VARIETIES WITH VANISHING HOLOMORPHIC EULER CHARACTERISTIC

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1. Introduction

A (smooth complex projective) variety X is said to have *maximal Albanese dimension* (mAd) if its Albanese morphism $a_X : X \to A$ is generically finite (onto its image). Equivalently, the sheaf of differentials Ω_X is generated by global sections at a general point.

Green and Lazarsfeld showed that such a variety satisfies $\chi(X, \omega_X) \geq 0$. Their method of proof is to introduce the cohomological loci

$$V_i(\omega_X) = \{\xi \in \widehat{A} \mid H^i(X, \omega_X \otimes a_X^* P_\xi) \neq 0\}$$

(For any smooth projective variety Y, we set $\widehat{Y} := \mathrm{Pic}^0(Y)$.) They prove

$$\forall i \geq 0 \qquad \mathrm{codim}_{\widehat{A}} V_i(\omega_X) \geq i.$$

This implies, for $\xi \in \widehat{A}$ general, $H^i(X, \omega_X \otimes a_X^* P_\xi) = 0$ for all $i > 0$, hence

$$\chi(X, \omega_X) = h^0(X, \omega_X \otimes a_X^* P_\xi) \geq 0.$$

Moreover, $\chi(X, \omega_X)$ vanishes if and only if $V_0(\omega_X) \neq \widehat{A}$.

Kollár then conjectured that if X is in addition of general type, one should have $\chi(X, \omega_X) > 0$ (this is the case in dimensions ≤ 2).

2. The Ein–Lazarsfeld examples

Let E_1, E_2, and E_3 be elliptic curves and let $\rho_j : C_j \twoheadrightarrow E_j$ be double coverings, where C_j is a smooth curve of genus ≥ 2 and $\rho_{j*}\omega_{C_j} \simeq \mathscr{O}_{E_j} \oplus L_j$. Denote by ι_j the corresponding involution of C_j. Set $A := E_1 \times E_2 \times E_3$ and $Z := (C_1 \times C_2 \times C_3)/\iota_1 \times \iota_2 \times \iota_3$, smooth except at finitely many points where it has rational singularities. The variety Z is minimal of general type. Let $\varepsilon : X \to Z$ be a desingularization. We have:

$$\begin{tikzcd}
 & X \arrow[d, two heads, "\varepsilon"] \arrow[dr, two heads, "a_X"] & \\
C_1 \times C_2 \times C_3 \arrow[r, two heads, "\mathbf{Z}/2\mathbf{Z}"'] & Z \arrow[r, two heads, "(\mathbf{Z}/2\mathbf{Z})^2"'] & A
\end{tikzcd}$$

and

$$a_{X*}\omega_X \simeq \mathscr{O}_A \oplus (L_1 \otimes L_2) \oplus (L_1 \otimes L_3) \oplus (L_2 \otimes L_3),$$

hence

$$V_0(\omega_X, a_X) = V_1(\omega_X, a_X) = (\widehat{E}_1 \times \widehat{E}_2 \times \{0\}) \cup (\widehat{E}_1 \times \{0\} \times \widehat{E}_3) \cup (\{0\} \times \widehat{E}_2 \times \widehat{E}_3),$$

whereas $V_2(\omega_X, a_X) = V_3(\omega_X, a_X) = \{0\}$.

This provides three-dimensional examples for which the Albanese mapping is birationally a $(\mathbf{Z}/2\mathbf{Z})^2$-covering. The same construction works starting from double coverings $\rho_j : X_j \twoheadrightarrow A_j$ of abelian varieties with smooth ample branch loci and provides examples in all dimensions ≥ 3. One can also extend it to any *odd* number $2r+1$ of factors and get examples where the Albanese mapping is birationally a $(\mathbf{Z}/2\mathbf{Z})^{2r}$-covering (when the number of factors is even, $\chi(X, \omega_X) = \prod_j \chi(E_j, L_j) > 0$).[1]

3. Varieties with $\chi(X, \omega_X) = 0$

We are interested in describing the structure of (smooth projective) varieties X of mAd (and of general type) with $\chi(X, \omega_X) = 0$. This class of varieties is stable by taking:

- modifications;
- étale covers;
- if $h : Y \twoheadrightarrow X$ is generically finite and X has mAd, we have $\chi(Y, \omega_Y) \geq \chi(X, \omega_X)$ (essentially because the inclusion $\omega_X \to h_*\omega_Y$ is split by the trace map) hence if $\chi(Y, \omega_Y)$ vanishes, so does $\chi(X, \omega_X)$;
- products with any other variety of mAd (and of general type);
- more generally, if X has mAd with a fibration whose general fiber F satisfies $\chi(F, \omega_F) = 0$, then $\chi(X, \omega_X) = 0$.

The last result (Chen and Hacon) is proved using generic vanishing theorems and (an extension of) a theorem of Kollár's: if $f : X \twoheadrightarrow Y$, then $R^i f_*(\omega_X \otimes P_\xi) = 0$ for all $i \geq 0$ and ξ general torsion in $\widehat{X}$, because it is torsion-free (Kollár–we need ξ torsion here) and supported on a proper subset (because $H^i(F, \omega_F \otimes P_\xi) = 0$ by generic vanishing (for $i > 0$) and hypothesis (for $i = 0$)[2]). Then

$$\chi(X, \omega_X) = \chi(X, \omega_X \otimes P_\xi) = \sum_i (-1)^i \chi(Y, R^i f_*(\omega_X \otimes P_\xi)) = 0.$$

Warning: in the situation $X \twoheadrightarrow Y$ above, $\chi(Y, \omega_Y)$ and $\chi(F, \omega_F)$ may be both positive (unless Y is a curve) (e.g., in the Ein-Lazarsfeld example, where we have an isotrivial morphism $X \to (C_1 \times C_2)/(\iota_1, \iota_2)$: the surface $S_3 := (C_1 \times C_2)/(\iota_1, \iota_2)$ is of general type, hence $\chi(S_3, \omega_{S_3}) > 0$, and the general fibers are C_3).

[1] One also checks that this construction does not work with abelian groups other than $\mathbf{Z}/2\mathbf{Z}$. For example, if one starts from cyclic triple covers $\rho_j : C_j \twoheadrightarrow E_j$ and set $\rho_{j*}\omega_{C_j} \simeq \mathscr{O}_{E_j} \oplus L_j \oplus L_j^2$, one will find a direct factor $L_1 \otimes L_2 \otimes L_3$ in $a_{X*}\omega_X$, hence $V_0(\omega_X, a_X) = \widehat{A}$.

[2] Here we need the results and reasoning used in the introduction, but for the generically finite morphism $F \to A$, because ξ is general in $\widehat{X} = \widehat{A}$, not in $\widehat{F}$.

This list shows that it might be difficult to describe all (smooth projective) varieties X of general type and mAd with $\chi(X, \omega_X) = 0$ when $\dim(X)$ is large. We suggest the following conjecture:

Conjecture. *Let X be a smooth projective variety of mAd, of general type, with $\chi(X, \omega_X) = 0$. There exist a smooth projective variety X', a morphism $X' \to X$ which is a composition of modifications and abelian étale covers, and a fibration $g : X' \twoheadrightarrow Y$ with general fiber F, such that $0 < \dim(Y) < \dim(X)$ and*

a) *either g is isotrivial;*
b) *or $\chi(F, \omega_F) = 0$.*

The condition in case a) is of course not sufficient to have $\chi(X, \omega_X) = 0$, whereas it is in case b). The Ein-Lazarsfeld example falls into case a) of the conjecture, and not into case b). We can construct examples that falls into case b), but not into case a); they are basically nonisotrivial fibrations whose general fibers are Ein-Lazarsfeld threefolds.

4. Main theorem (for threefolds)

Theorem 4.1. *Let X be a smooth projective threefold of mAd and of general type, with $\chi(X, \omega_X) = 0$. Some abelian étale cover of X is an Ein-Lazarsfeld threefold, and the Albanese mapping a_X is birationally a $(\mathbf{Z}/2\mathbf{Z})^2$-covering.*

Note also that a_X is not finite (we will come back to that later (Proposition 7.3): ω_X cannot be ample and a_X must contract rational curves on X) but that it is finite on the canonical model of X.

5. Varieties with $P_1 = 1$

Let X be a smooth projective variety of mAd; we have $P_1(X) \geq 1$ and we want to study those varieties for which $P_1(X) = 1$. Ueno proved the following:

a) We have
$$a_X^* : \bigwedge^{\bullet} H^0(A, \Omega_A) \simeq H^0(X, \Omega_X^{\bullet}).$$
In particular, we have $h^j(X, \mathscr{O}_X) = \binom{\dim(X)}{j}$ for all j, hence $\chi(X, \omega_X) = 0$.
b) $a_X : X \to \mathrm{Alb}(X)$ is surjective.
c) The point 0 is isolated in $V_0(\omega_X, a_X)$.

An example *of general type)* was constructed by Chen and Hacon as follows. With the same notation as above, choose points of order two $\xi_j \in \widehat{E}_j$ and consider the induced double étale covers $C'_j \to C_j$, with associated involution σ_j, and $E'_j \to E_j$. The involution ι_j on C_j pulls back to an involution ι'_j on C'_j (with quotient E'_j). Set

$$Z' := (C'_1 \times C'_2 \times C'_3)/\langle \mathrm{id}_1 \times \sigma_2 \times \iota'_3, \iota'_1 \times \mathrm{id}_2 \times \sigma_3, \sigma_1 \times \iota'_2 \times \mathrm{id}_3, \sigma_1 \times \sigma_2 \times \sigma_3 \rangle$$

and let $\varepsilon' : X' \to Z'$ be a desingularization. There is a morphism $f' : X' \to A$ of degree 4, the Albanese map of X' is $a_{X'} = f' \circ \varepsilon'$, and

$$a_{X'*}\omega_{\widetilde{X}'} \simeq \mathscr{O}_A \oplus (L_1 \otimes L_2^\xi \otimes P_{\xi_3}) \oplus (L_1^\xi \otimes P_{\xi_2} \otimes L_3) \oplus (P_{\xi_1} \otimes L_2 \otimes L_3^\xi),$$

where $L_j^\xi = L_j \otimes P_{\xi_j}$. In particular, $P_1(X') = 1$, and

$$V_0(\omega_{X'}, a_{X'}) = V_1(\omega_{X'}, a_{X'}) = \{0\} \cup (\widehat{E}_1 \times \widehat{E}_2 \times \{\xi_3\}) \cup (\widehat{E}_1 \times \{\xi_2\} \times \widehat{E}_3) \cup (\{\xi_1\} \times \widehat{E}_2 \times \widehat{E}_3).$$

Of course, the étale cover $E_1' \times E_2' \times E_3' \to E_1 \times E_2 \times E_3$ pulls back to an étale cover $X'' \to X'$, where X'' is an Ein-Lazarsfeld threefold:

$$\begin{tikzcd}
 & Z'' \arrow[d, two heads, "(\mathbf{Z}/2\mathbf{Z})^3"'] \arrow[dr, two heads, "(\mathbf{Z}/2\mathbf{Z})^2"] & \\
C_1' \times C_2' \times C_3' \arrow[ur, two heads, "\mathbf{Z}/2\mathbf{Z}"] \arrow[r, two heads, "(\mathbf{Z}/2\mathbf{Z})^4"'] \arrow[d, two heads, "(\mathbf{Z}/2\mathbf{Z})^3"'] & Z' \arrow[dr, two heads, "(\mathbf{Z}/2\mathbf{Z})^2"] & A' \arrow[d, two heads, "(\mathbf{Z}/2\mathbf{Z})^3"] \\
C_1 \times C_2 \times C_3 \arrow[r, two heads, "\mathbf{Z}/2\mathbf{Z}"] & Z \arrow[r, two heads, "(\mathbf{Z}/2\mathbf{Z})^2"] & A.
\end{tikzcd}$$

(The isogeny $A' \to A$ (or rather $\widehat{A} \to \widehat{A}'$) kills the 2-torsion points ξ_j in V_0.) This construction still works starting from double coverings of abelian varieties with smooth ample branch loci and provides examples in all dimensions ≥ 3. As in the Ein-Lazarsfeld example, one can extend it to any *odd* number $2r+1$ of factors and get examples where the Albanese mapping is birationally a $(\mathbf{Z}/2\mathbf{Z})^{2r}$-covering.

Our main theorem for threefolds allows us to prove the following.

Theorem 5.1. *Any smooth projective threefold X of mAd and of general type, with $P_1(X) = 1$, is a modification of an abelian étale cover of a Chen-Hacon threefold.*

In other words, to obtain all such threefolds, one starts from a Chen-Hacon threefold and take an abelian étale cover, such that no ξ_j is in the corresponding kernel.

It is difficult tostudies these varieties in higher dimensions (they are not stable in étale covers). For example, do they fall into case a) of our conjecture?

6. Ideas of proof

Here are further properties of the $V_i(\omega_X)$, for X of mAd:

(1) Each component of $V_i(\omega_X)$ is an abelian subvariety of $\widehat{A}$ (of codimension $\geq i$) translated by a torsion point (Simpson).
(2) There is a chain of inclusions (Green)

$$\{0\} = V_n(\omega_X) \subset V_{n-1}(\omega_X) \subset \cdots \subset V_0(\omega_X) \subset \widehat{A}.$$

(3) If $V_0(\omega_X)$ has a component V of codimension i, this component is contained in (hence is an irreducible component of) $V_i(\omega_X)$, so that we have $i \leq n$. Moreover, if K is the (i-dimensional) kernel of $A \twoheadrightarrow \widehat{V}$, we have $a_X(X) = a_X(X) + K$ (Ein-Lazarsfeld).

(4) The variety X is of general type if and only if $V_0(\omega_X)$ generates $\widehat{A}$ (Chen-Hacon).

6.1. **General facts.**

Theorem 6.1. *Let X be a smooth projective variety of dimension n with generically finite Albanese mapping $a_X : X \to A$. Assume that for some $i \in \{0, \ldots, n\}$, the locus $V_0(\omega_X)$ has a component V of codimension i in $\widehat{A}$. The Stein factorization of $X \to A \twoheadrightarrow \widehat{V}$ is $X \twoheadrightarrow Y \to \widehat{V}$, where Y is smooth of dimension $n-i$, of general type, with $\chi(Y, \omega_Y) > 0$.*

(This is not entirely accurate: one needs to take first a suitable abelian cover of X to assume that A splits as a product.)

We now deduce some consequences of Theorem 6.1 on the possible components of $V_0(\omega_X)$ and the number of simple factors of the abelian variety A.

Corollary 6.2. *Let X be a smooth projective variety of mAd with $\chi(X, \omega_X) = 0$.*

a) *The locus $V_0(\omega_X)$ does not have complementary components (by that, we mean components such that the sum morphism induces an isogeny from their product onto $\widehat{A}$).*
b) *If X is in addition of general type, A has at least three simple factors.*

Proof. If $V_0(\omega_X)$ has complementary components $V_1, \ldots, V_r$, we may assume $A = \prod_j \widehat{V}_j$. The image $a_X(X)$ is then stable by translation by $\prod_{j \neq i} \widehat{V}_j$ for each i (by property (3) above), hence a_X is surjective if $r \geq 2$. We obtain from Theorem 6.1 a generically finite surjective map $X \twoheadrightarrow Y_1 \times \cdots \times Y_r$, with $\chi(Y_i, \omega_{Y_i}) > 0$ for all i. Since $\chi(X, \omega_X) \geq \prod_i \chi(Y_i, \omega_{Y_i})$, this is absurd. This proves a). Item b) then follows from the fact that $V_0(\omega_X)$ generates $\widehat{A}$ (property (4) above). □

6.2. **Case when A has three simple factors.**

Proposition 6.3. *Let X be a smooth projective variety of general type and mAd with $\chi(X, \omega_X) = 0$, whose Albanese variety A has exactly three simple factors A_1, A_2, and A_3.*

a) *The map a_X is surjective.*
b) *After passing to an étale cover, we may assume $A = A_1 \times A_2 \times A_3$ and that $\widehat{A}_1 \times \widehat{A}_2 \times \{0\}$, $\widehat{A}_1 \times \{0\} \times \widehat{A}_3$, and $\{0\} \times \widehat{A}_2 \times \widehat{A}_3$ are irreducible components of $V_0(\omega_X)$.*

Proof. We just explain that a) follows from b) since $a_X(X)$ is then stable by translation by each A_i hence is equal to A. □

As shown by considering the product with a curve of genus ≥ 2 of any variety X of general type and mAd with $\chi(X, \omega_X) = 0$, the map a_X is not surjective in general as soon as A has at least four simple factors.

Finally, we have (after replacing X with a suitable modification), for each $\{i,j,k\} = \{1,2,3\}$, Stein factorizations

$$p_{jk}\circ f : X \xrightarrow{\ g_i\ } S_i \xrightarrow{(h_{ij},h_{ik})} A_j\times A_k,$$

where S_i is smooth of general type with $\chi(S_i,\omega_{S_i})>0$. Moreover, $f_j : X \twoheadrightarrow A_j$ has connected fibers, hence so does $h_{ij} : S_i \twoheadrightarrow A_j$. All in all, we have for each $\{i,j,k\} = \{1,2,3\}$ a commutative diagram:

(1)
$$\begin{tikzcd}
 & & X \arrow[dll, two heads, bend right, "f_j"'] \arrow[dl, two heads, "g_i"] \arrow[dd, two heads, "f_k"'] \arrow[dr, two heads, "g_j"'] \arrow[drr, two heads, bend left, "f_i"] & & \\
 & S_i \arrow[dl, two heads, "h_{ij}"] \arrow[dr, two heads, "h_{ik}"'] & & S_j \arrow[dl, two heads, "h_{jk}"] \arrow[dr, two heads, "h_{ji}"'] & \\
A_j & & A_k & & A_i,
\end{tikzcd}$$

where all the morphisms are fibrations.

Question 6.4. Are the h_{ij} isotrivial? Are the f_i isotrivial? Is X rationally dominated by a product $X_1\times X_2\times X_3$, where X_i dominates and is generically finite over A_i? We are inclined to think that the answers to all these questions should be affirmative, but we were only able to go further in the case where the A_i are all elliptic curves.

6.3. **The 3-dimensional case.** We now come to our main result, which completely describes all smooth projective threefolds X of mAd and of general type, with $\chi(X,\omega_X)=0$.

Proof of the theorem. By Proposition 6.3.a), a_X is surjective. Moreover, $\mathrm{Alb}(X)$ is isogeneous to the product of three elliptic curves and, after passing to an étale cover, we may assume that a_X can be written as

$$a_X : X \xrightarrow{(f_1,f_2,f_3)} E_1\times E_2\times E_3,$$

where each $f_i : X \twoheadrightarrow E_i$ is a fibration, and that $\widehat{E}_1\times\widehat{E}_2\times\{0\}$, $\widehat{E}_1\times\{0\}\times\widehat{E}_3$, and $\{0\}\times\widehat{E}_2\times\widehat{E}_3$ are irreducible components of $V_0(\omega_X)$.

The proof follows the following steps.

(1) *The fibrations $h_{ij} : S_i \twoheadrightarrow E_j$ are all isotrivial.* We denote by C_{ij} a general (constant) fiber of h_{ij}.
(2) *There exist finite groups G_i acting on C_{ij} such that $C_{ij}/G_i\simeq E_j$ and $C_{ik}/G_i\simeq E_k$, the surface S_i is birational to the quotient $(C_{ij}\times C_{ik})/G_i$ for the diagonal action of G_i, and h_{ij} and h_{ik} are identified with the two projections.* Let Y_j be a resolution of singularities of the irreducible threefold $S_i\times_{E_j}S_k$ and let Y be a resolution of singularities of the component of $Y_1\times_{E_2\times E_3}S_1$ that dominates both Y_1 and Y_3. After modification of X,

we obtain a diagram

$$\begin{tikzcd}
 & X \arrow[rr, two heads, "g"] & & Y \arrow[dl, two heads] \arrow[dr, two heads] & & & \\
 & & Y_1 \arrow[dl, two heads, "g_{13}"'] \arrow[dr, two heads, "g_{12}"] & & Y_3 \arrow[dl, two heads, "g_{32}"'] \arrow[dr, two heads, "g_{31}"] & & \\
 & S_3 \arrow[dl, two heads, "h_{32}"] \arrow[dr, two heads, "h_{31}"'] & \square & S_2 \arrow[dl, two heads, "h_{21}"] \arrow[dr, two heads, "h_{23}"'] & \square & S_1 \arrow[dl, two heads, "h_{12}"] \arrow[dr, two heads, "h_{13}"'] & \\
E_2 & & E_1 & & E_3 & & E_2
\end{tikzcd}$$

where the squares are birationally cartesian.

(3) *The threefold Y is dominated by a product of three curves.* After passing to an étale cover, we may and will assume, from now on, that *all the irreducible components of $V_0(\omega_X)$ pass through* 0.

(4) *We have $V_0(\omega_X) = (\widehat{E}_1 \times \widehat{E}_2 \times \{0\}) \cup (\widehat{E}_1 \times \{0\} \times \widehat{E}_3) \cup (\{0\} \times \widehat{E}_2 \times \widehat{E}_3)$.* We already know that $V_0(\omega_X)$ contains the right-hand-side and we must prove that it has no other components.

(5) *The morphism $g : X \twoheadrightarrow Y$ is birational.*

(6) *The groups G_i are all isomorphic to* $\mathbf{Z}/2\mathbf{Z}$*.* The idea is that if they had more than two elements, it would produce extra components of $V_0(\omega_X)$.

□

7. Volume and Euler characteristic

We end this talk with an exploration, for a variety X of general type, of possible relationships between the volume $\operatorname{vol}(\omega_X)$ of the canonical bundle and $\chi(X, \omega_X)$. We can prove the following.

Proposition 7.1. *Let X be a smooth projective variety of general type and of mAd. We have*

$$\operatorname{vol}(\omega_X) \geq \chi(X, \omega_X).$$

Because of the Ein-Lazarsfeld examples, there cannot exist any reverse inequality of the type $\chi(X, \omega_X) \geq a_n \operatorname{vol}(\omega_X)$ in general (with $a_n > 0$). However, one may ask the following question.

Question 7.2. Are there positive constants a_n such that, for any smooth projective variety X of dimension n with ω_X ample, we have

$$\chi(X, \omega_X) \geq a_n \omega_X^n?$$

With the additional hypothesis that Ω_X be nef, the weaker inequality $\chi(X, \omega_X) > 0$ was conjectured by Q. Zhang. He also showed that if $\dim(X) = 4$, Ω_X is *nef,* and ω_X is ample, then $\chi(X, \omega_X) \geq \frac{1}{1200}\omega_X^4$.

Proposition 7.3 (Q. Zhang)**.** *Question 7.2 has an affirmative answer in dimensions* ≤ 3.

Proof. Let X be a surface of general type. Noether's formula $\chi(X,\omega_X) = \frac{1}{12}(c_1^2 + c_2)$ and Miyaoka-Yau's inequality $c_2 \geq \frac{1}{3}c_1^2$ imply $\chi(X,\omega_X) \geq \frac{1}{9}\omega_X^2$.

In dimension 3, we have $\chi(X,\omega_X) = \frac{c_1c_2}{24}$ by Riemann-Roch, and Yau's inequality $c_1c_2 \geq \frac{3}{8}c_1^3$ implies $\chi(X,\omega_X) \geq \frac{1}{64}\omega_X^3$. ☐

Note that any smooth projective variety of general type with a *finite* map to an abelian variety contains no rational curves, hence has ample canonical bundle. In particular, a positive answer to Question 7.2 would imply that any smooth projective variety X of general type with a *finite* map to an abelian variety satisfies $\chi(X,\omega_X) > 0$. Proposition 7.3 shows that this is the case in dimensions ≤ 3 (see also Theorem 4.1).