SOME VARIETIES WITH $\mathrm{PSL}(2, \mathbf{F}_{11})$-ACTIONS

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ABSTRACT. Starting from an explicit EPW sextic with a faithful action of $\mathrm{PSL}(2, \mathbf{F}_{11})$ found by Giovanni Mongardi in his thesis, we construct various varieties with faithful actions of this finite simple group of order 660. This leads to constructions of further interesting varieties, such as explicit smooth irrational Gushel-Mukai threefolds and their 10-dimensional principally polarized intermediate Jacobians. This is joint work with Giovanni Mongardi.

1. A bit of history

In this talk, G will be the unique simple group of order 660. It is isomorphic to $\mathrm{PSL}(2, \mathbf{F}_{11})$. In 1879, Felix Klein considered the smooth cubic threefold (now called the Klein cubic) with equation

$$x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1 = 0$$

in $\mathbf{P}^4$ and proved that it has a faithful action of the group G by projective transformations. This can be checked by starting from one of the two conjugate irreducible representations ξ of degree 5 of G and proving, by a character computation, that there is a unique (up to multiplication by a constant) nonzero invariant cubic polynomial which can then be computed explicitly. This is what Adler did in 1978. He also went on to prove that the automorphism group of the Klein cubic is exactly G and extended these results as follows: for any prime number $p \geq 11$ such that $p \equiv 3 \pmod 8$, there exists a unique cubic in $\mathbf{P}^{\frac{1}{2}(p-3)}$ acted on faithfully by the group $\mathrm{PSL}(2, \mathbf{F}_p)$ and its automorphism group is exactly $\mathrm{PSL}(2, \mathbf{F}_p)$.

As any smooth cubic, the Klein cubic has an intermediate Jacobian, a 5-dimensional principally polarized abelian variety which inherits a faithful G-action. Given a prime number p such that $p \equiv 3 \pmod 4$, Adler classified all abelian varieties (non necessarily principally polarized) of dimension $\frac{1}{2}(p-1)$ whose automorphism group contains $\mathrm{PSL}(2, \mathbf{F}_p)$. They are all of CM type and Adler showed that there is a bijection between the group of ideal classes of the field $\mathbf{Q}(\sqrt{-p})$ and the set of isomorphism classes of these abelian varieties. Moreover, these abelian varieties all have a unique $\mathrm{PSL}(2, \mathbf{F}_p)$-invariant principal polarization and it is indecomposable (Bennama–Bertin, 1997).

In particular, when the class number of $\mathbf{Q}(\sqrt{-p})$ is 1, that is, when

$$p \in \{3, 7, 11, 19, 43, 67, 163\},$$

there is a unique such abelian variety and it is isomorphic to the product of $\frac{1}{2}(p-1)$ copies of the (unique) elliptic curve whose endomorphism ring is the ring of integers of $\mathbf{Q}(\sqrt{-p})$. For example,

- ($p = 7$) the (3-dimensional) Jacobian of the Klein quartic plane curve with equation $x_1^3x_2+x_2^3x_3+x_3^3x_1 = 0$ is isomorphic to product of 3 copies of the elliptic curve $\mathbf{C}/\mathbf{Z}[\frac{1}{2}(1+\sqrt{-7})]$;
- ($p = 11$) the intermediate Jacobian of the Klein cubic threefold is isomorphic to the product of 5 copies of the elliptic curve $\mathbf{C}/\mathbf{Z}[\frac{1}{2}(1+\sqrt{-11})]$.

In general, Bennama–Bertin interpret the abelian varieties constructed above as intermediate Jacobians of mildly singular hypersurfaces of degree p in $\mathbf{P}^{\frac{1}{2}(p-3)}$. I do not know of any relation with Adler's $\mathrm{PSL}(2, \mathbf{F}_p)$-invariant cubics in $\mathbf{P}^{\frac{1}{2}(p-3)}$.

2. Eisenbud–Popescu–Walter sextics and Gushel–Mukai varieties

I will begin by introducing the various kinds of varieties we will be dealing with. My running notation for an m-dimensional complex vector space is V_m.

2.1. **Eisenbud–Popescu–Walter (EPW) sextics.** These are (singular) sextics $Y_A \subset \mathbf{P}(V_6)$ constructed from a Lagrangian $A \subset \bigwedge^3 V_6$ as the Lagrangian degeneracy loci

$$Y_A := \{[x] \in \mathbf{P}(V_6) \mid A \cap (x \wedge \textstyle\bigwedge^2 V_6) \neq 0\}$$

(the definition is not important). When A satisfies certain explicit genericity conditions, we will say that Y_A (or A) is *quasi-smooth;* its singular locus is then a smooth surface and there is a canonical double covering

$$\widetilde{Y}_A \longrightarrow Y_A$$

branched along that surface, where $\widetilde{Y}_A$ is a smooth hyperkähler fourfold (called a *double EPW sextic*).

2.2. **Gushel–Mukai (GM) varieties.** These are (smooth) Fano varieties of dimension $n \in \{3, 4, 5\}$ with Picard number 1 and index $n - 2$. They were classified by Mukai and most of them can be obtained as complete intersections

$$X := \mathsf{Gr}(2, V_5) \cap \mathbf{P}^{n+4} \cap (\text{quadric}) \subset \mathbf{P}(\textstyle\bigwedge^2 V_5).$$

They can also be obtained, by a complicated process, from the data of

- a quasi-smooth Lagrangian $A \subset \bigwedge^3 V_6$ and
- a hyperplane $V_5 \subset V_6$.

The dimension of the resulting smooth GM variety X_{A,V_5} is given by

$$n = 5 - \dim(A \cap \textstyle\bigwedge^3 V_5).$$

If $Y_A^\vee \subset \mathbf{P}(V_6^\vee)$ is the projective dual of Y_A (it is also an EPW sextic!), this dimension is

- 5 if $[V_5] \in \mathbf{P}(V_6^\vee) \smallsetminus Y_A^\vee$,
- 4 if $(Y_A^\vee)_{\mathrm{smooth}}$,
- 3 if $(Y_A^\vee)_{\mathrm{sing}}$.

2.3. **The Mongardi Lagrangian.** Let ξ be the degree-5 irreducible representation of G mentioned above. There is a unique G-invariant quadric

$$\mathsf{Q} \subset \mathbf{P}(\textstyle\bigwedge^2 V_\xi)$$

and

$$X^5_{\mathsf{A}} := \mathsf{Q} \cap \mathsf{Gr}(2, V_\xi)$$

is a (smooth) GM fivefold with a faithful G-action.

The Lagrangian A that appears in the notation can be constructed as follows. Consider the representation

$$V_6 := \mathbf{C}e_0 \oplus V_\xi,$$

where G acts trivially on e_0. One has a decomposition

$$\textstyle\bigwedge^3 V_6 = (e_0 \wedge \bigwedge^2 V_\xi) \oplus \bigwedge^3 V_\xi$$

into irreducible representations. Choose a G-isomorphism $v\colon \bigwedge^2 V_\xi \xrightarrow{\sim} \bigwedge^3 V_\xi$. The graph

$$\mathsf{A} := \{e_0 \wedge x + v(x) \mid x \in \textstyle\bigwedge^2 V_\xi\}$$

of v is a quasi-smooth Lagrangian and X^5_{A} is the GM fivefold X_{A,V_ξ} associated with A and the hyperplane $V_\xi \subset V_6$ by the procedure mentioned above.

2.4. **Automorphisms.** With the notation above, one has, for any quasi-smooth Lagrangian A,

$$\mathrm{Aut}(Y_A) = \{g \in \mathrm{PGL}(V_6) \mid (\textstyle\bigwedge^3 g)(A) = A\},$$
$$\mathrm{Aut}(X_{A,V_5}) = \{g \in \mathrm{Aut}(Y_A) \mid g(V_5) = V_5\}.$$

It can be shown that the inclusions

$$\mathsf{G} \subset \mathrm{Aut}(X^5_{\mathsf{A}}) \subset \mathrm{Aut}(Y_A)$$

are equalities (note that V_ξ is fixed by the G-action). The automorphism groups of the GM varieties that can be constructed from the Mongardi Lagrangian A and various hyperplanes $V_5 \subset V_6$ are therefore all subgroups of G.

One can for example construct an explicit GM threefold with cyclic automorphism group of order 11 (this is the largest possible order). It is the intersection of $\mathsf{Gr}(2,5)$ in its Plücker embedding with the quadric defined by the equations

$$x_{03} + x_{12} = x_{04} - x_{23} = x_{01}x_{02} - x_{13}x_{14} - x_{24}x_{34} = 0.$$

3. Hodge structures

There are relations between the Hodge structures of various varieties attached to an EPW sextic Y_A (such as the double EPW sextic $\widetilde{Y}_A$) and those of the GM varieties X_{A,V_5} constructed from A (and hyperplanes $V_5 \subset V_6$).

For example, we can construct a GM fourfold X_{A,V_5} with Hodge group $\mathrm{Hdg}^2(X_{\mathsf{A},V_5})$ of maximal rank 22 (note that the integral Hodge conjecture is known in that case by recent work of Perry).

In dimensions 3 and 5, GM varieties have 10-dimensional principally polarized intermediate Jacobians and Kuznetsov and I proved that there is a canonical isomorphism

$$\mathrm{Jac}(X_{A,V_5}) \xrightarrow{\sim} \mathrm{Alb}(\widetilde{Y}^2_A),$$

where $\widetilde{Y}^2_A \to (Y_A)_{\mathrm{sing}}$ is a canonical double étale cover. In particular, the Torelli property for GM varieties of dimension 3 or 5 does not hold, because the intermediate Jacobian depends only on A and not on the choice of the hyperplane V_5.

Automorphisms of Y_A induce automorphisms of the surfaces $(Y_A)_{\mathrm{sing}}$ and $\widetilde{Y}^2_A$, hence automorphisms of the principally polarized abelian varieties $\mathrm{Alb}(\widetilde{Y}^2_A)$ and $\mathrm{Jac}(X_{A,V_5})$.

One can use this to construct explicit irrational GM threefolds (it has long been known, by a degeneration argument, that general GM threefolds are irrational, a fact that is reproved by the theorem below).

Theorem 1. *Any smooth GM threefold constructed from the Mongardi Lagrangian* A *is irrational and there exists a complete family with maximal variation, parametrized by a projective surface, of irrational smooth GM threefolds.*

In particular, the explicit GM threefold with cyclic automorphism group of order 11 constructed above is irrational.

Proof. We apply the Clemens–Griffiths criterion: it suffices to prove that the intermediate Jacobian is not a Jacobian of curve or a product of such. Following Beauville, we will prove that it has too many automorphisms. By what we just saw, this intermediate Jacobian has a faithful G-action. Assume it is the Jacobian of a (say smooth) curve C of genus 10. The group G then embeds into the group of automorphisms of $(\mathrm{Jac}(C), \theta_C)$; by the Torelli theorem for curves, this group is isomorphic to $\mathrm{Aut}(C)$ if C is hyperelliptic and to $\mathrm{Aut}(C) \times \mathbf{Z}/2\mathbf{Z}$ otherwise. Since any morphism from G to $\mathbf{Z}/2\mathbf{Z}$ is trivial, we see that G is a subgroup of $\mathrm{Aut}(C)$. This contradicts the fact that the automorphism group of a curve of genus 10 has order at most 432. □

Remark 2. It is a general fact that all GM varieties of the same dimension constructed from the same Lagrangian A are birationally isomorphic. So their rationality only depends on A. We do not know whether GM fourfolds constructed from A are rational. One can see directly, by elementary geometric constructions, that all (smooth) GM fivefolds are rational.

4. A mysterious abelian variety

The 10-dimensional abelian variety $\mathsf{J} := \mathrm{Alb}(\widetilde{Y}^2_{\mathsf{A}})$ with its canonical principal polarization θ_{J} seems worth studying. The group G acts on it and the induced action on its Lie algebra (the so-called analytic representation) is $\bigwedge^2\xi$, one of the two 10-dimensional irreducible representations of G, which is defined over $\mathbf{Q}$. In particular, it has complex multiplication by the cyclotomic field $\mathbf{Q}(\zeta_{11})$.

Theorem 3 (Ekedahl–Serre 1993, Lange 2004)**.** *Let G be a finite group that acts on an abelian variety X of dimension n. Assume that the analytic representation of G is irreducible and defined over $\mathbf{Q}$. Then X is isogeneous to the product of n copies of an elliptic curve.*

Conversely, for any elliptic curve E there is an action of G on E^{10} whose associated analytic representation is $\bigwedge^2 \xi$: this representation is defined over $\mathbf{Q}$, hence over $\mathbf{Z}$ by a theorem of Burnside; since

$$\mathrm{End}(E^{10}) \simeq \mathscr{M}_{10}(\mathrm{End}(E)) \supset \mathscr{M}_{10}(\mathbf{Z})$$

one can easily achieve what we want. Moreover, if θ_0 is any polarization on E^{10} (for example the product principal polarization), $\sum_{g \in \mathsf{G}} g^* \theta_0$ will be a G-invariant polarization.

If we hope to characterize J by what we have, we need to hope that the presence of a G-invariant *principal polarization* is a strong condition (this is in contrast with the situation described in the first section, where the existence of the G-action on E^5 forced E to have complex multiplication by $\mathbf{Q}(\sqrt{-11})$, but the principal polarization came for free).

Another possibility would be to follow what was classically done for the modular curve $X(11)$ to analyze its simple factors (which are all elliptic curves) and look for quotients of the surface $\widetilde{Y}^2_{\mathsf{A}}$ by various subgroups of G and hope that one of them has irregularity 1.

We have a candidate for J that can be constructed as follows. If $\mathfrak{o} = \mathbf{Z}[\frac{1}{2}(1 + \sqrt{-11})]$ is the ring of integers of the field $\mathbf{Q}(\sqrt{-11})$ and E is the elliptic curve $\mathbf{C}/\mathfrak{o}$ (with complex multiplication by $\mathfrak{o}$), one can show that there is a principal polarization on E^{10} which is invariant by the G-action described above.

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