THE FANO THREEFOLD X_{10}

OLIVIER DEBARRE

This is joint work in progress with A. Iliev (Sofia) and L. Manivel (Grenoble).

1. Introduction

A Fano variety is a (complex) projective manifold X with $-K_X$ ample.

- Dimension 1: $\mathbf{P}^1$ (1 family).
- Dimension 2 (*del Pezzo surfaces*): $\mathbf{P}^2$ blown-up in at most 8 points, or $\mathbf{P}^1 \times \mathbf{P}^1$ (10 families).
- Dimension 3: all classified ($17 + 88$ families).
- Dimension n: finitely many families.

Examples:

- the projective space $\mathbf{P}^n$.
- smooth hypersurfaces in $\mathbf{P}^{n+1}$ of degree $\leq n+1$, such as a cubic threefold $X_3 \subset \mathbf{P}^{n+1}$ with $n \geq 2$.
- the threefold X_{10} intersection in $\mathbf{P}(\wedge^2 V_5) = \mathbf{P}_9$ of the Grassmannian $G(2, V_5)$, a quadric Ω, and a $\mathbf{P}_7$ (V_5 is a 5-dimensional vector space).

Fano threefolds X with Picard group $\mathbf{Z}[K_X]$ and $(-K_X)^3 = 10$ are anticanonically embedded in $\mathbf{P}_7$ and, for "almost all" of them, the image is described as above.

2. Rationality, unirationality

Proposition 2.1 (M. Noether)**.** *A smooth cubic hypersurface $X_3 \subset \mathbf{P}^{n+1}$ is unirational for $n \geq 2$.*

Proof. Since a smooth cubic surface contains 27 lines and $n \geq 2$, the hypersurface X contains a line ℓ. A line ℓ_x tangent to X at a point x of ℓ meets X in a third point. The induced map

$$\begin{array}{rcl} \mathbf{P}(T_X|_\ell) & \dashrightarrow & X \\ \ell_x \subset T_{X,x} & \longmapsto & \text{third point of intersection} \\ & & \text{of } \ell_x \text{ with } X \end{array}$$

is dominant, of degree 2: if $x' \in X$, the intersection of X with the plane $\langle \ell, x' \rangle$ is the union of ℓ and a conic c and the two preimages of x' are the two points of $\ell \cap c$.

Since $\mathbf{P}(T_X|_\ell) \overset{\sim}{\dashrightarrow} \mathbf{P}^n$, the variety X is unirational. □

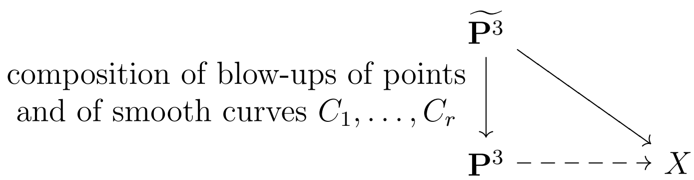
Most Fano threefolds (including X_{10}) are known to be unirational. However, smooth cubic hypersurfaces in $\mathbf{P}^4$ are *not* rational (Clemens–Griffiths, 1972). It is not known whether the general cubic of dimension ≥ 4 is rational.

3. The intermediate Jacobian

The *intermediate Jacobian* of a Fano threefold X is

$$J(X) = H^{2,1}(X)^\vee / \operatorname{Im} H_3(X, \mathbf{Z})$$

It is a principally polarized abelian variety: it contains a *theta divisor* Θ, uniquely defined up to translation, and the pair $(J(X), \Theta)$ carries information about X. For example, if X is rational, we have, by Hironaka's resolution of indeterminacies, a diagram



- $J(\widetilde{\mathbf{P}^3}) \simeq J(C_1) \times \cdots \times J(C_r)$;
- $J(X)$ is also a product of Jacobians of smooth curves, hence

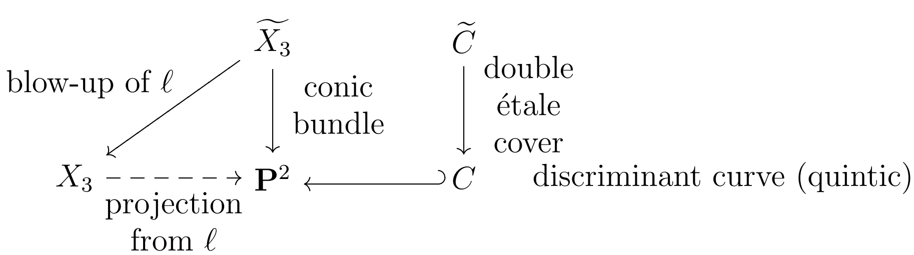
$$\boxed{\operatorname{codim}_{J(X)} \operatorname{Sing}(\Theta) \leq 4}$$

This works only in dimension 3!

Theorem 3.1. *A smooth cubic threefold $X_3 \subset \mathbf{P}^4$ is* not *rational.*

We will see later that a general X_{10} is also not rational.

Proof. Projection from a line $\ell \subset X_3$ yields



- $J(X_3)$ has dimension 5 and is isomorphic to the *Prym variety* of the cover $\widetilde{C} \to C$;
- Θ has a unique singular point o (Beauville).

It follows that X_3 is *not* rational (because $\operatorname{codim}_{J(X_3)} \operatorname{Sing}(\Theta) = 5$). $\square$

4. The Abel–Jacobi map

Geometric information about subvarieties of $J(X)$ is in general hard to get, in particular when X is *not* a conic bundle as before. One can construct subvarieties from families of curves on X:

- $(C_t)_{t\in T}$ (connected) family of curves on X, with a base-point $0 \in T$;
- C_t is algebraically, hence homologically, equivalent to C_0, so $C_t - C_0$ is the boundary of a (real) 3-chain Γ_t, defined modulo $H_3(X, \mathbf{Z})$;
- define the *Abel–Jacobi map*

$$\begin{array}{ccc} T & \longrightarrow & J(X) = H^{2,1}(X)^\vee / H_3(X, \mathbf{Z}) \\ t & \longmapsto & \left(\omega \mapsto \int_{\Gamma_t} \omega\right) \end{array}$$

5. The Torelli theorem for cubic threefolds

For example, lines on a smooth cubic threefold $X_3 \subset \mathbf{P}^4$ are parametrized by a smooth projective connected surface F. The Abel–Jacobi map $F \to J(X_3)$ is an embedding, and $\Theta = F - F$. From this, one can recover Beauville's results:

- o is a singular point of Θ, which is a triple point;
- the projectified tangent cone to Θ at o is isomorphic to X_3:

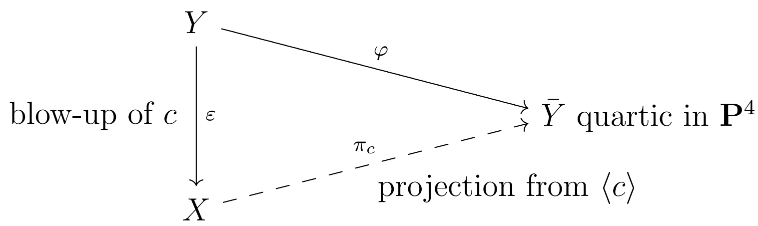
$$X_3 \simeq \mathbf{P}(TC_{\Theta,o}) \subset \mathbf{P}(T_{J(X_3),o}) \simeq \mathbf{P}^4$$

In particular, X_3 can be reconstructed from its intermediate Jacobian. This is the *Torelli theorem* for cubic threefolds.

6. The Torelli problem for X_{10}

For (a general) $X_{10} = G(2, V_5) \cap \mathbf{P}_7 \cap \Omega \subset \mathbf{P}(\wedge^2 V_5) = \mathbf{P}_9$, we will use *conics* to *disprove* the Torelli theorem. Conics on X_{10} are parametrized by a smooth projective connected surface $F(X)$ (Logachev) which is the blow-up at one point of a minimal surface of general type $F_m(X)$.

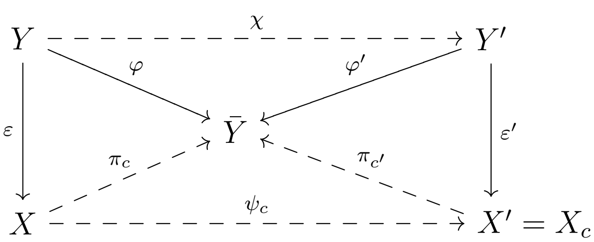
6.1. **Elementary transformation along a conic.** Let c be a general conic in a general X. Consider



The only curves contracted by φ are:

- the (strict transforms of the 20) lines in X that meet c;
- (the strict transform of) a conic $\iota(c)$ that meets c in 2 points.

It is a therefore *small contraction*. Furthermore, the exceptional divisor E of ε is φ-ample, hence one can perform a $(-E)$-flop:



where

- $\chi = \varphi' \circ \varphi^{-1}$ is an isomorphism outside curves contracted by φ or φ' and $\chi_*(E)$ is φ-antiample;
- Y' is a smooth projective threefold;
- X' is again a smooth Fano threefold of degree 10 in $\mathbf{P}^7$;
- ε' is the blow-up of a smooth conic c' in X';
- the picture is symmetric: $\psi_{c'} = \psi_c^{-1} : X' \dashrightarrow X$;
- the intermediate Jacobians of X and X' are isomorphic.

6.2. **The Torelli property does not hold for X_{10}.** It is enough to show that X and X' are *not* isomorphic. We construct

$$\varphi_c : F(X) \overset{\sim}{\dashrightarrow} F(X')$$

as follows:

- for $\bar{c} \in F(X)$ general, $\langle c, \bar{c} \rangle$ is a 5-plane in $\mathbf{P}^7$;
- $X \cap \langle c, \bar{c} \rangle$ is a section by a codimension-2 plane of the Fano variety X; by the adjunction formula, it is a canonically embedded genus-6, degree-10, curve $c + \bar{c} + \Gamma_{c,\bar{c}}$;

- $\Gamma_{c,\bar{c}}$ is a rational sextic meeting c and $\bar{c}$ in 4 points each, hence its image by ψ_c is a conic because $\psi_c \circ \varepsilon$ is given by a linear subsystem of $|3\varepsilon^* H - 4E|$ on Y;
- set $\varphi_c([\bar{c}]) = [\psi_c(\Gamma_{c,\bar{c}})]$.

The map φ_c is birational, hence induces an isomorphism

$$\varphi_c : F_m(X) \xrightarrow{\sim} F_m(X')$$

and $F(X')$ is isomorphic to the surface $F_m(X)$ blown up at the point $[c]$.

Since the automorphism group of a minimal surface of general type is finite, we have a 2-dimensional family of X_{10} with isomorphic intermediate Jacobians. In fact, one can show (Logachev) that the abstract surface $F(X)$ determines X up to isomorphism.

6.3. **Elementary transformations along a line.** Analogous constructions can be done by projecting from a line ℓ contained in X:

- we get a birational isomorphism $\psi_\ell : X \overset{\sim}{\dashrightarrow} X_\ell$;
- X_ℓ is again a smooth Fano threefold of degree 10 in $\mathbf{P}^7$;
- $J(X) \simeq J(X_\ell)$;
- there is a line $\ell' \subset X_\ell$ such that $\psi_{\ell'} = \psi_\ell^{-1} : X_\ell \dashrightarrow X$;
- for $c' \in F(X_\ell)$ general, $\psi_{\ell'}(c')$ is a rational quartic in X that meets ℓ in 2 points;
- the curve $\psi_{\ell'}(c') \cup \ell$ has trivial canonical sheaf and, applying Serre's construction, we get a birational isomorphism

  $$F(X_\ell) \overset{\sim}{\dashrightarrow} \mathscr{M}_X(2;1,5)$$

  where $\mathscr{M}_X(2;1,5)$ is the moduli space of semistable vector bundles of rank 2 on X with Chern numbers $c_1 = 1$ and $c_2 = 5$ (a smooth irreducible surface).

The transforms $(X_\ell)_c$, for $[c] \in F(X_\ell)$, again form a 2-dimensional family, parametrized by $\mathscr{M}_X(2;1,5)$, of Fano threefolds with same intermediate Jacobian as X.

6.4. **Fibers of the period map for X_{10}.** We expect that these two families (conic transforms and conic transforms of a line transform) should yield *all* Fano threefolds with same intermediate Jacobians as X, thereby describing the (general) fibers of the period map

$$\left\{\begin{matrix}\text{22-dim'l family of Fano}\\ \text{threefolds } X_{10}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{55-dim'l family of p.p.a.v.}\\ \text{of dimension 10}\end{matrix}\right\}$$

At the moment, here is what we can show.

- The kernel of the differential of the period map at *any* smooth X_{10} has dimension 2. In particular, fibers are smooth surfaces.

- For X *general,* we can extend the construction of the conic transform X_c to *all conics* $c \subset X$ (not just general ones).

In particular, this yields two 2-dimensional *projective* components of general fibers of the period map (except we cannot yet prove that they are different!). We expect that general fibers have just this two components: one isomorphic to $F(X)$ and one to $\mathscr{M}_X(2;1,5)$ (or rather their quotient by an involution, but we will ignore that). This would imply a *birational Torelli theorem: two general X_{10} with isomorphic intermediate Jacobians are birationally isomorphic.*

7. Nodal X_{10}

7.1. **The variety X_O.** Take now for Ω a (general singular) quadric with vertex a general point O of $W = G(2,V_5) \cap \mathbf{P}_7 \subset \mathbf{P}(\wedge^2 V_5) = \mathbf{P}_9$, so that $X = W \cap \Omega$ is smooth except for one node at O. Let $p_O : \mathbf{P}_7 \dashrightarrow \mathbf{P}^6_O$ be the projection from O.

- $W_O = p_O(W) \subset \mathbf{P}^6_O$ is the (singular) base-locus of a pencil of (singular) rank-6 quadrics (all such pencils are isomorphic, and were explicitly described by Hodge and Pedoe).
- $X_O \subset W_O \subset \mathbf{P}^6_O$ is the intersection of W_O with the general smooth quadric $\Omega_O = p_O(\Omega)$. It has 6 singular points corresponding to the 6 lines in X through O.

The intermediate Jacobian $J(X)$ is no longer an abelian variety, but rather an extension (called a rank-1 degeneration)

$$1 \to \mathbf{C}^\star \to J(X) \to J(\widetilde{X}) \to 0 \tag{1}$$

where $\widetilde{X} \to X$ is the blow-up of O.

We consider:

- the net Π of quadrics containing X_O;
- the discriminant curve $\Gamma_7 \subset \Pi$, union of the pencil Γ_1 of quadrics that contain W_O and a smooth sextic Γ_6 meeting Γ_1 transversely in 6 points $p_1, \dots, p_6$;
- the associated double étale cover $\pi : \widetilde{\Gamma}_7 \to \Gamma_7$, inducing $\pi : \widetilde{\Gamma}_6 \to \Gamma_6$, corresponding to the choice of a family of 3-planes contained in a quadric of rank 6 in $\mathbf{P}^6_O$.

In this situation, the choice of any line $\ell \subset X_O$ induces (by mapping a point $x \in X_O$ to the unique quadric in Π that contains the 2-plane $\langle \ell, x \rangle$) a birational conic bundle structure $p_\ell : X \dashrightarrow \Pi$ with discriminant cover $\pi : \widetilde{\Gamma}_6 \to \Gamma_6$. It follows from the general theory of conic bundles that there is an isomorphism

$$J(\widetilde{X}) \simeq \mathrm{Prym}(\widetilde{\Gamma}_6/\Gamma_6).$$

Proposition 7.1. *A general* X_{10} *is* not *rational.*

Proof. If X_{10} were rational, $J(X_{10})$ would be a product of Jacobians, and so would $J(\widetilde{X})$ (all rank-1 degenerations of (product of) Jacobians of curves are well-known). But this is not the case, since the theta divisor is not singular enough. □

Conversely, one can construct a nodal X as above from a double étale covering of a general reducible septic $\Gamma_7 = \Gamma_1 \cup \Gamma_6$, or equivalently, from a double étale covering of a general sextic Γ_6 and a general point $\tilde{p}_1 + \cdots + \tilde{p}_6$ in a special surface S associated to the linear system $|\mathscr{O}_{\Gamma_6}(1)|$. All these nodal X have isomorphic intermediate Jacobians.

This surface should be seen as the component $F(X)$ of the fiber of the period map extended to nodal X_{10} (one can show that S is indeed the (normalization of) the surface $F(X)$). Where is the other component?

7.2. **Another conic bundle structure on X.** The "double projection from O" is the rational map

$$p_W : X \dashrightarrow \mathbf{P}^2_W$$

induced on X by the projection from the 4-plane $\mathbf{T}_{W,O}$. One checks that it is birationally a conic bundle with discriminant a plane sextic $\Gamma_6^\star$ (isomorphic to the curve of conics in X passing through O). Once again,

$$J(\widetilde{X}) \simeq \operatorname{Prym}(\widetilde{\Gamma}_6^\star/\Gamma_6^\star)$$

although the sextics Γ_6 and $\Gamma_6^\star$ are in general not isomorphic.

7.3. **Verra's solid.** So we have double étale covers of two distinct plane sextics with the same Prym variety. This fits with the fact that the Prym map has degree 2 on plane sextics (Verra). The link with Verra's construction goes as follows.

Choose a line ℓ contained in the smooth quadric surface $p_O(\mathbf{T}_{W,O}) \cap X_O$ which passes through one of the six singular points of X_O. The image of the map $(p_W, p_\ell) : X \dashrightarrow \mathbf{P}^2_W \times \Pi$ is then a divisor of bidegree $(2,2)$.

This is precisely the starting point of Verra's construction: let Π_1 and Π_2 be two copies of $\mathbf{P}^2$ and let $T \subset \Pi_1 \times \Pi_2$ be a general (smooth) divisor of bidegree $(2,2)$. It is a Fano thereefold of index 1, and each projection $\rho_i : T \to \Pi_i$ makes it into a conic bundle with discriminant curve a smooth plane sextic $\Gamma_{6,i} \subset \Pi_i$ and associated connected double étale covering $\pi_i : \widetilde{\Gamma}_{6,i} \to \Gamma_{6,i}$. In particular, we have

$$J(T) \simeq \operatorname{Prym}(\widetilde{\Gamma}_{6,1}/\Gamma_{6,1}) \simeq \operatorname{Prym}(\widetilde{\Gamma}_{6,2}/\Gamma_{6,2}).$$

As explained above, this yields two components of the extended period map, parametrized by two special surfaces S_1 and S_2. We expect these two components to be the limits of the two components constructed in the smooth case.