FAKE PROJECTIVE SPACES AND FAKE TORI

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Abstract. Hirzebruch and Kodaira proved in 1957 that when n is odd, any compact Kähler manifold X which is homeomorphic to $\mathbf{P}^n$ is isomorphic to $\mathbf{P}^n$. This holds for all n by Aubin and Yau's proofs of the Calabi conjecture. One may conjecture that it should be sufficient to assume that the integral cohomology rings $H^\bullet(X,\mathbf{Z})$ and $H^\bullet(\mathbf{P}^n,\mathbf{Z})$ are isomorphic.

Catanese recently observed that complex tori are characterized among compact Kähler manifolds X by the fact that their integral cohomology rings are exterior algebras on $H^1(X,\mathbf{Z})$ and asked whether this remains true under the weaker assumptions that the rational cohomology ring is an exterior algebra on $H^1(X,\mathbf{Q})$ (we call the corresponding compact Kähler manifolds "rational cohomology tori").

We give a negative answer to Catanese's question by producing explicit examples. We also prove some structure theorems for rational cohomology tori. This is work in collaboration with Z. Jiang, M. Lahoz, and W. F. Sawin.

1. Characterizations of the complex projective space

Hirzebruch and Kodaira studied in 1957 the following question: let X be a compact Kähler manifold; if X is homeomorphic to $\mathbf{CP}^n$, is X biholomorphic to $\mathbf{CP}^n$?

This is certainly true for $n = 1$!

Theorem 1 (Hirzebruch–Kodaira, Yau). *Any compact Kähler manifold which is homeomorphic to $\mathbf{CP}^n$ is biholomorphic to $\mathbf{CP}^n$.*

Proof. Since X is Kähler, one can compute its Hodge numbers from the Betti numbers $b_{2i}(X) = 1$ (for $0 \le i \le n$) and $b_{2i+1}(X) = 0$. We obtain first $H^1(X,\mathscr{O}_X) = H^2(X,\mathscr{O}_X) = 0$, hence $\mathrm{Pic}(X) \simeq H^2(X,\mathbf{Z}) \simeq \mathbf{Z}$ and X is projective (Kodaira). We also have $H^{p,q}(X) = 0$ for $p \ne q$, hence $\chi(X,\mathscr{O}_X) = 1$.

If L is an ample generator of $\mathrm{Pic}(X) \simeq H^2(X,\mathbf{Z})$ and $f : X \to \mathbf{CP}^n$ is a homeomorphism, $f^*H = \pm L$. Since the Pontryagin classes are topological invariants (Novikov), we have

$$p_i(X) = f^*p_i(\mathbf{CP}^n) = f^*\big(\tbinom{n+1}{i}H^{2i}\big) = \tbinom{n+1}{i}L^{2i}.$$

Now write $c_1(X) = (n+1+2s)L$ (the number $2s$ is even because the Stiefel–Whitney class—which is the reduction mod 2 of $c_1(X)$—is invariant under homeomorphism). Using the Hirzebruch–Riemann–Roch theorem and the values of the Pontryagin classes given above, one can compute $\chi(X,\mathscr{O}_X) = \binom{n+s}{n}$. Since this number is 1,

- either $s = 0$, $c_1(X) = (n+1)L$, and X is a Fano variety;
- or $s = -n-1$, n is even, $c_1(X) = -(n+1)L$, and X is of general type.

In the first case, it is not difficult to see that X is biholomorphic to $\mathbf{CP}^n$ (Kobayashi–Ochiai). In the second case, we use a result of Aubin–Yau which says that if $c_1(X) < 0$, then X carries a Kähler–Einstein metric ω with $\mathrm{Ric}(\omega) = -\omega$. Moreover (Yau),

$$(-1)^n\big(\tfrac{2(n+1)}{n}c_2(X) - c_1^2(X)\big)\cdot c_1^{n-2}(X) \ge 0. \qquad (1)$$

These notes were written for a talk given at the "London G&T Seminar," Imperial College, London, UK.

In our case, one computes (using the known values for $c_1(X)$ and $p_1(X)$) that the left side vanishes. This implies that (X,ω) has constant negative holomorphic curvature, hence is covered by the unit ball in $\mathbf{C}^n$. But X, being homeomorphic to $\mathbf{CP}^n$, is simply connected, hence this is impossible. □

Possible extensions:

- *If we only assume that X has the same integral cohomology ring as $\mathbf{CP}^n$,* the conclusion still holds when $n \le 6$ and X is simply connected (i.e., when X has the same homotopy type as $\mathbf{CP}^n$, by Deligne–Griffiths–Morgan–Sullivan). This was proved by Fujita when $n \le 5$ and $c_1(X) > 0$ (which implies $\pi_1(X) = 0$) and Libgober–Wood when $n \le 6$ (the main new ingredient is a proof that $c_1(X)c_{n-1}(X)$ is determined by the Hodge numbers of X; in our case $c_1(X)c_{n-1}(X) = \frac{1}{2}n(n+1)^2$). Simple-connectedness of X may not be necessary:
  - when $n = 2$: all "fake projective planes" (complex surfaces other than $\mathbf{CP}^2$ with the same Betti numbers $(1,0,1,0,1)$ as $\mathbf{CP}^2$) have been classified and they all have torsion in their H^2;
  - when $n = 3$, the relation $c_1(X)c_2(X) = 24$ and Yau's inequality (1) imply $c_1(X) > 0$, hence $\pi_1(X) = 0$;
  - when $n \in \{4,5,6\}$, the Fujita–Libgober–Wood computations actually work without the hypothesis $\pi_1(X) = 0$; they imply that either $c_1(X) = n+1$, in which case $X \simeq \mathbf{CP}^n$ (Kobayashi–Ochiai), or $c_1(X) = -(n+1)$ and $n \in \{4,6\}$, and there is equality in (1), hence X is covered by the unit ball in $\mathbf{C}^n$ (Yau) and $\pi_1(X)$ is infinite.[1]
- *What if we only assume that X has the same rational cohomology ring as $\mathbf{CP}^n$?* In dimension 2, fake projective planes provide examples which are not isomorphic to $\mathbf{CP}^2$ (a generator L of $H^2(X,\mathbf{Z})/\mathrm{tors}$ satisfies $L^2 = 1$).

  In dimension 3, it still follows from the inequalities (1) and $c_1(X)c_2(X) > 0$ that X is a Fano variety. In the classification of Fano threefolds, we find that besides $\mathbf{CP}^3$, there are 3 other families with the same rational cohomology groups as $\mathbf{CP}^3$: the 3-dimensional quadric, for which an ample generator L of $H^2(X,\mathbf{Z})$ satisfies $L^3 = 2$, and two other families, with $L^3 = 5$ or $L^3 = 22$. Since in each case $\sqrt[3]{L^3} \notin \mathbf{Q}$, the rational cohomology rings are not isomorphic. The conclusion is therefore that X is isomorphic to $\mathbf{CP}^3$.

  More generally, a smooth odd-dimensional quadric $X \subset \mathbf{CP}^{2m}$ has the same integral cohomology groups as $\mathbf{CP}^{2m-1}$, but, if $m \ge 2$, a generator L of $H^2(X,\mathbf{Z})$ satisfies $L^{2m-1} = \pm 2$, hence again, the rational cohomology rings are not isomorphic.
- *If we do not assume that X is Kähler,* the conclusion still holds for $n \le 2$ (complex surfaces with even b_1 are Kähler) but nothing is known for $n \ge 3$. If $\mathbf{S}^6$ has a complex structure (necessarily non-Kähler since $b_2(\mathbf{S}^6) = 0$), its blow-up X at a point is a 3-dimensional complex manifold such that
$$X \underset{\text{diff}}{\sim} \mathbf{S}^6 \sharp \overline{\mathbf{CP}^3} \underset{\text{diff}}{\sim} \overline{\mathbf{CP}^3} \underset{\text{diff}}{\sim} \mathbf{CP}^3.$$
  However, $c_1(\mathbf{S}^6) = 0$ (because $b_2(\mathbf{S}^6) = 0$), hence $c_1(X) = -2E$ and $c_1(X)^3 = -8$, whereas $c_1(\mathbf{CP}^3)^3 = 4^3$, hence X is not biholomorphic to $\mathbf{CP}^3$. It follows that if the conclusion of the theorem is still valid without the Kähler assumption, there is no complex structure on $\mathbf{S}^6$.

[1]The only known fake $\mathbf{CP}^4$ (constructed in 2006 by Prasad and Yeung as quotients of the unit ball in $\mathbf{C}^4$) have torsion in their H^2.

- *In positive characteristics,* one can ask about smooth projective varieties with the same étale cohomology as $\mathbf{P}^n$, but nothing seems to be known about them.

2. Complex tori

Catanese asked a similar question: let X be a compact Kähler manifold; if X is homeomorphic to a complex torus, is X biholomorphic to a complex torus? Catanese proved that a stronger statement holds.

Theorem 2 (Catanese). *Let X be a compact Kähler manifold such that there is a ring isomorphism*

$$\bigwedge^\bullet H^1(X,\mathbf{Z}) \xrightarrow{\sim} H^\bullet(X,\mathbf{Z}). \tag{2}$$

Then X is biholomorphic to a complex torus.

Proof. Since X is Kähler, the Albanese map $a_X\colon X \to A_X$ induces an isomorphism

$$a_X^{*1}\colon H^1(A_X,\mathbf{Z}) \xrightarrow{\sim} H^1(X,\mathbf{Z})$$

and (2) implies that $a_X^*\colon H^\bullet(A_X,\mathbf{Z}) \to H^\bullet(X,\mathbf{Z})$ is also an isomorphism. Set $n := \dim(X)$; the fact that $a_X^{*2n}\colon H^{2n}(A_X,\mathbf{Z}) \xrightarrow{\sim} H^{2n}(X,\mathbf{Z})$ is an isomorphism implies that a_X is birational. Since we have an isomorphism of the whole cohomology rings and X is Kähler, a_X cannot contract any subvariety of X and is therefore finite. Thus, a_X is an isomorphism. □

The hypothesis "X Kähler" is necessary when $n \geq 3$, as we show in the next example (when $n = 2$, the condition (2) implies $b_1 = b_3 = \binom{b_1}{3}$, hence $b_1 = 0$ or 4, but complex surfaces with even b_1 are Kähler).

Example 3 (Blanchard, Ueno). Let E be an elliptic curve, let $\mathscr{L}$ be a very ample line bundle on E, and let φ and ψ be holomorphic sections of $\mathscr{L}$ with no common zeroes on E. Set

$$J_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J_3 := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \qquad J_4 := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

Since $\det\big(\sum_{i=1}^4 \lambda_i J_i\big) = \sum_{i=1}^4 \lambda_i^2$, the group

$$\Gamma := \sum_{i=1}^4 \mathbf{Z} J_i \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

is a relative lattice in the rank-2 vector bundle $\mathscr{V} := \mathscr{L} \oplus \mathscr{L}$ over E. The quotient $M := \mathscr{V}/\Gamma$ is a complex manifold of dimension 3 with a surjective holomorphic map $\pi\colon M \to E$. Each fiber of π is a complex 2-dimensional torus and its relative canonical bundle is $\omega_{M/E} = \pi^*\mathscr{L}^{-2}$.

One checks that M is diffeomorphic to a real torus, hence (2) holds, but M is not a complex torus, since it is not Kähler: if it were, $\pi_*\omega_{M/E}$ would be semipositive (Fujita).

Catanese then asked a further question. Assume that the compact Kähler manifold X is a *rational cohomology torus,* i.e.,

$$\bigwedge^\bullet H^1(X,\mathbf{Q}) \xrightarrow{\sim} H^\bullet(X,\mathbf{Q}). \tag{3}$$

Is X biholomorphic to a complex torus?

The answer to that question is negative already for surfaces, as we will show with a simple example. However, rational cohomology tori have interesting structures which we found worthwhile to investigate.

Example 4 (A rational cohomology torus which is not a torus)**.** Let $\rho\colon C \to E_1$ be a double cover of smooth projective curves, where C has genus $g \geq 2$ and E_1 is an elliptic curve. Let τ be the corresponding involution on C. Let $E_2' \to E_2$ be a degree-2 étale cover of elliptic curves and let σ be the corresponding involution on E_2'. Let X be the smooth surface $(C \times E_2')/\langle \tau \times \sigma \rangle$.

Then X is a rational cohomology torus. Indeed, since σ acts trivially on $H^\bullet(E_2', \mathbf{Q})$, we have

$$\begin{aligned} H^\bullet(X, \mathbf{Q}) &= H^\bullet(C \times E_2', \mathbf{Q})^{\tau\times\sigma} \\ &= \big(H^\bullet(C, \mathbf{Q}) \otimes H^\bullet(E_2', \mathbf{Q})\big)^{\tau\times\sigma} \\ &= H^\bullet(C, \mathbf{Q})^{\tau} \otimes H^\bullet(E_2', \mathbf{Q}) \\ &= H^\bullet(E_1, \mathbf{Q}) \otimes H^\bullet(E_2, \mathbf{Q}). \end{aligned}$$

However, X has Kodaira dimension 1 (its Iitaka fibration is $X \to E_1$), hence is not a torus. Alternatively, its (degree-2) Albanese map $X \to E_1 \times E_2$ induces an isomorphism $H^1(E_1 \times E_2, \mathbf{Z}) \xrightarrow{\sim} H^1(X, \mathbf{Z})$ but the image of the canonical map $\bigwedge^4 H^1(X, \mathbf{Z}) \to H^4(X, \mathbf{Z}) \simeq \mathbf{Z}$ has index 2.

The situation in the example is quite typical. Start from a compact Kähler manifold X of dimension n; its Albanese morphism $a_X\colon A \to A_X$ induces an isomorphism a_X^{1*} : $H^1(A_X, \mathbf{Z}) \xrightarrow{\sim} H^1(X, \mathbf{Z})$ so we can rephrase the property of being a rational cohomology torus as the property that

$$a_X^*\colon H^\bullet(A_X, \mathbf{Q}) \xrightarrow{\sim} H^\bullet(X, \mathbf{Q})$$

is an isomorphism. As Catanese already remarked, the bijectivity of a_X^{*2n} implies that a_X is surjective and generically finite and the injectivity of a_X^* that a_X is finite. More generally, one easily checks that X is a rational cohomology torus if and only if there is a finite morphism $f\colon X \to A$ to a torus A such that

$$f^*\colon H^\bullet(A, \mathbf{Q}) \xrightarrow{\sim} H^\bullet(X, \mathbf{Q})$$

is an isomorphism.

Given a finite morphism $f : X \to A$ to a torus, a theorem of Kawamata says that there are

- a subtorus K of A,
- a normal projective variety Y,

and a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{I_X} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \longrightarrow & A/K, \end{array}$$

where g is finite and I_X is an Iitaka fibration of X, i.e., it is the morphism induced by pluricanonical forms on X of high enough order, so that $\dim(Y)$ is the Kodaira dimension of X, and $X = Y$ if and only if X is of general type. General fibers of I_X are tori which are étale covers of K.

One then checks that X *is a rational cohomology torus if and only if* Y *is a rational cohomology torus.* The only problem is that Y might be singular. We will not discuss this issue here (the right thing to do is to allow rational singularities). One may then repeat the construction and obtain a sequence of morphisms

$$X \xrightarrow{I_X} X_1 \xrightarrow{I_{X_1}} X_2 \longrightarrow \cdots \longrightarrow X_{k-1} \xrightarrow{I_{X_{k-1}}} X_k,$$

with morphisms $f_i\colon X_i \to A_i$ to quotient tori of A, where X is a rational cohomology torus if, and only if, X_k is a rational cohomology torus, and

- either X_k is of general type and of positive dimension;
- or X_k is a point, in which case X is a rational cohomology torus which we call an an *Iitaka torus tower.*

Example 4 falls into the second category and one may ask whether all rational cohomology tori are Iitaka torus towers. This is equivalent to asking whether rational cohomology tori of general type do not exist (here we must allow singularities, and this makes a big difference). The answer is positive for curves and surfaces, but again negative in dimensions ≥ 3.

Example 5 (Rational cohomology tori which are of general type—hence not Iitaka torus towers)**.** For each $j \in \{1, 2, 3\}$, consider an elliptic curve E_j and a bielliptic curve $C_j \xrightarrow{2:1} E_j$ of genus ≥ 2. Pick double étale covers $E'_j \to E_j$ and pull them back to $C'_j \to C_j$, with associated involution σ_j of C'_j; let τ'_j be the involution of C'_j associated with the induced double cover $C'_j \to E'_j$. Let $C'_1 \times C'_2 \times C'_3 \twoheadrightarrow Z$ be the quotient by the group (isomorphic to $(\mathbf{Z}/2\mathbf{Z})^4$) of automorphisms generated by $\mathrm{id}_1 \times \tau'_2 \times \sigma_3$, $\sigma_1 \times \mathrm{id}_2 \times \tau'_3$, $\tau'_1 \times \sigma_2 \times \mathrm{id}_3$, and $\tau'_1 \times \tau'_2 \times \tau'_3$ and consider the tower

$$C'_1 \times C'_2 \times C'_3 \xrightarrow{g} Z \xrightarrow{f} E_1 \times E_2 \times E_3$$

of Galois covers of respective degrees 2^4 and 4. The threefold Z is of general type and has rational isolated singularities. One can show that it is a rational cohomology torus.

Building on this example, we can produce, starting from dimension 4, smooth rational cohomology tori that are not Iitaka torus towers.

Interestingly, the fact that the threefold Z in the example is singular is not an accident: there are no *smooth* rational cohomology tori of general type.

Theorem 6 (Sawin)**.** *Let $f\colon X \to A$ be a finite morphism from a smooth complex projective variety of general type X to an abelian variety A. Let n be the dimension of X. Then*

$$(-1)^n \chi_{\mathrm{top}}(X) > 0.$$

In particular, X is not a rational cohomology torus.

Proof. One checks that the pull-back to X of a general 1-form on A vanishes along a finite scheme, whose length is therefore $c_n(\Omega_X) = (-1)^n \chi_{\mathrm{top}}(X)$. By a result of Popa and Schnell, any 1-form on X vanishes at some point, hence the result. For the consequence, note that since the rational cohomology of a rational cohomology torus is the same as that of an abelian variety, its Euler characteristic must be the same as that of an abelian variety, which is zero. □

So where are we now? We proved that all rational cohomology tori are Iitaka torus towers "over" rational cohomology tori (with rational singularities) which are of general type, but little is known about the latter. Here are a couple of facts that we can prove. The proofs do not use the full knowledge of the rational cohomology algebra $H^\bullet(X, \mathbf{Q})$, but only the weaker consequences that the Albanese morphism a_X is finite surjective and that the Hodge numbers of any desingularization X' of X and the Albanese variety A_X are the same, so that

$$\begin{aligned}\chi(X,\mathscr{O}_X)=\chi(X',\mathscr{O}_{X'})&=\sum_{i=0}^{n}(-1)^i h^i(X',\mathscr{O}_{X'})=\sum_{i=0}^{n}(-1)^i h^{0,i}(X')\\&=\sum_{i=0}^{n}(-1)^i h^{0,i}(A_X)=\sum_{i=0}^{n}(-1)^i\binom{n}{i}=0.\end{aligned}$$

For any projective variety X with rational singularities, of general type, and with these two properties (a_X finite and $\chi(X,\mathscr{O}_X)=0$), we have:

- the degree of a_X is divisible by the square of a prime number;
- the number of simple factors of A_X is greater than the smallest prime number that divides $\deg(a_X)$.

In Example 5, the degree of a_X is 4 and A_X is the product of 3 elliptic curves.

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