

Cubic fourfolds

$X \subset \mathbb{P}(V_6)$ smooth cubic hypersurface / over \mathbb{C}

Aim. Uniformize the corresponding moduli space using the period map.

Classical case: elliptic curves

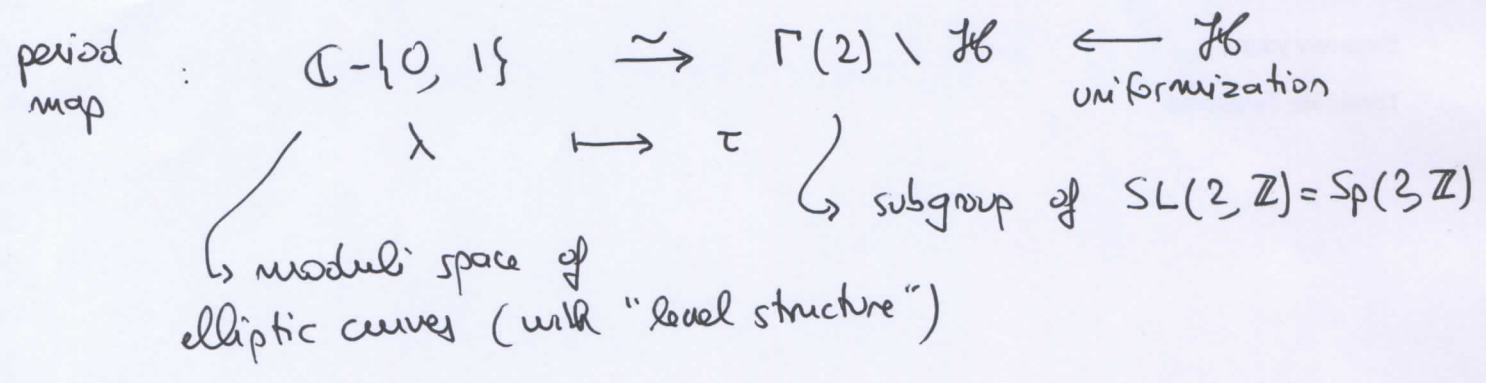
Family of elliptic curves

$$E_\lambda : y^2 = x(x-1)(x-\lambda) \quad \lambda \neq 0, 1$$

$\omega = \frac{dx}{y}$ is a holomorphic one-form

There exists a symplectic basis (σ, τ) of $H_1(E_\lambda, \mathbb{Z})$ such that

$$\int_\sigma \omega = 1 \quad \int_\tau \omega = \tau \in \mathfrak{H} \text{ Siegel upper half-plane.}$$



1. Lines on a smooth cubic

$L(X) \subset G(2, V_6)$ scheme of lines contained in X .

Prop $L(X)$ is a smooth projective fourfold

Proof The general theory says that the tangent space at L to $L(X)$ is $H^0(L, N_{L/X})$ and obstructions are in $H^1(L, N_{L/X})$.
 $N_{L/X}$ has rank 3 and degree $-K_X \cdot L + \deg K_L = 3 - 2 = 1$

$N_{L/X} \subset N_{L/\mathbb{P}^5} = \mathcal{O}_L(1)^4$ hence only two possibilities
 $\mathcal{O}_L \oplus \mathcal{O}_L \oplus \mathcal{O}_L(1)$ ← general case since a line thru a genl point must be "free"
 $\mathcal{O}_L(-1) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(1)$

In both cases, $h^0 = 4$ and $h^1 = 0$. \square

Theorem (Beauville-Donagi) $L(X)$ is a HK fourfold

Proof A HK manifold is simply connected, with a nowhere degenerate holomorphic 2-form. In particular, the canonical bundle is trivial.

This can be easily seen:

$L(X)$ is the zero set on $G(2, V_6)$ of the section of $\text{Sym}^3 \mathcal{J}^V$ defined by an equation of X

→ expected codimension 4

→ $K = K_G + c_1(\text{Sym}^3 \mathcal{J}^V) = K_G + 6c_1(\mathcal{J}^V) = 0$

The Borel-Weil theorem gives (using Koszul and normal sequences)
 $h^0(\Omega_{L(X)}^2) = 1$ and this is enough to prove what we want (by classification results) but this is not how BD proved this result.

They use the fact that any smooth deformation of

a HK manifold remains a HK manifold and exhibit an example, that of a Pfaffian cubic:

$\Delta \subset \mathbb{P}(\Lambda^2 W_6^V) = \mathbb{P}^{14}$ parametrizing degenerate skew-symmetric forms on W_6 , is singular along locus of forms of rank 2, $G(2, W_6^V)$ of codim 6. a cubic hypersurface defined by the vanishing of the Pfaffian, which is

$X := \Delta \cap (\text{general } \mathbb{P}^5)$ is a smooth cubic threefold, defined by a skew-symmetric matrix of general linear forms the Pfaffian of

Consider

$$S = G(2, W_6) \cap (\mathbb{P}^5)^\perp \subset \mathbb{P}(\Lambda^2 W_6) \quad \text{This is a smooth K3 surface of degree 14}$$

(= \mathbb{P}^8)

we define $S^2 \rightarrow L(X)$

$p, q \in S$ correspond to $P, Q \subset W_6$ of dim 2 and in general, $P+Q \subset W_6$ has dim 4

Let $\varphi \in \mathbb{P}^5$, a 2-form on W_6 .
 $\varphi|_P \equiv 0 \quad \varphi|_Q \equiv 0$ because $P, Q \in (\mathbb{P}^5)^\perp$

hence $\varphi|_{P+Q} \equiv 0$ is 4 linear conditions in \mathbb{P}^5
 \rightarrow this is a line L

Any $\varphi \in L$ is degenerate hence $L \subset X$.

One checks that this define a morphism $S^{[2]} \rightarrow L(X)$

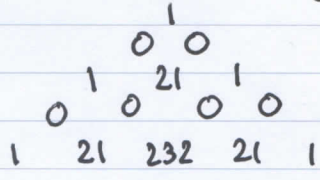
and which is an isomorphism. And $S^{[2]}$ is known to be HK

$$\begin{array}{ccc} S^{[2]} & \longrightarrow & S^2/c \\ \uparrow 2:1 & & \uparrow 2:1 \\ \tilde{S}^2 & \xrightarrow[\text{up}]{\text{blow}} & S^2 \end{array}$$

Remark: One can recover X from $L(X)$.

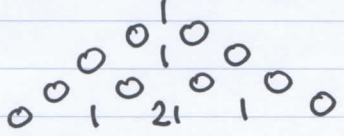
2. Hodge structures

This can also be used to compute the Hodge numbers of $L(X)$, since those of $S^{[2]}$ are known (this is the blow-up of the diagonal in S^2 divided by involution)



$$H^4(L(X), \mathbb{Q}) \cong \text{Sym}^2 H^2(L(X), \mathbb{Q})$$

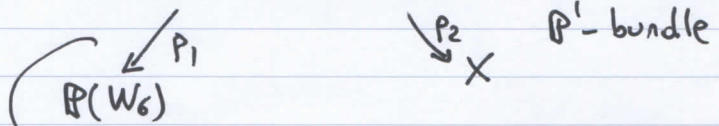
On the other hand, the Hodge numbers of X are



"same" as $H^2(L(X))$

Note: Pfaffian cubics are rational:

$$\{ (v, \varphi) \in \mathbb{P}(W_6) \times X \mid v \in \ker \varphi \}$$

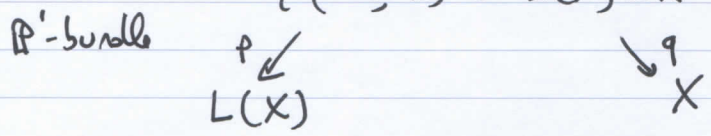


fibers: we need $\varphi(v, W_6) = 0$ 5 linear conditions on $\varphi \in \mathbb{P}^5$ hence in general a point. $(\varphi(v, v) = 0!)$

For $H \subset \mathbb{P}(W_6)$ general hyperplane, p_2 induces birational. $p_1^{-1}(H) \rightarrow X$

Since $L(X)$ parametrizes cycles on X , there is an Abel-Jacobi map

$$\{ (L, \alpha) \in L(X) \times X \mid \alpha \in L \}$$



$$\alpha := p_* q^* : H^4(X, \mathbb{Z}) \rightarrow H^2(L(X), \mathbb{Z})$$

We will be interested in primitive cohomologies

$$H^4(X, \mathbb{Z})_0 := \text{orthogonal of } h_X^2 \text{ for intersection product}$$

$$H^2(L(X), \mathbb{Z})_0 := \text{orthogonal of } g_{L(X)}^3$$

both endowed with quadratic forms

- the intersection product
- the product $u \cdot v = \frac{1}{6} q^2 uv$ (BB form)

Theorem (BD) α induces an isomorphism of polarized HS

$$H^4(X, \mathbb{Z})_0 \xrightarrow{\sim} H^2(L(X), \mathbb{Z})_0 (-1)$$

\hookrightarrow this changes the sign of the form

As lattices, they are

$$\Lambda_0 = 2E_8 \oplus 2U \oplus A_2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ signature } (2, 2)$$

Proof By direct calculation, using the facts that p is a \mathbb{P}^1 -bundle and that these lattices are known to have same discriminant 3

3. Period maps

$X \rightarrow T$ family of smooth cubic fourfolds (e.g. T Kuranishi space ($h^1(T_x) = 20$, $h^2(T_x) = 0$)).

This is locally \mathbb{C}^∞ trivial.

If T is simply connected, we can identify all lattices $H^4(X_t, \mathbb{Z})_0$ with Λ_0 .

Inside $\Lambda_0 \otimes_{\mathbb{Z}} \mathbb{C}$, $H^{3,1}(X_t)$ defines a line. More precisely,

by Riemann's relations, a point of

$$\mathcal{Q} = \{ \omega \in \mathbb{P}(\Lambda_0 \otimes \mathbb{C}) \mid \underbrace{q(\omega) = 0}_{\text{int. form}}, q(\omega + \bar{\omega}) > 0 \}$$

2 components (signature (2, 2))

\mathcal{Q}^+ , \mathcal{Q}^- bounded symmetric domains of type IV

Local Torelli theorem (Griffiths). Assume T is the Kuranishi space

The map $T \rightarrow \mathcal{Q}$ is étale (bijective differential)

When T is not simply connected, there might be nontrivial monodromy and to define a period map, one needs to take a quotient by a group of automorphisms of Λ_0 .

We have

$$H^4(X, \mathbb{Z})_0 \hookrightarrow H^4(X, \mathbb{Z}) \text{ unimodular}$$

$$\Lambda_0 \hookrightarrow \Lambda = I_{21,2}$$

and we should consider

$$\{g \in O(\Lambda_0) \mid g \text{ lifts to an isometry of } \Lambda \text{ fixing } h^2\}$$

In this case, this is the whole of $O(\Lambda_0)$ (this is a lattice-theoretic property), so we set

$$\mathcal{D} = O(\Lambda_0) \setminus \mathcal{Q}$$

quasi-projective variety by Baily-Borel theory, with compactification $\overline{\mathcal{D}}^{\text{BB}}$ (projective normal)

4. Moduli spaces

On the other side, GIT allows us to construct a moduli space for cubic fourfolds

$U \subset \mathbb{P}(\text{Sym}^3 V_6^\vee)$, parametrizing smooth cubics, is affine (its complement is the discriminant divisor) and $\text{PGL}(V_6)$ invariant

Since $\text{PGL}(V_6)$ is reductive, the invariant functions on U form a \mathbb{C} -algebra of finite type and one may consider

$$\mathcal{X} = \text{Spec}(\Gamma(U, \mathcal{O}_U)^{\text{PGL}(V_6)}) \quad \text{affine}$$

Since the stabilizers are finite ($\text{Aut}(X)$ finite because $H^0(X, T_X) = 0$), \mathcal{X} is a geometric quotient (its points are in 1:1 correspondence with \uparrow isom. classes of cubics), orbits, i.e. with

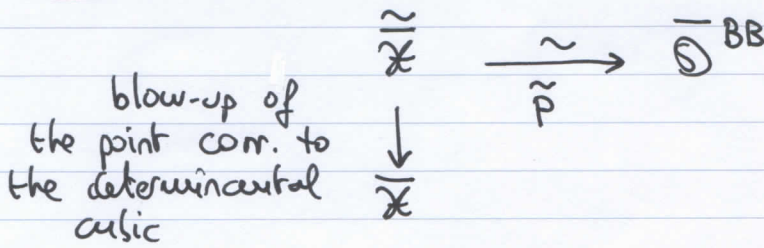
GIT also provides us with a compactification $X \subset \bar{X}$ projective

Conclusion We have an algebraic étale period map $X \xrightarrow{p} \mathbb{D}$ between 2D dim'l varieties.

Theorem (Voisin) p is an open embedding.

But the proof had a gap, which was filled only recently thanks to work of Laza, who proved the following remarkable theorem.

Theorem (Laza) p extends to an isomorphism



↳ given by $\begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} = 0$

singular along the locus where the rank is ≤ 1 , s-stable (not stable) (a Veronese surface)

Moreover, Laza proves that the image of p is the complement in $\bar{\mathbb{D}}^{BB}$ of two Heegner divisors

$\mathbb{D}_6 = \bar{\tilde{p}} \text{ (nodal cubics)}$

$\mathbb{D}_2 = \bar{\tilde{p}} \text{ (exc. divisor)}$

where a Heegner divisor is the image of $x^\perp \subset \mathbb{Q}$ in \mathbb{D} , the closure of for some $x \in \Lambda_0$.

More precisely, if $\mathbb{Z}h^2 \subset K$ rank-2 saturated $\subset I_{2,2}$
lattice

$$\mathcal{Q}_K = \{ \omega \in \mathcal{Q} \mid \omega \cdot K = 0 \} \subset \mathcal{Q}$$

Its image in \mathcal{D} only depends on $d := \text{disc}(K)$ and
 \mathcal{D}_d $d > 0$, $d \equiv 0, 2 \pmod{6}$

In geometrical terms, since the Hodge conjecture holds for cubic
fourfolds,

$$X \in \text{some } \mathcal{D}_d \Leftrightarrow X \text{ contains a surface with class} \\ \notin \mathbb{Z}h^2$$

Examples

$$\mathcal{D}_8 = p \{ \text{cubics containing a plane} \}$$

$$\mathcal{D}_{12} = p \{ \text{cubic surface scroll} \}$$

$$\mathcal{D}_{14} = p \{ \text{Pfaffian cubics} \}$$

The proof of the theorem is very long and technical: Loza
determines all types of cubics (ie the type of their singularities)
that correspond to semi-stable points.

He then studies the extension of the period map over these
points by a case-by-case analysis.

Remark As for elliptic curves, this provides a uniformization of
the moduli space of smooth cubic fourfolds by the
complement in a bounded symmetric domain of two hyperplane
arrangements.

What about our HK fourfolds $L(X)$?

They are polarized (by the restriction of the polarization
of $G(2, V_6)$)

and one can also construct by GIT a quasi-projective coarse moduli space \mathcal{Z} .

But no description of boundary points is available!

However, the BD theorem gives a commutative diagram

$$\begin{array}{ccc}
 \mathcal{X} & \hookrightarrow & \mathcal{Z} \\
 \downarrow \eta & & \downarrow q \\
 \overline{\mathcal{D}} - \mathcal{D}_2 - \mathcal{D}_6 & \hookrightarrow & \overline{\mathcal{D}}
 \end{array}$$

this period map is known to be also an open immersion by a theorem of Verbitsky

\mathcal{Z} contains a divisor corresponding to $S^{[2]}$ ^{smooth} S^V K3 surface of degree 2, which maps isomorphically into \mathcal{D}_2 .

Quadratic line complexes

We want to set up a similar picture for another family of projective varieties

$X = G(2, V_5) \cap Q \subset \mathbb{P}(\Lambda^2 V_5) = \mathbb{P}^9$
classically called quadratic line complexes. They are Fano
fivefolds of index 3 and degree 10 (almost all such
Fanos are of this type, by Mukai's classification)

1. Singular quadrics containing a quadratic line complex

This is work of O'Grady and Iliev-Mavrel.

$$I_X(2) = I_G(2) \oplus \mathbb{C}Q \quad \text{6 divisors}$$

\hookrightarrow Plücker quadrics $P_V(\omega) = \omega \wedge \omega \wedge v$
 $= V_5$ rank 6

Singular quadrics form a degree-10 hypersurface in $|I_X(2)| \simeq \mathbb{P}^5$
which splits as

$$4 |I_G(2)| + Y_X$$

hyperplane sextic

Y_X is singular (along locus where corank is ≥ 2)
It is called an EPW sextic

When X is smooth, $\text{Sing}(Y_X)$ is an integral surface with
finitely many singular points (smooth for X general).

There is a canonical double cover

$$\tilde{Y}_X \rightarrow Y_X \quad \text{branched along } \text{Sing}(Y_X)$$

and \tilde{Y}_X is a HK fourfold (in general smooth, or obtained
from a $S^{[2]}$ by contracting finitely many Lagrangian
planes)

2. EPW sextics

They are constructed from the following data
 $\Lambda^3 V_6$ with symplectic form given by wedge product
 (choose $\Lambda^6 V_6 \simeq \mathbb{C}$)

$A \subset \Lambda^3 V_6$ 10 dim'l Lagrangian

$$Y_A := \{ [V] \in \mathbb{P}(V_6) \mid A \cap (\Lambda^2 V_6 \wedge V) \neq 0 \}$$

Lagrangian

Standard parametrization of $LG(\Lambda^3 V_6)$, Lagrangian Grassmannian

Fix $V_6 = V_5 \oplus \mathbb{C}v_0$. Then
 $\Lambda^3 V_6 = \Lambda^3 V_5 \oplus (\Lambda^2 V_5 \wedge v_0)$
 but Lagrangians

One parametrizes the open subset of $LG(\Lambda^3 V_6)$ consisting of
 Lagrangians transverse to $\Lambda^3 V_5$ by

quadratic
forms on
 $\Lambda^2 V_5$

$$S^2(\Lambda^2 V_5^\vee) \rightarrow LG(\Lambda^3 V_6)$$

$$Q \mapsto \{ Q(x, \cdot) + x \wedge v_0 \mid x \in \Lambda^2 V_5 \}$$

\hookrightarrow linear form on $\Lambda^2 V_5 \simeq \Lambda^3 V_5$

$$(0 \mapsto \Lambda^2 V_5 \wedge v_0)$$

For $[A]$ in this open set (which depends on V_5, v_0), we
 obtain a quadratic form Q and X_Q

Remarks 1) X_Q smooth fivefold $\Leftrightarrow [A] \notin \Sigma$, an irreducible
 hypersurface of LG independent
 of choice of V_5, v_0

2) The isomorphism class of X_Q does not depend on choice of
 $v_0 \notin V_6 - V_5$

3) Y_{X_Q} (defined earlier) is Y_A (via $\begin{matrix} I_X(2) & \longleftrightarrow & V_6 \\ I_G(2) & \longleftrightarrow & V_5 \\ Q & \longleftrightarrow & -v_0 \end{matrix}$)

Can one recover X from γ_x ? No! Just count parameters

parameters for γ_A (or A) = $\dim LG - \dim PGL(V_6)$
= $55 - 35 = 20$

parameters for $X = \dim S^2(\Lambda^2 V_5^v) - \dim(\text{Pfaffian quadrics}) - \dim GL(V_5)$
= $55 - 5 - 25 = 25$

But we get the right number of parameters for pairs $(A, [V_5])$

For the choice of $[V_5] \in P(V_6^v)$, we only need $\Lambda^3 V_5 \cap A = 0$.

The bad choices are

$\{ [V_5] \in P(V_6^v) \mid \Lambda^3 V_5 \cap A \neq 0 \}$

This is also an EPW sextic (in the dual space)!

$A \subset \Lambda^3 V_6 \rightsquigarrow A^\perp \subset \Lambda^3 V_6^v$ still Lagrangian. This is γ_{A^\perp}

(for A general, this is the projective dual of γ_A).

Theorem $\mathcal{X} = \{ \text{smooth quadratic line complexes} \} / \text{isom.}$, affine

GIT moduli space, 25-dim'l

$\bar{\mathcal{Y}} = \{ \text{EPW sextics} \} / \text{isom.}$, projective moduli space, 20 dim'l

Consider the morphism

$f: \mathcal{X} \rightarrow \bar{\mathcal{Y}}$. Its image is the affine

variety $\bar{\mathcal{Y}} - \Sigma$ and the fiber of $[A]$ is $P(V_6^v) - \gamma_{A^\perp}$

Remark The image of f does not consist entirely of smooth double EPW sextics (some have finitely many Lagrangian singularities).

3. Hodge structures X quadratic line complex

The Hodge numbers for $H^5(X)$ are $0 \ 0 \ 10 \ 10 \ 0 \ 0$

This defines a (principally polarized) 10 dim'l intermediate Jacobian, but not a HS structure of the same type as

$$H^2(\tilde{Y}_X, \mathbb{Z}) \quad \begin{matrix} 1 & 21 & 1 \\ & 20 & \text{for primitive cohomology} \end{matrix}$$

Instead, we use a classical trick

$\tilde{X} \rightarrow G(3, V_5)$ double cover branched along X . This is a Fano 6-fold (also described by Mukai) and the Hodge numbers for $H^6(\tilde{X})$ are

$$\begin{matrix} 0 & 0 & 1 & 22 & 1 & 0 & 0 \\ & & & 20 & \text{for vanishing cohomology} \end{matrix}$$

Moreover, as lattices,

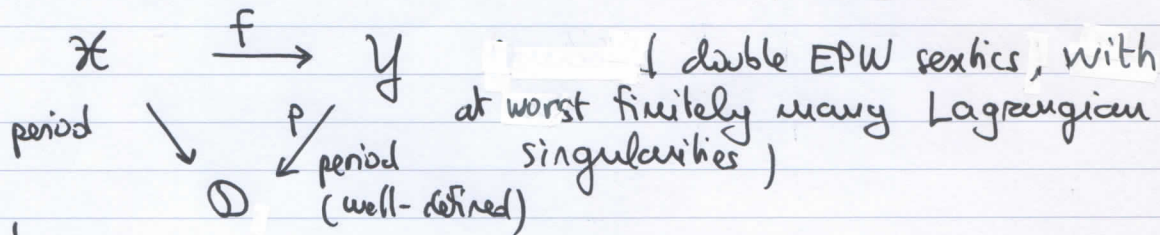
$$H^6(\tilde{X}, \mathbb{Z})_{\text{van}} \xrightarrow{\sim} H^2(\tilde{Y}_X, \mathbb{Z})_0$$

$$= \Lambda_1 := 2E_8 \oplus 2U \oplus 2A_1 \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \text{ signature } (20, 2)$$

So we have again period maps to

$$\mathbb{D} := \Gamma_1 \backslash \mathbb{Q}$$

But we don't know whether the corresponding diagram



commutes!

Remarks 1) $H^6(\tilde{X}, \mathbb{Z})_{\text{van}} \hookrightarrow H^6(\tilde{X}, \mathbb{Z})$ unimodular
 \parallel $\Lambda_1 \hookrightarrow \Lambda = I_{22,2}$

$$\Gamma_1 = \{ g \in O(\Lambda_1) \mid g \text{ lifts to an isometry of } \Lambda \text{ fixing } H^6(G, \mathbb{Z}) \}$$

has index 2 in $O(\Lambda_1)$.

Hence there is an involution τ on \mathbb{Q} , associated with the double cover

(Q5)

$$\mathbb{Q}_1 = \Gamma_1 \backslash \mathbb{Q} \rightarrow \mathcal{O}(1,1) \backslash \mathbb{Q}$$

There is also an involution ι on Y induced by the duality $A \mapsto A^\perp$. O'Grady proved

$$\tau \circ \rho = \rho \circ \iota$$

2) By Verbitsky's theorem, ρ is an open embedding.

4. The image of the period map

Unfortunately, nothing is known about extending the period maps to the GIT compactifications of X and Y . This seems extremely complicated (see O'Grady's work). Some partial results are known.

We can also define Heegner divisors $\mathbb{Q}_K \subset \mathbb{Q}$, for

$$\begin{array}{ccc} H^0(G, \mathbb{Z}) & \subset & K \\ \text{rank-2} & & \text{rank-3 saturated} \\ & & \text{lattice} \end{array} \subset I_{2,2}$$

For $d := \text{disc}(K)$ fixed, one needs $d > 0$, $d \equiv 0, 2, 4 \pmod{8}$ and

$$d \equiv 0, 4$$

\mathbb{Q}_d irreducible

$$d \equiv 2$$

two components \mathbb{Q}'_d and \mathbb{Q}''_d exchanged by involution τ

① The image of Y^{sm} is contained in

$$\overline{\mathbb{D}}^{\text{BB}} - \mathbb{Q}_2 - \mathbb{Q}_4 - \mathbb{Q}_8 - \mathbb{Q}_{10} \\ \hookrightarrow 2 \text{ components} \hookrightarrow$$

Note: $Y^{\text{sm}} = \overline{Y} - \Delta - \Sigma$ is affine hence its complement in $\overline{\mathbb{D}}^{\text{BB}}$ is a divisor

and one expects equality (difficult)

② The hypersurface Δ corresponding to smooth X but singular Y_X maps onto \mathbb{Q}'_{10}

(hence image of X should be

$$\overline{\mathbb{D}}^{\text{BB}} - \mathbb{Q}_2 - \mathbb{Q}_4 - \mathbb{Q}_8 - \mathbb{Q}''_{10})$$

③ The hypersurface Σ maps onto \mathbb{D}_8

④ As in the cubic fourfold case, there are some very singular EPW sextics which still correspond to semi-stable (not stable) points of \bar{Y} at which the period map is not defined

- $Y=2$ (discriminant cubic) is a (degenerate) EPW sextic in two ways (i.e. for two families of Lagrangians A not in the same $PGL(V_6)$ -orbit) and p "maps" these two points onto \mathbb{D}'_2 and \mathbb{D}''_2
- $Y=3$ (smooth quadric) is also a degenerate EPW sextic and p maps it onto \mathbb{D}_4

5. Compactification of the fibers

GIT provides a compactification $\mathcal{X} \subset \bar{\mathcal{X}}$, but f will not extend to $\bar{\mathcal{X}}$ because $p(\bar{\mathcal{X}}) = 1$.

Instead, we can fill in the (affine) fibers of

$$f: \mathcal{X} \longrightarrow Y$$

fibers

$$\mathbb{P}(V_6^V) - Y_{A^+}$$

by adding lower-dimensional varieties.

Recall that in the correspondence

$$Q \longleftrightarrow A \text{ Lagrangian}$$

quadric

we needed A transverse to $\Lambda^3 V_5$.

Assume now for example $\dim(A \cap \Lambda^3 V_5) = 1$. (*) Then

$$W_q := \text{Im}(A \xrightarrow{pr_2} \Lambda^2 V_5 \wedge v_0)$$

and one gets as before $Q \in S^2 W_q^V$

$$X_Q = (Q=0) \cap \mathbb{P}(W_q) \subset \mathbb{P}(\Lambda^2 V_5)$$

a Fano fourfold

Condition (*) is equivalent to $[V_5] \in Y_{A^\perp}^{sm}$

Similarly, if $[V_5] \in \text{Sing}(Y_{A^\perp})^{sm}$, we obtain a 3-fold, and if $\text{Sing}(Y_{A^\perp})^{sing}$ is non-empty, we obtain a (degree -10) K3 surface.

Of course, these are not flat degenerations of our 5-dim'l line complexes (one needs to consider cones instead), but the final picture we obtain is a \mathbb{P}^5 -bundle

$$\overline{X} \longrightarrow Y$$

In the fiber $\mathbb{P}_A^5 = \mathbb{P}(V_6^v)$, A general:

elements of $\mathbb{P}(V_6^v) - Y_{A^\perp}$ correspond to 5 folds (25 par.)

$Y_{A^\perp}^{sm}$ correspond to 4 folds (24 parameters)

$\text{Sing}(Y_{A^\perp})$ — 3 folds (22 parameters)