On nodal prime Fano threefolds of degree 10

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From now on, $X \subset \mathbf{P}^7$ will be such a nodal Fano threefold.

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- $\mathrm{Sing}(X_O) = \mathrm{Sing}(W_O) \cap \Omega_O$ consists of six points (corresponding to the six lines in X through O).

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- the double étale cover

$$\pi : \widetilde{\Gamma}_6 \cup \Gamma_1^1 \cup \Gamma_1^2 \longrightarrow \Gamma_6 \cup \Gamma_1$$

  corresponding to the choice of a family of 3-planes contained in a quadric of rank 6 in Π ($\mathbf{P}^3_W$ defines the component Γ_1^1).

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The discriminant curve is $\Gamma_7 = \Gamma_6 \cup \Gamma_1$.

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There is a birational isomorphism

$$\mathscr{X}_{10}^{\mathrm{nodal}} \stackrel{\sim}{\dashrightarrow} \left\{ \begin{matrix} \textit{triples} \\ (\Gamma_6, \Gamma_1, M) \end{matrix} \right\} \Big/ \textit{isom.}$$

where M is an even invertible theta-characteristic on $\Gamma_6 \cup \Gamma_1$.

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- $X_O \subset \mathbf{P}^6_O$ is determined up to projective isomorphism by the pair (Γ_7, M_X).

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Its inverse image under the birational map $W \dashrightarrow W_O$ is a threefold X_{10} with a single node at O. ∎

We now reinterpret the right-hand side in

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where S^{even} and S^{odd} are smooth, connected, with an involution σ. When the set-up comes from X, the divisor $\Gamma_1^1 \cdot \widetilde{\Gamma}_6$ defines a point s_X of S^{odd}.

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where θ is an open embedding and maps even *(resp.* odd*) theta-characteristics to S^{odd}/σ (resp. S^{even}/σ).*
Furthermore,

$$\theta(M_X) = s_X .$$

We obtain a birational isomorphism

$$\mathscr{X}_{10}^{\mathrm{nodal}} \stackrel{\sim}{\dashrightarrow} \left\{ \textit{pairs } (\pi : \widetilde{\Gamma}_6 \to \Gamma_6, s) \right\} \Big/ \textit{isom.}$$

where $s \in S^{\mathrm{odd}}/\sigma$.

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The projections induce two conic bundle structures $T \to \Pi$ and $T \to \Pi^\star$ with discriminant curves sextics $\Gamma_6 \subset \Pi$ and $\Gamma_6^\star \subset \Pi^\star$, and double étale covers

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T depends on 19 parameters (same as plane sextics).

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induce a birational isomorphism

$$(p_W, p_\ell) : X \dashrightarrow T \subset \mathbf{P}^2_W \times \Pi,$$

where T is a general Verra threefold.

The intermediate Jacobian

$$J(T) := H^3(T,\mathbf{C})/\big(F^2H^3(T,\mathbf{C}) + H^3(T,\mathbf{Z})\big)$$

is a 9-dim'l ppav

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   is *generically 2-to-1* onto the same image.

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Since $\widetilde{X}$ is birational to a Verra threefold T, we have

$$J(\widetilde{X}) \simeq J(T).$$

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A general fiber of the period map J is birationally the union of the surfaces S^{odd}/σ and $S^{\star,\text{odd}}/\sigma$.

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Furthermore, the surface $\widetilde{F}_g(X)$ contains a single exceptional curve and its contraction $\widetilde{F}_m(X)$ is isomorphic to S^{odd}.

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3. for each point p of $L_c \cap \Gamma_6$, the 3-plane

$$\langle p_O(c), \mathrm{Vertex}(\Omega_p) \rangle \subset \Omega_p$$

   (when defined) defines a point $\tilde{p} \in \widetilde{\Gamma}_6$ above p.

This defines a point $\rho([c]) \in S$.

So we have another interpretation of the fiber of the period map

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is the union of finitely many disjoint (pairs of) smooth irreducible *projective* surfaces of the type $F_m(X)/\sigma$.

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To any line contained in X, one can also associate another threefold of the same type in a different component of the fiber.
These correspond the two components that we described at the boundary.
We conjecture that these are the only two components of a general fiber of the period map

$$J : \mathscr{X}_{10} \longrightarrow \mathscr{A}_{10}.$$