On some Fano fourfolds

Olivier DEBARRE (joint with Atanas ILIEV and Laurent MANIVEL)

École normale supérieure de Paris, France

Yeosu, February 2013

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- π is dominant of degree 2: for $x \in X$ general, $\langle \ell, x \rangle \cap X$ is the union of ℓ and a conic; the two intersection points are $\pi^{-1}(x)$.

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No smooth cubic fourfold has yet been proven to be irrational

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is a cubic hypersurface with singular locus of codimension 6 in $\mathbf{P}^{14}$. A general linear section $X \cap \mathbf{P}^5$ is a (smooth) *Pfaffian cubic.*

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$$\mathbf{P}(\mathbf{C}^6) \xleftarrow{p_1} \{(v,\eta) \in \mathbf{P}(\mathbf{C}^6) \times \mathrm{Pf}(\mathbf{C}^6) \mid v \in \ker \eta\} \xrightarrow{p_2} \mathrm{Pf}(\mathbf{C}^6),$$

p_1 is a $\mathbf{P}^9$-bundle, p_2 is generically a $\mathbf{P}^1$-bundle.

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If $H \subset \mathbf{P}(\mathbf{C}^6)$ hyperplane,

$$H \xleftarrow{1:1} \{(v,\eta) \in H \times X \mid v \in \ker \eta\} \xrightarrow{1:1} X.$$

Period domain

$X \subset \mathbf{P}^5$ smooth cubic fourfold

Primitive Hodge structure is "of K3 type"

$$\begin{array}{ccccccccc} H^4(X, \mathbf{C})_0 & = & H^{1,3}(X) & \oplus & H^{2,2}(X)_0 & \oplus & H^{3,1}(X) \\ \text{dimensions} & & 1 & & 20 & & 1 \end{array}$$

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Lattices

$$\begin{array}{rcl} H^4(X,\mathbf{Z}) & \simeq & I_{21,2} \\ H^4(X,\mathbf{Z})_0 & \simeq & 2E_8 \oplus 2U \oplus A_2 =: \Lambda_0 \end{array}$$

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Local period domain (20-dimensional, bounded symmetric domain of type IV)

$$\mathscr{Q}_0 := \{\omega \in \mathbf{P}(\Lambda_0 \otimes \mathbf{C}) \mid \omega \cdot \omega = 0, \omega \cdot \bar{\omega} < 0\}$$

Period map

Moduli space is the 20-dimensional affine GIT quotient

$$\mathscr{M} := \mathbf{P}(H^0(\mathbf{P}^5, \mathscr{O}_{\mathbf{P}^5}(3)))^0 /\!/ \mathrm{SL}(\mathbf{C}^6)$$

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- *image of p known (Looijenga, Laza).*

Noether-Lefschetz locus

Dominance of period map implies

$$H^4(X,\mathbf{Z}) \cap H^{2,2}(X,\mathbf{Z}) = \mathbf{Z}h^2 \qquad \text{for } X \text{ very general} \qquad (1)$$

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The Noether-Lefschetz locus (where (1) does not hold) is

$$p^{-1}\Big(\bigcup_{d \in \mathbf{Z}} \mathscr{D}_0^d\Big)$$

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2) If X is a Pfaffian cubic, it contains a quartic scroll T and $T^2 = 10$: we are in $\mathscr{Q}_0^K$ with $K = \begin{pmatrix} 3 & 4 \\ 4 & 10 \end{pmatrix}$. The period is a general point of $\mathscr{D}_0^{14}$.

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- So cubics whose period is a general point of $\mathscr{D}_0^{14}$ are rational.
- Infinitely many divisors in $\mathscr{D}_0^8$ correspond to rational cubics (Hassett), but one expects a general cubic containing a plane to be irrational.

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- $\mathscr{D}_0^d \neq \varnothing$ iff $d > 0$ and $d \equiv 0, 2 \pmod 6$ (Hassett),
- image of p is $\overline{\mathscr{D}_0}^{\mathrm{BB}} - \overline{\mathscr{D}_0^2} - \overline{\mathscr{D}_0^6}$ (Laza).

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Here, the group $H^2(F(X), \mathbf{Z})_0$ is endowed with the Beauville quadratic form.

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The construction of a double cover $\mathbf{P}^4 \dashrightarrow X$ is a bit involved but elementary.

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from P is birational onto its image, a smooth cubic containing a cubic scroll (expected to be irrational in general).

Period domain

$X \subset \mathbf{P}^8$ as above.

"Vanishing" Hodge structure is of K3 type

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2) What are the fibers?

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- image $\mathscr{D}_1^d \subset \mathscr{D}_1$: it depends only on $d := \mathrm{disc}(K)$, except when $d \equiv 2 \pmod 8$, where it has two components $\mathscr{D}_1'^d$ and $\mathscr{D}_1''^d$.

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1) If X contains a plane $P := \mathbf{P}(V_1 \wedge V_4)$, we are in $\mathscr{Q}_1^K$ with $K = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. The period is a general point of $\mathscr{D}_1^{\prime\prime 10}$.

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3) If X contains a plane $P := G(2, V_3)$, the period is a general point of $\mathscr{D}_1^{12}$.

Image of the period map

Conjecture

The image of p should be

$$\overline{\mathscr{D}}_1^{\mathrm{BB}} - \overline{\mathscr{D}}_1^2 - \overline{\mathscr{D}}_1^4 - \overline{\mathscr{D}}_1^8.$$

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Theorem

- *$\widetilde{Y}_X$ is an irreducible symplectic fourfold (double EPW sextic).*
- *The incidence correspondance induces a factorization*

$$H^4(X, \mathbf{Z})_{\mathrm{van}} \xrightarrow{a} H^2(\widetilde{Y}_X, \mathbf{Z})_0(-1) \stackrel{\beta^*}{\hookrightarrow} H^2(F(X), \mathbf{Z})(-1)$$

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The group $H^2(\widetilde{Y}_X, \mathbf{Z})_0$ is endowed with the Beauville quad. form.

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We want to reconstruct X from (Y_X, h_X), hence parametrize the fiber of the period map by Y_X.

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In particular, general fibers of the period map are proper.

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We expect the induced finite morphism $Y_X \to F_X$ to be an isomorphism.

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Conjecture

The moduli space for fourfolds as above is the universal EPW sextic.

Problem: we don't even know that the moduli functor is separated...