Cubic fourfolds

$X \subset \mathbb{P}(V_6)$ smooth cubic hypersurface / over $\mathbb{C}$

Aim. Uniformize the corresponding moduli space using the period map.

Classical case: elliptic curves

Family of elliptic curves

E_λ : $y^2 = x(x-1)(x-\lambda)$ $\quad \lambda \neq 0, 1$

$\omega = \frac{dx}{y}$ is a holomorphic one-form

There exists a symplectic basis (δ, γ) of $H_1(E_\lambda, \mathbb{Z})$ such that

$\int_\delta \omega = 1 \qquad \int_\gamma \omega = \tau \in \mathcal{H}$ Siegel upper half-plane.

period map : $\mathbb{C} - \{0, 1\} \xrightarrow{\sim} \Gamma(2) \backslash \mathcal{H} \longleftarrow \mathcal{H}$ uniformization

$\lambda \longmapsto \tau$

↳ subgroup of $SL(2, \mathbb{Z}) = Sp(2, \mathbb{Z})$

↳ moduli space of elliptic curves (with "level structure")

1. Lines on a smooth cubic

$L(X) \subset G(2, V_6)$ scheme of lines contained in X.

Prop $L(X)$ is a smooth projective fourfold

Proof The general theory says that the tangent space at L to $L(X)$ is $H^0(L, N_{L/X})$ and obstructions are in $H^1(L, N_{L/X})$.
$N_{L/X}$ has rank 3 and degree $-K_X \cdot L + \deg K_L = 3-2 = 1$

$N_{L/X} \subset N_{L/\mathbb{P}^5} = \mathcal{O}_L(1)^4$ hence only two possibilities
$\mathcal{O}_L \oplus \mathcal{O}_L \oplus \mathcal{O}_L(1)$ ← general case since a line thru a gen'l point must be "free"
$\mathcal{O}_L(-1) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(1)$
In both cases, $h^0 = 4$ and $h^1 = 0$. ∎

Theorem (Beauville-Donagi) $L(X)$ is a HK fourfold

Proof A HK manifold is simply connected, with a nowhere degenerate holomorphic 2-form. In particular, the can'l bundle is trivial.
This can be easily seen:
$L(X)$ is the zero set on $G(2, V_6)$ of the section of $\mathrm{Sym}^3 \mathcal{S}^\vee$ defined by an equation of X
→ expected codimension 4
→ $K = K_G + c_1(\mathrm{Sym}^3 \mathcal{S}^\vee) = K_G + 6c_1(\mathcal{S}^\vee) = 0$

The Borel-Weil theorem gives (using Koszul and normal sequences) $h^0(\Omega^2_{L(X)}) = 1$ and this is enough to prove what we want (by classification results) but this is not how BD proved this result.
They use the fact that any smooth deformation of

a HK manifold remains a HK manifold and exhibit an example, that of a Pfaffian cubic:

$\Delta \subset \mathbb{P}(\Lambda^2 W_6^\vee) = \mathbb{P}^{14}$ parametrizing degenerate skew-symmetric forms on W_6, is a cubic hypersurface defined by the vanishing of the Pfaffian which is singular along locus of forms of rank 2, $G(2, W_6^\vee)$ of codim 6

$X := \Delta \cap (\text{general } \mathbb{P}^5)$ is a smooth cubic threefold, defined by the Pfaffian of a skew-symmetric matrix of general linear forms

Consider

$S = G(2, W_6) \cap \underset{(=\mathbb{P}^8)}{(\mathbb{P}^5)^\perp} \subset \mathbb{P}(\Lambda^2 W_6)$ This is a smooth K3 surface of degree 14

we define $S^2 \longrightarrow L(X)$

$p, q \in S$ correspond to $P, Q \subset W_6$ of dim 2 and in general, $P+Q \subset W_6$ has dim 4

Let $\varphi \in \mathbb{P}^5$, a 2-form on W_6.

$\varphi|_P \equiv 0$ $\varphi|_Q \equiv 0$ because $P, Q \in (\mathbb{P}^5)^\perp$

hence $\varphi|_{P+Q} \equiv 0$ is 4 linear conditions on $\mathbb{P}^5$
$\rightsquigarrow$ this is a line L

Any $\varphi \in L$ is degenerate hence $L \subset X$.

One checks that this define a morphism $S^{[2]} \longrightarrow L(X)$

and which is an isomorphism. And $S^{[2]}$ is known to be HK ∎

$$\begin{array}{ccc} S^{[2]} & \longrightarrow & S^2/_{\mathfrak{S}_2} \\ {\scriptstyle 2:1}\uparrow & & \uparrow{\scriptstyle 2:1} \\ \tilde{S}^2 & \xrightarrow[\text{up}]{\text{blow}} & S^2 \end{array}$$

<u>Remark</u>: One can recover X from $L(X)$.

2. Hodge structures

This can also be used to compute the Hodge numbers of $L(X)$, since those of $S^{[2]}$ are known (this is the blow-up of the diagonal in S^2 divided by involution)

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 1 & & 21 & & 1 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & 21 & & 232 & & 21 & & 1 \end{array}$$

$$H^4(L(X), \mathbb{Q}) \simeq \mathrm{Sym}^2 H^2(L(X), \mathbb{Q})$$

On the other hand, the Hodge numbers of X are

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 21 & & 1 & & 0 \end{array}$$

"same" as $H^2(L(X))$

<u>Note</u> : Pfaffian cubics are rational:

$$\begin{tikzcd} & \{(v,\varphi) \in \mathbb{P}(W_6) \times X \mid v \in \operatorname{Ker}\varphi\} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}(W_6) & & X \end{tikzcd}$$

$\mathbb{P}^1$-bundle

fibers : we need $\varphi(v, W_6) = 0$ 5 linear conditions on $\varphi \in \mathbb{P}^5$ hence in general a point. ($\varphi(v,v) = 0$!)

For $H \subset \mathbb{P}(W_6)$ general hyperplane, p_2 induces

$$p_1^{-1}(H) \longrightarrow X \quad \text{birational.}$$

Since $L(X)$ parametrizes cycles on X, there is an <u>Abel-Jacobi map</u>

$$\begin{tikzcd} & \{(L, x) \in L(X) \times X \mid x \in L\} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ L(X) & & X \end{tikzcd}$$

$\mathbb{P}^1$-bundle

$$\alpha := p_* q^* : H^4(X, \mathbb{Z}) \to H^2(L(X), \mathbb{Z})$$

We will be interested in <u>primitive cohomologies</u>

$H^4(X, \mathbb{Z})_0 :=$ orthogonal of h_X^2 for intersection product
$H^2(L(X), \mathbb{Z})_0 :=$ ——— $g^3_{L(X)}$ ———

both endowed with quadratic forms
- the intersection product
- the product $u \cdot v = \frac{1}{6} g^2 uv$ (BB form)

<u>Theorem</u> (BD) α induces an isomorphism of polarized HS
$$H^4(X,\mathbb{Z})_0 \xrightarrow{\sim} H^2(L(X),\mathbb{Z})_0(-1)$$
↳ this changes the sign of the form

As lattices, they are
$$\Lambda_0 = 2E_8 \oplus 2U \oplus A_2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{signature } (20,2)$$

<u>Proof</u> By direct calculation, using the facts that p is a $\mathbb{P}^1$-bundle and that these lattices are known to have same discriminant 3

3. <u>Period maps</u>

$\mathcal{X} \to T$ family of smooth cubic fourfolds (e.g. T Kuranishi space ($h^1(T_X) = 20$ $h^2(T_X) = 0$)).
This is locally $\mathcal{C}^\infty$ trivial.
If T is simply connected, we can identify all lattices $H^4(X_t,\mathbb{Z})_0$ with Λ_0.

Inside $\Lambda_0 \underset{\mathbb{Z}}{\otimes} \mathbb{C}$, $H^{3,1}(X_t)$ defines a line. More precisely, by Riemann's relations, a point of
$$Q = \{ \omega \in \mathbb{P}(\Lambda_{0\mathbb{C}}) \mid \underset{\text{int. form}}{q(\omega) = 0}, \; q(\omega + \bar{\omega}) > 0 \}$$
2 components (signature (20,2))
Q^+, Q^- bounded symmetric domains of type IV

<u>Local Torelli theorem</u> (Griffiths). Assume T is the Kuranishi space
The map $T \to Q$ is étale (bijective differential)

When T is not simply connected, there might be nontrivial monodromy and to define a period map, one needs to take a quotient by a group of automorphisms of Λ_0.

We have

$$\begin{array}{ccc} H^4(X,\mathbb{Z})_0 & \hookrightarrow & H^4(X,\mathbb{Z}) \text{ unimodular} \\ \| & & \| \\ \Lambda_0 & \hookrightarrow & \Lambda = I_{21,2} \end{array}$$

and we should consider

$$\{ g \in O(\Lambda_0) \mid g \text{ lifts to an isometry of } \Lambda \text{ fixing } h^2 \}$$

In this case, this is the whole of $O(\Lambda_0)$ (this is a lattice-theoretic property), so we set

$$\mathcal{D} = O(\Lambda_0) \backslash Q$$

quasi-projective variety by Baily-Borel theory, with compactification

$$\overline{\mathcal{D}}^{BB} \quad \text{(projective normal)}$$

4. Moduli spaces

On the other side, GIT allows us to construct a moduli space for cubic fourfolds

$U \subset \mathbb{P}(\mathrm{Sym}^3 V_6^\vee)$, parametrizing smooth cubics, is affine (its complement is the discriminant divisor) and $PGL(V_6)$ invariant

Since $PGL(V_6)$ is reductive, the invariant functions on U form a $\mathbb{C}$-algebra of finite type and one may consider

$$\mathcal{X} = \mathrm{Spec}\left(\Gamma(U, \mathcal{O}_U)^{PGL(V_6)} \right) \qquad \text{affine}$$

Since the stabilizers are finite ($\mathrm{Aut}(X)$ finite because $H^0(X, T_X) = 0$), $\mathcal{X}$ is a geometric quotient (its points are in 1:1 correspondance with orbits, i.e. with isom. classes of cubics).

GIT also provides us with a projective compactification $\mathcal{X} \subset \overline{\mathcal{X}}$

<u>Conclusion</u> We have an algebraic étale period map $\mathcal{X} \xrightarrow{p} \mathcal{D}$ between 20 dim'l varieties.

<u>Theorem</u> (Voisin) p is an open embedding.

But the proof had a gap, which was filled only recently thanks to work of Laza, who proved the following remarkable theorem.

<u>Theorem</u> (Laza) p extends to an isomorphism

$$\begin{tikzcd} \widetilde{\overline{\mathcal{X}}} \arrow[r, "\sim", "\tilde{p}"'] \arrow[d] & \overline{\mathcal{D}}^{BB} \\ \overline{\mathcal{X}} & \end{tikzcd}$$

blow-up of the point corr. to the determinantal cubic

↳ given by $\begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} = 0$ singular along the locus where the rank is 1, s-stable not stable (a Veronese surface)

Moreover, Laza proves that the image of p is the complement in $\overline{\mathcal{D}}^{BB}$ of two <u>Heegner divisors</u>

$$\mathcal{D}_6 = \overline{\tilde{p}\,(\text{nodal cubics})}$$

$$\mathcal{D}_2 = \overline{\tilde{p}\,(\text{exc. divisor})}$$

where a Heegner divisor is the closure of the image of $x^{\perp} \subset \mathcal{Q}$ in $\mathcal{D}$, for some $x \in \Lambda_0$.

More precisely, if $\mathbb{Z}h^2 \subset K$ rank-2 saturated lattice $\subset I_{21,2}$

$$Q_K = \{ \omega \in Q \mid \omega \cdot K = 0 \} \subset Q$$

Its image in $\mathcal{D}$ only depends on $d := \mathrm{disc}(K)$ and
$\mathcal{D}_d$ $d > 0$, $d \equiv 0, 2 \mod 6$

In geometrical terms, since the Hodge conjecture holds for cubic fourfolds,

$X \in$ some $\mathcal{D}_d \iff X$ contains a surface with class $\notin \mathbb{Z}h^2$

Examples $\mathcal{D}_8 = p\{$ cubics containing a plane $\}$

$\mathcal{D}_{12} = p\{$ ———— cubic surface scroll $\}$

$\mathcal{D}_{14} = p\{$ Pfaffian cubics $\}$

The proof of the theorem is very long and technical: Laza determines all types of cubics (ie the type of their singularities) that correspond to semi-stable points.
He then studies the extension of the period map over these points by a case-by-case analysis.

Remark As for elliptic curves, this provides a uniformization of the moduli space of smooth cubic fourfolds by the complement in a bounded symmetric domain of two hyperplane arrangements.

What about our HK fourfolds $L(X)$?

They are polarized (by the restriction of the polarization of $G(3, V_6)$)

and one can also construct by GIT a quasi-projective coarse moduli space $\mathcal{Z}$.

But no description of boundary points is available!

However, the BD theorem gives a commutative diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[r, hook] \arrow[d, phantom, "\wr"] & \mathcal{Z} \arrow[d, "q"] \\
\overline{\mathcal{D}} - \mathcal{D}_2 - \mathcal{D}_6 \arrow[r, hook] & \overline{\mathcal{D}}
\end{tikzcd}$$

this period map is known to be also an open immersion by a theorem of Verbitsky

$\mathcal{Z}$ contains a divisor corresponding to $S^{[2]}$ S smooth K3 surface of degree 2, which maps isomorphically into $\mathcal{D}_2$.

Quadratic line complexes

We want to set up a similar picture for another family of projective varieties

$$X = G(2, V_5) \cap Q \subset \mathbb{P}(\Lambda^2 V_5) = \mathbb{P}^9$$

classically called <u>quadratic line complexes</u>. They are Fano fivefolds of index 3 and degree 10 (almost all such Fanos are of this type by Mukai's classification)

1. Singular quadrics containing a quadratic line complex

This is work of O'Grady and Iliev-Manivel.

$$I_X(2) = I_G(2) \oplus \mathbb{C}Q \qquad 6 \text{ dim'l}$$

$\hookrightarrow$ Plücker quadrics $\simeq V_5$ $\qquad P_v(\omega) = \omega \wedge \omega \wedge v$ rank 6

Singular quadrics form a degree-10 hypersurface in $|I_X(2)| \simeq \mathbb{P}^5$ which splits as

$$\underset{\text{hyperplane}}{4\,|I_G(2)|} + \underset{\text{sextic}}{Y_X}$$

Y_X is singular (along locus where corank is ≥ 2). It is called an <u>EPW sextic</u>

When X is smooth, $\mathrm{Sing}(Y_X)$ is an integral surface with finitely many singular points (smooth for X general).

There is a canonical double cover

$$\tilde{Y}_X \to Y_X \quad \text{branched along } \mathrm{Sing}(Y_X)$$

and $\tilde{Y}_X$ is a HK fourfold (in general smooth, or obtained from a $S^{[2]}$ by contracting finitely many Lagrangian planes)

2. EPW sextics

They are constructed from the following data

$\Lambda^3 V_6$ with symplectic form given by wedge product (choose $\Lambda^6 V_6 \simeq \mathbb{C}$)

$A \subset \Lambda^3 V_6$ 10 dim'l Lagrangian

$$Y_A := \{ [v] \in \mathbb{P}(V_6) \mid A \cap \underbrace{(\Lambda^2 V_6 \wedge v)}_{\text{Lagrangian}} \neq 0 \}$$

Standard parametrization of $LG(\Lambda^3 V_6)$, Lagrangian Grassmannian

Fix $V_6 = V_5 \oplus \mathbb{C} v_0$. Then

$$\Lambda^3 V_6 = \Lambda^3 V_5 \oplus (\Lambda^2 V_5 \wedge v_0)$$

but Lagrangians

One parametrizes the open subset of $LG(\Lambda^3 V_6)$ consisting of Lagrangians transverse to $\Lambda^3 V_5$ by

quadratic forms on $\Lambda^2 V_5$
$$S^2(\Lambda^2 V_5^\vee) \longrightarrow LG(\Lambda^3 V_6)$$

$$Q \longmapsto \{ Q(x, \cdot) + x \wedge v_0 \mid x \in \Lambda^2 V_5 \}$$

$\hookrightarrow$ linear form on $\Lambda^2 V_5 \simeq \Lambda^3 V_5$

$$(0 \longmapsto \Lambda^2 V_5 \wedge v_0)$$

For $[A]$ in this open set (which depends on V_5, v_0), we obtain a quadratic form Q and X_Q

__Remarks__ 1) X_Q smooth fivefold $\iff$ $[A] \notin \Sigma$, an irreducible hypersurface of LG independent of choice of V_5, v_0

2) The isomorphism class of X_Q does not depend on choice of $v_0 \notin V_6 - V_5$

3) Y_{X_Q} (defined earlier) is Y_A $\left(\text{via } \begin{array}{l} I_X(2) \longleftrightarrow V_6 \\ I_G(2) \longleftrightarrow V_5 \\ Q \longleftrightarrow -v_0 \end{array} \right)$

Can one recover X from Y_X? No! Just count parameters

parameters for Y_A (or A) = $\dim LG - \dim PGL(V_6)$
$= 55 - 35 = 20$

parameters for X = $\dim S^2(\Lambda^2 V_5^\vee) - \dim$ (Pfaffian quadrics) $- \dim GL(V_5)$
$= 55 - 5 - 25 = 25$

__But__ we get the right number of parameters for pairs $(A, [V_5])$

For the choice of $[V_5] \in \mathbb{P}(V_6^\vee)$, we only need $\Lambda^3 V_5 \cap A = 0$.

The bad choices are

$$\{ [V_5] \in \mathbb{P}(V_6^\vee) \mid \Lambda^3 V_5 \cap A \neq 0 \}$$

This is also an EPW sextic (in the dual space)!

$A \subset \Lambda^3 V_6 \rightsquigarrow A^\perp \subset \Lambda^3 V_6^\vee$ still Lagrangian. This is $Y_{A^\perp}$

(for A general, this is the projective dual of Y_A).

__Theorem__ $\mathcal{X} = \{$ smooth quadratic line complexes $\}$ / isom., affine GIT moduli space, 25-dim'l

$\overline{\mathcal{Y}} = \{$ EPW sextics $\}$ / isom., projective moduli space, 20 dim'l

Consider the morphism

$$f : \mathcal{X} \longrightarrow \overline{\mathcal{Y}}, \quad [X] \longmapsto [Y_X]$$

Its image is the affine variety $\overline{\mathcal{Y}} - \Sigma$ and the fiber of $[A]$ is $\mathbb{P}(V_6^\vee) - Y_{A^\perp}$

__Remark__ The image of f does not consist entirely of smooth double EPW sextics (some have finitely many Lagrangian singularities).

Q4

3. <u>Hodge structures</u> X quadratic line complex

The Hodge numbers for $H^5(X)$ are $0 \quad 0 \quad 10 \quad 10 \quad 0 \quad 0$

This defines a (principally polarized) 10 dim'l intermediate Jacobian, but not a HS structure of the same type as $H^2(\tilde{Y}_X, \mathbb{Z})$ $\quad 1 \quad 21 \quad 1$

20 for primitive cohomology

Instead, we use a classical trick

$\tilde{X} \to G(2, V_5)$ double cover branched along X. This is a Fano 6-fold (also described by Mukai) and the Hodge numbers for $H^6(\tilde{X})$ are

$0 \quad 0 \quad 1 \quad 22 \quad 1 \quad 0 \quad 0$

20 for vanishing cohomology

Moreover, as lattices,

$$H^6(\tilde{X}, \mathbb{Z})_{van} \xrightarrow{\sim} H^2(\tilde{Y}_X, \mathbb{Z})_0$$

$$\simeq \Lambda_1 := 2E_8 \oplus 2U \oplus 2A_1 \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{signature } (20,2)$$

So we have again period maps to

$$\mathcal{D} := \Gamma_1 \backslash Q$$

<u>But</u> we don't know whether the corresponding diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[dr, "\text{period}"'] & \mathcal{Y} \arrow[d, "p"] \arrow[d, "\substack{\text{period} \\ \text{(well-defined)}}"'] \\ & \mathcal{D} \end{tikzcd}$$

(double EPW sextics, with at worst finitely many Lagrangian singularities)

commutes!

<u>Remarks</u> 1)
$$\begin{array}{ccc} H^6(\tilde{X}, \mathbb{Z})_{van} & \hookrightarrow & H^6(\tilde{X}, \mathbb{Z}) \text{ unimodular} \\ \| & & \| \\ \Lambda_1 & \hookrightarrow & \Lambda = I_{22,2} \end{array}$$

$\Gamma_1 = \{ g \in O(\Lambda_1) \mid g$ lifts to an isometry of Λ fixing $H^6(G, \mathbb{Z}) \}$ has index 2 in $O(\Lambda_1)$.

Hence there is an involution r on $\mathcal{D}_1$ associated with the double cover

$$\mathcal{D}_1 = \Gamma_1 \backslash Q \longrightarrow O(\Lambda_1) \backslash Q$$

There is also an involution ι on $\mathcal{Y}$ induced by the duality $A \mapsto A^\perp$. O'Grady proved

$$r \circ p = p \circ \iota$$

2) By Verbitsky's theorem, p is <u>an open embedding</u>.

4. <u>The image of the period map</u>

Unfortunately, nothing is known about extending the period maps to the GIT compactifications of $\mathcal{X}$ and $\mathcal{Y}$. This seems extremely complicated (see O'Grady's work). Some partial results are known.

We can also define Heegner divisors $Q_K \subset Q$, for

$$\underset{\text{rank}-2}{H^6(G,\mathbb{Z})} \subset \underset{\substack{\text{rank-3 saturated}\\ \text{lattice}}}{K} \subset I_{22,2}$$

For $d := \operatorname{disc}(K)$ fixed, one needs $d>0$, $d \equiv 0, 2, 4 \pmod 8$ and

$d \equiv 0, 4$ $\quad \mathcal{D}_d$ irreducible

$d \equiv 2$ $\quad$ two components $\mathcal{D}'_d$ and $\mathcal{D}''_d$ exchanged by involution r

① The image of $\mathcal{Y}^{sm}$ is contained in

$$\overline{\mathcal{D}}^{BB} - \mathcal{D}_2 - \mathcal{D}_4 - \mathcal{D}_8 - \mathcal{D}_{10}$$

$\hookrightarrow$ 2 components $\hookleftarrow$

and one expects equality (difficult)

<u>Note</u>: $\mathcal{Y}^{sm} = \overline{\mathcal{Y}} - \Delta - \Sigma$ is affine hence its complement in $\overline{\mathcal{D}}^{BB}$ is a divisor

② The hypersurface Δ corresponding to smooth X but singular Y_X maps onto $\mathcal{D}'_{10}$

(hence image of $\mathcal{X}$ should be

$$\overline{\mathcal{D}}^{BB} - \mathcal{D}_2 - \mathcal{D}_4 - \mathcal{D}_8 - \mathcal{D}''_{10})$$

③ The hypersurface Σ maps onto $\mathcal{D}_8$

④ As in the cubic fourfold case, there are some very singular EPW sextics which still correspond to semi-stable (not stable) points of $\overline{\mathcal{Y}}$ at which the period map is not defined

- $Y = 2$ (discriminant cubic) is a (degenerate) EPW sextics in two ways (i.e for two families of Lagrangians A not in the same $PGL(V_6)$-orbit) and p "maps" these two points onto $\mathcal{D}'_2$ and $\mathcal{D}''_2$
- $Y = 3$ (smooth quadric) is also a degenerate EPW sextic and p maps it onto $\mathcal{D}_4$

5. Compactification of the fibers

GIT provides a compactification $\mathcal{X} \subset \overline{\mathcal{X}}$, but f will not extend to $\overline{\mathcal{X}}$ because $\rho(\overline{\mathcal{X}}) = 1$.

Instead, we can fill in the (affine) fibers of

$$f : \mathcal{X} \longrightarrow \mathcal{Y}$$

fibers

$$\mathbb{P}(V_6^\vee) - Y_{A^\perp}$$

by adding lower-dimensional varieties.

Recall that in the correspondence

$$\underset{\text{quadric}}{Q} \longleftrightarrow A \text{ Lagrangian}$$

we needed A transverse to $\Lambda^3 V_5$.

Assume now for example $\dim(A \cap \Lambda^3 V_5) = 1$.(*) Then

$$W_q := \mathrm{Im}(A \xrightarrow{pr_2} \Lambda^2 V_5 \wedge v_0)$$

and one gets as before $Q \in S^2 W_q^\vee$

$$X_Q = (Q = 0) \cap \mathbb{P}(W_q) \subset \mathbb{P}(\Lambda^2 V_5)$$

a Fano fourfold

(27)

Condition (*) is equivalent to $[V_5] \in Y_{A^\perp}^{sm}$

Similarly, if $[V_5] \in \mathrm{Sing}(Y_{A^\perp})^{sm}$, we obtain a 3-fold, and if $\mathrm{Sing}(Y_{A^\perp})^{sing}$ is non-empty, we obtain a (degree-10) K3 surface.

Of course, these are not flat degenerations of our 5-dim'l line complexes (one needs to consider cones instead), but the final picture we obtain is a $\mathbb{P}^5$-bundle

$$\bar{\bar{\mathcal{X}}} \longrightarrow \mathcal{Y}$$

In the fiber $\mathbb{P}^5_A = \mathbb{P}(V_6^\vee)$, A general:

elements of $\mathbb{P}(V_6^\vee) - Y_{A^\perp}$ correspond to 5 folds (25 par.)

$Y_{A^\perp}^{sm}$ correspond to 4 folds (24 parameters)

$\mathrm{Sing}(Y_{A^\perp})$ —— 3 folds (22 parameters)