL, Theorem 5.2.4]). If $u \in \operatorname{End}(X)$, one has $\varphi_{u^*\theta'} = \widehat{u} \circ \varphi_{\theta'} \circ u$ hence

$$\iota_\theta(u^*\theta') = \varphi_\theta^{-1} \circ \varphi_{u^*\theta'} = \varphi_\theta^{-1} \circ \widehat{u} \circ \varphi_{\theta'} \circ u = u' \circ \varphi_\theta^{-1} \circ \varphi_{\theta'} \circ u = u' \circ \iota_\theta(\theta') \circ u. \tag{32}$$

Set $\operatorname{NS}_{\mathbf{Q}}(X) = \operatorname{NS}(X) \otimes \mathbf{Q}$ and $\operatorname{End}_{\mathbf{Q}}(X) = \operatorname{End}(X) \otimes \mathbf{Q}$ (both are finite-dimensional $\mathbf{Q}$-vector spaces). If the polarization θ is no longer principal, or if $\theta \in \operatorname{NS}_{\mathbf{Q}}(X)$ is only a $\mathbf{Q}$-polarization, the Rosati involution is still defined on $\operatorname{End}_{\mathbf{Q}}(X)$ by the same formula and we may view ι_θ as an injective morphism

$$\iota_\theta \colon \operatorname{NS}_{\mathbf{Q}}(X) \hookrightarrow \operatorname{End}_{\mathbf{Q}}(X)$$

with image $\operatorname{End}^s_{\mathbf{Q}}(X)$ ([BL, Remark 5.2.5]). Formula (32) remains valid for $u \in \operatorname{End}(X)$ and $\theta' \in \operatorname{NS}_{\mathbf{Q}}(X)$.

We will also need the so-called *analytic* representation

$$\rho_a \colon \operatorname{End}_{\mathbf{Q}}(X) \hookrightarrow \operatorname{End}_{\mathbf{C}}(T_{X,0}).$$

It sends an endomorphism of X to its tangent map at 0.

C.1. **Q-actions on abelian varieties.** Let X be an abelian variety and let G be a finite group. A $\mathbf{Q}$-action of G on X is a morphism $\rho \colon \mathbf{Q}[G] \to \operatorname{End}_{\mathbf{Q}}(X)$ of $\mathbf{Q}$-algebras. The composition

$$G \xrightarrow{\rho} \operatorname{End}_{\mathbf{Q}}(X) \xrightarrow{\rho_a} \operatorname{End}_{\mathbf{C}}(T_{X,0})$$

is called the analytic representation of G.

Proposition C.1. *Let X be an abelian variety of dimension n with a $\mathbf{Q}$-action of a finite group G. Assume that the analytic representation of G is irreducible and defined over $\mathbf{Q}$. Then X is isogeneous to the product of n copies of an elliptic curve.*

Proof. This follows from [ES, (3.1)–(3.4)] (see also [KR, Section 1] and [BL, Proposition 13.6.2]). This reference assumes that we have a faithful action $G \to \operatorname{Aut}(X)$ but only uses the induced morphism $\mathbf{Q}[G] \to \operatorname{End}_{\mathbf{Q}}(X)$ of $\mathbf{Q}$-algebras. □

In the situation of Proposition C.1, we prove that any G-invariant $\mathbf{Q}$-polarization is essentially unique.

Lemma C.2. *Let X be an abelian variety with a $\mathbf{Q}$-action of a finite group G and let θ be a G-invariant polarization on X. If the analytic representation of G is irreducible, any G-invariant $\mathbf{Q}$-polarization on X is a rational multiple of θ.*

Proof. Let $g \in G$, which we view as an invertible element of $\operatorname{End}_{\mathbf{Q}}(X)$. Since θ is g-invariant, identity (32) (applied with $\theta' = \theta$ and $u = g$) implies $g' \circ g = \operatorname{Id}_X$. Let $\theta' \in \operatorname{NS}_{\mathbf{Q}}(X)$. Applying (32) again, we get

$$\iota_\theta(g^*\theta') = g' \circ \iota_\theta(\theta') \circ g = g^{-1} \circ \iota_\theta(\theta') \circ g.$$

If θ' is G-invariant, we obtain $\iota_\theta(\theta') = g^{-1} \circ \iota_\theta(\theta') \circ g$ for all $g \in G$. If the analytic representation of G is irreducible, $\rho_a(\iota_\theta(\theta'))$ must, by Schur's lemma, be a multiple of the identity, hence θ' must be a multiple of θ. □

C.2. **Polarizations on self-products of elliptic curves.** Let E be an elliptic curve, so that $\mathfrak{o}_E := \mathrm{End}(E)$ is either $\mathbf{Z}$ or an order in an imaginary quadratic extension of $\mathbf{Q}$. We have

$$\mathrm{End}(E^n) \simeq \mathscr{M}_n(\mathfrak{o}_E) \quad \text{and} \quad \mathrm{End}_{\mathbf{Q}}(E^n) \simeq \mathscr{M}_n(\mathfrak{o}_E \otimes \mathbf{Q}),$$

and ρ_a is the embedding of these matrix rings into the ring $\mathscr{M}_n(\mathbf{C})$ induced by the analytic representation $\mathfrak{o}_E \hookrightarrow \mathbf{C}$ of E.

Polarizations on E^n were studied in particular by Lange in [L]. We denote by θ_0 the product principal polarization on E^n.

Proposition C.3. *Let E be an elliptic curve.*

- *The Rosati involution defined by θ_0 on $\mathrm{End}(E^n)$ corresponds to the involution $M \mapsto \overline{M}^T$ on $\mathscr{M}_n(\mathfrak{o}_E)$.*
- *Via the embedding ι_{θ_0}, polarizations θ on E^n correspond to positive definite Hermitian matrices $M_\theta \in \mathscr{M}_n(\mathfrak{o}_E)$ and the degree of the polarization θ is $\det(M_\theta)$.*
- *The group of automorphisms $\mathrm{Aut}(E^n, \theta)$ is the unitary group*

$$\mathbf{U}(n, M_\theta) := \{M \in \mathscr{M}_n(\mathfrak{o}_E) \mid \overline{M}^T M_\theta\, M = M_\theta\}.$$

Proof. If we write $E = \mathbf{C}/(\mathbf{Z} \oplus \tau\mathbf{Z})$, the period matrix for E^n is $\begin{pmatrix} I_n & \tau I_n \end{pmatrix}$. The first item then follows from [L, Lemma 2.3] and elements of $\mathrm{NS}(E^n)$ correspond to Hermitian matrices. By [BL, Theorem 5.2.4], polarizations correspond to positive definite Hermitian matrices and the degree of the polarization is the determinant of the matrix. More precisely, one has ([BL, Proposition 5.2.3])

$$\det(TI_n - M_\theta) = \sum_{j=0}^{n} (-1)^{n-j} \frac{\theta_0^j \cdot \theta^{n-j}}{j!(n-j)!} T^j.$$

The last item follows from (32). □

Remark C.4. Let G be a finite group with a $\mathbf{Q}$-representation $\rho\colon \mathbf{Q}[G] \to \mathscr{M}_n(\mathbf{Q})$. For any elliptic curve E, this defines a $\mathbf{Q}$-action of G on E^n. It follows from the proposition that any positive definite symmetric matrix $M_\theta \in \mathscr{M}_n(\mathbf{Q})$ such that, for all $g \in G$,

$$\rho(g)^T M_\theta\, \rho(g) = M_\theta$$

defines a G-invariant $\mathbf{Q}$-polarization on E^n. Such a matrix always exists: take for example $M_\theta := \sum_{g\in G} \rho(g)^T \rho(g)$ (it corresponds to the $\mathbf{Q}$-polarization $\sum_{g\in G} g^*\theta_0$).

The analytic representation is $\rho_{\mathbf{C}}\colon \mathbf{C}[G] \to \mathscr{M}_n(\mathbf{C})$. If it is irreducible, every G-invariant $\mathbf{Q}$-polarization on E^n is, by Lemma C.2, a rational multiple of θ.

We end this section with the construction of an explicit abelian variety of dimension 10 with a $\mathbb{G}$-action, such that the associated analytic representation is the irreducible representation $\bigwedge^2\xi$, together with a $\mathbb{G}$-invariant *principal* polarization. Set $\lambda := \frac{1}{2}(-1 + \sqrt{-11})$ and consider the elliptic curve $E_\lambda := \mathbf{C}/\mathbf{Z}[\lambda]$, which has complex multiplication by $\mathbf{Z}[\lambda]$.

Proposition C.5. *There exists a principal polarization θ on the abelian variety E_λ^{10} and a faithful action $\mathbb{G} \hookrightarrow \mathrm{Aut}(E_\lambda^{10}, \theta)$ such that the associated analytic representation is the irreducible representation $\bigwedge^2\xi$ of $\mathbb{G}$.*

Proof. By [S, Table 1]), there is a positive definite unimodular $\mathbf{Z}[\lambda]$-sesquilinear Hermitian form H' on $\mathbf{Z}[\lambda]^5$ with an automorphism of order 11. Its Gram matrix in the canonical $\mathbf{Z}[\lambda]$-basis $(e_1, \ldots, e_5)$ of $\mathbf{Z}[\lambda]^5$ is

$$\begin{pmatrix} 3 & 1-\bar\lambda & -\lambda & 1 & -\bar\lambda \\ 1-\lambda & 3 & -1 & -\lambda & 1 \\ -\bar\lambda & -1 & 3 & \lambda & -1+\lambda \\ 1 & -\bar\lambda & \bar\lambda & 3 & 1-\bar\lambda \\ -\lambda & 1 & -1+\bar\lambda & 1-\lambda & 3 \end{pmatrix}$$

and its unitary group has order $2^3 \cdot 3 \cdot 5 \cdot 11 = 1\,320$ ([S]).

By Proposition C.3, this form defines a principal polarization θ' on the abelian variety E_λ^5 and the group $\mathrm{Aut}(E_\lambda^5, \theta')$ has order $1\,320$; in particular, it contains an element of order 11. It follows from [BBe] that the group $\mathrm{Aut}(E_\lambda^5, \theta')$ is isomorphic to $\mathbb{G} \times \{\pm 1\}$ and the faithful representation $\mathbb{G} \hookrightarrow \mathrm{Aut}(E_\lambda^5, \theta') \hookrightarrow \mathbf{U}(5, H')$ given by Proposition C.3 is ξ.[3]

The Hermitian form H' on $\mathbf{Z}[\lambda]^5$ induces a positive definite unimodular Hermitian form H on $\bigwedge^2 \mathbf{Z}[\lambda]^5 = \mathbf{Z}[\lambda]^{10}$ by the formula

$$H(x_1 \wedge x_2, x_3 \wedge x_4) := H'(x_1, x_3)H'(x_2, x_4) - H'(x_1, x_4)H'(x_2, x_3).$$

The matrix of H (in the basis $(e_{12}, e_{13}, e_{14}, e_{15}, e_{23}, e_{24}, e_{25}, e_{34}, e_{35}, e_{45})$) is

$$\begin{pmatrix} 4 & 2\lambda & -1-2\lambda & -1-\lambda & -2+2\lambda & -\lambda & -1-2\lambda & -2-\lambda & 1 & -2 \\ 2\bar\lambda & 6 & -1+2\lambda & -1+2\lambda & 6+2\lambda & -2+\lambda & -4+\lambda & \lambda & -\lambda & 2+\lambda \\ -1-2\bar\lambda & -1+2\bar\lambda & 8 & 5+2\lambda & -2-2\lambda & 5+2\lambda & 3+2\lambda & 1-2\lambda & 1 & -1-2\lambda \\ -1-\bar\lambda & -1+2\bar\lambda & 5+2\bar\lambda & 6 & -1-2\lambda & 4 & 5+2\lambda & -1-\lambda & -1-\lambda & -1-\lambda \\ -2+2\bar\lambda & 6+2\bar\lambda & -2-2\bar\lambda & -1-2\bar\lambda & 8 & 2\lambda & -2+3\lambda & 2\lambda & -2-\lambda & 3+\lambda \\ -\bar\lambda & -2+\bar\lambda & 5+2\bar\lambda & 4 & 2\bar\lambda & 6 & 5+2\lambda & 0 & -1 & -\lambda \\ -1-2\bar\lambda & -4+\bar\lambda & 3+2\bar\lambda & 5+2\bar\lambda & -2+3\bar\lambda & 5+2\bar\lambda & 8 & 2 & -1+\lambda & -1-2\lambda \\ -2-\bar\lambda & \bar\lambda & 1-2\bar\lambda & -1-\bar\lambda & 2\bar\lambda & 0 & 2 & 6 & 2+2\lambda & -2\lambda \\ 1 & -\bar\lambda & 1 & -1-\bar\lambda & -2-\bar\lambda & -1 & -1+\bar\lambda & 2+2\bar\lambda & 4 & -2 \\ -2 & 2+\bar\lambda & -1-2\bar\lambda & -1-\bar\lambda & 3+\bar\lambda & -\bar\lambda & -1-2\bar\lambda & -2\bar\lambda & -2 & 4 \end{pmatrix}. \tag{33}$$

By Proposition C.3 again, the form H defines a principal polarization θ on the abelian variety E_λ^{10}, the group $\mathrm{Aut}(E_\lambda^{10}, \theta)$ contains $\mathbb{G}$, and the corresponding analytic representation is $\bigwedge^2 \xi$. $\square$

The $\mathbb{G}$-action on E_λ^{10} in the proposition is not the $\mathbb{G}$-action described in Remark C.4 (otherwise, since $\mathbb{G}$-invariant polarizations are proportional, the matrix (33) would, by Lemma C.2, have rational coefficients): these actions are only conjugate by a $\mathbf{Q}$-automorphism of E_λ^{10}.

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