UNEXPECTED ISOMORPHISMS BETWEEN HYPERKÄHLER FOURFOLDS

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Abstract. Using Verbitsky's Torelli theorem, we show the existence of various isomorphisms between certain hyperkähler fourfolds. This is joint joint with Emanuele Macrì.

1. Introduction: some "classical" isomorphisms

1.1. **The Beauville and O'Grady involutions.** Let $S \subset \mathbf{P}^3$ be a smooth (complex) quartic surface containing no lines (a general K3 surface with a degree-4 polarization). The Beauville involution σ of the Hilbert square $S^{[2]}$ takes a pair of points of S to the residual intersection with S of the line joining them. We have a diagram

$$\begin{array}{ccc} S^{[2]} & \xrightarrow{\ \sigma\ } & S^{[2]} \\ & {\scriptstyle 6:1}\searrow \quad \swarrow{\scriptstyle 6:1} & \\ & \mathsf{Gr}(2,4). & \end{array}$$

Let now $S = \mathsf{Gr}(2, V_5) \cap Q \cap \mathbf{P}(V_6) \subset \mathbf{P}(\bigwedge^2 V_5)$ be a general K3 surface with a degree-10 polarization. A general point of $S^{[2]}$ corresponds to $V_2, W_2 \subset V_5$. Then

$$\mathsf{Gr}(2, V_2 \oplus W_2) \cap S = \mathsf{Gr}(2, V_2 \oplus W_2) \cap Q \cap \mathbf{P}(V_6 \cap \textstyle\bigwedge^2 (V_2 \oplus W_2)) \subset \mathbf{P}^2$$

consists of 4 points. The (birational) O'Grady involution $S^{[2]} \dashrightarrow S^{[2]}$ takes the pair of points $([V_2], [W_2])$ to the residual two points of this intersection.

1.2. **The Beauville–Donagi isomorphism.** Let $W \subset \mathbf{P}^5$ be a general (smooth) Pfaffian cubic fourfold (this means that an equation of W is given by the Pfaffian of a 6×6 skew-symmetric matrix of linear forms; in the terminology that we will introduce later, W is a "general cubic fourfold of discriminant 14"). Beauville & Donagi showed that the variety $F(W)$ of lines contained in W is isomorphic to the Hilbert square $S^{[2]}$ of a general polarized K3 surface of degree 14.

Our aim in this talk is to produce other "unexpected" isomorphisms between various kinds of hyperkähler fourfolds: Hilbert squares of K3 surfaces, varieties of lines contained in a cubic fourfold, double EPW sextics.

2. Special hyperkähler fourfolds

Polarized hyperkähler fourfolds (F,h) of $\mathrm{K3}^{[2]}$-type admit a moduli space. Gritsenko–Hulek–Sankaran showed that its irreducible components $\mathscr{M}_{2n}^{(\gamma)}$ are all quasi-projective of dimension 20 and are labelled by the degree $2n := q_{BBJ}(h)$ of the polarization h and its *divisibility* $\gamma \in \{1,2\}$ (defined by $h \cdot H^2(F,\mathbf{Z}) = \gamma\mathbf{Z}$). A very general element in $\mathscr{M}_{2n}^{(\gamma)}$ has Picard number 1; its (birational) automorphism group is trivial (with the notable exception of the case $e=\gamma=1$) and nothing unexpected happens (note however that a fourfold with Picard number 1 and an automorphism of order 23 was constructed by Boissière–Camere–Mongardi–Sarti, but it is not very general).

When the Picard number is 2 (or ≥ 2; the fourfold F is then called *special*), an important supplementary invariant is the *discriminant* of the rank-2 lattice $(\mathrm{Pic}(F), q_{BBJ})$ (actually, it is rather the opposite of the discriminant of its orthogonal in $H^2(F,\mathbf{Z})$; they differ by a factor of -2). We let $\mathscr{C}_{2n,d}^{(\gamma)} \subset \mathscr{M}_{2n}^{(\gamma)}$ be the (closure of the) locus of fourfolds F with rank-2 discriminant-d lattice $(\mathrm{Pic}(F), q_{BBJ})$. Their union over all d is called the Noether–Lefschetz locus in $\mathscr{M}_{2n}^{(\gamma)}$.

When n is a prime number $p \equiv -1 \pmod 4$ and $\gamma = 2$, the $\mathscr{C}_{2n,d}^{(\gamma)}$ are irreducible or empty. The situation was worked out by Hassett when $p=3$: (general) elements of $\mathscr{M}_6^{(2)}$ correspond to varieties of lines contained in a cubic fourfold $W \subset \mathbf{P}^5$ and, for small (even) d, general elements of $\mathscr{C}_{6,d}^{(2)}$ correspond to cubics W with special geometric features: W contains a plane ($d=8$), W contains a cubic scroll ($d=12$), W is a Pfaffian cubic ($d=14$), W contains a Veronese surface ($d=20$); the locus $\mathscr{C}_{6,6}^{(2)}$ is empty.

The case $n=11$ and $\gamma=2$ corresponds to fourfolds constructed by Debarre–Voisin as smooth zero loci of sections of the third exterior power of the canonical rank-6 subbundle over the Grassmannian $\mathsf{Gr}(6,10)$. Nothing is known about the loci $\mathscr{C}_{11,d}^{(2)}$.

The case $n=1$ was worked out by Debarre–Iliev–Manivel: (general) elements of $\mathscr{M}_2^{(1)}$ correspond to so-called double EPW sextics varieties and, for small (even) d, general elements of $\mathscr{C}_{2,d}^{(1)}$ have geometric descriptions. Not all $\mathscr{C}_{2,d}^{(1)}$ are irreducible.

We will show that there are isomorphisms between general elements of various $\mathscr{C}_{2p,2e}^{(2)}$, some Hilbert squares of K3 surfaces, and some double EPW sextics.

Why do we care about these loci? For any hyperkähler fourfold F of $\mathrm{K3}^{[2]}$-type, the lattice $(H^2(F,\mathbf{Z}), q_{BBF})$ is isomorphic to

$$L_{\mathrm{K3}^{[2]}} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus I_1(-2).$$

Given a class $h^0 \in L_{\mathrm{K3}^{[2]}}$ with square $2n$ and divisibility γ, one defines the period map

$$\wp_{\gamma,n} \colon \mathscr{M}_{2n}^{(\gamma)} \longrightarrow O(L_{\mathrm{K3}^{[2]}}, h_0)\backslash \Omega_{h_0},$$

where

$$\Omega_{h_0} := \{x \in \mathbf{P}(L_{\mathrm{K3}^{[2]}} \otimes \mathbf{C}) \mid x \cdot h_0 = 0,\ x \cdot x = 0,\ x \cdot \bar{x} > 0\}$$

and

$$O(L_{\mathrm{K3}^{[2]}},h_0) := \{g \in O(L_{\mathrm{K3}^{[2]}}) \mid g(h_0)=h_0\}.$$

The Torelli theorem (Verbitsky, Markman) says that each $\wp_{\gamma,n}$ is an open embedding and one would like to know exactly what its image is. The special loci $\mathscr{C}^{(\gamma)}_{2n,d}$ are inverse images by $\wp_{\gamma,n}$ of *Heegner divisors* $\mathscr{D}^{(\gamma)}_{2n,d}$ in the period domain and one expects that the image of the period map is the complement of a finite union of these divisors. This was proved by Laza for $n=3$ and $\gamma=2$: the image is the complement of $\mathscr{D}^{(2)}_{6,6}$.

3. The Torelli theorem

The Torelli-type theorem for hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type has the following precise form.

Theorem 1 (Verbitsky, Markman)**.** *Let F_1 and F_2 be hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type, with respective polarizations h_1 and h_2. Let $\varphi\colon H^2(F_1,\mathbf{Z}) \xrightarrow{\sim} H^2(F_2,\mathbf{Z})$ be an isometry of Hodge structures such that $\varphi(h_1)=h_2$. There exists an isomorphism $u\colon F_2 \xrightarrow{\sim} F_1$ such that $\varphi = u^*$.*

This theorem allows us to translate the question of the existence of an isomorphism between (possibly identical) hyperkähler fourfolds F_1 and F_2 with Picard number 2 into the question of extending an isomorphism between their Picard lattices $\mathrm{Pic}(F_1)$ and $\mathrm{Pic}(F_2)$ to an isomorphism between $H^2(F_1,\mathbf{Z})$ and $H^2(F_2,\mathbf{Z})$ (both abstractly isomorphic to $L_{\mathrm{K3}^{[2]}}$).

4. Nef cones and automorphisms of Hilbert squares of K3 surfaces

To apply the Torelli theorem, one needs to know what the ample classes are. In the case of the Hilbert square of a K3 surface, there is no preferred polarization and we describe the nef cone. Let (S,f_S) be a very general polarized K3 surface of degree $2e$, so that $\mathrm{Pic}(S)=\mathbf{Z}f_S$, with $f_S^2=2e$. We have $\mathrm{Pic}(S^{[2]})=\mathbf{Z}f\oplus\mathbf{Z}\delta$, where f the class on $S^{[2]}$ induced by f_S and 2δ is the class of the divisor $E_S\subset S^{[2]}$ parametrizing non-reduced length-2 subschemes of S. We have the following intersection products for the Beauville–Bogomolov–Fujiki form on $H^2(S^{[2]},\mathbf{Z})$:

$$f^2=2e\ ,\quad \delta^2=-2\ ,\quad (f,\delta)=0.$$

The corresponding discriminant is $2e$ (but we have not chosen a polarization yet!). Since f is nef non-ample, it spans one of the two extremal rays of the nef cone $\mathrm{Nef}(S^{[2]})\subset \mathrm{Pic}(S^{[2]})\otimes\mathbf{R}$. The other extremal ray is spanned by a class $f-\nu_S\delta$, where ν_S is a positive real number.

Given integers a and b, the square of the class $bf-a\delta$ is $2(b^2e-a^2)$. The lattice $(\mathrm{Pic}(S^{[2]}),q_{BBJ})$ therefore represents $-2t$ if and only if the Pell-type equation

$$\mathscr{P}_e(t):\quad a^2-eb^2=t$$

has (integral) solutions. A solution (a,b) of this equation is called positive if $a>0$ and $b>0$ and the positive solution with minimal a is called the minimal solution.

Theorem 2 (Bayer–Macrì)**.** *Let (S,f_S) be a polarized K3 surface of degree $2e$ with Picard group $\mathbf{Z}f_S$. The slope ν_S is equal to the rational number ν_e defined as follows.*

- *Assume that the equation $\mathscr{P}_{4e}(5)$ has no solutions.*
  a) *If e is a perfect square, $\nu_e := \sqrt{e}$.*
  b) *If e is not a perfect square and (a_1, b_1) is the minimal solution of the equation $\mathscr{P}_e(1)$, we have $\nu_e := e\frac{b_1}{a_1}$.*
- *Assume that equation $\mathscr{P}_{4e}(5)$ has a solution.*
  c) *If (a_5, b_5) is its minimal solution, we have $\nu_e := 2e\frac{b_5}{a_5}$.*

Example 3. (automorphisms of Hilbert squares of very general K3 surfaces; Boissière–Cattaneo–Nieper-Wißkirchen–Sarti) Let (S, f_S) be a very general polarized K3 surface of degree $2e$. An automorphism of $S^{[2]}$ must preserve the nef cone. This gives very strong restrictions: it acts either trivially or as an involution about a line spanned by a class $D := b_{-1}f - a_{-1}\delta$ of square 2. The full statement is the following

- either $\mathrm{Aut}(S^{[2]})$ is trivial,
- or $\mathrm{Aut}(S^{[2]})$ has two elements, the equation $\mathscr{P}_e(-1)$ is solvable (and (a_{-1}, b_{-1}) is its minimal solution) but not $\mathscr{P}_{4e}(5)$.

In the second case, D is ample of square 2, so we get a 19-dimensional subvariety of $\mathscr{C}^{(1)}_{2,2e}$: these are double EPW sextics, hence the quotient of $S^{[2]}$ by its nontrivial involution is an EPW sextic in $\mathbf{P}^5$.

When $e = 2$, we recover the Beauville involution (note that in this case, the EPW sextic is 3 times the quadric $\mathsf{Gr}(2,4)$). For $e \in \{10, 13, 17, 26, \dots\}$, there is no further geometric description of the involution.

An analogous description of the movable cone[1] of $S^{[2]}$ can be used to describe the birational automorphism group $\mathrm{Bir}(S^{[2]})$. Nothing new happens: one has $\mathrm{Bir}(S^{[2]}) = \mathrm{Aut}(S^{[2]})$, except when $e = 5$, when $S^{[2]}$ has the birational O'Grady involution!

5. Nef cones and automorphisms of special hyperkähler fourfolds

Let F be a special hyperkähler fourfolds of degree $2p$, where $p \equiv -1 \pmod 4$ is prime, divisibility 2, and discriminant pe'. In a suitable basis (h, τ), the lattice $\mathrm{Pic}(F)$ has intersection matrix $\begin{pmatrix} 2p & 0 \\ 0 & -2e' \end{pmatrix}$. One shows that the two extremal rays of the nef cone of F are spanned by $h \pm \nu'_e \tau$, where ν'_e is a real number (possibly irrational, but then equal to $\sqrt{e}$) which depends only on e. Its value can be explicitly computed from the minimal solutions of the equations $\mathscr{P}_e(-p)$ and $\mathscr{P}_{4e}(-5p)$ (when they exist). As above, one can then find the groups $\mathrm{Aut}(F)$ and $\mathrm{Bir}(F)$. I will only state the result when $p = 3$.

Proposition 4. *Let F be a hyperkähler fourfold of* $\mathrm{K3}^{[2]}$*-type with a polarization of degree* 6 *and divisibility 2. Assume that the lattice* $(\mathrm{Pic}(F), q_{BBF})$ *has rank 2 and discriminant* $2e$ *divisible by* 3.

[1]This is the closed cone spanned by classes of divisors whose base locus has codimension ≥ 2. It is stable under the action of the birational automorphism group.

a) *If both equations $\mathscr{P}_e(-3)$ and $\mathscr{P}_{4e}(-15)$ are not solvable and e is not a perfect square, the groups* $\mathrm{Aut}(F)$ *and* $\mathrm{Bir}(F)$ *are equal. They are infinite cyclic, except when the equation $\mathscr{P}_e(3)$ is solvable, in which case they are infinite dihedral.*
b) *If the equation $\mathscr{P}_{4e}(-15)$ is solvable, the group* $\mathrm{Aut}(F)$ *is trivial and the group* $\mathrm{Bir}(F)$ *is infinite cyclic, except when the equation $\mathscr{P}_e(3)$ is solvable, in which case it is infinite dihedral.*
c) *If the equation $\mathscr{P}_e(-3)$ is solvable or if e is a perfect square, the group* $\mathrm{Bir}(F)$ *is trivial.*

The case $p = 3$ and $e = 6$ where ($\mathrm{Aut}(F) = \{\mathrm{Id}\}$ and $\mathrm{Bir}(F) \simeq \mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z} \simeq \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$) was obtained earlier by Hassett and Tschinkel using very elegant geometric constructions (recall that in that case, F is the variety of lines contained in a cubic fourfold containing a cubic scroll surface). When $3 \nmid e$, other groups may occur. Here is a table for some small values of e.

e	6	7	9	12	13	15	18	21	24	27	30	31
$\mathrm{Aut}(F)$	Id	Id	Id	Id	$\mathbf{Z}/2$	Id	$\mathbf{Z}$	Id	Id	$\mathbf{Z}$	$\mathbf{Z}$	Id
$\mathrm{Bir}(F)$	$\mathbf{Z} \rtimes \mathbf{Z}/2$	Id	Id	Id	$\mathbf{Z}/2$	$\mathbf{Z}$	$\mathbf{Z}$	Id	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	Id

When $e = 3(3m^2 - 1)$, the pair $(3m, 1)$ is a solution of $\mathscr{P}_e(3)$, but neither $\mathscr{P}_e(-3)$ nor, when $m \not\equiv \pm 1 \pmod 5$, $\mathscr{P}_{4e}(-15)$ are solvable (reduce modulo 3 and 5). Therefore, we have $\mathrm{Aut}(F) = \mathrm{Bir}(F) \simeq \mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$.

Remark 5. A very elegant result of Oguiso gives a general description of the regular and birational automorphism groups of any projective hyperkähler manifold X with Picard number 2:

- if both extremal rays of the nef cone are rational, $\mathrm{Aut}(X)$ is finite;
- if both extremal rays of the nef cone are irrational, $\mathrm{Aut}(X)$ (hence also $\mathrm{Bir}(X)$) is infinite;
- if both extremal rays of the movable cone are rational, $\mathrm{Bir}(X)$ (hence also $\mathrm{Aut}(X)$) is finite;
- if both extremal rays of the movable cone are irrational, $\mathrm{Bir}(X)$ is infinite.

We see from the examples above that all cases occur in dimension 4.

6. Unexpected isomorphisms between special hyperkähler fourfolds and Hilbert squares of K3 surfaces

We can use our knowledge of the ample cone of the Hilbert square of a K3 surface (S, f_S) of degree $2e$ to construct other polarizations of various squares: if a and b are positive integers such that

$$\frac{a}{b} < \nu_e, \tag{1}$$

the class $bf - a\delta$ is ample; its square is $2(b^2e - a^2)$ and its divisibility is $\gamma = 2$ if b is even and $\gamma = 1$ if b is odd. We formalize this remark in the following proposition.

Proposition 6. *Let n and e be positive integers and assume that the equation $\mathscr{P}_e(-n)$ has a positive solution (a,b) which satisfies the inequality (1). If $\mathscr{F}_{2e}$ is the moduli space of polarized K3 surfaces of degree $2e$, the rational map*

$$\begin{array}{rccc} \varpi\colon & \mathscr{F}_{2e} & \dashrightarrow & \mathscr{M}_{2n}^{(\gamma)} \\ & (S,f_S) & \longmapsto & (S^{[2]}, bf-a\delta), \end{array}$$

induces a birational isomorphism onto an irreducible component of $\mathscr{C}^{(\gamma)}_{2n,2e}$. In particular, if n is prime and b is even, it induces a birational isomorphism

$$\mathscr{F}_{2e} \stackrel{\sim}{\dashrightarrow} \mathscr{C}^{(2)}_{2n,2e}.$$

This generalizes a result of Hassett in the case $n=3$ and $e=m^2+m+1$ (one suitable solution is then $(a,b)=(2m+1,2)$). The proposition implies that for each $n\geq 1$, there are always infinitely many distinct explicit hypersurfaces $\mathscr{C}^{(1)}_{2n,2(m^2+n)}\subset \mathscr{M}^{(1)}_{2n}$ (resp. $\mathscr{C}^{(2)}_{2n,2\left(m^2+m+\frac{n+1}{4}\right)}\subset \mathscr{M}^{(2)}_{2n}$ when $n\equiv -1 \pmod 4$) whose general points correspond to Hilbert squares of K3 surfaces. In particular, $\mathscr{M}^{(1)}_{2n}$ (resp. $\mathscr{M}^{(2)}_{2n}$ when $n\equiv -1 \pmod 4$) is non-empty.

Example 7. Assume $n=3$. For $e=7$, one computes $\nu_7=\frac{21}{8}$. The only (positive) solution to the equation $\mathscr{P}_e(-n)$ with b even that satisfy (1) is $(5,2)$. A general element (F,h) is therefore isomorphic to the Hilbert square of a polarized K3 surface of degree 14. This is explained geometrically by the Beauville–Donagi construction.

Example 8. Assume $n=3$. For $e=13$, one computes $\nu_{13}=\frac{2340}{649}$. The only (positive) solutions to the equation $\mathscr{P}_e(-n)$ with b even that satisfy (1) are $(7,2)$ and $(137,38)$. A general element (F,h) is therefore isomorphic to the Hilbert square of a polarized K3 surface (S,f_S) of degree 26 in two different ways: h is mapped either to $h_1:=2f-7\delta$ or to $h_2:=38f-137\delta$. This is explained as follows: the equation $\mathscr{P}_{13}(-1)$ has a solution $((18,5))$, hence $S^{[2]}$ has a non-trivial involution σ; one isomorphism $S^{[2]}\xrightarrow{\sim}F$ is obtained from the other by composing it with σ (so that $\sigma^*(2f-7\delta)=38f-137\delta$). This is a case where F is the variety of lines on a cubic fourfold, the Hilbert square of a K3 surfaces, and a double EPW sextic! There is no geometric explanation for this remarkable fact (which also happens for $e\in\{73,157\}$).

7. Varieties of lines contained in a cubic fourfold

I mentioned earlier that the nef cones of elements of $\mathscr{M}^{(2)}_{6,6e'}$ (where $e'\geq 2$) with Picard number 2 (they are varieties of lines contained in a cubic fourfold) are known. One can therefore play the same game and construct components of Noether–Lefschetz loci that consists mostly of varieties of this type.

Proposition 9. *Let n be a positive integer such that the equation $\mathscr{P}_{3e'}(3n)$ has a solution $(3a,b)$ with $a>0$ and $\gcd(a,b)=1$ that satisfies the inequality*

$$\frac{|b|}{a}<\nu'_e. \tag{2}$$

The rational map

$$\begin{aligned} \varpi\colon \mathscr{C}^{(2)}_{6,6e'} &\dashrightarrow \mathscr{M}^{(\gamma)}_{2n} \\ (F,g) &\longmapsto (F, ah+b\tau), \end{aligned}$$

where $\gamma = 2$ *if* b *is even and* $\gamma = 1$ *if* b *is odd, induces a birational isomorphism onto an irreducible component of* $\mathscr{C}^{(\gamma)}_{2n,6e'}$. *In particular, if* n *is prime and* b *is even, it induces a birational isomorphism*

$$\mathscr{C}^{(2)}_{6,6e'} \overset{\sim}{\dashrightarrow} \mathscr{C}^{(2)}_{2n,6e'}.$$

Again, for each $n \geq 1$, there are always infinitely many distinct explicit hypersurfaces $\mathscr{C}^{(1)}_{2n,6(3m^2-n)} \subset \mathscr{M}^{(1)}_{2n}$ (resp. $\mathscr{C}^{(2)}_{2n,6\left(3m^2+3m-\frac{n-3}{4}\right)} \subset \mathscr{M}^{(2)}_{2n}$ when $n \equiv -1 \pmod 4$) whose general points correspond to varieties of lines contained in a cubic fourfold.

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