On the geometry of certain Fano threefolds

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Fano manifolds

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- Dimension n: finitely many families.

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  - a cubic surface $X_3 \subset \mathbf{P}^3$;
  - a cubic threefold $X_3 \subset \mathbf{P}^4$.
- The threefold X_{10} intersection in $\mathbf{P}(\wedge^2 \mathbf{C}^5) = \mathbf{P}^9$ of the Grassmannian $G(2,5)$, a quadric, and a $\mathbf{P}^7$.

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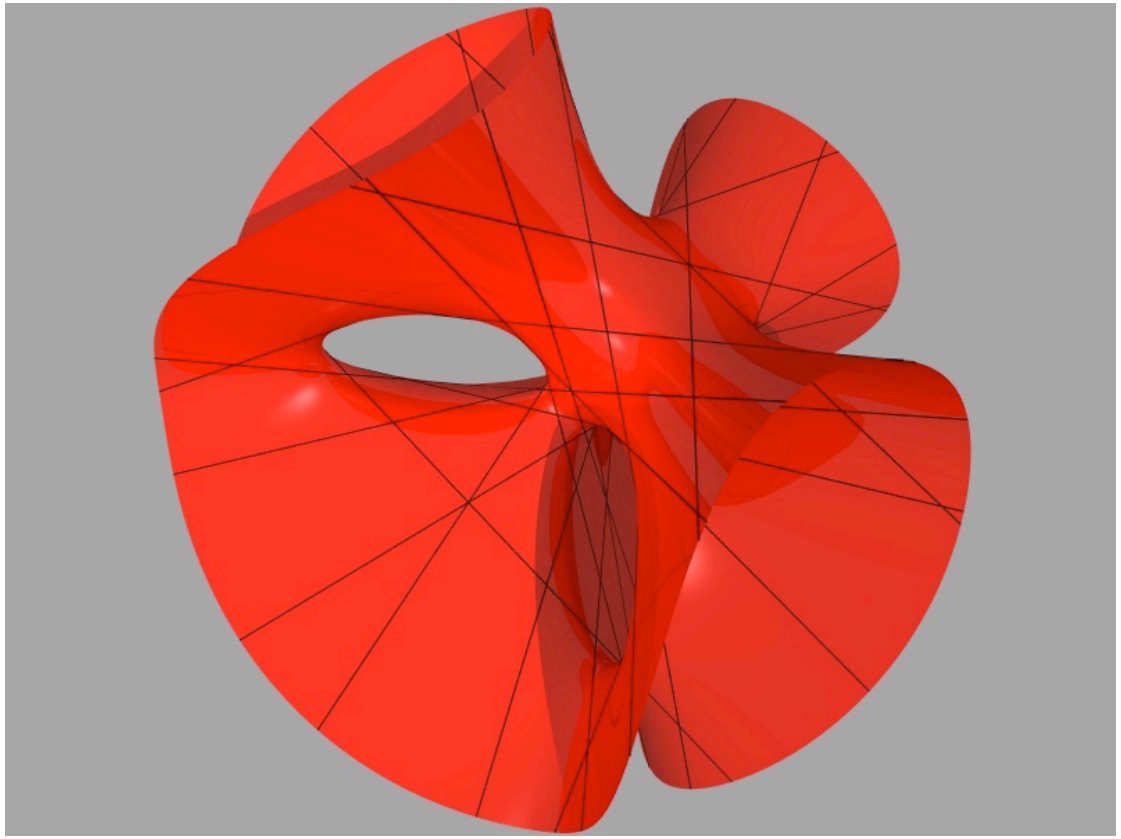

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The surface X is in fact $\mathbf{P}^2$ blown-up in 6 points hence rational (as all del Pezzo surfaces).

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- A (smooth) cubic hypersurface in $\mathbf{P}^{n+1}$ is unirational for $n \geq 2$.[2]

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Some (smooth) cubic hypersurfaces in $\mathbf{P}^n$, $n \geq 5$, such as

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...but the rationality of general cubics is unknown!

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The pair $(J(X), \Theta)$ carries information about X.

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$$\operatorname{codim}_{J(X)} \operatorname{Sing}(\Theta) \leq 4$$

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- Θ has a unique singular point o, which is a triple point (Beauville);
- X_3 is *not* rational (because $\operatorname{codim}_{J(X_3)} \operatorname{Sing}(\Theta) = 5$).

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$$\begin{array}{ccc} T & \longrightarrow & J(X) = H^{2,1}(X)^\vee / H_3(X, \mathbf{Z}) \\ t & \longmapsto & \left(\omega \mapsto \int_{\Gamma_t} \omega \right) \end{array}$$

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[1] Ruggiero Torelli (Napoli 1884 – Isonzo, 1915) showed that a *curve* C is uniquely determined by its Jacobian $(J(C), \Theta)$.

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- for which (by Prym theory) $\operatorname{codim}_{J(X')} \operatorname{Sing}(\Theta) > 4$.
- This implies the same for a *general* X_{10}, which is therefore *not rational.*

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$$\chi : Y \xrightarrow{\varphi} \bar{Y} \xleftarrow{\varphi'} Y' \quad \text{smooth projective threefold}$$

(isomorphism outside curves contracted by φ or φ').

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Are X and X' isomorphic?

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To answer this question (negatively), we study $F(X')$.

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The transforms $(X_\ell)_c$, for $c \in F(X_\ell)$, again form a 2-dimensional family of Fano threefolds with same intermediate Jacobian as X.

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(General) fibers should be the union of two surfaces.

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- Relax condition $\det(T_X)$ ample to $\det(T_X)$ big and nef (not all are degenerations of Fano varieties).