Cubic hypersurfaces over finite fields

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Theorem (Chevalley–Warning)

Any subscheme of $\mathbf{P}^m_{\mathbf{F}_q}$ defined by equations of degrees $d_1, \ldots, d_s$ with $d_1 + \cdots + d_s \leq m$ has an $\mathbf{F}_q$-point.

$\rightarrow$ any cubic $\mathbf{F}_q$-hypersurface of dimension ≥ 2 contains an $\mathbf{F}_q$-point.

What about $\mathbf{F}_q$-lines?

The scheme $F(X)$

- $X \subset \mathbf{P}_{\mathbf{k}}^{n+1}$ cubic of dimension n.
- $F(X) \subset \mathrm{Gr}(1, \mathbf{P}_{\mathbf{k}}^{n+1})$ projective scheme of lines contained in X.

- $F(X)$ connected if $n \geq 3$.
- $\mathrm{Sing}(X)$ finite $\implies$ $F(X)$ lci of dimension $2n-4$ and $\omega_{F(X)} = \mathscr{O}_{F(X)}(4-n)$.
- X smooth $\implies$ $F(X)$ smooth.

High dimensions

Theorem

Any $\mathbf{F}_q$-cubic of dimension ≥ 5 contains an $\mathbf{F}_q$-line.

Proof. Let $x \in X(\mathbf{F}_q)$. The scheme of lines through x contained in X is the intersection in $\mathbf{P}^n$ of hyperplane, a quadric, and a cubic
$\to$ Chevalley–Warning when $n \geq 6$.
When X is smooth, $F(X)$ is Fano when $n \geq 5$
$\to$ Esnault, and Fakhruddin–Rajan to extend to all X. $\square$

We now look at cubic surfaces, threefolds, and fourfolds.

Cubic surfaces

The diagonal cubic surface

$$x_1^3 + x_2^3 + x_3^3 + ax_4^3 = 0$$

contains no $\mathbf{F}_q$-lines when $a \in \mathbf{F}_q$ is not a cube.

a exists whenever $q \equiv 1 \pmod 3$

There are smooth cubic $\mathbf{F}_q$-surfaces with no $\mathbf{F}_q$-lines for q arbitrarily large.

The Galkin–Shinder "beautiful formula"

- $X \subset \mathbf{P}^{n+1}_{\mathbf{F}_q}$ $\mathbf{F}_q$-cubic
- $F(X) \subset \mathrm{Gr}(1, \mathbf{P}^{n+1}_{\mathbf{F}_q})$ scheme of lines contained in X
- $N_r(X) := \mathrm{Card}(X(\mathbf{F}_{q^r}))$

$$N_r(F(X)) = \frac{N_r(X)^2 - 2(1+q^{nr})N_r(X) + N_{2r}(X)}{2q^{2r}} + q^{(n-2)r} N_r(\mathrm{Sing}(X)).$$

This formula comes from a relation between the classes of X and $F(X)$ in the Grothendieck ring of varieties.

Y smooth projective scheme defined over $\mathbf{F}_q$

$$P_i(Y,T) := \det(\mathrm{Id} - TF^*, H^i(\overline{Y}, \mathbf{Q}_\ell)) =: \prod_{j=1}^{b_i(Y)} (1 - T\omega_{ij}) \in \mathbf{Z}[T]$$

where $|\omega_{ij}| = q^{i/2}$. The trace formula ($n := \dim(Y)$)

$$N_r(Y) = \sum_{0\le i\le 2n} (-1)^i \operatorname{Tr}(F^{*r}, H^i(\overline{Y}, \mathbf{Q}_\ell)) = \sum_{0\le i\le 2n} (-1)^i \sum_{j=1}^{b_i(Y)} \omega_{ij}^r$$

implies, for the zeta function,

$$Z(Y,T) := \exp\Big(\sum_{r\ge 1} N_r(Y)\frac{T^r}{r}\Big) = \prod_{0\le i\le 2n} P_i(Y,T)^{(-1)^{i+1}}.$$

Zeta function of $F(X)$

$X \subset \mathbf{P}^4_{\mathbf{F}_q}$ smooth cubic.The trace formula reads

$$Z(X,T) = \frac{\prod_{1\le j\le 10}(1-q\omega_j T)}{(1-T)(1-qT)(1-q^2T)(1-q^3T)},$$

with ω_j algebraic integers and $|\omega_j| = q^{1/2}$.

The Galkin–Shinder formula implies

$$Z(F(X),T) = \frac{\prod_{1\le j\le 10}(1-\omega_j T)\prod_{1\le j\le 10}(1-q\omega_j T)}{(1-T)(1-q^2T)\prod_{1\le j<k\le 10}(1-\omega_j\omega_k T)}$$

Cohomology of $F(X)$

This formula gives the Betti numbers of $F(X)$. Actually, the full Galkin–Shinder relation gives isomorphisms of $\mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$-modules

$$H^3(\overline{X},\mathbf{Q}_\ell) \xrightarrow{\ \sim\ } H^1(\overline{F(X)},\mathbf{Q}_\ell(1))$$

$$\begin{array}{ccc} \bigwedge^2 H^1(\overline{F(X)},\mathbf{Q}_\ell) & \xrightarrow{\ \sim\ } & H^2(\overline{F(X)},\mathbf{Q}_\ell) \\ \wr\uparrow & & \wr\uparrow \\ \bigwedge^2 H^1(\overline{A(F(X))},\mathbf{Q}_\ell) & \xrightarrow{\sim} & H^2(\overline{A(F(X))},\mathbf{Q}_\ell) \end{array}$$

The first one can also be obtained using the incidence correspondence.
The second one can also be deduced by smooth and proper base change from the statement in char. 0.

Existence of lines

Theorem

Any smooth $\mathbf{F}_q$*-cubic threefold contains at least 10* $\mathbf{F}_q$*-lines if* $q \geq 11$*.*

Proof. Write the Frobenius eigenvalues as $\omega_1, \dots, \omega_5, \overline{\omega}_1, \dots, \overline{\omega}_5$. Set $r_j := \omega_j + \overline{\omega}_j \in [-2\sqrt{q}, 2\sqrt{q}]$. Use the trace formula

$$\begin{aligned} N_1(F(X)) &= 1 - \sum_{1 \leq j \leq 5} r_j - \sum_{1 \leq j \leq 5} q r_j + q^2 \\ &\quad + 5q + \sum_{1 \leq j < k \leq 5} (\omega_j \omega_k + \overline{\omega}_j \omega_k + \omega_j \overline{\omega}_k + \overline{\omega}_j \overline{\omega}_k) \\ &= 1 + 5q + q^2 - (q+1) \sum_{1 \leq j \leq 5} r_j + \sum_{1 \leq j < k \leq 5} r_j r_k \end{aligned}$$

and study the minimum of this real function... $\square$

Examples

We found smooth cubic threefolds over $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$, and $\mathbf{F}_5$ with no lines.
The cubic threefold in $\mathbf{P}^4_{\mathbf{F}_5}$ with equation

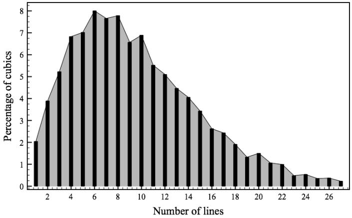
$$x_1^3 + 2x_2^3 + x_2^2x_3 + 3x_1x_3^2 + x_1^2x_4 + x_1x_2x_4 + x_1x_3x_4 + 3x_2x_3x_4 + 4x_3^2x_4 + x_2x_4^2 + 4x_3x_4^2 + 3x_2^2x_5 + x_1x_3x_5 + 3x_2x_3x_5 + 3x_1x_4x_5 + 3x_4^2x_5 + x_2x_5^2 + 3x_5^3$$

is smooth and contains no $\mathbf{F}_5$-lines and 126 $\mathbf{F}_5$-points.

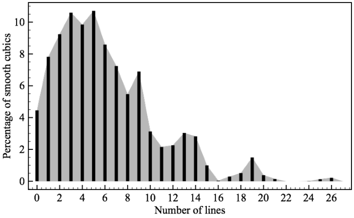
Remains $\mathbf{F}_7$, $\mathbf{F}_8$, and $\mathbf{F}_9$...

Average numbers of lines

Average numbers of lines computed on random samples of 10^5 $\mathbf{F}_2$-cubic threefolds



all cubics; average ~ 9.651



smooth cubics; average ~ 6.963.

Singular threefolds

Let $X \subset \mathbf{P}^4_{\mathbf{F}_q}$ $\mathbf{F}_q$-cubic with a single singular point, of type A_1 or A_2.
The curve C of lines in X through the singular point is smooth of genus 4 and canonically embedded in $\mathbf{P}^3_{\mathbf{F}_q}$. It has two pencils g^1_3 and h^1_3, used to embed C in $C^{(2)}$ by

$$x \mapsto g^1_3 - x$$

Clemens–Griffiths, Kouvidakis–van der Geer

$F(X)$ is the non-normal surface obtained by gluing in $C^{(2)}$ these two copies of C.

Lines on singular threefolds

Theorem

For any $r \geq 1$, set $n_r := \mathrm{Card}(C(\mathbf{F}_{q^r}))$. We have

$$\mathrm{Card}(F(X)(\mathbf{F}_q)) = \begin{cases} \frac{1}{2}(n_1^2 + n_2) - n_1 & \text{if } g_3^1 \neq h_3^1 \text{ are defined over } \mathbf{F}_q; \\ \frac{1}{2}(n_1^2 + n_2) + n_1 & \text{if } g_3^1 \neq h_3^1 \text{ are not defined over } \mathbf{F}_q; \\ \frac{1}{2}(n_1^2 + n_2) & \text{if } g_3^1 = h_3^1. \end{cases}$$

Again, this can also be obtained with the Galkin–Shinder method.

Lines on singular threefolds

Corollary

When $q \geq 4$, any cubic threefold $X \subset \mathbf{P}^4_{\mathbf{F}_q}$ defined over $\mathbf{F}_q$ with a single singular point, of type A_1 or A_2, contains an $\mathbf{F}_q$-line.

Proof. We need to exclude

- $n_1 = n_2 = 1$ and $g^1_3 \neq h^1_3$ defined over $\mathbf{F}_q$. Write $C(\mathbf{F}_q) = C(\mathbf{F}_{q^2}) = \{x\}$, and $g^1_3 \equiv x + x' + x''$.

$$\begin{aligned} g^1_3 \text{ is defined over } \mathbf{F}_q \quad &\Rightarrow \quad x' + x'' \text{ defined over } \mathbf{F}_q \\ &\Rightarrow \quad x', x'' \text{ are defined over } \mathbf{F}_{q^2} \\ &\Rightarrow \quad x' = x'' = x \\ &\Rightarrow \quad g^1_3 \equiv 3x \equiv h^1_3 \end{aligned}$$

Contradiction.

Lines on singular threefolds

- $n_1 = n_2 = 0$. Then $q \leq 7$ (Howe–Lauter–Top).
  Weil conjectures for C:
  - Frobenius roots $\omega_1, \ldots, \omega_4, \bar{\omega}_1, \ldots, \bar{\omega}_4$, with $|\omega_j| = \sqrt{q}$;
  - H monic with (real) roots $r_j := \omega_j + \bar{\omega}_j$ with $|r_j| \leq 2\sqrt{q}$ has integral coefficients;
  - $\sum_{1\leq j\leq 4} r_j = \sum_{1\leq j\leq 8} \omega_j = q + 1 - n_1 = q + 1$;
  - $\sum_{1\leq j\leq 4} r_j^2 = \sum_{1\leq j\leq 8}(\omega_j^2 + 2q) = q^2 + 8q + 1 - n_2 = q^2 + 8q + 1$.

  Hence

$$H(T) = T^4 - (q+1)T^3 - 3qT^2 + aT + b,$$

  with $|b| = |r_1 r_2 r_3 r_4| \leq 16q^2$ and $|a| = |\sum_{j=1}^4 b/r_j| \leq 32q^{3/2}$ integral.
  Computer search: such polynomials with 4 such real roots and $q \in \{2, 3, 4, 5, 7\}$ only exist for $q \leq 3$. □

Examples

We found nodal cubics threefolds over $\mathbf{F}_2$ and $\mathbf{F}_3$ with no lines.

Over $\mathbf{F}_2$:

$$x_2^3 + x_2^2 x_3 + x_3^3 + x_1 x_2 x_4 + x_3^2 x_4 + x_4^3 + x_1^2 x_5 + x_1 x_3 x_5 + x_2 x_4 x_5$$

contains no $\mathbf{F}_2$-lines and

$$H(T) = T^4 - 3T^3 - 6T^2 + 24T - 15.$$

Over $\mathbf{F}_3$:

$$2x_1^3 + 2x_1^2 x_2 + x_1 x_2^2 + 2x_2 x_3^2 + 2x_1 x_2 x_4 + x_2 x_3 x_4 \\ + x_1 x_4^2 + 2x_4^3 + x_2 x_3 x_5 + 2x_3^2 x_5 + x_2 x_5^2 + x_5^3$$

contains no $\mathbf{F}_3$-lines and

$$H(T) = T^4 - 4T^3 - 9T^2 + 47T - 32.$$

Zeta function of $F(X)$

$X \subset \mathbf{P}^5_{\mathbf{F}_q}$ smooth cubic.The trace formula reads

$$Z(X,T) = \frac{1}{(1-T)(1-qT)(1-q^3T)(1-q^4T)\prod_{j=1}^{23}(1-q\omega_j T)},$$

with ω_j algebraic integers, $|\omega_j| = q$, and $\omega_{23} = q$.

The Galkin–Shinder formula implies

$$Z(F(X),T) = \frac{1}{(1-T)(1-q^4T)\prod_j((1-\omega_j T)(1-q^2\omega_j T))\prod_{j<k}(1-\omega_j\omega_k T)}.$$

Cohomology of $F(X)$

Again, the Galkin–Shinder relation implies that there are isomorphisms of $\mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$-modules

$$H^4(\overline{X},\mathbf{Q}_\ell) \xrightarrow{\ \sim\ } H^2(\overline{F(X)},\mathbf{Q}_\ell(1))$$

$$\mathrm{Sym}^2 H^2(\overline{F(X)},\mathbf{Q}_\ell) \xrightarrow{\sim} H^4(\overline{F(X)},\mathbf{Q}_\ell)$$

The first also follows using the incidence correspondence.

The second one can also be deduced by smooth and proper base change from statement in char. 0 (Beauville–Donagi, Bogomolov).

Existence of lines

Theorem

Any smooth $\mathbf{F}_q$-cubic fourfold contains at least 26 $\mathbf{F}_q$-lines if $q \geq 5$.

One can use another trace formula (Katz). If $\mathrm{Sing}(X)$ finite, the cohomology of $\mathscr{O}_{F(X)}$ is very simple (Altman–Kleiman):

$$\dim_{\mathbf{F}_q} H^j(F(X), \mathscr{O}_{F(X)}) = 1$$

for $j \in \{0, 2, 4\}$, the others are 0, and the multiplication

$$H^2(F(X), \mathscr{O}_{F(X)}) \otimes H^2(F(X), \mathscr{O}_{F(X)}) \to H^4(F(X), \mathscr{O}_{F(X)})$$

is an isomorphism of Galois modules (Serre duality).

The Katz trace formula

$$\begin{aligned} N_1(F(X)) &\equiv \sum_{j=0}^{4}(-1)^j \operatorname{Tr}(F, H^j(F(X), \mathscr{O}_{F(X)})) \pmod p \\ &\equiv 1 + t + t^2 \pmod p \end{aligned}$$

Corollary

Assume $q \equiv 2 \pmod 3$. Any $\mathbf{F}_q$-cubic fourfold with finite singular set contains an $\mathbf{F}_q$-line.

This applies to $q = 2$ and leaves only the cases $q \in \{3, 4\}$ open for the existence of a line on a smooth cubic fourfold.

Final question

The computer found a smooth cubic fourfold over $\mathbf{F}_2$ with a single line.

Question

We found no cubic fourfolds without lines. Do they exist?