FANO VARIETIES*

Olivier Debarre

February 10, 2003

Contents

*These are notes from a minicourse given at the Instructional Conference "Higher-Dimensional Varieties and Rational Points" held at the Alfréd Rényi Institute of Mathematics in Budapest, September 2–7, 2001. Szeretném megköszönni a konferencia szervezőinek a meghívást.

There are few objects canonically attached to a smooth projective algebraic variety X of dimension n. One of them is the invertible sheaf of differential n-forms, or equivalently the corresponding divisor class, for which one chooses a representant K_X, called a *canonical divisor*. One can start a classification according to "how ample" K_X is, and consider the two extreme classes of varieties: those for which K_X is ample, and those for which $-K_X$ is ample. Varieties X for which K_X is ample are called *varieties of general type* and, as their name suggests, there are so many of them that a classification is impossible. In constrast, there are "very few" varieties X for which $-K_X$ is ample; they are called *Fano varieties.* In many cases, such as in Mori's classification program of algebraic varieties known as the Minimal Model Program, it is necessary to allow some kind of singularities, which we describe below.

The purpose of these notes is to survey the state of the present knowledge concerning some of the geometrical properties of Fano varieties.

1 Definitions and examples

A (smooth) Fano variety is usually defined as a smooth projective variety defined over an algebraically closed field, whose anticanonical divisor is ample. As explained above, it is necessary to allow some kind of singularities, which we now describe.

Let X be a normal variety defined over an algebraically closed field of characteristic zero such that some multiple of the Weil canonical divisor[1] K_X is a Cartier divisor (we say that K_X is a $\mathbf{Q}$-Cartier divisor). For any smooth variety Y and proper morphism $f : Y \to X$, write

$$K_Y \sim f^*K_X + \sum_i a_i E_i \tag{1}$$

where the E_i are f-exceptional divisors, the a_i are rational numbers, and $\sim$ denotes numerical equivalence.[2] The *discrepancy* of X is the minimum of the a_i for all possible f.

[1]For any projective variety X that is nonsingular in codimension 1, we define a canonical (Weil) divisor class K_X as the class of the closure of a canonical divisor of the smooth locus of X (see [H], II, Proposition 6.5(b)).

[2]This means that the degree of both sides on any curve is the same.

The discrepancy can be computed from any resolution f whose exceptional locus has normal crossings (these resolutions exist by Hironaka's theorem) by the formula

$$\operatorname{discrep}(X) = \begin{cases} \min(1, a_i) & \text{if the } a_i \text{ are all } \geq -1 \\ -\infty & \text{otherwise} \end{cases}$$

([KM], Corollary 2.32(2)). It is therefore either $-\infty$ or a rational number in $[-1, 1]$, and is 1 if X is smooth. We say that X has *log terminal singularities* if its discrepancy is > -1.

Definition 1 *A Fano variety X is a normal projective variety with log terminal singularities such that the anticanonical divisor $-K_X$ is an ample* **Q***-Cartier divisor.*

We will always make the implicit assumption that the characteristic is zero when we consider singular Fano varieties (this is necessary for the definition of log terminal to make sense).

As mentioned in the introduction, Fano varieties, although they are important for many reasons, are quite rare in nature. For example, as we will see later, there are only finitely many deformation types of smooth Fano varieties in any given dimension. Let us give a few examples.

Examples 2 (1) A projective curve is a Fano variety if and only if it is smooth and rational.

(2) A smooth Fano surface is usually called a del Pezzo surface. They have been classified and fall into 10 deformations types, to wit $\mathbf{P}^1 \times \mathbf{P}^1$ and the blow-up of $\mathbf{P}^2$ in at most 8 points. These points need to be in general position (e.g., no three are colinear). Singular del Pezzo surfaces are studied in [Dz]. A normal complex surface germ has log terminal singularities if and only if it is (analytically) the quotient of a smooth germ by a finite subgroup of $\mathrm{GL}(2, \mathbf{C})$.

(3) Smooth Fano threefolds have been classified in characteristic zero: there are 17 families with Picard number 1 ([I1], [I2],[3] [Sh][4]), and 88 other

[3] Iskovskikh's incomplete classification was completed by Mukai and Umemura in [MU] and by Cutkosky in [Cu].

[4] Iskovskikh's and Shokurov's arguments were greatly simplified by Takeuchi in [Ta].

families.[5] In positive characteristic, see [SB], where Fano threefolds with Picard number 1 are classified, and [Me] for additional results.

(4) A finite product of Fano varieties is a Fano variety.

(5) Let Y be a smooth subvariety of $\mathbf{P}^n$, with hyperplane divisor H, such that $K_Y \sim qH$ for some rational number q (this is the case for example when Y is a smooth curve). Assume further that Y is projectively normal, so that the cone X in $\mathbf{P}^{n+1}$ over Y is normal. Let $\tilde{X}$ be the $\mathbf{P}^1$-bundle[6] $\pi : \mathbf{P}(\mathscr{O}_Y \oplus \mathscr{O}_Y(H)) \to Y$ and let Y_0 be the section corresponding the quotient $\mathscr{O}_Y(H)$ of $\mathscr{O}_Y \oplus \mathscr{O}_Y(H)$, so that $Y_0|_{Y_0} \equiv -H$. The cone X is obtained as the contraction $f : \tilde{X} \to X$ of Y_0, and ([H], V, Corollary 2.11)

$$K_{\tilde{X}} \sim -2Y_0 + \pi^*(K_Y - H) = -2Y_0 + \pi^*((q-1)H)$$

hence

$$K_X \sim (q-1)H$$

(these two Weil divisors coincide outside of the vertex of X). The anti-canonical bundle of X is therefore ample if and only if $q < 1$. Writing $K_{\tilde{X}} \sim f^*K_X + aY_0$ as in (1), we get by restricting to Y_0 the relation

$$\big(-2Y_0 + \pi^*((q-1)H)\big)|_{Y_0} = aY_0|_{Y_0}$$

hence $a = -1 - q$. The singularities of X are therefore log terminal if and only if $q < 0$. It follows that *X is a Fano variety if and only if Y is a Fano variety.*

(6) Let X be a normal complete intersection in $\mathbf{P}^{n+s}$ defined by equations of degrees $d_1 \geq \cdots \geq d_s \geq 2$. A canonical divisor is $(d_1 + \cdots + d_s - n - s - 1)H$, where H is a hyperplane section; X is therefore a Fano variety if and only if it has log terminal singularities and $d_1 + \cdots + d_s \leq n + s$. Note that in any given dimension n, there are only finitely many choices for the degrees $d_1, \ldots, d_s$ (because $s \leq \sum(d_i - 1) \leq n$).

(7) A canonical divisor of a normal cyclic covering $X \to \mathbf{P}^n$ of degree $d > 1$ branched along a hypersurface of degree de is $(-n - 1 + (d-1)e)H$, where H is the inverse image of a hyperplane; X is therefore a Fano variety if and only if it has log terminal singularities and $(d-1)e \leq n$ (again, there are only finitely many choices for d and e when n is fixed).

[5] Mori and Mukai recently noticed that the family of blow-ups of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ along a curve of tridegree $(1, 1, 3)$ is missing from the list in [MM].

[6] We follow Grothendieck's notation: for a vector bundle $\mathscr{E}$, the projectivization $\mathbf{P}(\mathscr{E})$ is the space of *hyperplanes* in the fibers of $\mathscr{E}$.

(8) In characteristic zero, a projective variety acted on transitively by a connected linear algebraic group (such as a flag variety) is a smooth Fano variety (see [K1], Theorem V.1.4).

(9) If Y is a Fano variety and $D_1, \dots, D_r$ are nef Cartier divisors on Y such that $-K_Y - D_1 - \cdots - D_r$ is ample, $X = \mathbf{P}(\bigoplus_{i=1}^r \mathscr{O}_Y(D_i))$ is a Fano variety. Indeed, if D is a divisor associated with the line bundle $\mathscr{O}_Y(1)$ and π is the canonical map $X \to Y$, one gets as in [H], V, Lemma 2.10,

$$-K_X = rD + \pi^*(-K_Y - D_1 - \cdots - D_r)$$

Since each D_i is nef, the divisor D is nef on X. Since each $-K_Y - D_1 - \cdots - D_r + D_i$ is ample, the divisor $D + \pi^*(-K_Y - D_1 - \cdots - D_r)$ is ample (because a direct sum of nef (resp. ample) line bundles is a vector bundle that has the same property). It follows that $-K_X$ is ample, and the discrepancy of X is equal to the discrepancy of Y. Hence X is a Fano variety.

2 Toric Fano varieties

Although toric varieties (whose definition is given below in 2.1) are very special among all varieties, they are nevertheless well worth studying for several reasons. First, they provide very intricate examples against which one can test conjectures. For example, they have proved an efficient testing ground for Mori's Minimal Model Program ([R], [W], [Ca2]). The survey [Co] (with its very impressive bibliography) shows how active an area of research toric varieties are.

In another direction, the link that the toric construction establishes between the geometry of projective toric varieties and the combinatorial properties of polytopes has been the source of many fruitful interactions between these two fields (see the survey [F] for a sample of results).

Smooth toric Fano varieties are also very special among general smooth Fano varieties, but they again provide many interesting examples. For instance, their anticanonical hyperplane sections are Calabi–Yau varieties and this construction alone provides in the region of 400 million Calabi–Yau threefolds accessible through popular websites. Their birational geometry is also very rich ([Ca1], [Ca3]). Finally, there is hope that certain bounds (rank of the Picard group, degree,...) for smooth toric Fano varieties fall in the right ballpark for smooth Fano varieties in general.

This section is organized as follows. After quickly going through the basics of the toric construction in 2.1 and 2.2, we prove in 2.3 a couple of (nearly optimal) bounds on the Picard number and degree of smooth toric Fano varieties. The first bound implies that in each dimension, there are only finitely many isomorphism types of smooth toric Fano varieties (Corollary 7), a property that is still valid for singular toric Fano varieties whose discrepancy is bounded away from -1 (Corollary 13). This is a result that is conjectured to hold for general Fano varieties.

2.1 Toric varieties

The algebraic torus of dimension n over the algebraically closed field $\mathbf{k}$ is the affine variety

$$T = \operatorname{Spec}(\mathbf{k}[U_1, U_1^{-1}, \ldots, U_n, U_n^{-1}])$$

A *toric variety* over $\mathbf{k}$ is a normal algebraic variety X defined over $\mathbf{k}$ with an action of T and a dense open orbit isomorphic to T. Toric varieties can all be constructed from very explicit combinatorial data, as we explain below. This makes their study very accessible, and the whole theory is at the same time elementary and very rich.

Affine toric varieties. Let M be a free abelian group of rank n, with dual $N = \operatorname{Hom}_{\mathbf{Z}}(M, \mathbf{Z})$. Set $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$ and $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$.

For each cone σ in $N_{\mathbf{R}}$ generated by a finite number of vectors in N and containing no lines, the dual cone

$$\sigma^\vee = \{u \in M_{\mathbf{R}} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}$$

is also generated by a finite number of elements of M. Moreover, the semigroup $\sigma^\vee \cap M$ is finitely generated (Gordon's lemma) and we may define an affine variety by

$$X_\sigma = \operatorname{Spec}(\mathbf{k}[\sigma^\vee \cap M])$$

If τ is a face of σ, the semigroup $\sigma^\vee \cap M$ is contained in $\tau^\vee \cap M$, so there is a map $X_\tau \to X_\sigma$. In fact, X_τ is a principal open subset of X_σ: if we choose $u \in \sigma^\vee \cap M$ such that $\tau = \sigma \cap u^\perp$, we have $X_\tau = (X_\sigma)_u$.

For example,

$$X_{\{0\}} = \operatorname{Spec}(\mathbf{k}[M]) = T$$

The torus T acts on X_σ: the action corresponds to the map

$$\begin{array}{rcl} \mathbf{k}[\sigma^\vee \cap M] & \to & \mathbf{k}[\sigma^\vee \cap M] \otimes \mathbf{k}[M] \\ u & \mapsto & u \otimes u \end{array}$$

on algebras and is compatible with the inclusion $X_\tau \to X_\sigma$ above.

General toric varieties. Now if we consider a *finite* set Δ of cones in N as above, with the property that every face of a cone in Δ is in Δ and the intersection of any two cones in Δ is a face of each (such a set is called a *fan*), we construct an irreducible $\mathbf{k}$-scheme X_Δ with an action of T by gluing the affine varieties X_σ and $X_{\sigma'}$ along the open set $X_{\sigma\cap\sigma'}$, for all σ and σ' in Δ. It turns out that X_Δ is *irreducible, separated and normal* ([F], pp. 21 and 29) and has dimension n. Each X_σ, hence also X_Δ, contain the open set $X_{\{0\}} = T$.

All toric varieties are obtained via this construction.

Examples 3 Let $(e_1, \ldots, e_n)$ be a basis for N and let $U_1, \ldots, U_n$ be the corresponding elements of $\mathbf{k}[M]$.

(1) We have

$$X_{\langle e_1,\ldots,e_n\rangle} = \operatorname{Spec}(\mathbf{k}[U_1, \ldots, U_n]) = \mathbf{A}^n_{\mathbf{k}}$$

and similarly,

$$X_{\langle e_1,\ldots,e_m\rangle} = \operatorname{Spec}(\mathbf{k}[U_1, \ldots, U_m, U_{m+1}, U_{m+1}^{-1}, \ldots, U_n, U_n^{-1}])$$

is isomorphic to the product of $\mathbf{A}^m_{\mathbf{k}}$ with the $(n-m)$-dimensional torus.

(2) The toric variety associated with the set of cones generated by any proper subset of $\{e_0, e_1, \ldots, e_n\}$, where $e_0 = -e_1 - \cdots - e_n$, is isomorphic to the projective space $\mathbf{P}^n_{\mathbf{k}}$. The action of the torus T is given by

$$(t_1, \ldots, t_n) \cdot (x_0, \ldots, x_n) = (x_0, t_1 x_1, \ldots, t_n x_n)$$

Many geometric properties of the variety X_Δ can be read off the fan Δ.

- By Example (1) above, if a cone σ is generated by part of a basis of N, the corresponding variety X_σ is smooth. It turns out that the converse holds: the toric variety X_Δ is smooth if and only if each cone in Δ can be generated by part of a basis of N ([F], p. 29).

- The toric variety X_Δ is projective if and only if Δ is the set of cones spanned by the faces of a polytope[7] Q with vertices in N and the origin in its interior ([F], p. 72). We denote it also by X_Q.
- The toric variety X_Q is $\mathbf{Q}$-factorial (i.e., every Weil divisor is a $\mathbf{Q}$-Cartier divisor) if and only if Q is simplicial (i.e., each facet has n vertices). The Picard group of X_Q is then free abelian of rank ([F], p. 65)
$$\rho_{X_Q} = \mathrm{Card}(\{\text{vertices of } Q\}) - n \tag{2}$$
- The toric variety X_Q is a Fano variety if the vertices of the polytope Q are primitive vectors in N;[8] in this case, we say that Q is a *Fano polytope*. All toric Fano varieties are obtained in this way. It is a smooth Fano variety if each facet of Q is the convex hull of a basis of N; in this case, we say that Q is a *smooth Fano polytope*. All smooth toric Fano varieties are obtained in this way.

2.2 Toric Fano varieties

Let Q be a Fano polytope in the vector space $N_{\mathbf{R}}$, with vertices primitive vectors in N and associated toric Fano variety X. Its dual
$$P = Q^\vee = \{u \in M_{\mathbf{R}} \mid \langle u, v\rangle \geq -1 \text{ for all } v \in Q\}$$
(this is the opposite of the traditional definition) is a polytope with vertices in $M_{\mathbf{Q}}$ (in M when Q is smooth).

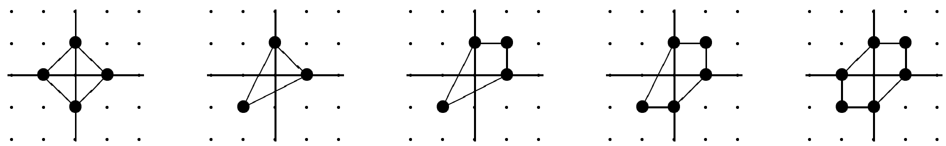
We have ([F], p. 70, Corollary p. 74, p. 75)
$$(-K_X)^n = n!\,\mathrm{vol}(P) \tag{3}$$
where the volume is computed with respect to M (i.e., the volume of the cube determined by a basis of M is 1).

The divisor $-mK_X$ is a Cartier divisor for all integers m such that the vertices of mP are in M and for all such positive integers m, we have ([F], p. 111)
$$h^0(X, -mK_X) = \chi(X, -mK_X) = \mathrm{Card}(M \cap mP) \tag{4}$$

[7] A *polytope* is the convex hull $\mathrm{Conv}(S)$ of a finite set S. Note that different polytopes may define the same fan.

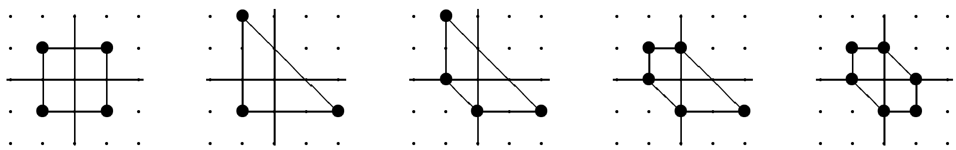
[8] The condition for the canonical divisor of a toric variety to be $\mathbf{Q}$-Cartier will be worked out in Proposition 12, where it is also proved that the singularities are then automatically log terminal; the rest follows from [F], p. 66, p. 72, and the proposition p. 85.

Examples 4 (1) Up to the action of GL(N), there are only 5 smooth Fano polytopes indimension 2. They are



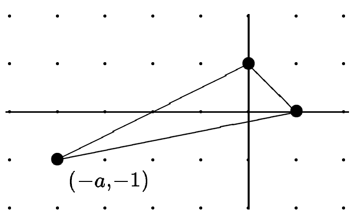
$\mathbf{P}^1 \times \mathbf{P}^1$	$\mathbf{P}^2$	$\mathbf{P}^2$ blown up at a fixed point	$\mathbf{P}^2$ blown up at 2 fixed points	$\mathbf{P}^2$ blown up at 3 fixed points

(the three fixed points of the action of T on $\mathbf{P}^2$ are the coordinate points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$). The dual polytopes are

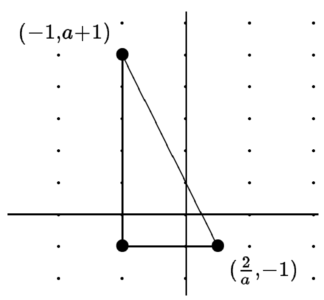


and formulas (3) and (4) are easily checked in these cases.

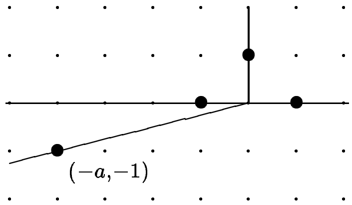
(2) For any positive integer a, the polytope



is a Fano polytope which is singular for $a \geq 2$ and whose dual is



The corresponding Fano variety X_a has a single singular point. Its blow up is the toric variety associated with the fan



By [F], p. 8, this is the Hirzebruch surface $\mathbf{P}(\mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}(a))$ and X_a is obtained from it by contracting the unique curve with self-intersection $-a$. This is a particular case of Example 2(5), where we computed that the discrepancy of X_a is $-1 + \frac{2}{a}$. The smallest positive integer m such that mK_X is a Cartier divisor is a if a is odd, $a/2$ otherwise.

This shows that there are, in each dimension at least 2, infinitely many isomorphism classes of toric Fano varieties.

2.3 Smooth toric Fano varieties

Let Q be a smooth Fano polytope in $N_{\mathbf{R}}$ with $\rho + n$ vertices, associated smooth Fano variety X, and dual polytope P. From (2), we know that the Picard number of X is ρ. The vertices of Q are in N and the vertices of P are in M.

Remarks 5 (1) There is a one-to-one correspondence between the vertices of Q (resp. of P) and the facets of P (resp. of Q). For any vertex u of P and any vertex v of Q, we have $\langle u, v\rangle \geq -1$ and there is equality if and only if v is on the facet F_u of Q that corresponds to the vertex u of P (or viceversa). Otherwise, we have $\langle u, v\rangle \geq 0$, because u is in M and v in N.

(2) Let $e_1, \dots, e_n$ be the vertices of F_u, so that $\langle u, e_i\rangle = -1$ for each i. They form a basis for N (see 2.1) and coordinates will be taken in this basis. Let v^i be the vertex of Q such that the face

$$F_i = \mathrm{Conv}(\{e_1, \dots, e_{i-1}, v^i, e_{i+1}, \dots, e_n\})$$

is adjacent to F_u. We have $v^i_i < 0$ (because the origin and e_i (resp. and v^i) are on the same side of the affine hyperplane generated by F_i (resp. by F_u)) hence $v^i_i = -1$ since the vertices of F_i form a basis for N. It follows that the affine hyperplane spanned by F_i has equation

$$\langle u, x\rangle + (1 + \langle u, v^i\rangle)x_i = -1$$

where $\langle u, v^i \rangle \geq 0$ because v^i is not on F_u. Let v be a vertex of Q such that $\langle u, v \rangle = 0$. If $v \neq v^i$, the vertex v is not on F_i hence

$$0 \leq \langle u, v \rangle + (1 + \langle u, v^i \rangle) v_i = (1 + \langle u, v^i \rangle) v_i$$

and $v_i \geq 0$. This cannot happen for all i, hence v must be among $v^1, \ldots, v^n$. In particular, there are at most n vertices v of Q such that $\langle u, v \rangle = 0$.

Theorem 6 ([VK]) *The Picard number of a smooth toric Fano variety of dimension n is at most $2n^2 - n$.*

Before we begin the proof, let us mention two elementary theorems about the convex hull $\mathrm{Conv}(S)$ of a subset S of $\mathbf{R}^n$ (see [DGK]). The first is Helly's theorem: *any point of* $\mathrm{Conv}(S)$ *is in the convex hull of at most $n+1$ points of S.* The second is Steinitz's theorem : *any point of the interior of* $\mathrm{Conv}(S)$ *is in the interior of the convex hull of at most $2n$ points of S.*

Proof of the Theorem. Since the origin is in the interior of P, it is in the interior of the convex hull of at most $2n$ vertices by Steinitz's theorem. This means that there is a relation

$$0 = \lambda_1 u_1 + \cdots + \lambda_d u_d$$

with $\lambda_1, \ldots, \lambda_d$ positive, where the vertices $u_1, \ldots, u_d$ of P generate the vector space $M_{\mathbf{R}}$ and $n < d \leq 2n$. Let v be a vertex of Q. Since $u_1, \ldots, u_d$ generate $M_{\mathbf{R}}$ and v is nonzero, we must have $\langle u_i, v \rangle \neq 0$ for at least one i, hence $\langle u_i, v \rangle < 0$ for at least one i. This means that v is on the facet F_{u_i} of Q corresponding to this u_i (Remark 5(1)). Since each facet of Q has n vertices, it follows that $\rho + n \leq nd \leq 2n^2$, hence the theorem. □

Corollary 7 ([VK]) *In each dimension, there are only finitely many isomorphism types of smooth toric Fano varieties defined over* $\mathbf{k}$.

Proof. It is enough to prove that up to the action of $\mathrm{GL}(N)$, there are only finitely many smooth Fano polytopes in $N_{\mathbf{R}}$. We know that the number of vertices of such a polytope Q is at most $2n^2$. Since each facet of Q has n vertices, their number is at most $\binom{2n^2}{n}$, hence the volume of Q is at most $\frac{1}{n!}\binom{2n^2}{n}$. Fix the vertices $e_1, \ldots, e_n$ of a facet of Q and denote by S the simplex $\mathrm{Conv}(\{0, e_1, \ldots, e_n\})$. For any other vertex v of Q, the volume of

$\mathrm{Conv}(S \cup \{v\})$ is at most $\frac{1}{n!}\binom{2n^2}{n}$, hence the absolute value of each coordinate of v in the basis $(e_1, \ldots, e_n)$ is bounded by $\binom{2n^2}{n}$. The integral vector v belongs to some bounded set hence may take only finitely many values (at most $\left(2\binom{2n^2}{n}\right)^n$). $\square$

Going back to the Picard number ρ of a smooth toric Fano variety of dimension n, Batyrev conjectured the inequalities[9]

$$\rho \leq \begin{cases} 2n & \text{if } n \text{ is even} \\ 2n-1 & \text{if } n \text{ is odd} \end{cases} \tag{5}$$

for all smooth toric Fano varieties. The inequality $\rho \leq n^2 - n + 1$ for $n \geq 3$ is proved in [VK] and using the same ideas, it is possible to (asymptotically) improve their bound.

Theorem 8 *The Picard number of a smooth toric Fano variety of dimension n is at most*

$$2 + 2\sqrt{(n^2-1)(2n-1)}$$

PROOF. Let u be a vertex of P. The half line $\mathbf{R}^- u$ meets the boundary of P at a point that belongs to some face of dimension $k \leq n-1$ (choose k minimal) hence is in the relative interior of the convex hull of some vertices $u_1, \ldots, u_d$, with $d \leq k+1$ by Helly's theorem. One can therefore write

$$0 = \lambda u + \lambda_1 u_1 + \cdots + \lambda_d u_d$$

with $\lambda_1 \geq \cdots \geq \lambda_d > 0$ and $\lambda > 0$. Let $a \in \{0, \ldots, d\}$, let $\mathscr{V}_a(Q)$ be the set of vertices of Q that are on the facets $F_u, F_{u_1}, \ldots, F_{u_a}$ or on a facet adjacent to them, and let u be a vertex of P in the complement $\mathscr{V}_a(Q)^c$ of $\mathscr{V}_a(Q)$ in the set $\mathscr{V}(Q)$ of all vertices of Q. By Remark 5(2), we have

$$\langle u, v \rangle \geq 1 \qquad \langle u_i, v \rangle \geq 1$$

[9]This holds for $n \leq 4$ thanks to Batyrev's classification ([B2] and [Sa]) and for $n = 5$ (where there is no classification) by [Ca3], Theorem 3.2. Equality is attained for n even by $S^{n/2}$ and for n odd by $S^{(n-1)/2} \times \mathbf{P}^1$, where S is $\mathbf{P}^2$ blown up at the 3 fixed points of the torus action (see Example 4(1)).

for $1 \leq i \leq a$, hence

$$\begin{aligned} -\lambda_{a+1}\langle u_{a+1}, v\rangle - \cdots - \lambda_d\langle u_d, v\rangle &= \lambda\langle u, v\rangle + \sum_{i=1}^{a} \lambda_i\langle u, v_i\rangle \\ &\geq \lambda + \sum_{i=1}^{a} \lambda_i > \sum_{i=1}^{a} \lambda_i \end{aligned}$$

This implies that the integer $\langle u_i, v\rangle$ must be equal to -1 for at least $a+1$ indices i in $\{a+1, \ldots, d\}$, i.e., that v must be on at least $a+1$ faces among $F_{u_{a+1}}, \ldots, F_{u_d}$. Consider the set

$$I = \{(v, i) \in \mathscr{V}_a(Q)^c \times \{a+1, \ldots, d\} \mid v \in F_{u_i}\}$$

The fiber of i for the second projection $I \to \{a+1, \ldots, d\}$ consists of vertices that are on F_{v_i} but not on the intersection $\bigcap_{i=1}^{d} F_{u_i}$, which is an $(n-k-1)$-dimensional face of Q, hence has $n-k$ vertices. It follows that

$$\operatorname{Card}(I) \leq k(d-a)$$

Since we proved that each fiber of the first projection has at least $a+1$ elements, we get

$$\operatorname{Card}(\mathscr{V}_a(Q)^c) \leq \frac{\operatorname{Card}(I)}{a+1} \leq \frac{k(d-a)}{a+1}$$

For $i \in \{2, \ldots, a\}$, the intersection $F_{u_1} \cap F_{u_i}$ is a face of Q of dimension at least $n-k-1$, hence contains at least $n-k$ vertices. It follows that

$$\begin{array}{rcll} \operatorname{Card}(\mathscr{V}_a(Q)) & \leq & 2n & \text{vertices on } F_u \text{ and } F_{u_1} \\ & + & (a-1)k & \text{vertices on } F_{u_2} \cup \cdots \cup F_{u_a} - F_{u_1} \\ & + & (a+1)n & \text{vertices adjacent to } F_u,\ F_{u_1}, \ldots, F_{u_a} \end{array}$$

hence

$$\begin{aligned} \operatorname{Card}(\mathscr{V}(Q)) &\leq \max_{1 \leq d \leq k+1 \leq n} \left(\min_{0 \leq a \leq d} \left(2n + (a-1)k + (a+1)n + \frac{k(d-a)}{a+1} \right) \right) \\ &\leq \max_{1 \leq d \leq k+1 \leq n} \left(\min_{0 \leq a \leq d} \left(3n - 2k + a(k+n) + \frac{k(d+1)}{a+1} \right) \right) \end{aligned}$$

Taking

$$a = \left[\sqrt{\frac{k(d+1)}{k+n}} \right]$$

we get

$$
\begin{aligned}
\mathrm{Card}(\mathscr{V}(Q)) &\le \max_{1\le d\le k+1\le n}\Big(3n-2k+2\sqrt{k(d+1)(k+n)}\Big)\\
&= n+2+2\sqrt{(n^2-1)(2n-1)}
\end{aligned}
$$

□

We now turn our attention towards the integer

$$(-K_X)^n = n!\,\mathrm{vol}(P)$$

often called the *degree* of the smooth Fano variety X. Batyrev gave in [B1] an explicit (but very large) bound for $\mathrm{vol}(P)$.

Note that[10]

$$\mathrm{Int}(P)\cap M = \{0\}$$

There is a bound in [LZ] for the volume of an integral polytope with only one integral interior point (see 2.4); it is doubly exponential in n and is almost optimal in general. For duals of smooth Fano polytopes, we prove that there is a much better bound.

Theorem 9 *Assume $\rho \ge 2$ and let Q be a smooth Fano polytope in $N_{\mathbf{R}}$ with $\rho + n$ vertices and dual P. We have*

$$\mathrm{vol}(P) \le \left(\frac{(n-1)^{\rho+1}-1}{n-2}\right)^n \le n^{\rho n}$$

[10]Indeed, any point u of P is in a cone spanned by a facet F of P with vertices $u_1,\dots,u_s$. It can therefore be written as $u=\sum_{i=1}^s r_iu_i$ with $r_i\in\mathbf{R}^+$ and $\sum r_i \le 1$. This facet corresponds to a vertex v of Q that satisfies $\langle u_i, v\rangle = -1$ for all i, hence

$$\langle u,v\rangle = \sum_{i=1}^s r_i\langle u_i,v\rangle = -\sum_{i=1}^s r_i$$

If u is in M, this is an integer ≥ -1, hence 0 or -1. If it is 0, all the r_i are 0 and $u=0$. If it is -1, the point u is on F.

Equivalently, a smooth toric Fano variety X of dimension n and Picard number $\rho \geq 2$ satisfies

$$(-K_X)^n \leq n! n^{\rho n} \tag{6}$$

A complete toric variety X of dimension n with $\rho = 1$ is isomorphic to $\mathbf{P}^n$, hence

$$(-K_X)^n = (n+1)^n$$

Smooth complete toric varieties with $\rho = 2$ have been classified in [KP]: they are all projectivizations of a decomposable bundle over a projective space of smaller dimension. A tedious check shows that a smooth toric Fano variety X of dimension n with $\rho = 2$ satisfies

$$(-K_X)^n \leq n^{2n}$$

For any integers $\rho \geq 2$ and $n \geq 4$ such that $\frac{n}{\log n} \geq 2^{\rho-2}$, examples of smooth toric Fano varieties X of dimension n and Picard number ρ with

$$(-K_X)^n \geq \left(\frac{n^\rho}{2^{\rho^2-1}(\log n)^{\rho-1}} \right)^n$$

are constructed in [De], Proposition 5.22, so the bound (6) is not too far off.

The theorem is a consequence of the following two lemmas.

Lemma 10 *Let P be a polytope in $M_{\mathbf{R}}$ whose only integral interior point is the origin. Let b be a positive number such that for each vertex u of P and each vertex v of $P^\vee$, we have*

$$-1 \leq \langle u, v \rangle \leq b$$

Then

$$\operatorname{vol}(P) \leq (b+1)^n$$

PROOF. This is Theorem 2.5 from [LZ], but we reproduce the proof for the convenience of the reader. The hypothesis implies $-\frac{1}{b}P \subset P$. Assume to the contrary $\operatorname{vol}(P) > (b+1)^n$ and set

$$P' = \frac{1-\varepsilon}{b+1} P$$

We have $\mathrm{vol}(P') > 1$ for ε positive small enough hence, by van der Corput's theorem, P' contains distinct points p and p' such that $p - p' \in M$. Since P' is convex and $-\frac{1}{b}P' \subset P'$, the point

$$\frac{p - p'}{b+1} = \frac{p + b\left(-\frac{1}{b}p'\right)}{b+1}$$

is in P', hence $p - p'$ is in $(b+1)P'$, which is contained in the interior of P. This contradicts the fact that the origin is the only integral interior point of P. □

Lemma 11 *Let Q be a smooth Fano polytope in $N_{\mathbf{R}}$ with $\rho + n$ vertices. For each vertex u of P and each vertex v of Q, we have*

$$-1 \le \langle u, v\rangle \le \frac{n-1}{n-2}((n-1)^\rho - 1)$$

when $\rho > 1$.

Proof. Consider a facet of Q, with vertices a basis $(e_1, \dots, e_n)$ of N, and the corresponding vertex u of P. It takes the value -1 on each e_i (and nonnegative values on other vertices of Q). Write the distinct values of u on the vertices of Q as an increasing sequence

$$-1 < b_1 < \cdots < b_k$$

with $k \le \rho$. Let j be an integer in $\{1, \dots, k\}$ and let v be a vertex of Q such that $\langle u, v\rangle = b_j$.

Consider a minimal subset I of $\{1, \dots, n\}$ such that v and e_i $(i \in I)$ are *not* all on the same facet of Q. We may and will assume $I = \{1, \dots, r\}$ for some positive integer r. The vector $v + e_1 + \cdots + e_r$ is in the cone spanned by some facet F of Q. The vertices of F form a basis of N, so we may write[11]

$$v + e_1 + \cdots + e_r = v_1 + \cdots + v_s + e'_1 + \cdots + e'_t$$

where

- $v_1, \dots, v_s, e'_1, \dots, e'_t$ are vertices of F (possibly nondistinct);

[11] This is what Batyrev calls a *primitive relation* in [B2].

- $v_1, \ldots, v_s \notin \{e_1, \ldots, e_n\}$ (so that $\langle u, v_i\rangle \geq 0$);
- $e'_1, \ldots, e'_t \in \{e_{r+1}, \ldots, e_n\}$, (by minimality of I).

The vector $\frac{1}{r+1}(v + e_1 + \cdots + e_r)$ is in the interior of Q, hence $s + t \leq r$.

If $r = n$, the polytope Q is the standard polytope of $\mathbf{P}^n$ (see Example 3(2)) and $\rho = 1$, so we have $r < n$.

If $s = 0$, we have $\langle u, v\rangle = r - t \leq n - 1$. If $s > 0$, we have for each i in $\{1, \ldots, s\}$

$$\langle u, v\rangle - r = \langle u, v_1\rangle + \cdots + \langle u, v_s\rangle - t \geq \langle u, v_i\rangle - t$$

hence $\langle u, v_i\rangle \leq \langle u, v\rangle - r + t \leq \langle u, v\rangle - s < \langle u, v\rangle$ (in particular, this case cannot happen when $j = 1$, hence $b_1 \leq n - 1$). It follows that $\langle u, v_i\rangle \leq b_{j-1}$ for each i in $\{1, \ldots, s\}$, hence

$$\begin{aligned} b_j = \langle u, v\rangle &= \langle u, v_1\rangle + \cdots + \langle u, v_s\rangle + r - t \\ &\leq s b_{j-1} + n - 1 \\ &\leq (n-1)(b_{j-1} + 1) \end{aligned}$$

From this relation, we get

$$b_j \leq b_1(n-1)^{j-1} + \frac{n-1}{n-2}\big((n-1)^{j-1} - 1\big) \leq \frac{n-1}{n-2}((n-1)^j - 1)$$

by induction on j. □

2.4 Singular toric Fano varieties

As we saw in Example 4(2), there are infinitely many isomorphism types of toric Fano varieties in any given dimension ≥ 2. We show here (Corollary 13) that by bounding the discrepancy away from -1, we get finitely many isomorphism types, as in the smooth case (Corollary 7).

We first show how to compute the discrepancy of a toric variety in terms of the geometry of its fan Δ and the lattice N.

Proposition 12 *Let X be the toric variety associated with a fan Δ in $N_{\mathbf{R}}$. The divisor K_X is a $\mathbf{Q}$-Cartier divisor if and only if, for each maximal cone*

σ in Δ, spanned by primitive vectors $v_1, \ldots, v_s$ in N, there exists an element u_σ of $M_{\mathbf{Q}}$ such that $\langle u_\sigma, v_i \rangle = 1$ for all i. In this case,

$$\operatorname{discrep}(X) = -1 + \min_{\substack{\sigma \text{ maximal cone in } \Delta \\ v \in \sigma \cap N - \{0, v_1, \ldots, v_s\}}} \langle u_\sigma, v \rangle$$

In particular, X has log terminal singularities.

PROOF. We may, by subdividing the cones in Δ, refine Δ into a fan $\tilde{\Delta}$ to obtain a T-equivariant resolution of singularities $f : \tilde{X} \to X$ whose exceptional locus has normal crossings ([F], Proposition p. 48).

Let σ be a maximal cone in Δ, spanned by primitive vectors $v_1, \ldots, v_s$ of N. The cones of $\tilde{\Delta}$ contained in σ are spanned by vectors among $v_1, \ldots, v_s$ and other primitive elements $w_1, \ldots, w_t$ of $\sigma \cap N$. The latter correspond bijectively to the exceptional divisors $E_1, \ldots, E_t$ of f that meet $f^{-1}(X_\sigma)$.

The complement of the dense orbit T in X_σ is the union of T-invariant irreducible hypersurfaces $D_1, \ldots, D_s$ (one for each v_i).

Any element u of M defines a regular function on T, hence a rational function h_u on X_σ whose divisor is $\sum_i \langle u, v_i \rangle D_i$ ([F], p. 61). Since $K_X = -\sum_i D_i$ ([F], Proposition p. 85), mK_X is a Cartier divisor if and only if there exists, for each maximal cone σ in Δ, an element u of M such that $\langle u, v_i \rangle = -m$ for all i ([F], Exercise p. 62).

The coefficient of the divisor $f^*(mK_X)$ on E_j is that of $f^*(\operatorname{div}(h_u))$, to wit $\langle u, w_j \rangle$, whereas the coefficient of $mK_{\tilde{X}}$ is $-m$. With the notation of (1), this means

$$a_j = -1 + \langle -\frac{1}{m} u, w_j \rangle$$

We have $u_\sigma = -\frac{1}{m} u$. Now each v in $\sigma \cap N$ is in a cone of $\tilde{\Delta}$ hence, since $\tilde{X}$ is smooth, is a linear combination with positive integral coefficients of vectors among $v_1, \ldots, v_s, w_1, \ldots, w_t$. If $v \notin \{0, v_1, \ldots, v_s\}$, the number $\langle u_\sigma, v \rangle$ is at least

$$\min\{2, \langle u_\sigma, w_1 \rangle, \ldots, \langle u_\sigma, w_t \rangle\} = 1 + \min\{1, a_1, \ldots, a_t\}$$

This proves the lemma. □

Let X be the projective toric variety associated with a polytope Q with vertices in N. By Proposition 12, the divisor K_X is a $\mathbf{Q}$-Cartier divisor if and only if, for each facet of Q, the primitive vectors of N in the direction of the vertices are on an affine hyperplane. Let σ be the cone spanned by this facet and let $\langle u_\sigma, \cdot\rangle = 1$ be the equation of this hyperplane. For any positive integer q, we have

$$\operatorname{discrep}(X) \geq -1 + \frac{1}{q} \;\Rightarrow\; \langle u_\sigma, v\rangle \geq \frac{1}{q} \text{ for any } v \text{ in } \sigma \cap N - \{0\}$$

For a *Fano* toric variety, the vertices of Q are primitive (see 2.1), so this last relation implies

$$\operatorname{Int}(Q) \cap qN = \{0\}$$

This remark was used in [BoB] to prove the following.

Corollary 13 *Given positive integers n and q, there are only finitely many isomorphism types of toric Fano varieties of dimension n and discrepancy $\geq -1 + \frac{1}{q}$ defined over $\mathbf{k}$.*

The authors appear to have been unaware of the earlier articles [He] and [LZ], from which the result follows immediately, with explicit bounds: Theorems 1 and 2 give the bound

$$\operatorname{vol}(Q) \leq q^n (7(q+1))^{n2^{n+1}} = c(n,q)$$

for any polytope Q in $\mathbf{R}^n$ such that $\operatorname{Int}(Q) \cap q\mathbf{Z}^n = \{0\}$, and the finiteness of the number of isomorphism classes of such polytopes under the action of $\operatorname{GL}(\mathbf{Z}^n)$. One can also get all kinds of explicit bounds: consider a face of Q with vertices $v_1, \ldots, v_s$ primitive in N and the corresponding vertex u of P. Assume that $v_1, \ldots, v_n$ are linearly independent. We have

$$\delta_u \det_N(v_1, \ldots, v_n) = n! \operatorname{vol}(\langle v_1, \ldots, v_n\rangle) \leq n! \operatorname{vol}(Q) \leq n! c(n,q)$$

Since $\delta_u u$ is in M, we see that

$$(n!c(n,q))! K_X \quad \text{is a Cartier divisor}$$

Let r be the number of facets of Q. Since Q has integral vertices, we have

$$\operatorname{vol}(Q) \geq \frac{r}{n!}$$

If X is $\mathbf{Q}$-factorial, each facet of Q has n vertices (see 2.1), hence the total number $\rho_X + n$ of vertices of Q is at most nr, and

$$\rho_X \leq n n! c(n,q)$$

If mK_X is a Cartier divisor, i.e., if mP is integral, the origin is again the only interior point of mP in mM and Theorem 1 of [LZ] yields explicit bounds on $(-K_X)^n$ and $h^0(X, -mK_X)$.

3 Rational curves on smooth Fano varieties

We now forget about toric varieties and study smooth Fano varieties in general. Our aim is to show that a result analogous to Corollary 7 holds. The method of course is of course very different, since there is no description of smooth Fano varieties in general (except in dimension at most 3). The main ingredient of this proof is the study of rational curves on smooth Fano varieties (which is of interest by itself) via Mori's bend-and-break lemmas.

3.1 The bend-and-break lemmas

Let $\mathbf{k}$ be an algebraically closed field.

Given varieties X and Y defined over $\mathbf{k}$, with X quasi-projective and Y projective, Grothendieck showed ([Gr1], 4(c)) that morphisms from Y to X are parametrized by a locally Noetherian scheme $\mathrm{Mor}(Y, X)$.

Let C be a curve (a projective connected reduced scheme of dimension 1 over $\mathbf{k}$). The scheme $\mathrm{Mor}(C, X)$ has in general countably many components, but if we fix an ample divisor H on X and an integer d, the subscheme $\mathrm{Mor}_d(C, X)$ of $\mathrm{Mor}(C, X)$ that parametrizes "curves" $f : C \to X$ with H-degree $f_*C \cdot H$ equal to d is quasi-projective over $\mathbf{k}$, and $\mathrm{Mor}(C, X)$ is the disjoint union of the $\mathrm{Mor}_d(C, X)$, for all integers d.

The local structure of the scheme $\mathrm{Mor}(C, X)$ around the point $[f]$ can be worked out: the Zariski tangent space is isomorphic to

$$H^0(C, \mathscr{H}om(f^*\Omega^1_X, \mathscr{O}_C))$$

and, *if X is smooth along $f(C)$*, locally around $[f]$, the scheme $\mathrm{Mor}(C, X)$ can be defined by $h^1(C, f^*T_X)$ equations in a nonsingular variety of dimension $h^0(C, f^*T_X)$. In particular, any irreducible component of $\mathrm{Mor}(C, X)$ through $[f]$ has dimension at least

$$h^0(C, f^*T_X) - h^1(C, f^*T_X)$$

Since C is a curve, this is equal by the Riemman-Roch theorem to

$$\chi(C, f^*T_X) = -K_X \cdot f_*C + (1 - g(C)) \dim(X)$$

Fixing a finite subset B of C, we can consider the subscheme $\mathrm{Mor}(C, X; f|_B)$ of $\mathrm{Mor}(C, X)$ that consists of morphisms that coincide with f on B. Fixing the value of a morphism at one point imposes $\dim(X)$ conditions, hence

$$\begin{aligned} \dim_{[f]} \operatorname{Mor}(C, X; f|_B) &\geq \dim_{[f]} \operatorname{Mor}(C, X) - \operatorname{Card}(B) \dim(X) \\ &\geq -K_X \cdot f_*C + (1 - g(C) - \operatorname{Card}(B)) \dim(X) \end{aligned} \tag{7}$$

We now prove the first bend-and-break lemma, which can be found in [Mo] (Theorems 5 and 6). It says that a curve deforming nontrivially, while keeping a point fixed, must break into an effective 1-cycle with a rational component passing through the fixed point.

Proposition 14 (First bend-and-break lemma) *Let X be a projective variety, let $f : C \to X$ be a smooth curve, and let c be a point on C. If $\dim_{[f]} \operatorname{Mor}(C, X; f|_{\{c\}}) \geq 1$, there exists a rational curve on X through $f(c)$.*

According to (7), *when X is smooth along $f(C)$*, the hypothesis is fulfilled whenever

$$-K_X \cdot f_*C - g(C) \dim(X) \geq 1$$

PROOF OF THE PROPOSITION. We may assume that $f(C)$ is irrational. Let T be the normalization of a 1-dimensional subvariety of $\operatorname{Mor}(C, X; f|_{\{c\}})$ passing through $[f]$ and let $\overline{T}$ be a smooth compactification of T. The indeterminacies of the rational evaluation map

$$\begin{aligned} \operatorname{ev} : \quad C \times \overline{T} &\dashrightarrow X \\ (x, t) &\longmapsto f_t(x) \end{aligned}$$

defined on $C \times T$ can be resolved by blowing up points to get a morphism

$$e : S \xrightarrow{\varepsilon} C \times \overline{T} \overset{\operatorname{ev}}{\dashrightarrow} X$$

If ev is defined at every point of $\{c\} \times \overline{T}$, take an affine open neighborhood U of $f(c)$ in X and an open neighborhood V of c in C such that $\operatorname{ev}(V \times \overline{T})$ is contained in U. For each v in V, the image of $\{v\} \times \overline{T}$ is a complete subvariety of the affine variety U, hence is a point. It follows that ev has infinitely many 1-dimensional fibers, so that its image must be the curve $f(C)$. In particular, for each t in T, the image of f_t must be $f(C)$. But $f(C)$ being irrational, there are only finitely many morphisms from C to $f(C)$ mapping c to $f(c)$, so this implies that the f_t remain the same, which is absurd.

Hence there exists a point t_0 in $\overline{T}$ such that ev is not defined at (c, t_0). The fiber of t_0 under the projection $S \to \overline{T}$ is the union of the strict transform of $C \times \{t_0\}$ and a (connected) exceptional rational 1-cycle E which is not entirely contracted by e and meets the strict transform of $\{c\} \times \overline{T}$. Since the latter is contracted by e to the point $f(c)$, the rational 1-cycle e_*E passes through $f(c)$. □

The same proof shows that the proposition still holds when X is a complex compact Kähler variety (because, by a result of Campana, the variety $\mathrm{Mor}(C, X; f|_{\{c\}})$ is *Moishezon*, hence contains algebraic curves as soon as its dimension is positive). However, it fails in general for curves on compact complex manifolds.

Once we know there is a rational curve, it may under certain conditions be broken up into several components. More precisely, if it deforms nontrivially while keeping two points fixed, it must break up (into an effective 1-cycle with rational components).

Proposition 15 (Second bend-and-break lemma) *Let X be a projective variety and let $f : \mathbf{P}^1 \to X$ be a rational curve. If $\dim_{[f]}(\mathrm{Mor}(\mathbf{P}^1, X; f|_{\{0,\infty\}})) \geq 2$, the 1-cycle $f_*\mathbf{P}^1$ is numerically equivalent to the sum of a least two rational curves whose union is connected and passes through $f(0)$ and $f(\infty)$.*

According to (7), *when X is smooth along $f(\mathbf{P}^1)$*, the hypothesis is fulfilled whenever

$$-K_X \cdot f_*\mathbf{P}^1 - \dim(X) \geq 2$$

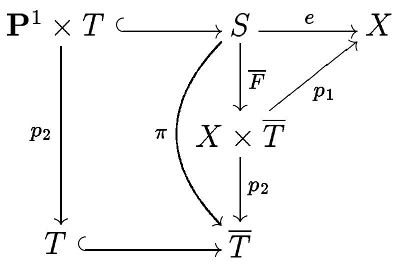
Proof of the Proposition. The group of automorphisms of $\mathbf{P}^1$ fixing two points is the multiplicative group $\mathbf{G}_m$. Let T be the normalization of a 1-dimensional subvariety of $\mathrm{Mor}(\mathbf{P}^1, X; f|_{\{0,\infty\}})$ passing through $[f]$ but not contained in its $\mathbf{G}_m$-orbit. The corresponding map

$$F : \mathbf{P}^1 \times T \to X \times T$$

is finite. Let $\overline{T}$ be a smooth compactification of T, let S be the normalization in the function field of $\mathbf{P}^1 \times T$ of the closure in $X \times \overline{T}$ of the image of F and let $\overline{F} : S \to X \times \overline{T}$ be the canonical finite morphism.[12] Since $\mathbf{P}^1 \times T$ is already

[12] This is constructed exactly as the standard normalization (see [H], II, Example 3.8) by patching up the spectra of the integral closures *in* $K(\mathbf{P}^1 \times T)$ of the coordinate rings of affine open subsets of $\overline{F(\mathbf{P}^1 \times T)}$. The fact that $\overline{F}$ is finite follows from the finiteness of integral closure ([H], I, Theorem 3.9A).

normal, we have $\overline{F}^{-1}(X \times T) = \mathbf{P}^1 \times T$ by uniqueness of normalization, so that there is a commutative diagram



The surface S might not be smooth. On the other hand, we know that no component of a fiber of π is contracted by e (because it would then be contracted by $\overline{F}$).

Since $\overline{T}$ is a smooth curve and S is integral, π is flat ([H], III, Proposition 9.7); hence each fiber C is a 1-dimensional projective scheme without embedded component, whose genus is constant hence equal to 0 ([H], III, Corollary 9.10). In particular, any component C_1 of C_{red} is smooth rational, because $\mathcal{O}_{C_1}$ is a quotient of $\mathcal{O}_C$; hence $H^1(C_1, \mathcal{O}_{C_1})$ is a quotient of $H^1(C, \mathcal{O}_C)$, and therefore vanishes. In particular, if C is integral, it is a smooth rational curve.

Assume all fibers of π are integral. Then S is a (minimal) ruled surface in the sense of [H], V, §2 (Hartshorne assumes that S is smooth, but this hypothesis is not used in the proofs, hence follows from the others). Let T_0 be the closure of $\{0\} \times T$ in S and let T_∞ be the closure of $\{\infty\} \times T$. These sections of π are contracted by e (to $f(0)$ and $f(\infty)$, respectively).

If H is an ample divisor on $e(S)$, which is a surface by construction, we have $(e^*H)^2 > 0$ and $e^*H \cdot T_0 = e^*H \cdot T_\infty = 0$; hence T_0^2 and T_∞^2 are negative by the Hodge index theorem.

However, since T_0 and T_∞ are both sections of π, their difference is linearly equivalent to the pull-back by π of a divisor on $\overline{T}$ ([H], V, Proposition 2.3). In particular,

$$0 = (T_0 - T_\infty)^2 = T_0^2 + T_\infty^2 - 2T_0 \cdot T_\infty < 0$$

which is absurd.

It follows that at least one fiber of π is not integral. Since none of its components is contracted by e, its direct image on X is the required 1-cycle. □

3.2 Rational curves on smooth Fano varieties

We will apply the bend-and-break lemmas to show that any smooth Fano variety X is covered by rational curves. We start from any curve $f : C \to X$ and want to show, using the estimate (7), that it deforms nontrivially while keeping a point x fixed. In positive characteristic, we use the Frobenius morphism to increase the degree of f without changing the genus of C. The first bend-and-break lemma gives in that case the required rational curve through x. Using the second bend-and-break lemma, we can bound the degree of this curve by a constant depending only on the dimension of X, and this will be essential for the remaining step: reduction of the characteristic zero case to positive characteristic.

Assume for a moment that X and x are defined over $\mathbf{Z}$. For almost all prime numbers p, the reduction of X modulo p is a Fano variety of the same dimension; hence there is a rational curve (defined over the algebraic closure of $\mathbf{F}_p$) through x. This means that the scheme $\bigcup_{d>0} \mathrm{Mor}_d(\mathbf{P}^1, X; 0 \mapsto x)$, which is defined over $\mathbf{Z}$, has a geometric point modulo almost all primes p. Since we can moreover bound the degree d of the curve by a constant independent of p, we are in fact dealing with a quasi-projective scheme, and this implies[13] that it has a point over $\bar{\mathbf{Q}}$, hence over $\mathbf{k}$.

In general, X and x are defined over some finitely generated ring and a similar reasoning yields the existence of a $\mathbf{k}$-point of $\bigcup_{d>0} \mathrm{Mor}_d(\mathbf{P}^1, X; 0 \mapsto x)$, that is, of a rational curve on X through x.

Theorem 16 *Let X be a smooth Fano variety of positive dimension n. Through any point of X there is a rational curve of $(-K_X)$-degree at most $n+1$.*

Proof. Let x be a point of X. To construct a rational curve through x, it is enough by the first bend-and-break lemma (Proposition 14) to produce a curve $f : C \to X$ and a point c on C with $\dim_{[f]} \mathrm{Mor}(C, X; f|_{\{c\}}) \geq 1$ and $f(c) = x$. By the dimension estimate (7), it is enough to have

$$-K_X \cdot f_*C - ng(C) \geq 1$$

[13]For a projective scheme, this is because a system of homogeneous polynomial equations with integral coefficients that has, for almost all primes p, a solution modulo p in $\bar{\mathbf{F}}_p$, has a solution in $\bar{\mathbf{Q}}$. One can modify this argument to work for quasi-projective schemes. The proof of this point presented below uses heavier machinery (Chevalley's theorem).

Unfortunately, there is no known way to achieve that, except in positive characteristic. Here is how it works.

Assume that the field $\mathbf{k}$ has characteristic $p > 0$. Choose a smooth curve $f : C \to X$ through x and a point c of C such that $f(c) = x$. Consider the ($\mathbf{k}$-linear) Frobenius morphism $C_1 \to C$ ([H], pp. 301–302). It has degree p, but C_1 and C being isomorphic as abstract schemes have the same genus. Iterating the construction, we get a morphism $F_m : C_m \to C$ of degree p^m between curves of the same genus. But

$$-K_X \cdot (f \circ F_m)_* C_m - ng(C_m) = -p^m K_X \cdot f_* C - ng(C)$$

is positive for m large enough. By the first bend-and-break lemma (Proposition 14), there exists a rational curve $f' : \mathbf{P}^1 \to X$, with say $f'(0) = x$. If

$$-K_X \cdot f'_* \mathbf{P}^1 - n \geq 2$$

the scheme $\mathrm{Mor}(\mathbf{P}^1, X; f'|_{\{0,1\}})$ has dimension at least 2 at $[f']$. By the second bend-and-break lemma (Proposition 15), one can break up the rational curve $f'(\mathbf{P}^1)$ into at least two rational pieces. The component passing through x has smaller $(-K_X)$-degree, and we can repeat the process as long as $-K_X \cdot \mathbf{P}^1 - n \geq 2$, until we get a rational curve of degree no more than $n+1$.

This proves the theorem in positive characteristic. Assume now that $\mathbf{k}$ has characteristic 0. Embed X in some projective space, where it is defined by a finite set of equations, and let R be the (finitely generated) subring of $\mathbf{k}$ generated by the coefficients of these equations and coordinates of x. There is a projective scheme $\mathscr{X} \to \mathrm{Spec}(R)$ with an R-point x_R, such that X is obtained from its generic fiber by base change from the quotient field of R to $\mathbf{k}$.

The geometric generic fiber is a Fano variety of dimension n. There is a dense open subset U of $\mathrm{Spec}(R)$ over which $\mathscr{X}$ is flat ([Gr2], Théorème 6.9.1), and even smooth of dimension n ([Gr3], Théorème 12.2.4(iii)). Since ampleness is an open property ([Gr3], Corollaire 9.6.4), we may even, upon shrinking U, assume that the relative dualizing sheaf $\omega_{\mathscr{X}_U/U}$ is ample on each fiber. It follows that for each maximal ideal $\mathfrak{m}$ of R in U, the (geometric) fiber $X_{\mathfrak{m}}$ is a Fano variety of dimension n. In the future, we will skip these steps when we use this process of reduction to positive characteristic.

Let us take a short break and use a little commutative algebra to show that the finitely generated ring R has the following properties:

- for each maximal ideal $\mathfrak{m}$ of R, the field $R/\mathfrak{m}$ is finite;

- maximal ideals are dense in Spec(R).

The first item is proved as follows. The field $R/\mathfrak{m}$ is a finitely generated $(\mathbf{Z}/\mathbf{Z}\cap\mathfrak{m})$-algebra, hence is finite over the quotient field of $\mathbf{Z}/\mathbf{Z}\cap\mathfrak{m}$ by a theorem of Zariski.[14] If $\mathbf{Z}\cap\mathfrak{m}=0$, the field $R/\mathfrak{m}$ is a finite-dimensional $\mathbf{Q}$-vector space with basis $(e_1,\dots,e_m)$. If $x_1,\dots,x_r$ generate the $\mathbf{Z}$-algebra $R/\mathfrak{m}$, there exists an integer q such that qx_j belongs to $\mathbf{Z}e_1\oplus\cdots\oplus\mathbf{Z}e_m$ for each j. This implies

$$\mathbf{Q}e_1\oplus\cdots\oplus\mathbf{Q}e_m=R/\mathfrak{m}\subset\mathbf{Z}[1/q]e_1\oplus\cdots\oplus\mathbf{Z}[1/q]e_m$$

which is absurd. Therefore, $\mathbf{Z}/\mathbf{Z}\cap\mathfrak{m}$ is finite and so is $R/\mathfrak{m}$.

For the second item, we need to show that the intersection of all maximal ideals of R is $\{0\}$. Let a be a nonzero element of R and let $\mathfrak{n}$ be a maximal ideal of the localization R_a. The field $R_a/\mathfrak{n}$ is finite by the first item; hence its subring $R/R\cap\mathfrak{n}$ is a finite domain, hence a field. Therefore, $R\cap\mathfrak{n}$ is a maximal ideal of R, which is in the open subset Spec(R_a) of Spec(R).

Now back to the proof of the theorem. One can show that there is a quasi-projective scheme

$$\rho:\mathrm{Mor}_{\le n+1}(\mathbf{P}^1_R,\mathscr{X};0\mapsto x_R)\to\mathrm{Spec}(R)$$

that parametrizes nonconstant morphisms of degree at most $n+1$.

Let $\mathfrak{m}$ be a maximal ideal of R. Since the field $R/\mathfrak{m}$ is finite, hence of positive characteristic, what we just saw implies that the (geometric) fiber over a closed point of the dense open subset U of Spec(R) is nonempty. It follows that the image of ρ, which is a constructible[15] subset of Spec(R) by Chevalley's theorem ([H], II, Example 3.19), contains all closed points of U. It is therefore dense by the second item, hence contains the generic point ([H], II, Example 3.18(b)). This implies that the generic fiber is nonempty. It has therefore a geometric point, which corresponds to a rational curve on X through x, of degree at most $n+1$, defined over an algebraic closure of the quotient field of R, hence over $\mathbf{k}$.[16] □

[14] This theorem says that if k is a field and K a finitely generated k-algebra which is a field, K is an algebraic hence finite extension of k; see [M], Theorem 5.2.

[15] Recall that a constructible subset is a finite union of locally closed subsets.

[16] It is important to remark that the "universal" bound on the degree of the rational curve is essential for the proof.

Using a stronger version of the first bend-and-break lemma, one can actually show that *through any point of a Fano*[17] *variety X of positive dimension n, there is a rational curve of $(-K_X)$-degree at most* $2n$ (see e.g., [De], Theorem 3.6). This means that X is *uniruled*: there exist a variety Y of dimension $n-1$ and a dominant morphism $\mathbf{P}^1 \times Y \to X$. A variety is *separably uniruled* if there is such a morphism which is in addition separable (i.e., generically smooth). The distinction is only important in positive characteristic. Unfortunately, some Fano varieties are *not* separably uniruled: in characteristic $p > 0$, this is the case for the (smooth Fano) cyclic cover of degree p of $\mathbf{P}^n$, with $n \geq p$, branched along a general hypersurface of degree p ([K1], Theorem V.5.11).

There is not much one can do with nonseparably uniruled varieties, hence we will assume in what follows that the characteristic of $\mathbf{k}$ is 0. In this case, on any normal uniruled variety X, there is a *free* rational curve, i.e., a morphism $f : \mathbf{P}^1 \to X$ whose image is contained in the smooth locus of X and such that f^*T_X is generated by global sections.

Free rational curves are essential in many constructions and theorems, such as smoothing results.

3.3 A bound on the degree of smooth Fano varieties in characteristic zero

We want to show that any two points of a smooth Fano variety can be joined by a chain of rational curves (I do not know whether this remains true for a singular Fano variety). The idea for the proof is the following.

First, one constructs, for any normal proper variety X, a kind of quotient $\rho : X \dashrightarrow R(X)$ for the equivalence relation "to be joined by a chain of rational curves;" this we will call the *rational quotient* of X. Then, when X is a smooth Fano variety, a souped-up version of Theorem 16 shows that if $R(X)$ is not a point, there must exist a rational curve on X not contracted by ρ, which is absurd.

As for the construction of the rational quotient (which goes back to [C1]), it is no more complicated to consider a family of subvarieties of X given by

[17] No conditions on the singularities of X are actually necessary beyond its normality. It is interesting to look at Example 2(5): if the base Y of the cone is an abelian variety, the anticanonical divisor of X is $\mathbf{Q}$-Cartier and ample, but the only rational curves that X contains are the lines through its vertex.

a diagram

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & X \\ \Big\downarrow{\scriptstyle \pi} & & \\ T & & \end{array} \tag{8}$$

and the associated equivalence relation: two points of X are $\mathscr{C}$-equivalent if they can be connected by a chain of subvarieties $F(\mathscr{C}_{t_1}) \cup \cdots \cup F(\mathscr{C}_{t_m})$. For a given point x of X, the sets $V_m(x)$ of points that can be connected to x by such a chain of fixed length m form an ascending chain of constructible subsets of X whose closures stabilize after at most n steps, where n is the dimension of X. For x general in X, the corresponding sets $\overline{V_n(x)}$ are either disjoint or equal, hence form the fibers of a rational map $\tau : X \dashrightarrow Y$ which is the required quotient for $\mathscr{C}$-equivalence. This process actually works only when F and π are *flat*. For more details, see for example [De], Section 5.2. The precise result is the following.

Proposition 17 *In the diagram (8), assume that π is flat with irreducible fibers and that F is flat. There exists a dense open subset X^0 of X, a variety Y^0 and a morphism $\tau : X^0 \to Y^0$ such that*

(a) *two points of X^0 have the same image by τ if and only if the closures in X of their $\mathscr{C}$-equivalence classes are the same;*

(b) *if x is in X^0, a general point of its fiber $\tau^{-1}(\tau(x))$ can be connected to x by a $\mathscr{C}$-chain of length* $\dim(X) - \dim(Y^0)$.

Smooth uniruled projective varieties with Picard number 1. An infinitesimal calculation shows that free deformations of a free rational curve $\mathbf{P}^1 \to X$ of degree d are parametrized by a smooth quasi-projective variety M and that the evaluation map $\mathrm{ev} : \mathbf{P}^1 \times M \to X$ is smooth (hence flat). There is therefore a quotient $\tau : X^0 \to Y^0$. Assume that X is smooth and has Picard number 1. Standard arguments show that any such morphism must be constant (essentially because any nonzero effective divisor, such as the inverse image of an effective ample divisor on Y^0, is ample). The proposition implies that a general pair of points of X can be connected by a chain of free rational curves of total degree at most $d\dim(X)$. Because we are dealing with free curves, this chain can be smoothed and we have "shown" the following.

Corollary 18 *If a smooth projective variety X with Picard number 1 has a free rational curve of degree d with respect to some ample divisor, there is a free rational curve of degree at most $d\dim(X)$ through a general pair of points of X.*

If X is smooth and separably uniruled with Picard number 1, it has a free rational curve. The corollary applies and produces a rational curve through a general pair of points of X. Varieties with the property that there is a rational curve through a general pair of points are called *rationally connected.*

Smooth Fano varieties with Picard number 1. In characteristic 0, the corollary applies to any smooth Fano variety X with Picard number 1, with $d = \dim(X) + 1$, where the degree is taken with respect to the anticanonical divisor. These varieties are therefore rationally connected. The corollary will also give us a bound on their degree, via the following elementary result.

Proposition 19 *Let X be a projective variety of dimension n, let H be an ample divisor on X, and let d be a nonnegative real number. Let x be a smooth point of X such that a general point of X can be connected to x by an irreducible curve of H-degree at most d. We have $H^n \leq d^n$.*

PROOF. By definition of the intersection number, $h^0(X, mH)$ is equivalent to

$$\frac{H^n}{n!}m^n = \frac{(m\sqrt[n]{H^n})^n}{n!}$$

for $m \gg 0$. Let $\mathfrak{m}_{X,x}$ be the maximal ideal of $\mathscr{O}_{X,x}$ and let r be a positive integer. Because of the exact sequence

$$0 \to H^0(X, \mathfrak{m}_{X,x}^r(mH)) \to H^0(X, mH) \to H^0(X, \mathscr{O}_{X,x}/\mathfrak{m}_{X,x}^r) \simeq \mathbf{k}^{\binom{n+r-1}{n}}$$

having a point of multiplicity at least r imposes at most $\binom{n+r-1}{n} \sim \frac{r^n}{n!}$ conditions on the elements of the linear system $|mH|$. It follows that given $\varepsilon > 0$, there exist a large positive integer m and a divisor D in $|mH|$ such that

$$\operatorname{mult}_x D \geq m\sqrt[n]{H^n} - m\varepsilon$$

Let C be an irreducible curve of H-degree at most d connecting x to a point outside of D. Since C is not contained in the support of D, we have

$$md \geq C \cdot (mH) = C \cdot D \geq \operatorname{mult}_x D \geq m\sqrt[n]{H^n} - m\varepsilon$$

and the proposition follows by letting ε go to 0. □

Corollary 20 *Assume the characteristic is zero. Any smooth Fano variety X of dimension n and Picard number 1 satisfies*

$$(-K_X)^n \leq (n(n+1))^n$$

This bound is not too far off (recall $(-K_{\mathbf{P}^n})^n = (n+1)^n$).

The rational quotient and general smooth Fano varieties. To deal with smooth Fano varieties with arbitrary Picard number, we need the construction of the full rational quotient, as explained at the beginning of the section. This requires a lot of technical work and I will only explain the main steps. Say that two points on a variety are $\mathscr{R}$-equivalent if they can be connected by a chain of rational curves. A variety is rationally chain-connected if two general points are $\mathscr{R}$-equivalent.

(1) Extend the construction of Proposition 17 to show that given a normal projective variety X, there exist a normal variety $R(X)$, called *the rational quotient of* X and a rational map $\rho : X \dashrightarrow R(X)$, defined and proper over a dense open subset of X, whose very general fibers are $\mathscr{R}$-equivalence classes.

(2) Extend Theorem 16 to show that for any smooth Fano variety X and nonconstant dominant rational map $\pi : X \dashrightarrow Y$ defined and proper over a dense open subset of X, there exists a rational curve on X that meets but is not contained in a general fiber of π. This implies that a smooth Fano variety is rationally chain-connected. More precisely, any two points of a Fano variety X of dimension n can be connected by a chain of rational curves of total $(-K_X)$-degree at most $(2^n-1)(n+1)$.

(3) In characteristic zero, use smoothing results to show that a rationally chain-connected smooth projective variety is rationally connected. In case of a smooth Fano variety X, any two points can be connected by a single rational curve of $(-K_X)$-degree at most $3(2^n-1)(n+1)^{(n+1)(2^n-1)}$.

Using Proposition 19, we get that in characteristic zero, *any smooth Fano variety X of dimension n is rationally connected and satisfies*

$$(-K_X)^n \leq (3(2^n-1)(n+1)^{(n+1)(2^n-1)})^n \tag{9}$$

This bound is probably very far off. In view of (6), it is tempting to conjecture the inequality

$$(-K_X)^n \leq (n^\rho(n+1))^n$$

for smooth Fano varieties with Picard number ρ (this holds if two general points of X can be joined by a chain of n^ρ free curves of $(-K_X)$-degree at most $n+1$). Getting a bound on ρ in terms of n is another story, although one might maybe expect a linear bound (if X is a del Pezzo surface with Picard number 9, the Fano variety X^m has dimension $n = 2m$ and Picard number $9n/2$).

3.4 Smooth Fano varieties form a bounded family

It is out of the question to prove anything here. We will just give a few references.

There are only finitely many deformation types of Fano varieties X with fixed Hilbert polynomial $P(m) = \chi(X, \mathscr{O}_X(mK_X))$ (see [Ma]). Using the Hodge index theorem and the Kodaira vanishing theorem, one proves that in characteristic zero, for any ample divisor H on X, the coefficients of the Hilbert polynomial $\chi(X, \mathscr{O}_X(mH))$ are bounded by a function which depends only on the dimension n of X, on H^n and on $K_X \cdot H^{n-1}$ (see [K1], Exercise VI.2.15.8). When $H = -K_X$, these numbers are bounded by a function that depends only on n by (9), hence there are finitely many deformation types of Fano varieties in any given dimension. One can even get an effective bound.

Theorem 21 *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic zero. There are at most* $(n+2)^{(n+2)^{n2^{3n}}}$ *deformation types of Fano varieties of dimension* n *defined over* $\mathbf{k}$.

As for singular Fano varieties, it is conjectured that if the dimension n is fixed and the discrepancy bounded away from -1, there should still be only finitely many deformation types.

This was proved in Corollary 13 in the toric case. The point is again to bound $(-K_X)^n$ *and* an integer j such that jK_X is a Cartier divisor. To my knowledge, the best general result in this direction appears in the recent preprint [Mc] where it is shown that given integers n and j, there are only finitely many deformation types of complex Fano varieties X (in the sense of Definition 1) of dimension n such that jK_X is a Cartier divisor. This is done by bounding $(-K_X)^n$ using techniques that are far beyond the scope of this report.

4 The differential–geometric point of view

If (X, ω) is a compact Kähler manifold, its Ricci form is closed by the Bianchi identity and its cohomology class is the first Chern class of X. Conversely, by Yau's proof of Calabi's conjecture ([Y1], [Y2], [Bou1]), given a closed $(1,1)$-form ρ representing $c_1(X)$, there exists a Kähler metric on X whose Ricci form is ρ.

It follows that the smooth complex Fano varieties are exactly the compact Kähler manifolds with positive Ricci curvature ([Be], 11.16(ii)). This implies in particular ([Be], Theorem 11.26) that they are simply connected (see also 5.1 and 5.2 for more general results).

Compact complex manifolds with a positive Kähler–Einstein metric (i.e., whose Ricci form is a positive (constant) multiple of the Kähler form ω) are trivially Fano varieties, but it is a difficult problem to find conditions on a smooth complex Fano variety so that a Kähler–Einstein metric exists. There are restrictions: for example, the existence of such a metric forces the automorphism group to be reductive[18] ([Be], Corollary 11.54) (this excludes for example the projective plane blown up at one or two points). There are also existence results for which the reader is refered to the excellent survey [Bou2] and, in the toric case, to [BS].

Classical methods of differential geometry yield a good bound on the degree of compact complex manifolds with a positive Kähler–Einstein metric.

Let X be a compact complex manifold of dimension n with a Kähler–Einstein form ω normalized so that its Ricci form is ω. By [Be], 11.5, we have

$$(-K_X)^n = \mathrm{vol}(X)\ \frac{n!}{(2\pi)^n}$$

From [BiC], Corollary 4, p. 257, it follows that the volume of X is less than the volume of a sphere of the same dimension and radius $\sqrt{2n-1}$, so that

$$\begin{aligned}(-K_X)^n &\leq \frac{n!}{(2\pi)^n}\ (\sqrt{2n-1})^{2n}\ \frac{2^{n+1}\pi^n}{1\cdot 3\cdot\cdots\cdot(2n-1)}\\ &= (2n-1)^n\ \frac{2^{n+1}(n!)^2}{(2n)!}\end{aligned}$$

This bound is equivalent to $2^{n+1}\sqrt{\frac{\pi n}{e}}\ n^n$.

[18] An algebraic group is reductive if it has no nontrivial connected unipotent abelian normal subgroups.

5 Simple connectedness of complex Fano varieties

5.1 Simple connectedness of smooth complex Fano varieties

Any complex complete toric variety is simply connected ([F], p. 56). More generally, any smooth complex projective rationally connected variety (such as a smooth complex Fano variety) is simply connected, as we now explain.

- **A finite connected étale cover of a smooth complex rationally connected variety is trivial.** We will only give the argument for a smooth complex Fano variety X, since this is the only case we will use. By Kodaira's vanishing theorem, $H^m(X, \mathscr{O}_X)$ vanishes for $m > 0$, hence $\chi(X, \mathscr{O}_X) = 1$. If $\pi : \tilde{X} \to X$ is a connected finite étale cover, $\tilde{X}$ is a smooth Fano variety, hence $\chi(\tilde{X}, \mathscr{O}_{\tilde{X}}) = 1$. But $\chi(\tilde{X}, \mathscr{O}_{\tilde{X}}) = \deg(\pi)\,\chi(X, \mathscr{O}_X)$ hence π is an isomorphism.

- **The fundamental group of a rationally connected variety is finite.** Let X be a rationally connected variety. Since a general point x of X can be connected to another general point of X by a rational curve, this means that there is a quasi-projective open subscheme M of $\mathrm{Mor}(\mathbf{P}^1, X; 0 \mapsto x)$ over which the evaluation map $\mathrm{ev} : \mathbf{P}^1 \times M \to X$ is dominant. We use the following fact: *let X and Y be complex algebraic varieties, with Y normal, and let $f : Y \to X$ be a dominant morphism. The image of the induced morphism $\pi_1(f) : \pi_1(Y) \to \pi_1(X)$ has finite index.* The composition of ev with the injection $\iota : \{0\} \times M \hookrightarrow \mathbf{P}^1 \times M$ is constant, hence
$$\pi_1(\iota) \circ \pi_1(\mathrm{ev}) = 0$$
Since $\mathbf{P}^1$ is simply connected, $\pi_1(\iota)$ is bijective, hence $\pi_1(\mathrm{ev}) = 0$. Since ev is dominant, the lemma implies that the image of $\pi_1(\mathrm{ev})$ has finite index. Therefore, the group $\pi_1(X)$ is finite.

Campana gives in [C2] a completely different proof of the simple connectedness of smooth complex rationally connected varieties. He shows more generally that the fundamental group of a compact Kähler manifold is finite if any two points can be connected by a chain of subvarieties whose fundamental group of the normalization has the same property.

In [K2], Theorem 4.13, it is proved that the fundamental group of a complex rationally chain-connected normal proper variety is finite (possibly nontrivial if the variety is singular).

5.2 Simple connectedness of complex Fano varieties

This is a particular case of a result of Takayama ([T]) whose proof proceeds along the same steps as in 5.1.

Let X be a complex Fano variety.

- **Any finite connected étale cover of X is trivial.** The argument is the same as in the smooth case, upon replacing Kodaira's vanishing theorem with the Kawamata–Viehweg vanishing theorem (see e.g., [De], Theorem 7.26), which works for varieties with log terminal singularities and yields $H^m(X, \mathscr{O}_X) = 0$ for $m > 0$.

- **The fundamental group of X is finite.** This is the difficult part. The "obvious" idea would be to prove that X is rationally connected (or even only rationally chain-connected), but this is still known only in dimension ≤ 3 by [KMM].

The argument for the second step is based on completely different analytic techniques. We first explain how the argument works for smooth Fano varieties.

Von Neumann dimensions and Atiyah's L^2 index theorem. First of all, recall that for a vector bundle E with a Hermitian metric on a compact Kähler manifold X of dimension n, Hodge theory tells us that the vector space $H^q(X, E)$ is isomorphic to the space $\mathscr{H}^{0,q}(X, E)$ of E-valued, $\mathscr{C}^\infty$, *harmonic* forms of type $(0, q)$. In particular, we have

$$\chi(X, \mathscr{E}) = \sum_{q=0}^{n} (-1)^q \dim \mathscr{H}^{0,q}(X, E) \tag{10}$$

where $\mathscr{E}$ is the sheaf of holomorphic sections of E.

If X is not necessarily compact (but still Kähler), we let

$$\mathscr{H}^{0,q}_{(2)}(X, E)$$

be the space of harmonic L^2 forms of type $(0,q)$ with values in E. Although the notation does not take this into account, these spaces depend very much on the choice of the metrics when X is not compact. Also, these spaces are usually infinite-dimensional, so an analog of (10) is not obvious, even to formulate.

Let us go back to the case where X is compact and let $\pi : \tilde{X} \to X$ be its universal cover. We can pull back the Kähler form, the vector bundle E, and its metric to $\tilde{X}$ and consider the vector spaces

$$\mathscr{H}^{p,q}_{(2)}(\tilde{X}, \pi^* E)$$

As it turns out, a lot of Hodge theory can be done on a *complete* Kähler manifold such as $\tilde{X}$. For example, on such a variety, *any harmonic L^2 function is d-closed, hence constant* ([G], § 1). Similarly, any harmonic L^2 form with values in $\pi^* E$ is $\bar{\partial}$- and $\bar{\partial}^*$-closed ([G], § 3) hence elements of $\mathscr{H}^{0,0}_{(2)}(\tilde{X}, \pi^* E)$ are *holomorphic* L^2 sections of $\mathscr{O}_{\tilde{X}}(\pi^* E)$.

There is a way to define, for the spaces $\mathscr{H}^{p,q}_{(2)}(\tilde{X}, \pi^* E)$, or for any closed space H of L^2 sections invariant under the action of $\pi_1(X)$, the von Neumann dimension $\dim_{\pi_1(X)} H$, which is a nonnegative real number that vanishes if and only if $H = 0$ (see, e.g., [K2], Chap. 6). The Atiyah index theorem ([At]) states

$$\chi(X, \mathscr{E}) = \sum_{q=0}^{n} (-1)^q \dim_{\pi_1(X)} \mathscr{H}^{0,q}_{(2)}(\tilde{X}, \pi^* E)$$

There is a Kodaira-type vanishing theorem due to Andreotti and Vesentini ([AV]) that says

$$\mathscr{H}^{0,q}_{(2)}(\tilde{X}, K_{\tilde{X}} + \pi^* D) = 0 \qquad \text{for} \quad q > 0$$

if D is an ample divisor on X. Combining this with the usual Kodaira theorem on X, we get

$$h^0(X, K_X + D) = \dim_{\pi_1(X)} \mathscr{H}^{0,0}_{(2)}(\tilde{X}, K_{\tilde{X}} + \pi^* D)$$

Finiteness of the fundamental group of smooth Fano varieties. Let X be a smooth Fano variety. We can apply the equality above with $D = -K_X$ to get

$$\dim_{\pi_1(X)} \mathscr{H}^{0,0}_{(2)}(\tilde{X}, \mathbf{C}) = 1$$

Therefore, there exists a nonzero harmonic L^2 function on $\tilde{X}$ which is, as we saw above, constant. This implies that $\tilde{X}$ has finite volume. Since this volume is computed with respect to the pull-back of the metric on X, it is equal to the volume of X times the degree of π. It follows that the latter, which is the cardinality of $\pi_1(X)$, is finite.

Finiteness of the fundamental group of Fano varieties. To adapt the proof to the case of an arbitrary Fano variety X, we consider a desingularization $f : Y \to X$ whose exceptional locus has normal crossings and the pull back $\pi : \tilde{Y} = Y \times_X \tilde{X} \to Y$ of the universal cover of X (the map $\pi_1(f)$ is surjective, hence $\tilde{Y}$ is connected). Write

$$K_Y \sim f^* K_X + \sum_i a_i E_i$$

as in (1), with $a_i > -1$ and set

$$\Delta = \sum_{a_i < 0} (-a_i) E_i + \sum_{a_i \geq 0} (\lceil a_i \rceil - a_i) E_i$$

so that

$$K_Y \sim f^* K_X - \Delta + E$$

where $E = \sum_{a_i \geq 0} \lceil a_i \rceil E_i$ is an effective (integral) f-exceptional divisor. Demailly proved in [Dm] (see also [K2], Theorem 11.5) a generalization of the Andreotti–Vesentini vanishing theorem that says in this situation

$$\mathscr{H}^{0,q}_{(2)}(\tilde{Y}, K_{\tilde{Y}} + \pi^*(D + \Delta)) = 0 \qquad \text{for} \quad q > 0$$

for any nef and big $\mathbf{Q}$-divisor D on Y such that $D + \Delta$ is integral. Taking $D = -f^* K_X$, we get

$$\mathscr{H}^{0,q}_{(2)}(\tilde{Y}, \pi^* E) = 0 \qquad \text{for} \quad q > 0$$

On Y, the Kodaira vanishing theorem is replaced with the Kawamata–Viehweg vanishing theorem. We get as above

$$1 = h^0(Y, E) = \dim_{\pi_1(X)} \mathscr{H}^{0,0}_{(2)}(\tilde{Y}, \pi^* E)$$

hence there is a nonzero *holomorphic* L^2 section s of $\mathscr{O}_{\tilde{Y}}(\pi^* E)$. Since the induced morphism $\tilde{f} : \tilde{Y} \to \tilde{X}$ is birational, this section induces a holomorphic function on the complement of $\tilde{f}(\pi^{-1}(E))$ in $\tilde{X}$, and since this set has codimension at least 2, this function extends to the whole of $\tilde{X}$ because the latter is normal. This means that s vanishes on the divisor π^*E.

Let t be a section of $\mathscr{O}_Y(E)$ with divisor E. The quotient s/π^*t is a holomorphic *function* on $\tilde{Y}$ and a local calculation (see, e.g., [K2], Lemma 11.2.2) shows that it is still L^2, hence constant. It follows that $\tilde{Y}$ has finite volume, hence the fundamental group of X is finite.

Conjectures. It is conjectured that

- the smooth locus of a complex Fano variety is simply connected;[19]
- a Fano variety is rationally chain-connected, or even rationally connected.[20]

Example 2(5) shows that normal varieties with discrepancy -1 and ample anticanonical bundle may not be rationally connected (take for Y an abelian variety as in footnote 17). Note however that these examples, as any projective cone in the complex projective space, are simply connected.

References

[AV] Andreotti, A. and Vesentini, E., Carleman estimates for the Laplace–Beltrami equation in complex manifolds, *Inst. Hautes Études Sci. Publ. Math.* **25** (1965), 81–130.

[At] Atiyah, M., *Elliptic operators, discrete groups and von Neumann algebras*, Astérisque **32–33** (1976), 43–72.

[B1] Batyrev, V., Boundedness of the degree of multidimensional toric Fano varieties (in Russian), *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, **37** (1982), 22–27. English transl.: *Moscow Univ. Math. Bull.* **37** (1982), 28–33.

[B2] Batyrev, V., On the classification of toric Fano 4-folds, *Algebraic geometry, 9. J. Math. Sci. (New York)* **94** (1999), 1021–1050.

[19] This holds in dimension 2 by [Z]. The algebraic fundamental group is shown to be finite in [Mc].

[20] The latter holds in characteristic 0 and dimension ≤ 3 by [KMM].

[BS] Batyrev, V. and Selivanova, E., Einstein–Kähler metrics on symmetric toric Fano manifolds, *J. reine angew. Math.* **512** (1999), 225–236.

[Be] Besse, A., *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete **10**, Springer-Verlag, New York, 1987.

[BiC] Bishop, R. and Crittenden, R., *Geometry of Manifolds*, Pure and Applied Mathematics **XV**, Academic Press, New York, 1964.

[BoB] Borisov, A. and Borisov, L., Singular toric Fano varieties, (in Russian) *Mat. Sb.* **183** (1992), 134–141. English transl.: *Russian Acad. Sci. Sb. Math.* **75** (1993), 277–283.

[Bou1] Bourguignon, J.-P., Premières formes de Chern des variétés kählériennes compactes, d'après E. Calabi, T. Aubin et S. T. Yau, *Séminaire Bourbaki, 1977/78*, Exp. 507, Lecture Notes in Mathematics **710**, Springer-Verlag, New York, 1979.

[Bou2] Bourguignon, J.-P., Métriques d'Einstein–Kähler sur les variétés de Fano : obstructions et existence (d'après Y. Matsushima, A. Futaki, S. T. Yau, A. Nadel et G. Tian), *Séminaire Bourbaki, 1996/97*, Exp. 830, Astérisque **245** (1997), 277–305.

[C1] Campana, F., Coréduction algébrique d'un espace analytique compact faiblement kählérien, *Invent. Math.* **63** (1981), 187–223.

[C2] Campana, F., Remarques sur le revêtement universel des variétés kählériennes compactes, *Bull. Soc. Math. Fr.* **122** (1994), 255–284.

[Ca1] Casagrande, C., On the birational geometry of toric Fano 4-folds, *C. R. Acad. Sci. Paris* **332** (2001), 1093–1098.

[Ca2] Casagrande, C., Contractible classes in toric varieties, Math. Zeit. **243** (2003), 99–126.

[Ca3] Casagrande, C., Toric Fano varieties and birational morphisms, e-print `math.AG/0112007`, to appear in Int. Math. Res. Not.

[Co] Cox, D., Update on toric geometry, *Geometry of Toric Varieties*, 1–41, L. Bonavero and M. Brion editors, Séminaires et Congrès **6**, Soc. Math. Fr., 2002.

[Cu] Cutkosky, S., On Fano 3-folds, *Manuscripta Math.* **64** (1989), 189–204.

[DGK] Dancer, L., Grjunbaum, B., and Klee, V., Helly's theorem and its relatives, *1963 Proc. Sympos. Pure Math.*, Vol. VII, pp. 101–180, Amer. Math. Soc., Providence, R.I.

[De] Debarre, O., *Higher-Dimensional Algebraic Geometry*, Universitext, Springer-Verlag, 2001.

[Dm] Demailly, J.-P., Singular Hermitian metrics on positive line bundles, in *Complex Algebraic Varieties*, 87–104, Lecture Notes in Mathematics **1507**, Springer-Verlag, New York, 1992.

[Dz] Demazure, M., *Séminaire sur les Singularités des Surfaces*, Centre de Mathématiques de l'École Polytechnique, Palaiseau, 1976–1977, M. Demazure, H. Pinkham, and B. Teissier editors, Lecture Notes in Mathematics **777**, Springer-Verlag, Berlin, 1980.

[F] Fulton, W., *Introduction to toric varieties*, Annals of Mathematics Studies, **131**. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[G] Gromov, M., Kähler hyperbolicity and L_2-Hodge theory, *J. Diff. Geom.* **33** (1991), 263–292.

[Gr1] Grothendieck, A., Techniques de construction et théorèmes d'existence en géométrie algébrique IV : les schémas de Hilbert, *Séminaire Bourbaki, 1960/61*, Exp. 221, Astérisque hors série **6**, Soc. Math. Fr., 1997.

[Gr2] Grothendieck, A., Éléments de Géométrie Algébrique IV, 2, *Inst. Hautes Études Sci. Publ. Math.* **24**, 1965.

[Gr3] Grothendieck, A., Éléments de Géométrie Algébrique IV, 3, *Inst. Hautes Études Sci. Publ. Math.* **28**, 1966.

[H] Hartshorne, R., *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag, New York, 1977.

[He] Hensley, D., Lattice vertex polytopes with interior lattice points, *Pacific J. Math.* **105** (1983), 183–191.

[I1] Iskovskikh, V., Fano 3–folds I (in Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **41** (1977), 516–562. English transl.: *Math. USSR Izvestiya* **11** (1977), 485–527.

[I2] Iskovskikh, V., Fano 3–folds II (in Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **42** (1978), 506–549. English transl.: *Math. USSR Izvestiya* **12** (1978), 469–506.

[IP] Iskovskikh, V. and Prokhorov, Yu., *Algebraic geometry. V. Fano varieties* (in Russian), A. Parshin and I. Shafarevich editors, Ross. Akad. Nauk, Vseross. Inst. Nauchn. i Tekhn. Inform., Moscow. English transl.: Encyclopaedia Math. Sci. **47**, Springer-Verlag, Berlin, 1999.

[KP] Kleinschmidt, P. and Sturmfels, B., Smooth toric varieties with small Picard number are projective, *Topology* **30** (1991), 289–299.

[K1] Kollár, J., *Rational Curves on Algebraic Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete **32**, Springer-Verlag, New York, 1996.

[K2] Kollár, J., *Shafarevich maps and automorphic forms*, Princeton University Press, 1995.

[KMM] Kollár, J., Miyaoka, Y., and Mori, S., Rationally connected varieties, *J. Alg. Geom.* **1** (1992), 429–448.

[KM] Kollár, J. and Mori, S., *Birational Geometry of Algebraic Varieties*, with collaboration of C. H. Clemens and A. Corti, Cambridge Tracts in Mathematics **134**, Cambridge University Press, Cambridge, 1998.

[LZ] Lagarias, J. and Ziegler, G., Bounds for lattice polytopes containing a fixed number of interior points in a sublattice, *Canad. J. Math.* **43** (1991), 1022–1035.

[M] Matsumura, H., *Commutative ring theory*, Second edition, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, Cambridge, 1989.

[Ma] Matsusaka, T., On canonically polarized varieties (II), *Amer. J. Math.* **92** (1970), 283–292.

[Mc] M^cKernan, J., Boundedness of log terminal Fano pairs of bounded index, eprint `math.AG/0205214`.

[Me] Megyesi, G., Fano threefolds in positive characteristic, *J. Alg. Geom.* **7** (1998), 207–218.

[Mo] Mori, S., Projective manifolds with ample tangent bundles, *Ann. of Math.* **110** (1979), 593–606.

[MM] Mori, S. and Mukai, S., Classification of Fano threefolds with $b_2 \geq 2$, *Manuscr. Math.* **36** (1981), 147–162.

[MU] Mukai, S. and Umemura, H., Minimal rational threefolds, *Algebraic geometry (Tokyo/Kyoto, 1982)*, 490–518, Lecture Notes in Math. **1016**, Springer-Verlag, Berlin, 1983.

[R] Reid, M., Decomposition of toric morphisms, *Arithmetic and geometry, Vol. II*, 395–418, Progr. Math. **36**, Birkhäuser, Boston, 1983.

[Sa] Sato, H., Toward the classification of higher-dimensional toric Fano varieties, *Tôhoku Math. J.* **52** (2000), 383–413.

[SB] Shepherd-Barron, N., Fano threefolds in positive characteristic, *Comp. Math.* **105** (1997), 237–265.

[Sh] Shokurov, V. V., The existence of a line on Fano varieties (in Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979) 922–964. English transl.: *Math. USSR Izvestiya* **14** (1980), 395–405.

[T] Takayama, S., Simple Connectedness of weak Fano varieties, *J. Alg. Geom.* **9** (2000), 403–407.

[Ta] Takeuchi, K., Some birational maps of Fano 3-folds, *Compositio Math.* **71** (1989), 265–283.

[VK] Voskresenskiĭ, V. E. and Klyachko, A. A., Toric Fano varieties and systems of roots (in Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), 237–263. English transl.: *Math. USSR Izv.* **24** (1985), 221–244.

[W] Wiśniewski, J., Toric Mori Theory and Fano Manifolds, *Geometry of Toric Varieties*, 249–272, L. Bonavero and M. Brion editors, Séminaires et Congrès **6**, Soc. Math. Fr., 2002.

[Y1] Yau, S. T., On Calabi's conjecture and some new results in algebraic geometry, *Proc. Nat. Acad. Sci. USA* **74** (1977), 1798–1799.

[Y2] Yau, S. T., On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I, *Comm. Pure and Appl. Math.* **31** (1978), 339–411.

[Z] Zhang, D. Q., Algebraic surfaces with nef and big anti-canonical divisor, *Math. Proc. Cambridge Philos. Soc.* **117** (1995), 161–163.