Curves of low degrees on Fano varieties

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Summary. We survey the period maps of some Fano varieties and the geometry of their spaces of curves of low genera and degrees.

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1 Introduction

We work over the field of complex numbers. If $X \subset \mathbf{P}^N$ is a smooth projective variety, we let $\mathscr{C}^g_d(X)$ be the (quasi-projective) subscheme of the Hilbert scheme of X corresponding to smooth, irreducible, genus-g, degree-d curves on X.

These schemes, and their compactifications $\overline{\mathscr{C}}^g_d(X)$, particularly for low d and g, have proved very useful in the study of the geometry and the period maps of some Fano varieties of low degrees.

In this informal survey article, we explain, mainly without proofs, what is known (or conjectured) in the following cases: cubic threefolds, Fano threefolds of degree 14 and index 1, cubic fourfolds, Fano varieties of degree 10, coindex 3, and dimensions 3, 4, or 5.

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In the case of a smooth cubic threefold $X \subset \mathbf{P}^4$ (§2), Hilbert schemes of curves have long be used to parametrize useful subvarieties of the intermediate Jacobian $J(X)$. For example, the (smooth projective) surface $\mathscr{C}_1^0(X)$ parametrizing lines contained in X was an essential ingredient in the proofs by Clemens & Griffiths of the non-rationality of X, and of the Torelli theorem (that $J(X)$ determines X). The Abel-Jacobi maps $\mathrm{aj} : \overline{\mathscr{C}}_d^g(X) \to J(X)$ have been since proved in some cases to be mrc fibrations (see §2 for definitions and results).

Curves on Fano threefolds can also be used to construct interesting birational correspondences. This is illustrated in §2.9, where it is explained that Fano threefolds of degree 14 and index 1 are birational to cubic threefolds. The situation is very rich and gives rise to a beautiful description of the period maps for these Fano threefolds.

In §3, we move on to smooth cubic fourfolds $Y \subset \mathbf{P}^5$, for which there are no intermediate Jacobians. Nevertheless, the mrc fibrations of $\overline{\mathscr{C}}_d^g(Y)$ should be substitutes for the Abel-Jacobi maps and should provide interesting information. In another direction, ideas of Mukai show how to construct *symplectic forms* on some moduli spaces of sheaves (§3.1). This can be used to prove that the space of lines $\mathscr{C}_1^0(Y)$ is a symplectic fourfold (Beauville & Donagi), whose Hodge structure is closely related to that of Y. This was the basis of Voisin's proof of the injectivity of the period map for smooth cubic fourfolds. This period map is now completely described by work of Looijenga and Laza and identifies the moduli space of smooth cubic fourfolds with an explicit open subset of the Baily-Borel compactification of the period domain (§3.2). Analogous results are available for the the moduli space of smooth cubic threefolds (§3.3).

Mukai's construction was also shown to apply to Fano fourfolds X_{10}^4 of degree 10 and index 2 (Example 3.3). In this case, the scheme $\overline{\mathscr{C}}_2^0(X_{10}^4)$ leads to the construction of a symplectic fourfold called a *double EPW sextic* (Iliev & Manivel, O'Grady).

In §4, we study in some details the period maps of these Fano varieties of degree 10, in dimensions 3, 4, and 5. There is beautiful geometry at work here, with intriguing interplay with that of EPW sextics (work in progress with Iliev & Manivel).

2 Cubic threefolds

The *maximal rationally connected fibration* (mrc fibration for short) of a smooth proper (complex) variety Y is a rational dominant map $\rho : Y \dashrightarrow R(Y)$ such that for $z \in R(Y)$ very general, the fiber $\rho^{-1}(z)$ is rationally connected (two general points can be joined by a rational curve), and any rational curve in Y that meets $\rho^{-1}(z)$ is contained in $\rho^{-1}(z)$. The mrc fibration exists and is unique up to birational equivalence. The variety $R(Y)$ is called the reduction of Y.

Let $X \subset \mathbf{P}^4$ be a general (although some results are known for *any smooth* X) cubic hypersurface. The intermediate Jacobian

$$J(X) := H^{2,1}(X)^\vee / H_3(X, \mathbf{Z})$$

is a 5-dimensional principally polarized abelian variety. We let $\mathscr{X}_3$ be the (irreducible, 10-dimensional) moduli space for smooth cubic threefolds, and we let $\mathscr{A}_n$ be the moduli space for principally polarized abelian varieties of dimension n. Taking X to $J(X)$ defines the *period map* $\mathscr{X}_3 \to \mathscr{A}_5$. We have:

- for $g = 0$ or $d \leq 5$, $\mathscr{C}_d^g(X)$ is integral of dimension $2d$ ([HRS3], Theorem 1.1);
- for $d \leq 5$, the *Abel-Jacobi map* $\mathrm{aj} : \overline{\mathscr{C}}_d^g(X) \to J(X)$ is the mrc fibration ([HRS2], Theorem 1.1);
- for $d \geq 4$, $\mathrm{aj} : \overline{\mathscr{C}}_d^0(X) \to J(X)$ is dominant with irreducible general fibers.

It is natural to ask whether the Abel-Jacobi map $\mathrm{aj} : \overline{\mathscr{C}}_d^0(X) \to J(X)$ is the mrc fibration for all d (i.e., are general fibers rationally connected?).

2.1 Lines

- $\mathscr{C}_1^0(X)$ (which parametrizes lines on X) is a smooth projective irreducible surface of general type;
- the image of $\mathrm{aj} : \mathscr{C}_1^0(X) \to J(X)$ is a surface S with minimal class $[\Theta]^3/3!$ and $S - S$ is a theta divisor Θ;
- aj induces an isomorphism between the Albanese variety of $\mathscr{C}_1^0(X)$ and $J(X)$ ([CG]).

The second item yields a proof of Torelli: *the period map* $\mathscr{X}_3 \to \mathscr{A}_5$ *is injective.*

2.2 Conics

- $\overline{\mathscr{C}}_2^0(X)$ is a smooth projective irreducible fourfold;
- $\mathrm{aj} : \overline{\mathscr{C}}_2^0(X) \to J(X)$ is a $\mathbf{P}^2$-bundle over the surface S of §2.1 (a conic is uniquely determined by the plane that it spans and the residual line in X). The mrc fibration is therefore $\overline{\mathscr{C}}_2^0(X) \to S$.

2.3 Plane cubics

- $\overline{\mathscr{C}}_3^1(X)$ is isomorphic to the Grassmannian $G(3, 5)$;
- the Abel-Jacobi map is therefore constant.

2.4 Twisted cubics

- $\mathrm{aj} : \mathscr{C}_3^0(X) \to J(X)$ is birational to a $\mathbf{P}^2$-bundle over a theta divisor Θ ([HRS2], §4). The mrc fibration is therefore $\overline{\mathscr{C}}_3^0(X) \to \Theta$.

2.5 Elliptic quartics

- $\mathrm{aj}:\mathscr{C}_4^1(X)\to J(X)$ is birational to a $\mathbf{P}^6$-bundle over the surface S of §2.1 ([HRS2], §4.1). The mrc fibration is therefore $\overline{\mathscr{C}}_4^1(X)\to S$.

2.6 Normal rational quartics

- $\mathrm{aj}:\overline{\mathscr{C}}_4^0(X)\to J(X)$ is dominant and the general fiber is birational to X ([IMa], Theorem 5.2), hence unirational. It is therefore the mrc fibration.

2.7 Normal elliptic quintics

- $\mathscr{C}_5^1(X)$ is an irreducible 10-fold;
- there is a factorization ([MT1], Theorem 5.6; [IMa], Theorem 3.2)

$$\mathrm{aj}:\overline{\mathscr{C}}_5^1(X)\xrightarrow{\alpha}M_X(2;0,2)\xrightarrow{\beta}J(X),$$

  where
  - $M_X(2;0,2)$ is some component of the moduli space of rank-2 stable vector bundles on X with Chern classes $c_1=0$ and $c_2=2$;
  - α is a $\mathbf{P}^5$-bundle over a dense open subset of $M_X(2;0,2)$;
  - β is birational (this is proved in [IMa] via ingenious geometrical constructions).

  In particular, $\overline{\mathscr{C}}_5^1(X)\to J(X)$ is therefore the mrc fibration.

This is seen as follows. The map α is obtained via the Serre construction: to $C\in\mathscr{C}_5^1(X)$, one associates a stable rank-2 vector bundle $\mathscr{E}_C$ on X with Chern classes $c_1=0$ and $c_2=2$ such that C is the zero-locus of a section of $\mathscr{E}_C(1)$.

The fibers of α are $\mathbf{P}(H^0(X,\mathscr{E}_C(1)))\simeq\mathbf{P}^5$ hence the Abel-Jacobi map factors through α.

According to Murre, the Chow group of algebraic 1-cycles of fixed degree on X modulo rational equivalence is canonically isomorphic to $J(X)$. The map β can then be defined directly as $\mathscr{E}\mapsto c_2(\mathscr{E})$.

2.8 Normal elliptic sextics

- $\mathscr{C}_6^1(X)$ is an irreducible 12-fold;
- $\mathrm{aj}:\overline{\mathscr{C}}_6^1(X)\to J(X)$ is the mrc fibration ([V], Theorem 2.1).

2.9 Fano threefolds of degree 14 and index 1

There is a very interesting relationship between cubic threefolds and Fano threefolds of degree 14 and index 1 ([MT1], [IMa], [K]). The latter are obtained as linear sections of $G(2,6) \subset \mathbf{P}^{14}$ by a $\mathbf{P}^9$.

Let $X \subset \mathbf{P}^4$ be a smooth cubic threefold, let C be general in $\mathscr{C}_5^1(X)$, and let $\pi : \widetilde{X} \to X$ be its blow-up, with exceptional divisor E. We have

$$-K_{\widetilde{X}} \underset{\text{lin}}{\equiv} -\pi^* K_X - E \underset{\text{lin}}{\equiv} 2\pi^* H - E.$$

This linear system induces a morphism $\widetilde{X} \to \mathbf{P}^4$ which induces a small contraction φ onto the normalization $\bar{X}$ of its image. Its non-trivial fibers are the strict transforms of the 25 lines bisecant to C: the divisor E is φ-ample hence there is a flop

$$\chi : \widetilde{X} \xrightarrow{\varphi} \bar{X} \xleftarrow{\varphi'} \widetilde{X}',$$

where $\widetilde{X}'$ is smooth projective and $\chi(E)$ is φ'-antiample.

We have $\rho(\widetilde{X}') = 2$. Since the extremal ray generated by the classes of curves contracted by φ' has $K_{\widetilde{X}'}$-degree 0 and $K_{\widetilde{X}'}$ is not nef ($-K_{\widetilde{X}'} \underset{\text{lin}}{\equiv} \varphi'^* \bar{H}$), the other extremal ray is $K_{\widetilde{X}'}$-negative and defines a contraction $\pi' : \widetilde{X}' \to X'$. One checks that:

- X' is a smooth Fano threefold of degree 14 and index 1, with Picard group generated by $H' := -K_{X'}$;
- π' is the blow-up of a smooth elliptic quintic curve $C' \subset X'$, with exceptional divisor $E' \underset{\text{lin}}{\equiv} 5\varphi'^* \bar{H} - 3\chi_*(E)$ and $\chi^* \pi'^* H' \underset{\text{lin}}{\equiv} 7H - 4E$.

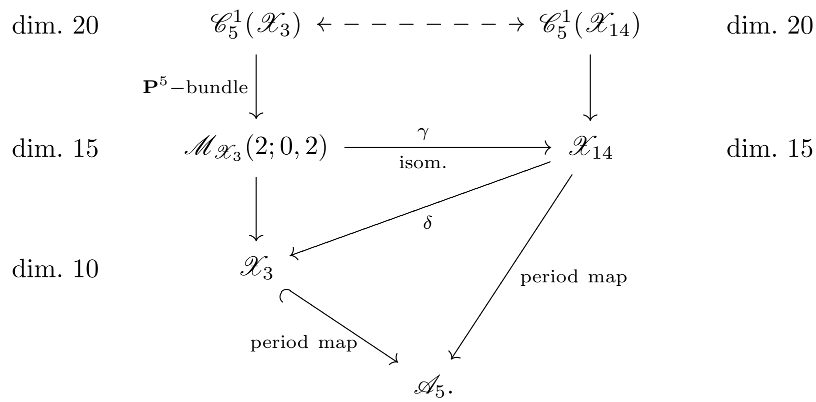
Conversely, given a general elliptic quintic curve C' in a smooth Fano threefold X' of degree 14 and index 1, one can reverse the construction above and obtain a cubic threefold X with a quintic curve $C \subset X$. In other words, if $\mathscr{C}_5^1(\mathscr{X}_3)$ denotes the moduli space of all pairs (X, C) as above, and $\mathscr{C}_5^1(\mathscr{X}_{14})$ the moduli space of all pairs (X', C'), we have a birational isomorphism

$$\mathscr{C}_5^1(\mathscr{X}_3) \dashrightarrow \mathscr{C}_5^1(\mathscr{X}_{14})$$

between (irreducible, 20-dimensional) varieties. We have the following two properties.

- *The intermediate Jacobians of X and X' are isomorphic* (this holds because $J(X) \times J(C) \simeq J(X') \times J(C')$, $J(X')$ is not a product, and $J(X)$ and $J(X')$ have the same dimension). Since we have Torelli for cubic threefolds (§2.1), the cubic obtained from a pair (X', C') only depends on X'.
- *The variety X' only depends on the vector bundle $\mathscr{E}_C$* defined in §2.7 (Kuznetsov proves in [K] that if $\mathscr{S}$ is the rank-2 tautological vector bundle on $G(2,6)$, the variety $\mathbf{P}(\mathscr{S}|_{X'})$ is obtained by a flop of $\mathbf{P}(\mathscr{E}_C)$).

So, if $\mathscr{X}_{14}$ is the moduli space for smooth Fano threefolds of degree 14 and index 1, and if $\mathscr{M}_{\mathscr{X}_3}(2;0,2)$ is the moduli space of pairs $(X,\mathscr{E})$, with $[\mathscr{E}] \in M_X(2;0,2)$ (see §2.7), we get a commutative diagram



The map γ is actually an isomorphism of stacks and the fiber of δ (between stacks) at $[X]$ is the (quasi-projective) moduli space of stable rank-2 vector bundles $\mathscr{E}$ such that $c_1(\mathscr{E}) = 0$ and $H^1(X,\mathscr{E}(-1)) = 0$ (instanton bundles), an open subset of $J(X)$ ([K], Theorem 2.9).

3 Cubic fourfolds

Let $Y \subset \mathbf{P}^5$ be a general cubic hypersurface.

The variety $\overline{\mathscr{C}}{}^0_d(Y)$ is integral of dimension $3d+1$ ([dJS], Proposition 2.4). To study these varieties, one could use, instead of the Abel-Jacobi map, their mrc fibration ([dJS]). Here is what is known, or conjectured:

- $\mathscr{C}^0_1(Y)$ is a symplectic 4-fold ([BD]) hence is its own reduction;
- the map $\overline{\mathscr{C}}{}^0_2(Y) \dashrightarrow \mathscr{C}^0_1(Y)$ given by residuation is a $\mathbf{P}^3$-bundle so this is the mrc fibration;
- $\overline{\mathscr{C}}{}^0_3(Y)$ is uniruled: a general cubic curve lies on a unique cubic surface and moves in a 2-dimensional linear system on it; so its reduction has dimension ≤ 8;
- there is a map $\overline{\mathscr{C}}{}^0_4(Y) \dashrightarrow \mathscr{J}(Y)$ (where $\mathscr{J}(Y) \to (\mathbf{P}^5)^\vee$ is the relative intermediate Jacobian of smooth hyperplane sections of Y) whose general fibers are these cubic threefolds (§2.6) hence are unirational; moreover, $\mathscr{J}(Y)$ should be non-uniruled (see also Example 3.2 below), so this should be the mrc fibration;
- for $d \geq 5$ odd, there is ([dJS], Theorem 1.2; [KM]) a holomorphic 2-form on (a smooth nonsingular model of) $\mathscr{C}^0_d(Y)$ which is *non-degenerate at a general point.* In particular, $\overline{\mathscr{C}}{}^0_d(Y)$ is not uniruled.

3.1 Constructing symplectic forms on moduli spaces

Mukai proved in 1984 that the moduli space $\mathscr{M}$ of simple sheaves on a K3 or abelian surface carries a closed non-degenerate holomorphic 2-form: the tangent space to $\mathscr{M}$ at a point $[\mathscr{F}]$ representing a simple sheaf $\mathscr{F}$ on a smooth projective variety Z is isomorphic to $\mathrm{Ext}^1(\mathscr{F},\mathscr{F})$. The Yoneda coupling

$$\mathrm{Ext}^1(\mathscr{F},\mathscr{F}) \times \mathrm{Ext}^1(\mathscr{F},\mathscr{F}) \longrightarrow \mathrm{Ext}^2(\mathscr{F},\mathscr{F})$$

is skew-symmetric whenever $[\mathscr{F}]$ is a *smooth* point of $\mathscr{M}$. When Z is a symplectic surface S with a symplectic holomorphic form $\omega \in H^0(S, \Omega_S^2)$, Mukai composes the Yoneda coupling with the map

$$\mathrm{Ext}^2(\mathscr{F},\mathscr{F}) \xrightarrow{\mathrm{Tr}} H^2(S, \mathscr{O}_S) \xrightarrow{\cup\,\omega} H^2(S, \Omega_S^2) = \mathbf{C},$$

and this defines the symplectic structure on the smooth locus of $\mathscr{M}$.

Over an n-dimensional variety Z such that $h^{q,q+2}(Z) \neq 0$ for some integer q, we

- pick a non-zero element $\omega \in H^{n-q-2}(Z, \Omega_Z^{n-q})$;
- use the exterior power $\mathrm{At}(\mathscr{F})^{\wedge q} \in \mathrm{Ext}^q(\mathscr{F}, \mathscr{F} \otimes \Omega_Z^q)$ of the Atiyah class[2] $\mathrm{At}(\mathscr{F}) \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{F} \otimes \Omega_Z^1)$;

and define

$$\mathrm{Ext}^2(\mathscr{F},\mathscr{F}) \xrightarrow{\mathrm{At}(\mathscr{F})^{\wedge q}\circ\bullet} \mathrm{Ext}^{q+2}(\mathscr{F}, \mathscr{F} \otimes \Omega_Z^q) \xrightarrow{\mathrm{Tr}} H^{q+2}(Z, \Omega_Z^q) \xrightarrow{\cup\,\omega} H^n(Z, \Omega_Z^n) \simeq \mathbf{C}.$$

Composing the Yoneda coupling with this map provides a closed (possibly degenerate) 2-form on the smooth locus of the moduli space.

Example 3.1 Let $Y \subset \mathbf{P}^5$ be a smooth cubic fourfold. Since $h^{1,3}(Y) = 1$, the construction provides a (unique) 2-form on the smooth loci of moduli spaces of sheaves on Y. Kuznetsov & Markushevich use this construction in a round-about way to produce a symplectic structure on the (smooth) fourfold $\mathscr{C}_1^0(Y)$ of lines $L \subset Y$ (originally constructed by Beauville and Donagi by a deformation argument; note that the simple-minded idea to look at sheaves of the form $\mathscr{O}_L$ does not work).

[2] Let $\Delta : Z \to Z \times Z$ be the diagonal embedding and let $\Delta(Z)^{(2)} \subset Z \times Z$ be the closed subscheme defined by the sheaf of ideals $\mathscr{I}^2_{\Delta(Z)}$. Since $\mathscr{I}_{\Delta(Z)}/\mathscr{I}^2_{\Delta(Z)} \sim \Omega_Z$, we have an exact sequence

$$0 \to \Delta_* \Omega_Z \to \mathscr{O}_{\Delta(Z)^{(2)}} \to \Delta_* \mathscr{O}_Z \to 0.$$

If $\mathscr{F}$ is a locally free sheaf on Z, we obtain an exact sequence

$$0 \to \mathscr{F} \otimes \Omega_Z \to p_{1*}(p_2^*(\mathscr{F} \otimes \mathscr{O}_{\Delta(Z)^{(2)}})) \to \mathscr{F} \to 0$$

hence an extension class $\mathrm{At}_{\mathscr{F}} \in \mathrm{Ext}^1(\mathscr{F}, \mathscr{F} \otimes \Omega_Z)$. The same construction can be extended to any coherent sheaf on Z by working in the derived category (Illusie).

Example 3.2 Let $Y \subset \mathbf{P}^5$ be a smooth cubic fourfold. Let $\mathscr{N}_Y$ be the (quasi-projective) moduli space of sheaves on Y of the form $i_*\mathscr{E}$, where $i : X \to Y$ is a non-singular hyperplane section of Y and $[\mathscr{E}] \in M_X(2; 0, 2)$. By §2.7, $\mathscr{N}_Y$ is a torsor under the (symplectic) relative intermediate Jacobian $\mathscr{J}(Y)$ of smooth hyperplane sections of Y. The Donagi-Markman symplectic structure ([DM], 8.5.2) on $\mathscr{J}(Y)$ induces a symplectic structure on $\mathscr{N}_Y$ which should be the same as the Kuznetsov-Markushevich structure ([MT2]; [KM], Theorem 7.3 and Remark 7.5). Note that since we do not know whether the Donagi-Markman symplectic form on $\mathscr{J}(Y)$ extends to a smooth compactification, we cannot deduce that $\mathscr{J}(Y)$ is not uniruled.

Example 3.3 Let X^4_{10} be a smooth Fano fourfold obtained by intersecting the Grassmannian $G(2, 5)$ in its Plücker embedding with a general hyperplane and a general quadric (see §4). We have $h^{1,3}(X^4_{10}) = 1$. The Hilbert scheme $\overline{\mathscr{C}}^0_2(X^4_{10})$ of (possibly degenerate) conics in X^4_{10} is smooth ([IM2]), hence it is endowed, by the construction above, with a canonical global holomorphic 2-form. Since $\overline{\mathscr{C}}^0_2(X^4_{10})$ has dimension five, this form must be degenerate.

There is a naturally defined morphism

$$\overline{\mathscr{C}}^0_2(X^4_{10}) \to \mathbf{P}(H^0(\mathscr{I}_{X^4_{10}}(2))) \simeq \mathbf{P}^5$$

whose image a *Eisenbud-Popescu-Walter* (EPW for short) *sextic hypersurface* $Z_{X^4_{10}} \subset \mathbf{P}^5$ (see [O1] for the definition of these sextics). In the Stein factorization

$$\overline{\mathscr{C}}^0_2(X^4_{10}) \to Y_{X^4_{10}} \to Z_{X^4_{10}},$$

the projective variety $Y_{X^4_{10}}$ is a smooth fourfold over which $\overline{\mathscr{C}}^0_2(X^4_{10})$ is (essentially) a smooth fibration in projective lines. The 2-form on $\overline{\mathscr{C}}^0_2(X^4_{10})$ thus descends to $Y_{X^4_{10}}$ and makes $Y_{X^4_{10}}$ into a holomorphic symplectic fourfold ([O1]; [IM2]) called a *double EPW sextic.*

3.2 Periods for cubic fourfolds

Let $Y \subset \mathbf{P}^5$ be a smooth cubic fourfold. Since Y has even dimension, it has no intermediate Jacobian, but still an interesting Hodge structure "of K3-type":

$$\begin{array}{ccccccc} H^4(Y, \mathbf{C})_{\text{prim}} = & H^{1,3}(Y) & \oplus & H^{2,2}(Y)_{\text{prim}} & \oplus & H^{3,1}(Y) \\ \text{dimensions:} & 1 & & 20 & & 1 \end{array}$$

with period domain

$$\mathscr{D}^{20} = \{[\omega] \in \mathbf{P}^{21} \mid Q(\omega, \omega) = 0,\ Q(\omega, \bar\omega) > 0\},$$

a 20-dimensional bounded symmetric domain of type IV (where Q is the (non-degenerate) intersection form on $H^4(Y, \mathbf{C})$). Let $\mathscr{Y}_3$ be the (irreducible, 20-dimensional) moduli space for smooth cubic fourfolds. We get a period map

$$\mathscr{Y}_3 \to \mathscr{D}^{20}/\Gamma$$

where Γ is an explicit discrete arithmetic group. Voisin proved that it is injective and its (open) image was determined in [L].

3.3 Periods for cubic threefolds

One can construct another period map for cubic threefolds (compare with §2): to such a cubic $X \subset \mathbf{P}^4$, we associate the cyclic triple cover $Y_X \to \mathbf{P}^4$ branched along X. It is a cubic fourfold, hence this construction defines a map $\mathscr{X}_3 \to \mathscr{Y}_3 \to \mathscr{D}^{20}/\Gamma$.

Because of the presence of an automorphism of Y_X of order 3, we can restrict the image and define a period map (Allcock-Carlson-Toledo)

$$\mathscr{X}_3 \to \mathscr{D}^{10}/\Gamma'$$

where

$$\mathscr{D}^{10} := \{\omega \in \mathbf{P}^{10} \mid Q(\omega, \bar{\omega}) < 0\} \simeq \mathbf{B}^{10}$$

and Γ' is a discrete arithmetic group. Again, it is an isomorphism onto an explictily described open subset of $\mathscr{D}^{10}/\Gamma'$.

4 Fano varieties of degree 10

Let V_5 be a 5-dimensional complex vector space. We define, for $k \in \{3, 4, 5\}$, a smooth, degree-10, coindex-3, Fano k-fold by

$$X_{10}^k := G(2, V_5) \cap \mathbf{P}^{k+4} \cap \Omega \subset \mathbf{P}(\wedge^2 V_5),$$

where $\mathbf{P}^{k+4}$ is a general $(k+4)$-plane and Ω a general quadric. Let $\mathscr{X}_{10}^k$ be the moduli stack for smooth varieties of this type.

Enriques proved that all (smooth) X_{10}^3, X_{10}^4, and X_{10}^5 are unirational. A general X_{10}^3 is not rational, whereas all (smooth) X_{10}^5 are rational (Semple, 1930). Some smooth X_{10}^4 are rational (Roth, 1949; Prokhorov), but the rationality of a general X_{10}^4 is an open question.

We have

$$I_{X_{10}^k}(2) \simeq \mathbf{C}\Omega \oplus V_5,$$

where V_5 corresponds to the rank-6 *Plücker quadrics* $\omega \mapsto \omega \wedge \omega \wedge v$ $(v \in V_5)$. In the 5-plane $\mathbf{P}(I_{X_{10}^k}(2))$, the degree-$(k+5)$ hypersurface corresponding to singular quadrics decomposes as

$$(k-1)\mathbf{P}(V_5) + Z^\vee_{X_{10}^k},$$

where $Z^\vee_{X^k_{10}} \subset \mathbf{P}(I_{X^k_{10}}(2))$ is an *EPW sextic* ([IM2], §2.2; this is indeed the projective dual of the sextic defined in Example 3.3 when $k = 4$).

If $\mathscr{EPW}$ is the (irreducible, 20-dimensional) moduli space of EPW sextics, we get morphisms

$$\mathrm{epw}^k : \mathscr{X}^k_{10} \to \mathscr{EPW}$$

which are *dominant* ([IM2], Corollary 4.17). Note that

$$\dim(\mathscr{X}^5_{10}) = 25,$$
$$\dim(\mathscr{X}^4_{10}) = 24,$$
$$\dim(\mathscr{X}^3_{10}) = 22.$$

Proposition 4.1 (Debarre-Iliev-Manivel) *For a general EPW sextic Z, with projective dual $Z^\vee$, the fiber $(\mathrm{epw}^3)^{-1}([Z])$ is isomorphic to the smooth surface $\mathrm{Sing}(Z^\vee)$.*

4.1 Gushel degenerations

The link between these Fano varieties of various dimensions can be made through a construction of Gushel analogous to what we did with cubics in §3.3.

Let $CG \subset \mathbf{P}(\mathbf{C} \oplus \wedge^2 V_5)$ be the cone, with vertex $v = \mathbf{P}(\mathbf{C})$, over the Grassmannian $G(2, V_5)$. Intersect CG with a general quadric $\Omega \subset \mathbf{P}(\mathbf{C} \oplus \wedge^2 V_5)$ and a linear space $\mathbf{P}^{k+4}$ to get a Fano variety T^k of dimension k. There are two cases:

- either $v \notin \mathbf{P}^{k+4}$, in which case T^k is isomorphic to the intersection X^k_{10} of $G(2, V_5)$ with the projection of $\mathbf{P}^{k+4}$ to $\mathbf{P}(\wedge^2 V_5)$ and a quadric;
- or $v \in \mathbf{P}^{k+4}$, in which case $\mathbf{P}^{k+4}$ is a cone over a $\mathbf{P}^{k+3} \subset \mathbf{P}(\wedge^2 V_5)$ and T^k is a double cover X^k_G of $G(2, V_5) \cap \mathbf{P}^{k+3}$ branched along its intersection X^{k-1}_{10} with a quadric.

The second case is a specialization of the first, and the EPW sextics $Z_{X^k_{10}}$ from the first case degenerate to the sextics $Z_{X^k_G}$ from the second case. Moreover, in the second case, *the sextics $Z_{X^k_G}$ and $Z_{X^{k-1}_{10}}$ are the same.*

Let $\mathscr{X}^k_G$ be the moduli stack for smooth varieties of type X^k_{10} and their Gushel degenerations. The Gushel constructions therefore yield morphisms $\mathscr{X}^k_{10} \to \mathscr{X}^{k+1}_G$ such that the diagrams

$$\begin{array}{ccc} \mathscr{X}^k_{10} & \longrightarrow & \mathscr{X}^{k+1}_G \\ & \searrow^{\mathrm{epw}^k} & \downarrow{\scriptstyle \mathrm{epw}^{k+1}_G} \\ & & \mathscr{EPW} \end{array}$$

commute.

We can perform a "'double Gushel construction" as follows. Let $CCG \subset \mathbf{P}(\mathbf{C}^2 \oplus \wedge^2 V_5)$ be the cone, with vertex $L = \mathbf{P}(\mathbf{C}^2)$, over the Grassmannian $G(2, V_5)$. Intersect CCG with a general quadric $\Omega \subset \mathbf{P}(\mathbf{C}^2 \oplus \wedge^2 V_5)$ and a codimension-2 linear space $\mathbf{P}^9$ to get a Fano variety T of dimension 5.

- If $L \cap \mathbf{P}^9 = \varnothing$, the variety T is smooth of type $\mathscr{X}^5_{10}$.
- If $L \subset \mathbf{P}^9$, in which case $\mathbf{P}^9$ is a cone over a $\mathbf{P}^7 \subset \mathbf{P}(\wedge^2 V_5)$, the corresponding variety T_0 meets L at two points p and q, and the projection from L induces a rational conic bundle $T_0 \dashrightarrow W^4_5 := G(2, V_5) \cap \mathbf{P}^7$, undefined at p and q, whose discriminant locus is a threefold X^3_{10}. Blowing up these two points, we obtain a conic bundle $\widehat{T}_0 \to W^4_5$ with two disjoint sections corresponding to the two exceptional divisors. These two sections trivialize the canonical double étale cover $\widetilde{X}^3_{10} \to X^3_{10}$ of the discriminant.

We call the singular variety T_0 a double-Gushel degeneration. If we let $\overline{\mathscr{X}}^5_{GG}$ be the moduli stack for smooth varieties of type X^5_{10} and their Gushel and double-Gushel degenerations, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathscr{X}^3_{10} & \longrightarrow & \mathscr{X}^4_G & \longrightarrow & \mathscr{X}^5_{GG} \\ & {\scriptstyle \mathrm{epw}^3} \searrow & \downarrow {\scriptstyle \mathrm{epw}^4_G} & \swarrow {\scriptstyle \mathrm{epw}^5_{GG}} & \\ & & \mathscr{E\!P\!W}. & & \end{array}$$

4.2 Period maps

Both $H^3(X^3_{10}, \mathbf{Q})$ and $H^5(X^5_{10}, \mathbf{Q})$ have dimension 20 and carry Hodge structures of weight 1. This gives rise to period maps

$$\wp^3 : \mathscr{X}^3_{10} \to \mathscr{A}_{10} \quad \text{and} \quad \wp^5 : \mathscr{X}^5_{10} \to \mathscr{A}_{10}.$$

By [DIM], the general fibers of $\wp^3$ are unions of smooth proper surfaces that come in pairs:

- $\mathscr{F}_{X^3_{10}}$, isomorphic to the quotient of $\overline{\mathscr{C}}^0_2(X^3_{10})$ by a fixed-point-free involution, and also to $\operatorname{Sing}(Z^\vee_{X^3_{10}})$;
- $\mathscr{F}^\star_{X^3_{10}}$, the analogous surface for any "line-transform" of X^3_{10}.

In particular, by Proposition 4.1, there is a commutative diagram

$$\begin{array}{ccccc} \mathscr{X}^3 & \xrightarrow{\mathrm{epw}^3} & \mathscr{E\!P\!W} & \longrightarrow & \mathscr{E\!P\!W}/\text{duality} \\ & {\scriptstyle \wp^3} \searrow & \downarrow {\scriptstyle \wp} & \swarrow {\scriptstyle \bar{\wp}} & \\ & & \mathscr{A}_{10}, & & \end{array}$$

where the map $\bar{\wp}$ is generically finite (presumably birational) onto its image. Since $J(X^3_{10})$ is isomorphic to the Albanese variety of the surface $\overline{\mathscr{C}}^0_2(X^3_{10})$ ([Lo]), the map $\wp$ is defined by sending the class of a general EPW sextic Z to the Albanese variety of the canonical double cover of its singular locus (the resulting principally polarized abelian varieties are isomorphic for Z and $Z^\vee$).

Theorem 4.2 *The image of $\wp^3$ and the image of $\wp^5$ have same closures.*

Proof. This is proved using a double Gushel degeneration (see §4.1): with the notation above, one proves that intermediate Jacobian $J(\widehat{T}_0)$ is still a 10-dimensional principally polarized abelian variety which is a limit of intermediate Jacobians of Fano fivefolds of type $\mathscr{X}^5_{10}$, hence belongs to the closure of $\mathrm{Im}(\wp^5)$.

Since, by Lefschetz theorem, we have $H^5(W^4_5,\mathbf{Q}) = H^3(W^4_5,\mathbf{Q}) = 0$, the following lemma implies that the intermediate Jacobians $J(\widehat{T}_0)$ and $J(X^3_{10})$ are isomorphic. This proves already that the image of $\wp^3$ is contained in the closure of the image of $\wp^5$.

Lemma 4.3 *Let T and W be smooth projective varieties. Assume*

$$H^k(W,\mathbf{Q}) = H^{k-2}(W,\mathbf{Q}) = 0.$$

Let $\pi : T \to W$ be a conic bundle with smooth irreducible discriminant divisor $X \subset W$. Assume further that $\pi^{-1}(X)$ is reducible. There is an isomorphism of polarized Hodge structures

$$H^k(T,\mathbf{Z})/\mathrm{tors} \simeq H^{k-2}(X,\mathbf{Z})/\mathrm{tors}.$$

To finish the proof of the theorem, we prove, by computing the kernel of its differential, that the fibers of $\wp^5$ have dimension at least 5. □

Question 4.4 Is $\wp^5$ equal to $\wp \circ \mathrm{epw}^5$?

Question 4.5 Can one extend $\wp$ to the GIT compactification of $\mathscr{EPW}$ studied in [O2], with values in a suitable compactification of $\mathscr{A}_{10}$?

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