Rational curves on algebraic varieties

Lecture notes for the GAeL XVIII conference
Coimbra, Portugal – June 6-11, 2010

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November 14, 2020

Contents

Chapter 1

Rational curves

1.1 Exceptional locus of a morphism

A *variety* is an integral and separated scheme of finite type over a field $\mathbf{k}$. If Y and X are varieties, a morphism $f : Y \to X$ is *birational* if there exists a dense open subset $U \subset X$ such that f induces an isomorphism between $f^{-1}(U)$ and U. The *exceptional locus* $\mathrm{Exc}(f)$ of f is $Y - f^{-1}(U)$, where U is the largest such open subset of X. If X is *normal,* Zariski's Main Theorem says that the fibers of the induced morphism $Y - f^{-1}(U) \to X - U$ are connected and everywhere positive-dimensional. In particular, $X - U$ has codimension ≥ 2 in X. If X is *smooth* (or more generally normal and locally $\mathbf{Q}$-factorial), every component of $\mathrm{Exc}(f)$ has codimension 1 in Y.

1.2 Rational curves and birational morphisms

A lot of the birational geometry of a smooth projective variety depends on how many rational curves it contains (later on, we will give several definitions to quantify what we mean by " how many"). Many tools have been introduced to study varieties with many rational curves, and they have had several striking consequences in algebraic and arithmetic geometry (see Chapter 4). We begin with a classical result which illustrates this principle.

Proposition 1.1 *Let X and Y be projective varieties, with X smooth, and let $f : Y \to X$ be a birational morphism which is not an isomorphism. Through a general point of each component of* $\mathrm{Exc}(f)$*, there exists a rational curve contracted by f.*

When X and Y are surfaces, the proposition follows immediately from the classical fact that (upon replacing Y with a desingularization) f is a composition of blow-ups of points.

PROOF. Set $E = \mathrm{Exc}(f)$. Upon replacing Y with its normalization, we may assume that Y is smooth in codimension 1.

Each component of E has codimension 1 (§1.1) hence, by shrinking Y, we may assume that Y and E are smooth and irreducible. Set $U_0 = X - \mathrm{Sing}(\overline{f(E)})$, so that the closure in U_0 of the image of $E \cap f^{-1}(U_0)$ is smooth, of codimension at least 2. Let $\varepsilon_1 : X_1 \to U_0$ be its blow-up; by the universal property of blow-ups ([H1], Proposition II.7.14), there exists a factorization

$$f|_{V_1} : V_1 \xrightarrow{f_1} X_1 \xrightarrow{\varepsilon_1} U_0 \subset X$$

where the complement of $V_1 = f^{-1}(U_0)$ in Y has codimension at least 2 and $\overline{f_1(E \cap V_1)}$ is contained in the support of the exceptional divisor of ε_1. If the codimension of $\overline{f_1(E \cap V_1)}$ in X_1 is at least 2, the divisor $E \cap V_1$ is contained in the exceptional locus of f_1 and, upon replacing V_1 by the complement V_2 of a closed subset of codimension at least 2 and X_1 by an open subset U_1, we may repeat the construction. After i steps, we get a factorization

$$f : V_i \xrightarrow{f_i} X_i \xrightarrow{\varepsilon_i} U_{i-1} \subset X_{i-1} \xrightarrow{\varepsilon_{i-1}} \cdots \xrightarrow{\varepsilon_2} U_1 \subset X_1 \xrightarrow{\varepsilon_1} U_0 \subset X$$

as long as the codimension of $\overline{f_{i-1}(E \cap V_{i-1})}$ in X_{i-1} is at least 2, where V_i is the complement in Y of a closed subset of codimension at least 2. Let $E_j \subset X_j$ be the exceptional divisor of ε_j. We have

$$\begin{aligned} K_{X_i} &= \varepsilon_i^* K_{U_{i-1}} + c_i E_i \\ &= (\varepsilon_1 \circ \cdots \circ \varepsilon_i)^* K_X + c_i E_i + c_{i-1} E_{i,i-1} + \cdots + c_1 E_{i,1}, \end{aligned}$$

where $E_{i,j}$ is the inverse image of E_j in X_i and

$$c_i = \mathrm{codim}_{X_{i-1}}(\overline{f_{i-1}(E \cap V_{i-1})}) - 1 > 0.$$

Since f_i is birational, $f_i^* \mathscr{O}_{X_i}(K_{X_i})$ is a subsheaf of $\mathscr{O}_{V_i}(K_{V_i})$. Moreover, since $f_j(E \cap V_j)$ is contained in the support of E_j, the divisor $f_j^* E_j - E|_{V_j}$ is effective, hence so is $E_{i,j} - E|_{V_i}$.

It follows that $\mathscr{O}_Y(f^* K_X + (c_i + \cdots + c_1)E)|_{V_i}$ is a subsheaf of $\mathscr{O}_{V_i}(K_{V_i}) = \mathscr{O}_Y(K_Y)|_{V_i}$. Since Y is normal and the complement of V_i in Y has codimension at least 2, $\mathscr{O}_Y(f^* K_X + (c_i + \cdots + c_1)E)$ is also a subsheaf of $\mathscr{O}_Y(K_Y)$. Since there are no infinite ascending sequences of subsheaves of a coherent sheaf on a Noetherian scheme, the process must terminate at some point: $\overline{f_i(E \cap V_i)}$ is a divisor in X_i for some i, hence $E \cap V_i$ is not contained in the exceptional locus of f_i (by §1.1 again). The morphism f_i then induces a birational isomorphism between $E \cap V_i$ and E_i, and the latter is ruled: more precisely, through every point of E_i there is a rational curve contracted by ε_i. This proves the proposition. □

Corollary 1.2 *Let Y and X be projective varieties. Assume that X is smooth and that Y contains no rational curves. Any rational map $X \dashrightarrow Y$ is defined everywhere.*

PROOF. Let $X' \subset X \times Y$ be the graph of a rational map $u : X \dashrightarrow Y$. The first projection induces a birational morphism $f : X' \to X$. Assume its exceptional locus $\operatorname{Exc}(f)$ is nonempty. By Proposition 1.1, there exists a rational curve on $\operatorname{Exc}(f)$ which is contracted by p. Since Y contains no rational curves, it must also be contracted by the second projection, which is absurd since it is contained in $X \times Y$. Hence $\operatorname{Exc}(f)$ is empty and u is defined everywhere. □

Corollary 1.3 *Let X be a smooth projective variety which contains no rational curve. Any birational automorphism of X is an automorphism of X.*

1.3 Parametrizing rational curves

Let $\mathbf{k}$ be a field. Any $\mathbf{k}$-morphism f from $\mathbf{P}^1_{\mathbf{k}}$ to $\mathbf{P}^N_{\mathbf{k}}$ can be written as

$$f(u,v) = (F_0(u,v), \dots, F_N(u,v)) \tag{1.1}$$

where $F_0, \dots, F_N$ are homogeneous polynomials in two variables, of the same degree d, with no nonconstant common factor in $\mathbf{k}[U,V]$ (or, equivalently, with no nonconstant common factor in $\bar{\mathbf{k}}[U,V]$, where $\bar{\mathbf{k}}$ is an algebraic closure of $\mathbf{k}$).

We are going to show that there exist universal integral polynomials in the coefficients of $F_0, \dots, F_N$ which vanish if and only if they have a nonconstant common factor in $\bar{\mathbf{k}}[U,V]$, i.e., a nontrivial common zero in $\mathbf{P}^1_{\bar{\mathbf{k}}}$. By the Nullstellensatz, the opposite holds if and only if the ideal generated by $F_0, \dots, F_N$ in $\bar{\mathbf{k}}[U,V]$ contains some power of the maximal ideal (U,V). This in turn means that for some m, the map

$$\begin{array}{ccc} (\bar{\mathbf{k}}[U,V]_{m-d})^{N+1} & \longrightarrow & \bar{\mathbf{k}}[U,V]_m \\ (G_0, \dots, G_N) & \longmapsto & \sum_{j=0}^N F_j G_j \end{array}$$

is surjective, hence of rank $m+1$ (here $\mathbf{k}[U,V]_m$ is the vector space of homogeneous polynomials of degree m). This map being linear and defined over $\mathbf{k}$, we conclude that $F_0, \dots, F_N$ have a nonconstant common factor in $\mathbf{k}[U,V]$ if and only if, for all m, all $(m+1)$-minors of some universal matrix whose entries are linear integral combinations of the coefficients of the F_i vanish. This defines a Zariski closed subset of the projective space $\mathbf{P}((\operatorname{Sym}^d \mathbf{k}^2)^{N+1})$, *defined over* $\mathbf{Z}$.

Therefore, morphisms of degree d from $\mathbf{P}^1_{\mathbf{k}}$ to $\mathbf{P}^N_{\mathbf{k}}$ are parametrized by a Zariski open set of the projective space $\mathbf{P}((\operatorname{Sym}^d \mathbf{k}^2)^{N+1})$; we denote this quasi-projective variety $\operatorname{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}})$. Note that these morphisms fit together into a *universal morphism*

$$\begin{array}{rccc} f^{\mathrm{univ}}: & \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}}) & \longrightarrow & \mathbf{P}^N_{\mathbf{k}} \\ & \big((u,v), f\big) & \longmapsto & \big(F_0(u,v), \dots, F_N(u,v)\big). \end{array}$$

Finally, morphisms from $\mathbf{P}^1_{\mathbf{k}}$ to $\mathbf{P}^N_{\mathbf{k}}$ are parametrized by the disjoint union

$$\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}}) = \bigsqcup_{d \geq 0} \mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}})$$

of quasi-projective $\mathbf{k}$-schemes.

Let now X be a (closed) subscheme of $\mathbf{P}^N_{\mathbf{k}}$ defined by homogeneous equations $G_1, \dots, G_m$. Morphisms of degree d from $\mathbf{P}^1_{\mathbf{k}}$ to X are parametrized by the subscheme $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X)$ of $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}})$ defined by the equations

$$G_j(F_0, \dots, F_N) = 0 \qquad \text{for all} \quad j \in \{1, \dots, m\}.$$

Again, morphisms from $\mathbf{P}^1_{\mathbf{k}}$ to X are parametrized by the disjoint union

$$\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X) = \bigsqcup_{d \geq 0} \mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X)$$

of quasi-projective $\mathbf{k}$-schemes. The same conclusion holds for any quasi-projective $\mathbf{k}$-variety X: embed X into some projective variety $\overline{X}$; there is a universal morphism

$$f^{\mathrm{univ}} : \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, \overline{X}) \longrightarrow \overline{X}$$

and $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ is the complement in $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, \overline{X})$ of the image by the (proper) second projection of the closed subscheme $(f^{\mathrm{univ}})^{-1}(\overline{X} - X)$.

If now X can be defined by homogeneous equations $G_1, \dots, G_m$ with coefficients in a subring R of $\mathbf{k}$, the scheme $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X)$ has the same property. If $\mathfrak{m}$ is a maximal ideal of R, one may consider the reduction $X_{\mathfrak{m}}$ of X modulo $\mathfrak{m}$: this is the subscheme of $\mathbf{P}^N_{R/\mathfrak{m}}$ defined by the reductions of the G_j modulo $\mathfrak{m}$. Because the equations defining the complement of $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}})$ in $\mathbf{P}((\mathrm{Sym}^d \mathbf{k}^2)^{N+1})$ are defined over $\mathbf{Z}$ and the same for all fields, $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X_{\mathfrak{m}})$ is the reduction of the R-scheme $\mathrm{Mor}_d(\mathbf{P}^1, X)$ modulo $\mathfrak{m}$. In fancy terms, one may express this as follows: if $\mathscr{X}$ is a scheme over $\mathrm{Spec}\, R$, the R-morphisms $\mathbf{P}^1_R \to \mathscr{X}$ are parametrized by (the R-points of) a locally Noetherian scheme

$$\mathrm{Mor}(\mathbf{P}^1_R, \mathscr{X}) \to \mathrm{Spec}\, R$$

and the fiber of a closed point $\mathfrak{m}$ is the space $\mathrm{Mor}(\mathbf{P}^1_{R/\mathfrak{m}}, \mathscr{X}_{\mathfrak{m}})$.

1.4 Parametrizing morphisms

1.4. The space $\mathrm{Mor}(Y, X)$**.** If X and Y are varieties defined over a field $\mathbf{k}$, with X quasi-projective and Y projective, Grothendieck showed ([G1], 4.c) that

k-morphisms from Y to X are parametrized by a locally Noetherian **k**-scheme $\mathrm{Mor}(Y,X)$. As we saw in the case $Y = \mathbf{P}^1_{\mathbf{k}}$ and $X = \mathbf{P}^N_{\mathbf{k}}$, this scheme will in general have countably many components. One way to remedy that is to fix an ample divisor H on X and a polynomial P with rational coefficients: the subscheme $\mathrm{Mor}_P(Y,X)$ of $\mathrm{Mor}(Y,X)$ which parametrizes morphisms $f: Y \to X$ with fixed *Hilbert polynomial*

$$P(m) = \chi(Y, mf^*H)$$

is now quasi-projective over **k**, and $\mathrm{Mor}(Y,X)$ is the disjoint (countable) union of the $\mathrm{Mor}_P(Y,X)$, for all polynomials P. Note that when Y is a curve, fixing the Hilbert polynomial amounts to fixing the degree of the 1-cycle f_*Y for the embedding of X defined by some multiple of H.

Let us make more precise this notion of *parameter space*. We ask as above that there be a *universal morphism* (also called *evaluation map*)

$$f^{\mathrm{univ}} : Y \times \mathrm{Mor}(Y,X) \to X$$

such that for any **k**-scheme T, the correspondance between

- morphisms $\varphi : T \to \mathrm{Mor}(Y,X)$ and
- morphisms $f : Y \times T \to X$

obtained by sending φ to

$$f(y,t) = f^{\mathrm{univ}}(y, \varphi(t))$$

is one-to-one.

1.5. The tangent space to $\mathrm{Mor}(Y,X)$**.** We will use the universal property to determine the Zariski tangent space to $\mathrm{Mor}(Y,X)$ at a **k**-point $[f]$. This vector space parametrizes morphisms from $\mathrm{Spec}\,\mathbf{k}[\varepsilon]/(\varepsilon^2)$ to $\mathrm{Mor}(Y,X)$ with image $[f]$, hence extensions of f to morphisms

$$f_\varepsilon : Y \times \mathrm{Spec}\,\mathbf{k}[\varepsilon]/(\varepsilon^2) \to X$$

which should be thought of as first-order infinitesimal deformations of f.

Proposition 1.6 *Let X and Y be varieties defined over a field* **k***, with X quasi-projective and Y projective, let $f : Y \to X$ be a* **k***-morphism, and let $[f]$ be the corresponding* **k***-point of* $\mathrm{Mor}(Y,X)$*. One has*

$$T_{\mathrm{Mor}(Y,X),[f]} \simeq H^0(Y, \mathscr{H}om(f^*\Omega_X, \mathscr{O}_Y)).$$

In particular, when X is smooth along the image of f,

$$T_{\mathrm{Mor}(Y,X),[f]} \simeq H^0(Y, f^*T_X).$$

Proof. Assume first that Y and X are affine and write $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$ (where A and B are finitely generated $\mathbf{k}$-algebras). Let $f^\sharp : A \to B$ be the morphism corresponding to f, making B into an A-algebra; we are looking for $\mathbf{k}$-algebra homomorphisms $f^\sharp_\varepsilon : A \to B[\varepsilon]$ of the type

$$\forall a \in A \quad f^\sharp_\varepsilon(a) = f(a) + \varepsilon g(a).$$

The equality $f^\sharp_\varepsilon(aa') = f^\sharp_\varepsilon(a) f^\sharp_\varepsilon(a')$ is equivalent to

$$\forall a, a' \in A \quad g(aa') = f^\sharp(a) g(a') + f^\sharp(a') g(a).$$

In other words, $g : A \to B$ is a $\mathbf{k}$-derivation of the A-module B, hence factors as $g : A \to \Omega_A \to B$ ([H1], §II.8). Such extensions are therefore parametrized by $\mathrm{Hom}_A(\Omega_A, B) = \mathrm{Hom}_B(\Omega_A \otimes_A B, B)$.

In general, cover X by affine open subsets $U_i = \mathrm{Spec}(A_i)$ and Y by affine open subsets $V_i = \mathrm{Spec}(B_i)$ such that $f(V_i)$ is contained in U_i. First-order extensions of $f|_{V_i} : V_i \to U_i$ are parametrized by

$$g_i \in \mathrm{Hom}_{B_i}(\Omega_{A_i} \otimes_{A_i} B_i, B_i) = H^0(V_i, \mathscr{H}om(f^*\Omega_X, \mathscr{O}_Y)).$$

To glue these, we need the compatibility condition

$$g_i|_{V_i \cap V_j} = g_j|_{V_i \cap V_j},$$

which is exactly saying that the g_i define a global section on Y. □

1.7. Parametrizing morphisms vs. parametrizing subvarieties. Grothendieck's construction of the scheme $\mathrm{Mor}(Y, X)$ is a consequence of his construction of the Hilbert scheme $\mathrm{Hilb}(Z)$ which parametrizes subschemes of a fixed projective scheme Z (a morphism $Y \to X$ is identified with its graph in $Y \times X$).

Let Y be a subscheme of a projective scheme Z, with ideal sheaf $\mathscr{I}_Y$, and let $N_{Y/Z} := \mathscr{H}om(\mathscr{I}_Y/\mathscr{I}_Y^2, \mathscr{O}_Y)$ be the normal sheaf to Y in Z. One can show as above that the Zariski tangent space to $\mathrm{Hilb}(Z)$ at $[Y]$ is isomorphic to $H^0(Y, N_{Y/Z})$.

The canonical exact sequence

$$\mathscr{I}_Y/\mathscr{I}_Y^2 \to f^*\Omega_Z \to \Omega_Y \to 0$$

dualizes to

$$0 \to \mathscr{H}om(\Omega_Y, \mathscr{O}_Y) \to \mathscr{H}om(f^*\Omega_Z, \mathscr{O}_Y) \to N_{Y/Z},$$

which yields

$$0 \to H^0(Y, \mathscr{H}om(\Omega_Y, \mathscr{O}_Y)) \to H^0(Y, \mathscr{H}om(f^*\Omega_Z, \mathscr{O}_Y)) \to H^0(Y, N_{Y/Z}).$$

By Proposition 1.6, the Zariski tangent space to $\mathrm{Aut}(Y)$ at $[\mathrm{Id}_Y]$ is isomorphic to the leftmost space in this sequence, and the maps in the sequence are the tangent maps to the natural maps

$$\begin{array}{rcccl} \mathrm{Aut}(Y) & \longrightarrow & \mathrm{Mor}(Y,Z) & \longrightarrow & \mathrm{Hilb}(Z) \\ g & \longmapsto & f\circ g,\ h & \longmapsto & h(Y). \end{array}$$

1.8. The local structures of $\mathrm{Mor}(Y,X)$ and $\mathrm{Hilb}(Z)$. Our next result provides a lower bound for the dimension of $\mathrm{Mor}(Y,X)$ at a point $[f]$, thereby allowing us in certain situations to produce many deformations of f. This lower bound is very accessible, via the Riemann-Roch theorem, when Y is a curve (see 1.11).

Theorem 1.9 *Let X and Y be varieties defined over a field $\mathbf{k}$, with Y projective and X quasi-projective, and let $f: Y \to X$ be a $\mathbf{k}$-morphism such that X is smooth along $f(Y)$. Locally around $[f]$, the scheme $\mathrm{Mor}(Y,X)$ can be defined by $h^1(Y, f^*T_X)$ equations in a smooth scheme of dimension $h^0(Y, f^*T_X)$. In particular, any (geometric) irreducible component of $\mathrm{Mor}(Y,X)$ through $[f]$ has dimension at least*

$$h^0(Y, f^*T_X) - h^1(Y, f^*T_X).$$

In particular, under the hypotheses of the theorem, a sufficient condition for $\mathrm{Mor}(Y,X)$ to be smooth at $[f]$ is $H^1(Y, f^*T_X) = 0$. We will give in 1.13 an example that shows that this condition is not necessary.

For the proof of the theorem, I refer to [D], Theorem 2.6. There is a similar result for the Hilbert scheme: *if Y is a locally complete intersection subscheme of a smooth projective k-variety Z, with (locally free) normal sheaf $N_{Y/Z}$, locally around $[Y]$, the scheme $\mathrm{Hilb}(Z)$ can be defined by $h^1(Y, N_{Y/Z})$ equations in a smooth scheme of dimension $h^0(Y, N_{Y/Z})$* ([S], Theorem 4.3.5).

1.5 Parametrizing morphisms with extra structure

1.10. Morphisms with fixed points. We will need a slightly more general situation: fix a finite subset $B = \{y_1, \ldots, y_r\}$ of Y and points $x_1, \ldots, x_r$ of X; then morphisms $f: Y \to X$ which map each y_i to x_i are parametrized by a subscheme $\mathrm{Mor}(Y, X; y_i \mapsto x_i)$ of $\mathrm{Mor}(Y,X)$ which is just the fiber over $(x_1, \ldots, x_r)$ of the map

$$\begin{array}{rcl} \rho : \mathrm{Mor}(Y,X) & \longrightarrow & X^r \\ [f] & \longmapsto & (f(y_1), \ldots, f(y_r)) \end{array}$$

By Theorem 1.9, *its irreducible components at* $[f]$ *are therefore all of dimension at least*

$$h^0(Y, f^*T_X) - h^1(Y, f^*T_X) - r\dim(X).$$

The tangent map to ρ at $[f]$ is the evaluation

$$H^0(Y, f^*T_X) \to \bigoplus_{i=1}^{r} (f^*T_X)_{y_i} \simeq \bigoplus_{i=1}^{r} T_{X,x_i}.$$

The tangent space to $\mathrm{Mor}(Y, X; y_i \mapsto x_i)$ at $[f]$ is therefore isomorphic to $H^0(Y, f^*T_X \otimes \mathscr{I}_{y_1,\dots,y_r})$ One can show that if X is smooth along $f(Y)$, *the scheme* $\mathrm{Mor}(Y, X; y_i \mapsto x_i)$ *can be defined by* $h^1(Y, f^*T_X \otimes \mathscr{I}_{y_1,\dots,y_r})$ *equations in a smooth scheme of dimension* $h^0(Y, f^*T_X \otimes \mathscr{I}_{y_1,\dots,y_r})$.

1.11. Morphisms from a curve. Everything takes a particularly simple form when Y is a curve C: for any $f : C \to X$ and $c_1, \dots, c_r \in C$, one has by Riemann-Roch

$$\begin{aligned} \dim_{[f]} \mathrm{Mor}(C, X; c_i \mapsto f(c_i)) &\geq \chi(C, f^*T_X) - r\dim(X) \\ &= (-K_X \cdot f_*C) + (1 - g(C) - r)\dim(X), \end{aligned} \tag{1.2}$$

where $g(C) = 1 - \chi(C, \mathscr{O}_C)$.

1.12. Relative situation. All this can be done over an irreducible Noetherian base scheme S ([M]; [K1], Theorem II.1.7): if $Y \to S$ is a projective flat S-scheme, with a subscheme $B \subset Y$ finite and flat over S, and $X \to S$ is a quasi-projective S-scheme with an S-morphism $g : B \to X$, the S-morphisms from Y to X that restrict to g on B can be parametrized by a locally Noetherian S-scheme $\mathrm{Mor}_S(Y, X; g)$. The universal property implies in particular that for any point s of S, one has

$$\mathrm{Mor}_S(Y, X; g)_s \simeq \mathrm{Mor}(Y_s, X_s; g_s).$$

In other words, the schemes $\mathrm{Mor}(Y_s, X_s; g_s)$ fit together to form a scheme over S ([M], Proposition 1, and [K1], Proposition II.1.5).

When moreover Y is a relative reduced curve C over S, with geometrically reduced fibers, and X is smooth over S, given a point s of S and a morphism $f : C_s \to X_s$ which coincides with g_s on B_s, we have

$$\begin{aligned} \dim_{[f]} \mathrm{Mor}_S(C, X; g) &\geq \chi(C_s, f^*T_{X_s} \otimes \mathscr{I}_{B_s}) + \dim(S) \\ &= (-K_{X_s} \cdot f_*C_s) + (1 - g(C_s) - \lg(B_s))\dim(X_s) + \dim(S). \end{aligned} \tag{1.3}$$

Furthermore, if $H^1(C_s, f^*T_{X_s} \otimes \mathscr{I}_{B_s})$ vanishes, $\mathrm{Mor}_S(C, X; g)$ is smooth over S at $[f]$ ([K1], Theorem II.1.7).

The situation is similar for Hilbert schemes (and as in the absolute situation, the morphism case reduces to this case by considering graphs): if $Z \to S$ is a projective morphism, subschemes of Z which are proper and flat over S can be parametrized by a locally Noetherian projective S-scheme $\mathrm{Hilb}_S(Z)$. For any point s of S, one has

$$\mathrm{Hilb}_S(Z)_s \simeq \mathrm{Hilb}(Z_s).$$

In other words, the schemes $\mathrm{Hilb}(Z_s)$ fit together to form a scheme over S.

1.6 Lines on a subvariety of a projective space

We will describe lines on complete intersections in a projective space *over an algebraically closed field* $\mathbf{k}$ to illustrate the concepts developed above.

Let X be a subvariety of $\mathbf{P}^N$ of dimension n. By associating its image to a rational curve, we define a morphism

$$\mathrm{Mor}_1(\mathbf{P}^1_{\mathbf{k}}, X) \to G(1, \mathbf{P}^N),$$

where $G(1, \mathbf{P}^N)$ is the Grassmannian of lines in $\mathbf{P}^N$ (this is a particular case of 1.7). Its image parametrizes lines in X; it has a natural scheme structure and we will denote it by $F(X)$. By 1.7, the tangent space to $F(X)$ at $[\ell]$ is $H^0(\ell, N_{\ell/X})$.

Similarly, given a point x on X, we let $F(X,x)$ be the subscheme of $F(X)$ consisting of lines passing through x and contained in X. Lines through x are parametrized by a hyperplane in $\mathbf{P}^N$ of which $F(X,x)$ is a subscheme. The tangent space to $F(X,x)$ at $[\ell]$ is isomorphic to $H^0(\ell, N_{\ell/X}(-1))$.

There is an exact sequence of normal bundles

$$0 \to N_{\ell/X} \to \mathscr{O}_\ell(1)^{\oplus(N-1)} \to (N_{X/\mathbf{P}^N})|_\ell \to 0. \tag{1.4}$$

Since any locally free sheaf on $\mathbf{P}^1_{\mathbf{k}}$ is isomorphic to a direct sum of invertible sheaf, we can write

$$N_{\ell/X} \simeq \bigoplus_{i=2}^{n} \mathscr{O}_\ell(a_i), \tag{1.5}$$

where $a_2 \geq \cdots \geq a_n$. By (1.4), we have $a_2 \leq 1$. In particular, if $f : \ell \to X$ is the inclusion,

$$f^*T_X \simeq \mathscr{O}_\ell(2) \oplus \bigoplus_{i=2}^{n} \mathscr{O}_\ell(a_i). \tag{1.6}$$

If $a_n \geq -1$, the scheme $\mathrm{Mor}(\ell, X)$ is smooth at $[f]$, hence $F(X)$ is smooth at $[\ell]$ (Theorem 1.9). Similarly, if $a_n \geq 0$, the scheme $F(X,x)$ is smooth at $[\ell]$ for any point x on ℓ (see 1.10).

1.13. Fermat hypersurfaces. The Fermat hypersurface X_N^d is the hypersurface in $\mathbf{P}^N$ defined by the equation

$$x_0^d + \cdots + x_N^d = 0.$$

It is smooth if and only if the characteristic p of $\mathbf{k}$ does not divide d. *Assume* $p>0$ *and* $d=p^r+1$ for some $r>0$. The line joining two points x and y is contained in X_N^d if and only if

$$\begin{aligned} 0 &= \sum_{j=0}^{N} (x_j+ty_j)^{p^r+1} \\ &= \sum_{j=0}^{N} (x_j^{p^r}+t^{p^r}y_j^{p^r})(x_j+ty_j) \\ &= \sum_{j=0}^{N} (x_j^{p^r+1}+tx_j^{p^r}y_j+t^{p^r}x_jy_j^{p^r}+t^{p^r+1}y_j^{p^r+1}) \end{aligned}$$

for all $t\in\bar{\mathbf{k}}$. It follows that the scheme

$$\{(x,y)\in X\times X \mid \langle x,y\rangle\subset X\}$$

is defined by the two equations

$$0=\sum_{j=0}^{n+1} x_j^{p^r}y_j=\Big(\sum_{j=0}^{n+1} x_j^{p^{-r}}y_j\Big)^{p^r}$$

in $X\times X$, hence has everywhere dimension $\geq 2N-4$. Since this scheme (minus the diagonal of $X\times X$) is fibered over $F(X_N^d)$ with fibers $\mathbf{P}^1_{\mathbf{k}}\times\mathbf{P}^1_{\mathbf{k}}$ (minus the diagonal), it follows that $F(X_N^d)$ has everywhere dimension $\geq 2N-6$. With the notation of (1.5), this implies

$$2N-6\leq \dim(T_{F(X_N^d),[\ell]})=h^0(\ell,N_{\ell/X_N^d})=\dim\sum_{a_i\geq 0}(a_i+1). \tag{1.7}$$

Since $a_i\leq 1$ and $a_1+\cdots+a_{N-2}=N-1-d$ by (1.4), the only possibility is, when $d\geq 4$,

$$N_{\ell/X_N^d}\simeq \mathscr{O}_\ell(1)^{\oplus(N-3)}\oplus\mathscr{O}_\ell(2-d)$$

and there is equality in (1.7). It follows that $F(X_N^d)$ *is everywhere smooth of dimension* $2N-6$*, although* $H^1(\ell,N_{\ell/X_N^d})$ *is nonzero.* Considering parametrizations of these lines, we get an example of a scheme $\mathrm{Mor}_1(\mathbf{P}^1_{\mathbf{k}},X_N^d)$ *smooth at all points* $[f]$ *although* $H^1(\mathbf{P}^1_{\mathbf{k}},f^*T_{X_N^d})$ *never vanishes.*

The scheme

$$\{(x,[\ell])\in X\times F(X_N^d)\mid x\in\ell\}$$

is therefore smooth of dimension $2N-5$, hence the fiber $F(X_N^d,x)$ of the first projection has dimension $N-4$ for x general in X.[1] On the other hand, the

[1] This is actually true for all $x\in X$.

calculation above shows that the scheme $F(X_N^d, x)$ is defined (in some fixed hyperplane not containing x) by the three equations

$$0 = \sum_{j=0}^{n+1} x_j^{p^r} y_j = \Big(\sum_{j=0}^{n+1} x_j^{p^{-r}} y_j\Big)^{p^r} = \sum_{j=0}^{n+1} y_j^{p^r+1}.$$

It is clear from these equations that the tangent space to $F(X_N^d, x)$ at every point has dimension $\geq N-3$. For $N \geq 4$, it follows that for x general in X, the scheme $F(X_N^d, x)$ *is nowhere reduced* and similarly, $\mathrm{Mor}_1(\mathbf{P}^1_{\mathbf{k}}, X_N^d; 0 \mapsto x)$ *is nowhere reduced.*

1.7 Bend-and-break

The title of this section refers to a series of results (originated by Mori; [M], Theorems 5 and 6) that say that a curve deforming nontrivially while keeping some points fixed must break into an effective 1-cycle with a rational component passing through one of the fixed points. We only present the here the strongest of these results.

Proposition 1.14 *Let X be a projective variety and let H be an ample Cartier divisor on X. Let $f : C \to X$ be a smooth curve and let B be a finite nonempty subset of C such that*

$$\dim_{[f]} \mathrm{Mor}(C, X; B \mapsto f(B)) \geq 1.$$

There exists a rational curve Γ on X which meets $f(B)$ and such that

$$(H \cdot \Gamma) \leq \frac{2(H \cdot f_*C)}{\mathrm{Card}(B)}.$$

According to (1.2), *when X is smooth along $f(C)$*, the hypothesis is fulfilled whenever

$$(-K_X \cdot f_*C) + (1 - g(C) - \mathrm{Card}(B)) \dim(X) \geq 1.$$

The proof actually shows that there exist a morphism $f' : C \to X$ and a nonzero effective rational 1-cycle Z on X such that

$$f_*C \sim_{\mathrm{num}} f'_*C + Z,$$

one component of which meets $f(B)$ and satisfies the degree condition above.

Proof. Set $B = \{c_1, \dots, c_b\}$. Let C' be the normalization of $f(C)$. If C' is rational and f has degree $\geq b/2$ onto its image, just take $\Gamma = C'$. From now on, we will assume that if C' is rational, f has degree $< b/2$ onto its image.

By 1.10, the dimension of the space of morphisms from C to $f(C)$ that send B to $f(B)$ is at most $h^0(C, f^*T_{C'} \otimes \mathscr{I}_B)$. When C' is irrational, $f^*T_{C'} \otimes \mathscr{I}_B$ has negative degree, and, under our assumption, this remains true when C' is rational. In both cases, the space is therefore 0-dimensional, hence any 1-dimensional subvariety of $\mathrm{Mor}(C, X; B \mapsto f(B))$ through $[f]$ corresponds to morphisms with varying images.

Let T be the normalization of such a subvariety and let $\overline{T}$ be a smooth compactification of T. The indeterminacies of the rational map

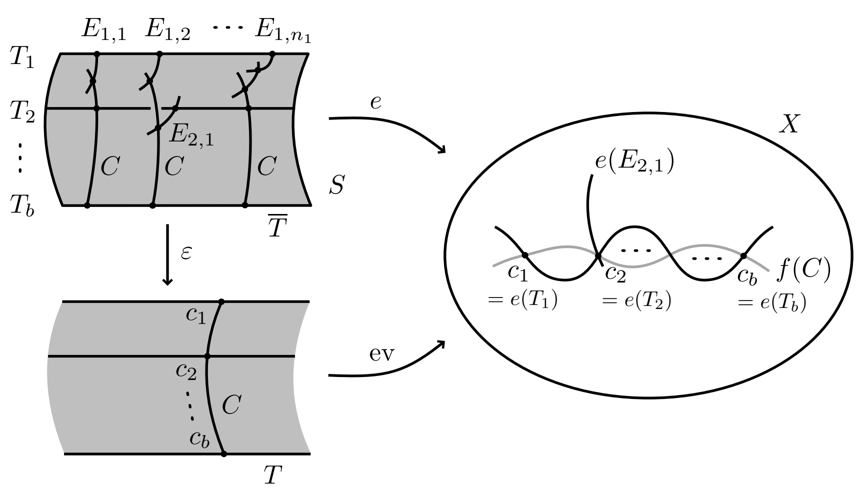
$$\mathrm{ev} : C \times \overline{T} \dashrightarrow X$$

coming from the morphism $T \to \mathrm{Mor}(C, X; B \mapsto f(B))$ can be resolved by blowing up points to get a morphism

$$e : S \xrightarrow{\varepsilon} C \times \overline{T} \overset{\mathrm{ev}}{\dashrightarrow} X$$

whose image is a *surface.*

For $i = 1, \dots, b$, we denote by $E_{i,1}, \dots, E_{i,n_i}$ the inverse images on S of the (-1)-exceptional curves that appear every time some point lying over $\{c_i\} \times \overline{T}$ is blown up.



The 1-cycle f_*C bends and breaks keeping $c_1, \dots, c_b$ fixed.

We have

$$(E_{i,j} \cdot E_{i',j'}) = -\delta_{i,i'}\delta_{j,j'}.$$

Write the strict transform T_i of $\{c_i\} \times \overline{T}$ as

$$T_i \sim_{\mathrm{num}} \varepsilon^*\overline{T} - \sum_{j=1}^{n_i} \varepsilon_{i,j} E_{i,j},$$

where $\varepsilon_{i,j} = (T_i \cdot E_{i,j})$ is 1 if the point blown-up is on the (smooth) strict transform of $\{c_i\} \times \overline{T}$, and 0 if it is not. Write also

$$e^*H \sim_{\text{num}} a\varepsilon^*C + d\varepsilon^*\overline{T} - \sum_{i=1}^{b}\sum_{j=1}^{n_i} a_{i,j}E_{i,j} + G,$$

where G is orthogonal to the $\mathbf{R}$-vector subspace of $N^1(S)_{\mathbf{R}}$ generated by ε^*C, $\varepsilon^*\overline{T}$ and the $E_{i,j}$. Note that e^*H is nef, hence

$$a = (e^*H \cdot \varepsilon^*\overline{T}) \geq 0 \quad , \quad a_{i,j} = (e^*H \cdot E_{i,j}) \geq 0.$$

Since T_i is contracted by e to $f(c_i)$, we have for each i

$$0 = (e^*H \cdot T_i) = a - \sum_{j=1}^{n_i} \varepsilon_{i,j}a_{i,j}.$$

Summing up over i, we get

$$ba = \sum_{i,j} \varepsilon_{i,j}a_{i,j}. \tag{1.8}$$

Moreover, since $(\varepsilon^*C \cdot G) = 0 = ((\varepsilon^*C)^2)$ and ε^*C is nonzero, the Hodge Index Theorem implies $(G^2) \leq 0$, hence (using (1.8))

$$\begin{aligned}
((e^*H)^2) &= 2ad - \sum_{i,j} a_{i,j}^2 + (G^2) \\
&\leq 2ad - \sum_{i,j} a_{i,j}^2 \\
&= \frac{2d}{b}\sum_{i,j} \varepsilon_{i,j}a_{i,j} - \sum_{i,j} a_{i,j}^2 \\
&\leq \frac{2d}{b}\sum_{i,j} \varepsilon_{i,j}a_{i,j} - \sum_{i,j} \varepsilon_{i,j}a_{i,j}^2 \\
&= \sum_{i,j} \varepsilon_{i,j}a_{i,j}\left(\frac{2d}{b} - a_{i,j}\right).
\end{aligned}$$

Since $e(S)$ is a surface, this number is positive, hence there exist indices i_0 and j_0 such that $\varepsilon_{i_0,j_0} > 0$ and $0 < a_{i_0,j_0} < \frac{2d}{b}$.

But $d = (e^*H \cdot \varepsilon^*C) = (H \cdot C)$, and $(e^*H \cdot E_{i_0,j_0}) = a_{i_0,j_0}$ is the H-degree of the rational 1-cycle $e_*(E_{i_0,j_0})$. The latter is nonzero since $a_{i_0,j_0} > 0$, and it passes through $f(c_{i_0})$ since E_{i_0,j_0} meets T_{i_0} (their intersection number is $\varepsilon_{i_0,j_0} = 1$) and the latter is contracted by e to $f(c_{i_0})$. This proves the proposition: take for Γ a component of $e_*E_{i_0,j_0}$ which passes through $f(c_{i_0})$. □

Remark 1.15 The stuation is different on compact complex manifolds. Let E be an elliptic curve, let L be a line bundle on E, and let s and s' be sections of L that generate it at each point. The sections (s, s'), $(is, -is')$, $(s', -s)$ and (is', is) of $L \oplus L$ are independent over $\mathbf{R}$ in each fiber. They generate a discrete subgroup of the total space of $L \oplus L$ and the quotient X is a compact complex threefold with a morphism $\pi : X \to E$ whose fibers are 2-dimensional complex tori. There is a 1-dimensional family of sections $\sigma_t : E \to X$ of π defined by $\sigma_t(x) = (ts(x), 0)$, for $t \in \mathbf{C}$, and they all pass through the points of the zero section where s vanishes. However, X contains no rational curves, because they would have to be contained in a fiber of π, and complex tori contain no rational curves. The variety X is not algebraic, and not even bimeromorphic to a Kähler manifold.

1.8 Rational curves on varieties whose canonical divisor is not nef

Let X be a smooth projective variety defined over an algebraically closed, with a curve $f : C \to X$ such that $(K_X \cdot f_*C) < 0$. We want to use Proposition 1.14 to show that X contains rational curves. For that, we would like to show, using the estimate (1.2), that C deforms nontrivially while keeping points fixed. We only know how to do that in positive characteristic, where the Frobenius morphism allows to increase the degree of f without changing the genus of C. This gives in that case the required rational curve, with a bound on its degree.

Standard arguments then prove that the results still holds over any algebraically closed. They goes roughly as follows. Assume for a moment that X, C, f, H and a point x of C are defined over $\mathbf{Z}$; for almost all prime numbers p, the reduction of X modulo p is a smooth variety hence there is a rational curve (defined over the algebraic closure of $\mathbf{Z}/p\mathbf{Z}$) through x. This means that the scheme $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \to x)$, which is defined over $\mathbf{Z}$, has a geometric point modulo almost all primes p. Since we can moreover bound the degree of the curve by a constant independent of p, we are in fact dealing with a quasi-projective scheme, and this implies that it has a point over $\bar{\mathbf{Q}}$, hence over $\mathbf{k}$. In general, X and x are defined over some finitely generated ring and a similar reasoning yields the existence of a $\mathbf{k}$-point of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \to x)$, i.e., of a rational curve on X through x.

It is important to remark that the "universal" bound on the degree of the rational curve is essential for the proof.

Note that there is no known proof of this theorem that uses only transcendental methods.

Theorem 1.16 (Miyaoka-Mori) *Let X be a projective variety, let H be an ample divisor on X, and let $f : C \to X$ be a smooth curve such that X is* *smooth along* $f(C)$ *and* $(K_X \cdot f_*C) < 0$. *Given any point* x *on* $f(C)$, *there exists a rational curve* Γ *on* X *through* x *with*

$$(H \cdot \Gamma) \leq 2\dim(X)\ \frac{(H \cdot f_*C)}{(-K_X \cdot f_*C)}.$$

It is useful to allow X to be singular. It implies for example that a normal projective variety X with ample ($\mathbf{Q}$-Cartier) anticanonical divisor is covered by rational curves of $(-K_X)$-degree at most $2\dim(X)$: it is *uniruled* in the sense of Definition 2.3.

Also, a simple corollary of this theorem is that the canonical divisor of a smooth projective variety which contains no rational curves is nef.

Proof. The idea is to take b as big as possible in Proposition 1.14, in order to get the lowest possible degree for the rational curve. We first assume that the characteristic of the ground field $\mathbf{k}$ is positive, and use the Frobenius morphism to construct sufficiently many morphisms from C to X.

Assume then that the characteristic of the base field is $p > 0$. Consider the ($\mathbf{k}$-linear) Frobenius morphism $C_1 \to C$;[2] it has degree p, but C_1 and C being isomorphic as abstract schemes have the same genus. Iterating the construction, we get a morphism $F_m : C_m \to C$ of degree p^m between curves of the same genera. Composing f with F_m, we get a morphism $f_m : C_m \to X$ of degree $p^m \deg(f)$ onto its image. For any subset B_m of C_m with b_m elements, we have by 1.11

$$\dim_{[f_m]} \operatorname{Mor}(C_m, X; B_m \mapsto f_m(B_m)) \geq p^m(-K_X \cdot f_*C) + (1 - g(C) - b_m)\dim(X),$$

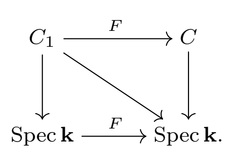
which is positive if we take

$$b_m = \left[\frac{p^m(-K_X \cdot f_*C)}{\dim(X)} - g(C)\right],$$

which is positive for m sufficiently large. This is what we need to apply Proposition 1.14. It follows that there exists a rational curve Γ_m through some point of $f_m(B_m)$, such that

$$(H \cdot \Gamma_m) \leq \frac{2(H \cdot (f_m)_*C_m)}{b_m} = \frac{2p^m}{b_m}(H \cdot f_*C).$$

[2] If $F : \mathbf{k} \to \mathbf{k}$ is the Frobenius morphism, the $\mathbf{k}$-scheme C_1 fits into the Cartesian diagram



In other words, C_1 is the scheme C, but $\mathbf{k}$ acts on $\mathscr{O}_{C_1}$ via pth powers.

As m goes to infinity, p^m/b_m goes to $\dim(X)/(-K_X \cdot f_*C)$. Since the left-hand side is an integer, we get

$$(H \cdot \Gamma_m) \leq \frac{2\dim(X)}{(-K_X \cdot f_*C)}(H \cdot f_*C) := M$$

for $m \gg 0$. By the lemma below, the set of points of $f(C)$ through which passes a rational curve of degree at most M is *closed* (it is the intersection of $f(C)$ and the image of the evaluation map); it cannot be finite since we could then take B_m such that $f_m(B_m)$ lies outside of that locus, hence it is equal to $f(C)$. This finishes the proof when the characteristic is positive.

This proves the theorem in positive characteristic. Assume now that $\mathbf{k}$ has characteristic 0. Embed X in some projective space by some multiple of H, where it is defined by finite sets of equations, and let R be the (finitely generated) subring of $\mathbf{k}$ generated by the coefficients of these equations and the coordinates of a point x of C. There is a projective scheme $\mathscr{X} \to \mathrm{Spec}(R)$ with an R-point x_R, such that X is obtained from its generic fiber by base change from the quotient field $K(R)$ of R to $\mathbf{k}$.

Consider now (§1.3) the quasi-projective scheme

$$\rho : \mathrm{Mor}_{\leq M}(\mathbf{P}^1_R, \mathscr{X}; 0 \mapsto x_R) \to \mathrm{Spec}(R)$$

which parametrizes nonconstant morphisms of H-degree at most M. Let $\mathfrak{m}$ be a maximal ideal of R. It is known from commutative algebra that the field $R/\mathfrak{m}$ is finite, hence of positive characteristic. What we just saw therefore implies that the (geometric) fiber of ρ over a closed point of the dense open subset U of $\mathrm{Spec}(R)$ over which $\mathscr{X}$ is smooth is nonempty. It follows that the image of ρ, which is a constructible[3] subset of $\mathrm{Spec}(R)$ by Chevalley's theorem ([H1], Exercise II.3.19), contains all closed points of U, which are dense in $\mathrm{Spec}(R)$ (this also follows from commutative algebra). It is therefore dense, hence contains the generic point ([H1], Exercise II.3.18.(b)). This implies that the generic fiber of ρ is nonempty; it has therefore a geometric point, which corresponds to a rational curve on X through x, of H-degree at most M, defined over an algebraic closure of the quotient field of R, hence over $\mathbf{k}$. □

Lemma 1.17 *Let X be a projective variety and let d be a positive integer. Let M_d be the quasi-projective scheme that parametrizes morphisms $\mathbf{P}^1_{\mathbf{k}} \to X$ of degree at most d with respect to some ample divisor. The image of the evaluation map*

$$\mathrm{ev}_d : \mathbf{P}^1_{\mathbf{k}} \times M_d \to X$$

is closed in X.

The image of ev_d is the set of points of X through which passes a rational curve of degree at most d.

[3] A constructible subset is a finite union of locally closed subsets.

Proof. The idea is that a rational curve can only degenerate into a union of rational curves of lower degrees.

Let x be a point in $\overline{\mathrm{ev}_d(\mathbf{P}^1_{\mathbf{k}} \times M_d)}$. Since M_d is a quasi-projective scheme, there exists an irreducible component M of M_d such that $x \in \overline{\mathrm{ev}_d(\mathbf{P}^1_{\mathbf{k}} \times M)}$ and a projective compactification $\overline{\mathbf{P}^1_{\mathbf{k}} \times M}$ such that ev_d extends to $\overline{\mathbf{P}^1_{\mathbf{k}} \times M}$ and $x \in \overline{\mathrm{ev}}_d(\overline{\mathbf{P}^1_{\mathbf{k}} \times M})$. Let $\overline{T}$ be the normalization of a curve in $\overline{\mathbf{P}^1_{\mathbf{k}} \times M}$ passing through a preimage of x and meeting $\mathbf{P}^1_{\mathbf{k}} \times M$.

Consider the rational map $\varphi : \overline{T} \dashrightarrow \mathbf{P}^1_{\mathbf{k}} \times M \to M \to M_d$. If it is constant with value $[f] \in M_d$, the point x is in the image of the corresponding morphism $f : \mathbf{P}^1_{\mathbf{k}} \to X$. Otherwise, the image of the rational map

$$\mathrm{ev} : \mathbf{P}^1_{\mathbf{k}} \times \overline{T} \dashrightarrow X$$

associated with φ is a surface and the indeterminacies of ev can be resolved by blowing up a finite number of points to get a morphism

$$e : S \xrightarrow{\varepsilon} \mathbf{P}^1_{\mathbf{k}} \times \overline{T} \overset{\mathrm{ev}}{\dashrightarrow} X.$$

The surface $e(S)$ contains x; it is covered by the images of the fibers of the projection $S \to \overline{T}$, which are unions of rational curves of degree at most d. This proves the lemma. □

Our next result shows that varieties with nef but not numerically trivial anticanonical divisor are covered by rational curves.

Theorem 1.18 *If X is a smooth projective variety with $-K_X$ nef,*

- *either K_X is numerically trivial,*
- *or there is a rational curve through any point of X.*

More precisely, in the second case, there exists an ample divisor H on X such that, through any point x of X, there exists a rational curve of H-degree $\leq 2n\frac{(H^n)}{(-K_X \cdot H^{n-1})}$: X is *uniruled* in the sense of Definition 2.3.

Proof. Let $n = \dim(X)$. If $(K_X \cdot H^{n-1}) = 0$ for *all* ample divisors H, it follows from the Hodge Index Theorem that K_X is numerically trivial.

Assume therefore $(K_X \cdot H^{n-1}) < 0$ for *some* very ample divisor H. Let x be a point of X and let C be the (smooth) intersection of $n-1$ general hyperplane sections through x. Since $(K_X \cdot C) = (K_X \cdot H^{n-1}) < 0$, by Theorem 1.16, there is a rational curve on X which passes through x. □

Note that the canonical divisor of an abelian variety X is trivial, and that X contains no rational curves.

Chapter 2

Varieties with many rational curves

2.1 Rational varieties

Let $\mathbf{k}$ be a field. A $\mathbf{k}$-variety X of dimension n is $\mathbf{k}$-*rational* if it is birationally isomorphic to $\mathbf{P}^n_{\mathbf{k}}$. It is *rational* if, for some algebraically closed extension $\mathbf{K}$ of $\mathbf{k}$, the variety $X_{\mathbf{K}}$ is $\mathbf{K}$-rational.

One can also say that a variety is $\mathbf{k}$-rational if its function field is a purely transcendental extension of $\mathbf{k}$.

A geometrically integral projective curve is rational if and only if it has genus 0. It is $\mathbf{k}$-rational if and only if it has genus 0 and has a $\mathbf{k}$-point.

2.2 Unirational and separably unirational varieties

Definition 2.1 A $\mathbf{k}$-variety X of dimension n is

- $\mathbf{k}$-*unirational* if there exists a dominant rational map $\mathbf{P}^n_{\mathbf{k}} \dashrightarrow X$;
- $\mathbf{k}$-*separably unirational* if there exists a dominant and separable[1] rational map $\mathbf{P}^n_{\mathbf{k}} \dashrightarrow X$.

[1] We say that a dominant rational map $f : Y \dashrightarrow X$ between integral schemes is *separable* if the field extension $K(Y)/K(X)$ is separable. Equivalently, f is smooth on a dense open subset of Y.

In characteristic zero, both definitions are equivalent. We say that X is *(separably) unirational* if for some algebraically closed extension $\mathbf{K}$ of $\mathbf{k}$, the variety $X_{\mathbf{K}}$ is $\mathbf{K}$-(separably) unirational (it is then true for all algebraically closed extensions of $\mathbf{k}$).

A variety is $\mathbf{k}$-(separably) unirational if its function field has a purely transcendental (separable) extension.

Rational points are Zariski-dense in a $\mathbf{k}$-unirational variety, hence any conic with no rational points is rational but not $\mathbf{k}$-unirational.

Example 2.2 (Fermat hypersurfaces) Recall from 1.13 that the Fermat hypersurface X_N^d is the hypersurface in $\mathbf{P}_{\mathbf{k}}^N$ defined by the equation

$$x_0^d + \cdots + x_N^d = 0.$$

Assume that the field $\mathbf{k}$ has characteristic $p > 0$ and contains an element ω such that $\omega^{p^r+1} = -1$ for some $r > 0$. One can show ([D], Exercise 2.5.1) that when $N \geq 3$, the hypersurface $X_N^{p^r+1}$ has a purely inseparable $\mathbf{k}$-rational cover of degree p^r.

In particular, X_N^e is unirational whenever $N \geq 3$ and e divides $p^r + 1$ for some positive integer r. However, when $e \geq N + 1$, the canonical class of X_N^e is nef, hence X_N^e is not separably unirational (not even separably uniruled; see Example 2.14).

Any unirational curve is rational (Lüroth theorem) and any separably unirational surface is rational. However, any smooth cubic hypersurface $X \subset \mathbf{P}_{\mathbf{k}}^4$ is unirational but not rational.

I will explain the classical construction of a double cover of X which is rational. Let ℓ be a line contained in X and consider the map $\varphi : \mathbf{P}(T_X|_\ell) \dashrightarrow X$ defined as follows:[2] let L be a tangent line to X at a point $x_1 \in \ell$; the divisor $X|_L$ can be written as $2x_1 + x$, and we set $\varphi(L) = x$. Given a general point $x \in X$, the intersection of the 2-plane $\langle \ell, x \rangle$ with X is the union of the line ℓ and a conic C_x. The points of $\varphi^{-1}(x)$ are the two points of intersection of ℓ and C_x, hence φ is dominant of degree 2.

Now $T_X|_\ell$ is a sum of invertible sheaves which are all trivial on the complement $\ell^0 \simeq \mathbf{A}_{\mathbf{k}}^1$ of any point of ℓ. It follows that $\mathbf{P}(T_X|_{\ell_0})$ is isomorphic to $\ell^0 \times \mathbf{P}_{\mathbf{k}}^2$ hence is rational. This shows that X is unirational. The fact that it is not rational is a difficult theorem of Clemens-Griffiths and Artin-Mumford.

[2] Here we *do not* follow Grothendieck's convention: $\mathbf{P}(T_X|_\ell)$ is the set of tangent directions to X at points of ℓ.

2.3 Uniruled and separably uniruled varieties

We want to make a formal definition for varieties that are "covered by rational curves". The most reasonable approach is to make it a "geometric" property by defining it over an algebraic closure of the base field: contrary to what happens with unirationality, it makes no difference being $\mathbf{k}$-uniruled or $\bar{\mathbf{k}}$-uniruled. Special attention has to be paid to the positive characteristic case, hence the two variants of the definition.

Definition 2.3 Let $\mathbf{k}$ be a field and let $\mathbf{K}$ be an algebraically closed extension of $\mathbf{k}$. A variety X of dimension n defined over a field $\mathbf{k}$ is

- *uniruled* if there exist a $\mathbf{K}$-variety M of dimension $n-1$ and a dominant rational map $\mathbf{P}^1_{\mathbf{K}} \times M \dashrightarrow X_{\mathbf{K}}$;
- *separably uniruled* if there exist a $\mathbf{K}$-variety M of dimension $n-1$ and a dominant and separable rational map $\mathbf{P}^1_{\mathbf{K}} \times M \dashrightarrow X_{\mathbf{K}}$.

These definitions do not depend on the choice of $\mathbf{K}$, and in characteristic zero, both definitions are equivalent.

In the same way that a "unirational" variety is dominated by a rational variety, a "uniruled" variety is dominated by a ruled variety; hence the terminology.

Of course, (separably) unirational varieties of positive dimension are (separably) uniruled. For the converse, uniruled curves are rational; separably uniruled surfaces are birationally isomorphic to a ruled surface. As explained in Example 2.2, in positive characteristic, some Fermat hypersurfaces are unirational (hence uniruled), but not separably uniruled.

Also, smooth projective varieties X with $-K_X$ nef and not numerically trivial are uniruled (Theorem 1.18), but there are Fano varieties that are not separably uniruled ([K2]).

Here are various other characterizations and properties of (separably) uniruled varieties.

Remark 2.4 A point is not uniruled. Any variety birationally isomorphic to a (separably) uniruled variety is (separably) uniruled. The product of a (separably) uniruled variety with any variety is (separably) uniruled.

Remark 2.5 A variety X of dimension n is (separably) uniruled if and only if there exist a a $\mathbf{K}$-variety M, an open subset U of $\mathbf{P}^1_{\mathbf{K}} \times M$ and a dominant (and separable) morphism $e : U \to X_{\mathbf{K}}$ such that for some point m in M, the set $U \cap (\mathbf{P}^1_{\mathbf{K}} \times m)$ is nonempty and not contracted by e.

Remark 2.6 Let X be a *proper* (separably) uniruled variety, with a rational map $e : \mathbf{P}^1_{\mathbf{K}} \times M \dashrightarrow X_{\mathbf{K}}$ as in the definition. We may compactify M then normalize it. The map e is then defined outside of a subvariety of $\mathbf{P}^1_{\mathbf{K}} \times M$ of codimension at least 2, which therefore projects onto a proper closed subset of M. By shrinking M, we may therefore assume that e is a *morphism.*

Remark 2.7 Assume $\mathbf{k}$ is algebraically closed. It follows from Remark 2.6 that there is a rational curve through a general point of a proper uniruled variety (actually, by Lemma 1.17, there is even a rational curve through *every* point). The converse holds *if* $\mathbf{k}$ *is uncountable;* in the definition, it is therefore often useful to choose an uncountable algebraically closed extension $\mathbf{K}$.

Indeed, the fact that there is a rational curve through a general point means that the image of the evaluation map $\mathrm{ev} : \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}_{>0}(\mathbf{P}^1_{\mathbf{k}}, X) \to X$ contains a dense open subset U of X. Since $\mathrm{Mor}_{>0}(\mathbf{P}^1_{\mathbf{k}}, X)$ has at most countably many irreducible components and U is not the union of countably many proper closed subsets, the image of the restriction of ev to at least one of these components must be dense in U, hence in X, hence X is uniruled by Remark 2.5.

Remark 2.8 Let $X \to T$ be a proper and equidimensional morphism with irreducible fibers. The set $\{t \in T \mid X_t \text{ is uniruled}\}$ is closed ([K1], Theorem 1.8.2).

Remark 2.9 A connected finite étale cover of a proper (separably) uniruled variety is (separably) uniruled.

Let X be a proper uniruled variety, let $e : \mathbf{P}^1_{\mathbf{K}} \times M \to X_{\mathbf{K}}$ be a dominant (and separable) morphism (Remark 2.6), and let $\pi : \tilde{X} \to X$ be a connected finite étale cover. Since $\mathbf{P}^1_{\mathbf{K}}$ is simply connected, the pull-back by e of $\pi_{\mathbf{K}}$ is an étale morphism of the form $\mathbf{P}^1_{\mathbf{K}} \times \tilde{M} \to \mathbf{P}^1_{\mathbf{K}} \times M$ and the morphism $\mathbf{P}^1_{\mathbf{K}} \times \tilde{M} \to \tilde{X}_{\mathbf{K}}$ is dominant (and separable).

2.4 Free rational curves and separably uniruled varieties

Let X be a $\mathbf{k}$-variety of dimension n and let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a nonconstant $\mathbf{k}$-morphism whose image is contained in the smooth locus of X. Since any locally free sheaf on $\mathbf{P}^1_{\mathbf{k}}$ is isomorphic to a direct sum of invertible sheaf, we can write

$$f^*T_X \simeq \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_n), \tag{2.1}$$

with $a_1 \geq \cdots \geq a_n$. If f is separable, f^*T_X contains $T_{\mathbf{P}^1_{\mathbf{k}}} \simeq \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(2)$ and $a_1 \geq 2$. In general, decompose f as $\mathbf{P}^1_{\mathbf{k}} \xrightarrow{h} \mathbf{P}^1_{\mathbf{k}} \xrightarrow{g} X$ where g is separable and h is a composition of r Frobenius morphisms. Then $a_1(f) = p^r a_1(g) \geq 2$.

If $H^1(\mathbf{P}^1_{\mathbf{k}}, f^*T_X)$ vanishes, the space $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ is smooth at $[f]$ (Theorem 1.9). This happens exactly when $a_n \geq -1$.

Definition 2.10 A $\mathbf{k}$-rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$ is *free* if its image is a curve contained in the smooth locus of X and f^*T_X is generated by its global sections.

With our notation, this means $a_n \geq 0$.

Examples 2.11 1) For any morphism $f : \mathbf{P}^1_{\mathbf{k}} \to X$ whose image is contained in the smooth locus of X, we have $\deg(f^*T_X) = -(K_X \cdot f_*\mathbf{P}^1_{\mathbf{k}})$: there are no free rational curves on a smooth variety whose canonical divisor is nef.

2) A rational curve with image C on a smooth surface is free if and only if $(C^2) \geq 0$.

Let $f : \mathbf{P}^1_{\mathbf{k}} \to C \subset X$ be the normalization and assume that f is free. Since

$$(K_X \cdot C) + (C^2) = 2h^1(C, \mathscr{O}_C) - 2,$$

we have, with the notation (2.1),

$$(C^2) = a_1 + a_2 + 2h^1(C, \mathscr{O}_C) - 2 \geq (a_1 - 2) + a_2 \geq a_2 \geq 0.$$

Conversely, assume $a := (C^2) \geq 0$. Since the ideal sheaf of C in X is invertible, there is an exact sequence

$$0 \to \mathscr{O}_C(-C) \to \Omega_X|_C \to \Omega_C \to 0$$

of locally free sheaves on C which pulls back to $\mathbf{P}^1_{\mathbf{k}}$ and dualizes to

$$0 \to \mathscr{H}om(f^*\Omega_C, \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}) \to f^*T_X \to f^*\mathscr{O}_X(C) \to 0. \tag{2.2}$$

There is also a morphism $f^*\Omega_C \to \Omega_{\mathbf{P}^1_{\mathbf{k}}}$ which is an isomorphism on a dense open subset of $\mathbf{P}^1_{\mathbf{k}}$, hence dualizes to an injection $T_{\mathbf{P}^1_{\mathbf{k}}} \hookrightarrow \mathscr{H}om(f^*\Omega_C, \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}})$. In particular, the invertible sheaf $\mathscr{H}om(f^*\Omega_C, \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}})$ has degree $b \geq 2$, and we have an exact sequence

$$0 \to \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(b) \to f^*T_X \to \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a) \to 0.$$

If $a_2 < 0$, the injection $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(b) \to f^*T_X$ lands in $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_1)$, and we have an isomorphism

$$\Big(\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_1)/\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(b)\Big) \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_2) \simeq \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a),$$

which implies $a_1 = b$ and $a = a_2 < 0$, a contradiction. So we have $a_2 \geq 0$ and f is free.

3) One can show ([D], 2.15) that the Fermat hypersurface (see 1.13) X^d_N of dimension at least 3 and degree $d = p^r + 1$ over a field of characteristic p is uniruled by lines, none of which are free (in fact, when $d > N$, there are no free rational curves on X by Example 2.11.1)). Moreover, $\mathrm{Mor}_1(\mathbf{P}^1_{\mathbf{k}}, X)$ is smooth, but the evaluation map

$$\mathrm{ev} : \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}_1(\mathbf{P}^1_{\mathbf{k}}, X) \longrightarrow X$$

is *not separable.*

Proposition 2.12 *Let X be a smooth quasi-projective variety defined over a field $\mathbf{k}$ and let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a $\mathbf{k}$-rational curve.*

a) *If f is free, the evaluation map*

$$\mathrm{ev} : \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X) \to X$$

is smooth at all points of $\mathbf{P}^1_{\mathbf{k}} \times \{[f]\}$.

b) *If there is a scheme M with a $\mathbf{k}$-point m and a morphism $e : \mathbf{P}^1_{\mathbf{k}} \times M \to X$ such that $e|_{\mathbf{P}^1_{\mathbf{k}} \times m} = f$ and the tangent map to e is surjective at some point of $\mathbf{P}^1_{\mathbf{k}} \times m$, the curve f is free.*

Geometrically speaking, item a) implies that the deformations of a free rational curve cover X. In b), the hypothesis that the tangent map to e is surjective is weaker than the smoothness of e, and does not assume anything on the smoothness, or even reducedness, of the scheme M.

The proposition implies that the set of free rational curves on a quasi-projective $\mathbf{k}$-variety X is a smooth open subset $\mathrm{Mor}^{\mathrm{free}}(\mathbf{P}^1_{\mathbf{k}}, X)$ of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$, possibly empty.

Finally, *when* $\mathrm{char}(\mathbf{k}) = 0$, and there is an irreducible $\mathbf{k}$-scheme M and a *dominant* morphism $e : \mathbf{P}^1_{\mathbf{k}} \times M \to X$ which does not contract one $\mathbf{P}^1_{\mathbf{k}} \times m$, the rational curves corresponding to points in some nonempty open subset of M are free (by generic smoothness, the tangent map to e is surjective on some nonempty open subset of $\mathbf{P}^1_{\mathbf{k}} \times M$).

Proof. The tangent map to ev at $(t, [f])$ is the map

$$\begin{aligned} T_{\mathbf{P}^1_{\mathbf{k}},t} \oplus H^0(\mathbf{P}^1_{\mathbf{k}}, f^*T_X) &\longrightarrow T_{X,f(t)} \simeq (f^*T_X)_t \\ (u, \sigma) &\longmapsto T_t f(u) + \sigma(t). \end{aligned}$$

If f is free, it is surjective because the evaluation map

$$H^0(\mathbf{P}^1_{\mathbf{k}}, f^*T_X) \longrightarrow (f^*T_X)_t$$

is. Moreover, since $H^1(\mathbf{P}^1_{\mathbf{k}}, f^*T_X)$ vanishes, $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ is smooth at $[f]$ (1.10). This implies that ev is smooth at $(t, [f])$ and proves a).

Conversely, the morphism e factors through ev, whose tangent map at $(t, [f])$ is therefore surjective. This implies that the map

$$H^0(\mathbf{P}^1_{\mathbf{k}}, f^*T_X) \to (f^*T_X)_t / \mathrm{Im}(T_t f) \tag{2.3}$$

is surjective. There is a commutative diagram

$$\begin{array}{ccc} H^0(\mathbf{P}^1_{\mathbf{k}}, f^*T_X) & \xrightarrow{a} & (f^*T_X)_t \\ \uparrow & & \uparrow{\scriptstyle T_t f} \\ H^0(\mathbf{P}^1_{\mathbf{k}}, T_{\mathbf{P}^1_{\mathbf{k}}}) & \xrightarrow{a'} & T_{\mathbf{P}^1_{\mathbf{k}},t}. \end{array}$$

Since a' is surjective, the image of a contains $\mathrm{Im}(T_t f)$. Since the map (2.3) is surjective, a is surjective. Hence f^*T_X is generated by global sections at one point. It is therefore generated by global sections and f is free. □

Corollary 2.13 *Let X be a variety defined over an algebraically closed field.*

a) If X contains a free rational curve, X is separably uniruled.

b) Conversely, if X is separably uniruled, smooth, and proper, there exists a free rational curve through a general point of X.

PROOF. By shrinking X, we may assume that it is quasi-projective and we may consider the scheme $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$. If $f : \mathbf{P}^1_{\mathbf{k}} \to X$ is free, the evaluation map ev is smooth at $(0, [f])$ by Proposition 2.12.a). It follows that the restriction of ev to the unique component of $\mathrm{Mor}_{>0}(\mathbf{P}^1_{\mathbf{k}}, X)$ that contains $[f]$ is separable and dominant and X is separably uniruled.

Assume conversely that X is separably uniruled, smooth, and proper. By Remark 2.6, there exists a **k**-variety M and a dominant and separable, hence generically smooth, morphism $\mathbf{P}^1_{\mathbf{k}} \times M \to X$. The rational curve corresponding to a general point of M passes through a general point of X and is free by Proposition 2.12.b). □

Example 2.14 By Example 2.11 and Corollary 2.13.b), a smooth proper variety whose canonical class is nef is not separably uniruled.

On the other hand, we proved in Theorem 1.18 that smooth projective varieties X with $-K_X$ nef and not numerically trivial are uniruled. However, Kollár constructed Fano varieties that are not separably uniruled ([K2]).

Corollary 2.15 *Let $X \to T$ be a smooth and proper morphism. The set $\{t \in T \mid X_t$ is separably uniruled$\}$ is open.*

Recall (Remark 2.8) that in characteristic zero, this set is also closed.

PROOF. Consider the T-scheme $\mathrm{Mor}_T(\mathbf{P}^1_T, X \times T)$ defined in 1.12. The subset parametrizing free morphisms is open and smooth over T, hence its image in T is open. □

Corollary 2.16 *If X is a smooth proper separably uniruled variety, the* plurigenera $p_m(X) := h^0(X, \mathscr{O}_X(mK_X))$ *vanish for all positive integers m.*

The converse is conjectured to hold: for curves, it is obvious since $p_1(X)$ is the genus of X; for surfaces, we have the more precise Castelnuovo criterion;

$p_{12}(X) = 0$ if and only if X is birationally isomorphic to a ruled surface; in dimension three, it is known in characteristic zero.

Proof. We may assume that the base field is algebraically closed. By Corollary 2.13.b), there is a free rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$ through a general point of X. Since f^*K_X has negative degree, any section of $\mathscr{O}_X(mK_X)$ must vanish on $f(\mathbf{P}^1_{\mathbf{k}})$, hence on a dense subset of X, hence on X. □

The next results says that a rational curve through a very general point (i.e., outside the union of a countable number of proper subvarieties) of a smooth variety is free (in characteristic zero).

Proposition 2.17 *Let X be a smooth quasi-projective variety defined over a field of characteristic zero. There exists a subset X^{free} of X which is the intersection of countably many dense open subsets of X, such that any rational curve on X whose image meets X^{free} is free.*

Proof. The space $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ has at most countably many irreducible components, which we denote by $(M_i)_{i\in\mathbf{N}}$. Let $e_i : \mathbf{P}^1_{\mathbf{k}} \times (M_i)_{\mathrm{red}} \to X$ be the morphisms induced by the evaluation maps.

By generic smoothness, there exists a dense open subset U_i of X such that the tangent map to e_i is surjective at each point of $e_i^{-1}(U_i)$ (if e_i is not dominant, one may simply take for U_i the complement of the closure of the image of e_i). We let X^{free} be the intersection $\bigcap_{i\in\mathbf{N}} U_i$.

Let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a curve whose image meets X^{free}, and let M_i be an irreducible component of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ that contains $[f]$. By construction, the tangent map to e_i is surjective at some point of $\mathbf{P}^1_{\mathbf{k}} \times \{[f]\}$, hence f is free by Proposition 2.12.b). □

The proposition is interesting only when X is uniruled (otherwise, the set X^{free} is more or less the complement of the union of all rational curves on X); it is also useless when the ground field is countable, because X^{free} may be empty.

Examples 2.18 1) If $\varepsilon : \widetilde{\mathbf{P}}^2_{\mathbf{k}} \to \mathbf{P}^2_{\mathbf{k}}$ is the blow-up of one point, $(\widetilde{\mathbf{P}}^2_{\mathbf{k}})^{\mathrm{free}}$ is the complement of the exceptional divisor E: for any rational curve C other than E, write $C \sim_{\mathrm{lin}} dH - mE$, where H is the inverse image of a line; we have $m = (C \cdot E) \geq 0$. The intersection of C with the strict transform of a line through the blown-up point, which has class $H - E$, is nonnegative, hence $d \geq m$. It implies $(C^2) = d^2 - m^2 \geq 0$, hence C is free by Example 2.11.2).

2) On the blow-up X of $\mathbf{P}^2_{\mathbf{C}}$ at nine general points, there are countably many rational curves with self-intersection -1 ([H1], Exercise V.4.15.(e)) hence X^{free} is not open.

The proposition will often be used together with the following remark. Let

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & X \\ \downarrow{\scriptstyle \pi} & & \\ T & & \end{array}$$

be a flat family of curves on X parametrized by a variety T. If *the base field is uncountable* (and of characteristic zero) and one of these curves meets X^{free}, the same is true for a very general curve in the family.

Indeed, X^{free} is the intersection of a countable *nonincreasing* family $(U_i)_{i\in\mathbf{N}}$ of open subsets of X. Let $\mathscr{C}_t$ be the curve $\pi^{-1}(t)$. The curve $F(\mathscr{C}_t)$ meets X^{free} if and only if $\mathscr{C}_t$ meets $\bigcap_{i\in\mathbf{N}} F^{-1}(U_i)$. We have

$$\pi\Big(\bigcap_{i\in\mathbf{N}} F^{-1}(U_i)\Big) = \bigcap_{i\in\mathbf{N}} \pi(F^{-1}(U_i)).$$

Let us prove this equality. The right-hand side contains the left-hand side. If t is in the right-hand side, the $\mathscr{C}_t \cap F^{-1}(U_i)$ form a nonincreasing family of nonempty open subsets of $\mathscr{C}_t$. Since the base field is uncountable, their intersection is nonempty. This means exactly that t is in the left-hand side.

Since π, being flat, is open ([G2], th. 2.4.6), this proves that the set of t such that $f_t(\mathbf{P}^1_{\mathbf{k}})$ meets X^{free} is the intersection of a countable family of open subsets of T.

This is expressed by the following principle:

2.19. A very general deformation of a curve which meets X^{free} has the same property.

2.5 Rationally connected and separably rationally connected varieties

We now want to make a formal definition for varieties for which there exists a rational curve through two general points. Again, this will be a geometric property.

Definition 2.20 Let $\mathbf{k}$ be a field and let $\mathbf{K}$ be an algebraically closed extension of $\mathbf{k}$. A $\mathbf{k}$-variety X is *rationally connected* (resp. *separably rationally connected*) if it is *proper* and if there exist a $\mathbf{K}$-variety M and a rational map $e : \mathbf{P}^1_{\mathbf{K}} \times M \dashrightarrow X_{\mathbf{K}}$ such that the rational map

$$\begin{array}{rcl} \operatorname{ev}_2 : \mathbf{P}^1_{\mathbf{K}} \times \mathbf{P}^1_{\mathbf{K}} \times M & \dashrightarrow & X_{\mathbf{K}} \times X_{\mathbf{K}} \\ (t, t', z) & \longmapsto & (e(t,z), e(t',z)) \end{array}$$

is dominant (resp. dominant and separable).

Again, this definition does not depend on the choice of the algebraically closed extension $\mathbf{K}$, and in characteristic zero, both definitions are equivalent. Moreover, the rational map e may be assumed to be a morphism (proceed as in Remark 2.6).

Of course, (separably) rationally connected varieties are (separably) uniruled, and (separably) unirational varieties are (separably) rationally connected. For the converse, rationally connected curves are rational, and separably rationally connected surfaces are rational. One does not expect, in dimension ≥ 3, rational connectedness to imply unirationality, but no examples are known!

It can be shown that Fano varieties are rationally connected,[3] although they are in general not even separably uniruled in positive characteristic (Example 2.2).

Remark 2.21 A point is separably rationally connected. (Separable) rational connectedness is a birational property (for proper varieties!); better, if X is a (separably) rationally connected variety and $X \dashrightarrow Y$ a (separable) dominant rational map, with Y proper, Y is (separably) rationally connected. A (finite) product of (separably) rationally connected varieties is (separably) rationally connected. A (separably) rationally connected variety is (separably) uniruled.

Remark 2.22 In the definition, one may replace the condition that ev_2 be dominant (resp. dominant and separable) by the condition that the map

$$\begin{array}{ccc} M & \dashrightarrow & X_{\mathbf{K}} \times X_{\mathbf{K}} \\ z & \longmapsto & (e(0,z), e(\infty,z)) \end{array}$$

be dominant (resp. dominant and separable).

Indeed, upon shrinking and compactifying X, we may assume that X is projective. The morphism e then factors through an evaluation map $\mathrm{ev} : \mathbf{P}^1_{\mathbf{K}} \times \mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{K}}, X) \to X_{\mathbf{K}}$ for some $d > 0$ and the image of

$$\mathrm{ev}_2 : \mathbf{P}^1_{\mathbf{K}} \times \mathbf{P}^1_{\mathbf{K}} \times \mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{K}}, X) \to X_{\mathbf{K}} \times X_{\mathbf{K}}$$

is then the same as the image of

$$\begin{array}{ccc} \mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{K}}, X) & \to & X_{\mathbf{K}} \times X_{\mathbf{K}} \\ z & \longmapsto & (e(0,z), e(\infty,z)) \end{array}$$

(This is because $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{K}}, X)$ is stable by reparametrizations, i.e., by the action of $\mathrm{Aut}(\mathbf{P}^1_{\mathbf{K}})$; for separable rational connectedness, there are some details to check.)

[3] This is a result due independently to Campana and Kollár-Miyaoka-Mori; see for example [D], Proposition 5.16.

Remark 2.23 Assume $\mathbf{k}$ is algebraically closed. On a rationally connected variety, a general pair of points can be joined by a rational curve.[4] The converse holds *if* $\mathbf{k}$ *is uncountable* (with the same proof as in Remark 2.7).

Remark 2.24 Any proper variety which is an étale cover of a (separably) rationally connected variety is (separably) rationally connected (proceed as in Remark 2.9). We will see in Corollary 3.7 that any such a cover of a smooth proper separably rationally connected variety is in fact trivial.

2.6 Very free rational curves and separably rationally connected varieties

Definition 2.25 Let X be a $\mathbf{k}$-variety. A $\mathbf{k}$-rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$ is *r-free* if its image is contained in the smooth locus of X and $f^*T_X \otimes \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(-r)$ is generated by its global sections.

In particular, 0-free curves are free curves. We will say "very free" instead of "1-free". For easier statements, we will also agree that a constant morphism $\mathbf{P}^1_{\mathbf{k}} \to X$ is very free if and only if X is a point. Note that given a very free rational curve, its composition with a (ramified) finite map $\mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^1_{\mathbf{k}}$ of degree r is r-free.

Examples 2.26 1) Any $\mathbf{k}$-rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^n_{\mathbf{k}}$ is very free. This is because $T_{\mathbf{P}^n_{\mathbf{k}}}$ is a quotient of $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(1)^{\oplus(n+1)}$, hence its inverse image by f is a quotient of $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(d)^{\oplus(n+1)}$, where $d > 0$ is the degree of $f^*\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(1)$. With the notation of (2.1), each $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_i)$ is a quotient of $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(d)^{\oplus(n+1)}$ hence $a_i \geq d$.

2) A rational curve with image C on a smooth surface is very free if and only if $(C^2) > 0$ (proceed as in Example 2.11.2)).

Informally speaking, the freer a rational curve is, the more it can move while keeping points fixed. The precise result is the following. It generalizes Proposition 2.12 and its proof is similar.

Proposition 2.27 *Let X be a smooth quasi-projective $\mathbf{k}$-variety, let r be a non-negative integer, let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a $\mathbf{k}$-rational curve and let B be a finite subset of $\mathbf{P}^1_{\mathbf{k}}$ of cardinality b.*

a) *If f is r-free, for any integer s such that $0 < s \leq r + 1 - b$, the evaluation map*

$$\begin{array}{rccc} \mathrm{ev}_s : & (\mathbf{P}^1_{\mathbf{k}})^s \times \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; f|_B) & \longrightarrow & X^s \\ & (t_1, \dots, t_s, [g]) & \longmapsto & (g(t_1), \dots, g(t_s)) \end{array}$$

[4] In characteristic zero, we will prove in Theorem 2.49 that *any* two points of a *smooth* projective rationally connected variety can be joined by a rational curve.

is smooth at all points $(t_1, \dots, t_s, [f])$ *such that* $\{t_1, \dots, t_s\} \cap B = \varnothing$.

b) *If there is a* $\mathbf{k}$*-scheme* M *with a* $\mathbf{k}$*-point* m *and a morphism* $\varphi : M \to \operatorname{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; f|_B)$ *such that* $\varphi(m) = [f]$ *and the tangent map to the corresponding evaluation map*

$$\operatorname{ev}_s : (\mathbf{P}^1_{\mathbf{k}})^s \times M \longrightarrow X^s$$

is surjective at some point of $\mathbf{P}^1_{\mathbf{k}} \times m$ *for some* $s > 0$*, the rational curve* f *is* $\min(2, b + s - 1)$*-free.*

Geometrically speaking, item a) implies that the deformations of an r-free rational curve keeping b points fixed ($b \le r$) pass through $r+1-b$ general points of X.

The proposition implies that the set of very free rational curves on X is a smooth open subset $\operatorname{Mor}^{\text{vfree}}(\mathbf{P}^1_{\mathbf{k}}, X)$ of $\operatorname{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$, possibly empty.

In §2.4, we studied the relationships between separable uniruledness and the existence of free rational curves on a smooth projective variety. We show here that there is an analogous relationship between separable rational connectedness and the existence of very free rational curves.

Corollary 2.28 *Let* X *be a proper variety defined over an algebraically closed field.*

a) If X *contains a very free rational curve, there is a very free rational curve through a general finite subset of* X*. In particular,* X *is separably rationally connected.*

b) Conversely, if X *is separably rationally connected and smooth, there exists a very free rational curve through a general point of* X*.*

The result will be strengthened in Theorem 2.49 where it is proved that in characteristic zero, there is on a smooth projective rationally connected variety a very free rational curve through *any* given finite subset.

Proof. Assume there is a very free rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$. By composing f with a finite map $\mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^1_{\mathbf{k}}$ of degree r, we get an r-free curve. By Proposition 2.12.a) (applied with $B = \varnothing$), there is a deformation of this curve that passes through $r+1$ general points of X. The rest of the proof is the same as in Corollary 2.13. □

Corollary 2.29 *Let* $X \to T$ *be a smooth and proper morphism. The set* $\{t \in T \mid X_t$ *is separably rationally connected*$\}$ *is open.*

Proof. Proceed as in the proof of Corollary 2.15. □

In characteristic zero, this set is also closed (Theorem 2.49 and Remark 2.48).

Exercise 2.30 Construct a projective flat family $X \to \mathbf{A}^1_{\mathbf{k}}$ for which all the fibers but one are separably rationally connected. (*Hint:* consider a degeneration of a smooth cubic surface to a cone over a plane section.)

Corollary 2.31 *If X is a smooth proper separably rationally connected variety, $H^0(X, (\Omega^p_X)^{\otimes m})$ vanishes for all positive integers m and p. In particular, in characteristic zero, $\chi(X, \mathscr{O}_X) = 1$.*

A converse is conjectured to hold (at least in characteristic zero): if $H^0(X, (\Omega^1_X)^{\otimes m})$ vanishes for all positive integers m, the variety X should be rationally connected. This is proved in dimensions at most 3 in [KMM], Theorem (3.2).

Note that the conclusion of the corollary does not hold in general for unirational varieties: some Fermat hypersurfaces X are unirational with $H^0(X, K_X) \neq 0$ (see Example 2.2).

Proof of the Corollary. For the first part, proceed as in the proof of Corollary 2.16. For the second part, $H^p(X, \mathscr{O}_X)$ then vanishes for $p > 0$ by Hodge theory,[5] hence $\chi(X, \mathscr{O}_X) = 1$. □

Corollary 2.32 *Let X be a proper normal rationally connected variety defined over an algebraically closed field $\mathbf{k}$.*

a) The algebraic fundamental group of X is finite.

b) If $\mathbf{k} = \mathbf{C}$ and X is smooth, X is topologically simply connected.

When X is smooth and separably rationally connected, Kollár proved that X is in fact algebraically simply connected (Corollary 3.7).

Proof of the Corollary. By Remark 2.22, there exist a variety M and a point x of X such that the evaluation map

$$\mathrm{ev} : \mathbf{P}^1_{\mathbf{k}} \times M \longrightarrow X$$

is dominant and satisfies $\mathrm{ev}(0 \times M) = x$. The composition of ev with the injection $\iota : 0 \times M \hookrightarrow \mathbf{P}^1_{\mathbf{k}} \times M$ is then constant, hence

$$\pi_1(\mathrm{ev}) \circ \pi_1(\iota) = 0.$$

Since $\mathbf{P}^1_{\mathbf{k}}$ is simply connected, $\pi_1(\iota)$ is bijective, hence $\pi_1(\mathrm{ev}) = 0$. Since ev is dominant, the following lemma implies that the image of $\pi_1(\mathrm{ev})$ has finite index. This proves a).

[5] For a smooth separably rationally connected variety X, the vanishing of $H^m(X, \mathscr{O}_X)$ for $m > 0$ is not known in general.

Lemma 2.33 *Let X and Y be $\mathbf{k}$-varieties, with Y normal, and let $f : X \to Y$ be a dominant morphism. For any geometric point x of X, the image of the morphism $\pi_1(f) : \pi_1^{\mathrm{alg}}(X, x) \to \pi_1^{\mathrm{alg}}(Y, f(x))$ has finite index.*

When $\mathbf{k} = \mathbf{C}$, the same statement holds with topological fundamental groups.

Sketch of proof. The lemma is proved in [De] (lemme 4.4.17) when X and Y are smooth. The same proof applies in our case ([CL]).

We will sketch the proof when $\mathbf{k} = \mathbf{C}$. The first remark is that if A is an irreducible analytic space and B a proper closed analytic subspace, $A - B$ is connected. The second remark is that the universal cover $\pi : \tilde{Y} \to Y$ is irreducible; indeed, Y being normal is locally irreducible in the classical topology, hence so is $\tilde{Y}$. Since it is connected, it is irreducible.

Now if Z is a proper subvariety of Y, its inverse image $\pi^{-1}(Z)$ is a proper subvariety of $\tilde{Y}$, hence $\pi^{-1}(Y - Z)$ is connected by the two remarks above. This means exactly that the map $\pi_1(Y - Z) \to \pi_1(Y)$ is surjective. So we may replace Y with any dense open subset, and assume that Y is smooth.

We may also shrink X and assume that it is smooth and quasi-projective. Let $\overline{X}$ be a compactification of X. We may replace X with a desingularization $\overline{\overline{X}}$ of the closure in $\overline{X} \times Y$ of the graph of f and assume that f is *proper*. Since the map $\pi_1(X) \to \pi_1(\overline{\overline{X}})$ is surjective by the remark above, this does not change the cokernel of $\pi_1(f)$.

Finally, we may, by generic smoothness, upon shrinking Y again, assume that f is smooth. The finite morphism in the Stein factorization of f is then étale; we may therefore assume that the fibers of f are connected. It is then classical that f is locally $\mathscr{C}^\infty$-trivial with fiber F, and the long exact homotopy sequence

$$\cdots \to \pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to \pi_0(F) \to 0$$

of a fibration gives the result. □

If $\mathbf{k} = \mathbf{C}$ and X is smooth, we have $\chi(X, \mathscr{O}_X) = 1$ by Corollary 2.31. Let $\pi : \tilde{X} \to X$ be a connected finite étale cover; $\tilde{X}$ is rationally connected by Remark 2.24, hence $\chi(\tilde{X}, \mathscr{O}_{\tilde{X}}) = 1$. But $\chi(\tilde{X}, \mathscr{O}_{\tilde{X}}) = \deg(\pi)\, \chi(X, \mathscr{O}_X)$ ([L], Proposition 1.1.28) hence π is an isomorphism. This proves b). □

We finish this section with an analog of Proposition 2.17: on a *smooth* projective variety defined over an algebraically closed field *of characteristic zero,* a rational curve through a fixed point and a very general point is very free.

Proposition 2.34 *Let X be a smooth quasi-projective variety defined over an algebraically closed field of characteristic zero and let x be a point in X. There exists a subset X_x^{vfree} of $X - \{x\}$ which is the intersection of countably many dense open subsets of X, such that any rational curve on X passing through x and whose image meets X_x^{free} is very free.*

PROOF. The space $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto x)$ has at most countably many irreducible components, which we will denote by $(M_i)_{i\in\mathbf{N}}$. Let $e_i : \mathbf{P}^1_{\mathbf{k}} \times (M_i)_{\mathrm{red}} \to X$ be the morphisms induced by the evaluation maps.

Denote by U_i a dense open subset of $X - \{x\}$ over which e_i is smooth and let X_x^{vfree} be the intersection of the U_i. Let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a curve with $f(0) = x$ whose image meets X_x^{vfree}, and let M_i be an irreducible component of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto x)$ that contains $[f]$. By construction, the tangent map to e_i is surjective at some point of $\mathbf{P}^1_{\mathbf{k}} \times \{[f]\}$, hence so is the tangent map to ev; it follows from Proposition 2.27 that f is very free. □

Again, this proposition is interesting only when X is rationally connected and the ground field is uncountable.

2.7 Smoothing trees of rational curves

2.7.1 Trees

As promised, we are now on our way to prove that rational chain connectedness implies rational connectedness for *smooth* varieties in *characteristic zero* (this will be proved in Theorem 2.49). For that, we will need to smooth a chain of rational curves connecting two points, and this can be done when the links of the chain are free.

We assume now that the field $\mathbf{k}$ *is algebraically closed.*

Definition 2.35 A *rational* $\mathbf{k}$*-tree* is a connected projective nodal $\mathbf{k}$-curve C such that $\chi(C, \mathscr{O}_C) = 1$.

Equivalently, the irreducible components of a tree are smooth rational curves and they can be numbered as $C_0, \dots, C_m$ in such a way that C_0 is any given component and, for each $0 \le i \le m-1$, the curve C_{i+1} meets $C_0 \cup \cdots \cup C_i$ transversely in a single smooth point. We will always assume that the components of a rational tree are numbered in this fashion.

It is easy to construct a *smoothing* of a rational $\mathbf{k}$-tree C: let $T = \mathbf{P}^1_{\mathbf{k}}$ and blow up the smooth surface $C_0 \times T$ at the point $(C_0 \cap C_1) \times 0$, then at $((C_0 \cup C_1) \cap C_2) \times 0$ and so on. The resulting flat projective T-curve $\mathscr{C} \to T$ has fiber C above 0 and $\mathbf{P}^1_{\mathbf{k}}$ elsewhere. Given a smooth $\mathbf{k}$-variety X and a rational $\mathbf{k}$-tree C, any morphism $f : C \to X$ defines a $\mathbf{k}$-point $[f]$ of the T-scheme $\mathrm{Mor}_T(\mathscr{C}, X \times T)$ above $0 \in T(\mathbf{k})$. By 1.12, if $H^1(C, f^*T_X) = 0$, this T-scheme is smooth at $[f]$. This means that f can be smoothed to a rational curve $\mathbf{P}^1_{\mathbf{k}} \to X_{\mathbf{k}}$.

It will often be useful to be able to fix points in this deformation. If $p_1, \dots, p_r$ are smooth points of C, one can construct sections $\sigma_1, \dots, \sigma_r$ of the smoothing

$\mathscr{C} \to T$ such that $\sigma_i(0) = p_i$;[6] upon shrinking T, we may assume that they are disjoint. Let

$$g : \bigsqcup_{i=1}^{r} \sigma_i(T) \to X \times T$$

be the morphism $\sigma_i(t) \mapsto (f(p_i), t)$. Now, T-morphisms from $\mathscr{C}$ to $X \times T$ extending g are parametrized by the T-scheme $\mathrm{Mor}_T(\mathscr{C}, X \times T; g)$ whose fiber at 0 is $\mathrm{Mor}(C, X; p_i \mapsto f(p_i))$, and this scheme is smooth over T at $[f]$ when $H^1(C, (f^*T_X)(-p_1 - \cdots - p_r))$ vanishes.

It is therefore useful to have a criterion which ensures that this group vanish.

Lemma 2.36 *Let $C = C_0 \cup \cdots \cup C_m$ be a rational* **k***-tree. Let $\mathscr{E}$ be a locally free sheaf on C such that $(\mathscr{E}|_{C_i})(1)$ is nef for $i = 0$ and ample for each $i \in \{1, \ldots, m\}$. We have $H^1(C, \mathscr{E}) = 0$.*

Proof. We show this by induction on m, the result being obvious for $m = 0$. Set $C' = C_0 \cup \cdots \cup C_{m-1}$ and $C' \cap C_m = \{q\}$. There are exact sequences

$$0 \to (\mathscr{E}|_{C_m})(-q) \to \mathscr{E} \to \mathscr{E}|_{C'} \to 0$$

and

$$H^1(C_m, (\mathscr{E}|_{C_m})(-q)) \to H^1(C, \mathscr{E}) \to H^1(C', \mathscr{E}|_{C'}).$$

By hypothesis and induction, the spaces on both ends vanish, hence the lemma. □

2.7.2 Smoothing trees with free components

Proposition 2.37 *Let X be a smooth projective variety, let C be a rational tree, both defined over an algebraically closed field, and let $f : C \to X$ be a morphism whose restriction to each component of C is free.*

a) *The morphism f is smoothable, keeping any smooth point of C fixed, into a free rational curve.*

b) *If moreover f is r-free on one component C_0 ($r \geq 0$), f is smoothable, keeping fixed any r points of C_0 smooth on C and any smooth point of $C - C_0$, into an r-free rational curve.*

[6] Given a smooth point p of C, let C_1' be the component of C that contains p. Each connected component of $\overline{C - C_1'}$ is a rational tree hence can be blown-down, yielding a birational T-morphism $\varepsilon : \mathscr{C} \to \mathscr{C}'$, where $\mathscr{C}'$ is a ruled smooth surface over T, with fiber of 0 the curve $\varepsilon(C_1')$. Take a section of $\mathscr{C}' \to T$ that passes through $\varepsilon(p)$; its strict transform on $\mathscr{C}$ is a section of $\mathscr{C} \to T$ that passes through p.

PROOF. Item a) is a particular case of item b) (case $r = 0$). Let $p_1, \dots, p_r$ be smooth points of C on C_0 and let q be a smooth point of C, on the component C_i, with $i \neq 0$. The locally free sheaf $\big((f^*T_X)(-p_1 - \cdots - p_r - q)\big)|_{C_j}(1)$ is nef for $j = i$ and ample for $j \neq i$. The lemma implies $H^1(C, (f^*T_X)(-p_1 - \cdots - p_r - q)) = 0$, hence, by the discussion in §2.7.1,

- f is smoothable, keeping $f(p_0), \dots, f(p_r), f(q)$ fixed, to a rational curve $h : \mathbf{P}^1_{\mathbf{k}} \to X$;
- by semi-continuity, we may assume $H^1(\mathbf{P}^1_{\mathbf{k}}, (h^*T_X)(-r-1)) = 0$, hence h is r-free.

This proves the proposition. □

2.7.3 Smoothing combs

Definition 2.38 A $\mathbf{k}$-*comb* is a rational $\mathbf{k}$-tree with a distinguished irreducible component C_0 (the *handle*) such that all the other irreducible components (the *teeth*) meet C_0 (transversely in a single point).

Proposition 2.37 tells us that a morphism f from a comb C to a smooth variety can be smoothed when the restriction of f to each component of C is free. When C is a comb, we can relax this assumption: we only assume that the restriction of f to each tooth is free, and we get a smoothing of a subcomb if there are enough teeth.

Theorem 2.39 *Let C be a rational comb with m teeth and let $p_1, \dots, p_r$ be points on its handle C_0 which are smooth on C. Let X be a smooth projective variety and let $f : C \to X$ be a morphism.*

a) *Assume that the restriction of f to each tooth of C is free, and that*

$$m > (K_X \cdot f_*C_0) + (r-1)\dim(X) + \dim_{[f|_{C_0}]} \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; f|_{\{p_1,\dots,p_r\}}).$$

There exists a subcomb C' of C with at least one tooth such that $f|_{C'}$ is smoothable, keeping $f(p_1), \dots, f(p_r)$ fixed.

b) *Let s be a nonnegative integer such that $((f^*T_X)|_{C_0})(s)$ is nef. Assume that the restriction of f to each tooth of C is very free and that*

$$m > s + (K_X \cdot f_*C_0) + (r-1)\dim(X) + \dim_{[f|_{C_0}]} \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; f|_{\{p_1,\dots,p_r\}}).$$

There exists a subcomb C' of C with at least one tooth such that $f|_{C'}$ is smoothable, keeping $f(p_1), \dots, f(p_r)$ fixed, to a very free curve.

Proof. We construct a "universal" smoothing of the comb C as follows. Let $\mathscr{C}_m \to C_0 \times \mathbf{A}^m_{\mathbf{k}}$ be the blow-up of the (disjoint) union of the subvarieties $\{q_i\} \times \{y_i = 0\}$, where $y_1, \ldots, y_m$ are coordinates on $\mathbf{A}^m_{\mathbf{k}}$. Fibers of $\pi : \mathscr{C}_m \to \mathbf{A}^m_{\mathbf{k}}$ are subcombs of C, the number of teeth being the number of coordinates y_i that vanish at the point. Note that π is projective and flat, because its fibers are curves of the same genus 0. Let m' be a positive integer smaller than m, and consider $\mathbf{A}^{m'}_{\mathbf{k}}$ as embedded in $\mathbf{A}^m_{\mathbf{k}}$ as the subspace defined by the equations $y_i = 0$ for $m' < i \le m$. The inverse image $\pi^{-1}(\mathbf{A}^{m'}_{\mathbf{k}})$ splits as the union of $\mathscr{C}_{m'}$ and $m - m'$ disjoint copies of $\mathbf{P}^1_{\mathbf{k}} \times \mathbf{A}^{m'}_{\mathbf{k}}$. We set $\mathscr{C} = \mathscr{C}_m$.

Let σ_i be the constant section of π equal to p_i, and let

$$g : \bigsqcup_{i=1}^{r} \sigma_i(\mathbf{A}^m_{\mathbf{k}}) \to X \times \mathbf{A}^m_{\mathbf{k}}$$

be the morphism $\sigma_i(y) \mapsto (f(p_i), y)$. Since π is projective and flat, there is an $\mathbf{A}^m_{\mathbf{k}}$-scheme (1.12)

$$\rho : \operatorname{Mor}_{\mathbf{A}^m_{\mathbf{k}}}(\mathscr{C}, X \times \mathbf{A}^m_{\mathbf{k}}; g) \to \mathbf{A}^m_{\mathbf{k}}.$$

We will show that a neighborhood of $[f]$ in that scheme is not contracted by ρ to a point. Since the fiber of ρ at 0 is $\operatorname{Mor}(C, X; f|_{\{p_1,\ldots,p_r\}})$, it is enough to show

$$\dim_{[f]} \operatorname{Mor}(C, X; f|_{\{p_1,\ldots,p_r\}}) < \dim_{[f]} \operatorname{Mor}_{\mathbf{A}^m_{\mathbf{k}}}(\mathscr{C}, X \times \mathbf{A}^m_{\mathbf{k}}; g). \tag{2.4}$$

By the estimate (1.3), the right-hand side of (2.4) is at least

$$(-K_X \cdot f_*C) + (1 - r)\dim(X) + m.$$

The fiber of the restriction

$$\operatorname{Mor}(C, X; f|_{\{p_1,\ldots,p_r\}}) \to \operatorname{Mor}(C_0, X; f|_{\{p_1,\ldots,p_r\}})$$

is $\prod_{i=1}^m \operatorname{Mor}(C_i, X; f|_{\{q_i\}})$, so the left-hand side of (2.4) is at most

$$\begin{aligned}
& \dim_{[f|_{C_0}]} \operatorname{Mor}(C_0, X; f|_{\{p_1,\ldots,p_r\}}) + \sum_{i=1}^m \dim_{[f]} \operatorname{Mor}(C_i, X; f|_{\{q_i\}}) \\
= \; & \dim_{[f|_{C_0}]} \operatorname{Mor}(C_0, X; f|_{\{p_1,\ldots,p_r\}}) + \sum_{i=1}^m (-K_X \cdot f_*C_i) \\
< \; & m - (K_X \cdot f_*C) - (r - 1)\dim(X),
\end{aligned}$$

where we used first the local description of $\operatorname{Mor}(C_i, X; f|_{\{q_i\}})$ given in 1.10 and the fact that $f|_{C_i}$ being free, $H^1(C_i, f^*T_X(-q_i)|_{C_i})$ vanishes, and second the hypothesis. So (2.4) is proved.

Let T be the normalization of a 1-dimensional subvariety of $\mathrm{Mor}_{\mathbf{A}^m_{\mathbf{k}}}(\mathscr{C}, X \times \mathbf{A}^m_{\mathbf{k}}; g)$ passing through $[f]$ and not contracted by ρ. The morphism from T to $\mathrm{Mor}_{\mathbf{A}^m_{\mathbf{k}}}(\mathscr{C}, X \times \mathbf{A}^m_{\mathbf{k}}; g)$ corresponds to a morphism

$$\mathscr{C} \times_{\mathbf{A}^m_{\mathbf{k}}} T \to X.$$

After renumbering the coordinates, we may assume that $\{m'+1, \dots, m\}$ is the set of indices i such that y_i vanishes on the image of $T \to \mathbf{A}^m_{\mathbf{k}}$, where m' is a *positive* integer. As we saw above, $\mathscr{C} \times_{\mathbf{A}^m_{\mathbf{k}}} T$ splits as the union of $\mathscr{C}' = \mathscr{C}_{m'} \times_{\mathbf{A}^{m'}_{\mathbf{k}}} T$, which is flat over T, and some other "constant" components $\mathbf{P}^1_{\mathbf{k}} \times T$. The general fiber of $\mathscr{C}' \to T$ is $\mathbf{P}^1_{\mathbf{k}}$, its central fiber is the subcomb C' of C with teeth attached at the points q_i with $1 \le i \le m'$, and $f|_{C'}$ is smoothable keeping $f(p_1), \dots, f(p_r)$ fixed. This proves a).

Under the hypotheses of b), the proof of a) shows that there is a smoothing $\mathscr{C}' \to T$ of a subcomb C' of C with teeth $C'_1, \dots, C'_{m'}$, where $m' > s$, a section $\sigma' : T \to \mathscr{C}'$ passing through a point of C_0, and a morphism $F : \mathscr{C}' \to X$. Assume for simplicity that $\mathscr{C}'$ is smooth[7] and consider the locally free sheaf

$$\mathscr{E} = (F^*T_X)\Big(\sum_{i=1}^{s+1} C'_i - 2\sigma'(T)\Big)$$

on $\mathscr{C}'$. For $i \in \{1, \dots, s+1\}$, we have $((C'_i)^2) = -1$, hence the restriction of $\mathscr{E}$ to C'_i is nef, and so is $\mathscr{E}|_{C_0} \simeq (f^*T_X|_{C_0})(s-1)$. Using the exact sequences

$$0 \to \bigoplus_{i=1}^{m'} (\mathscr{E}|_{C'_i})(-1) \to \mathscr{E}|_{C'} \to \mathscr{E}|_{C_0} \to 0$$

and

$$0 = \bigoplus_{i=1}^{m'} H^1(C'_i, (\mathscr{E}|_{C'_i})(-1)) \to H^1(C', \mathscr{E}|_{C'}) \to H^1(C_0, \mathscr{E}|_{C_0}) = 0,$$

we obtain $H^1(C', \mathscr{E}|_{C'}) = 0$. By semi-continuity, this implies that a nearby smoothing $h : \mathbf{P}^1_{\mathbf{k}} \to X$ (keeping $f(p_1), \dots, f(p_r)$ fixed) of $f|_{C'}$ satisfies $H^1(\mathbf{P}^1_{\mathbf{k}}, (h^*T_X)(-2)) = 0$, hence h is very free. □

We saw in Corollary 2.28 that on a smooth separably rationally connected projective variety X, there is a very free rational curve through a *general* finite subset of X. We now show that we can do better.

Theorem 2.40 *Let X be a smooth separably rationally connected projective variety defined over an algebraically closed field. There is a very free rational curve through any finite subset of X.*

[7]For the general case, one needs to analyze precisely the singularities of $\mathscr{C}$ and proceed similarly, replacing C'_i by a suitable Cartier multiple.

Proof. We first prove that there is a very free rational curve through any point of X. Proceed by contradiction and assume that the set Y of points of X through which there are no very free rational curves is nonempty. Since X is separably rationally connected, by Corollary 2.28, its complement U is dense in X, and, since it is the image of the smooth morphism

$$\begin{array}{rcl} \mathrm{Mor}^{\mathrm{vfree}}(\mathbf{P}^1_{\mathbf{k}}, X) & \to & X \\ [f] & \mapsto & f(0), \end{array}$$

it is also open in X. By Remark 2.47, any point of Y can be connected by a chain of rational curves to a point of U, hence there is a rational curve $f_0 : \mathbf{P}^1_{\mathbf{k}} \to X$ whose image meets U and a point y of Y. Choose distinct points $t_1, \dots, t_m \in \mathbf{P}^1_{\mathbf{k}}$ such that $f_0(t_i) \in U$ and, for each $i \in \{1, \dots, m\}$, choose a very free rational curve $\mathbf{P}^1_{\mathbf{k}} \to X$ passing through $f_0(t_i)$. We can then assemble a rational comb with handle f_0 and m very free teeth. By choosing m large enough, this comb can by Theorem 2.39.b) be smoothed to a very free rational curve passing through y. This contradicts the definition of Y.

Let now $x_1, \dots, x_r$ be points of X. We proceed by induction on r to show the existence of a very free rational curve through $x_1, \dots, x_r$. Assume $r \geq 2$ and consider such a curve passing through $x_1, \dots, x_{r-1}$. We can assume that it is $(r-1)$-free and, by Proposition 2.27.a), that it passes through a general point of X. Similarly, there is a very free rational curve through x_r and any general point of X. These two curves form a chain that can be smoothed to an $(r-1)$-free rational curve passing through $x_1, \dots, x_r$ by Proposition 2.37.b). □

Remark 2.41 By composing it with a morphism $\mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^1_{\mathbf{k}}$ of degree s, this very free rational curve can be made s-free, with s greater than the number of points. One can then prove that a general deformation of that curve keeping the points fixed is an immersion if $\dim(X) \geq 2$ and an embedding if $\dim(X) \geq 3$ ([K1], Theorem II.1.8).

With a little more work, one can also find, on a smooth projective separably rationally connected variety, a very free rational curve passing through given points with prescribed tangents.

Theorem 2.42 *Let X be a smooth separably rationally connected projective variety defined over an algebraically closed field $\mathbf{k}$. Given distincts points $p_1, \dots, p_r$ of X and tangent directions $\ell_1, \dots, \ell_r$ at each of these points, there exists an unramified very free rational curve $\mathbf{P}^1_{\mathbf{k}} \to X$ with tangent ℓ_i at p_i.*

2.7.4 Another smoothing result for combs

We present here another approach to smoothing combs that was discovered in [GHS]. Instead of allowing loss of teeth as in §2.7.3, which is not convenient for certain applications, we assume that the teeth are general. The trick is to not deform the morphism from the comb to X, but to deform the corresponding subscheme of X.

Let $C = C_0 \cup C_1 \cup \cdots \cup C_m$ be a comb (with possible nonrational handle C_0) embedded in a smooth variety X. The tangent space to $\mathrm{Hilb}(X)$ at $[C]$ is isomorphic to $H^0(C, N_{C/X})$ (1.7) and $\mathrm{Hilb}(X)$ is smooth at $[C]$ if $H^1(C, N_{C/X}) = 0$ (1.8).

Theorem 2.43 *Let X be a smooth projective variety defined over an algebraically closed field, let m_0 be an integer, and let $C \subset X$ be a comb with handle C_0 and m teeth. If $m \gg 0$ and the teeth of C are very free and attached to general points of C_0 with general tangents, the comb $C \subset X$ deforms to a smooth projective curve $C' \subset X$ such that*

$$H^1(C', N_{C'/X}(-D')) = 0$$

for all divisors D' on C' of degree $\leq m_0$.

PROOF. Let $C_0 \cap C_i = \{q_i\}$ and set $Q = q_1 + \cdots + q_m$. A local calculation gives an exact sequence

$$0 \to N_{C_0/X} \to (N_{C/X})|_{C_0} \to \bigoplus_{i=1}^{m} T_{C_i, q_i} \to 0$$

and the first-order deformation of $C \subset X$ corresponding to an element of $H^0(C, N_{C/X})$ smoothes the double point q_i of C if and only if its image in T_{C_i, q_i} is nonzero.

Let L be the union of the teeth of C and consider the exact sequence

$$0 \to N_{L/X} \to (N_{C/X})|_L \to \bigoplus_{i=1}^{m} T_{C_0, q_i} \to 0.$$

Since $N_{L/X}(-Q)$ is nef on each tooth, we obtain $H^1(L, (N_{C/X})|_L(-Q)) = 0$. Using the exact sequence

$$0 \to \mathscr{O}_L(-Q) \to \mathscr{O}_C \to \mathscr{O}_{C_0} \to 0$$

tensored with $N_{C/X}$, we obtain that the restriction

$$H^0(C, N_{C/X}) \to H^0(C_0, (N_{C/X})|_{C_0})$$

is surjective and

$$H^1(C, N_{C/X}) \to H^1(C_0, (N_{C/X})|_{C_0}) \tag{2.5}$$

is bijective. We also have a diagramm

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (N_{C/X})|_{C_0}(-p_i) & \longrightarrow & (N_{C/X})|_{C_0} & \longrightarrow & N_{C/X,q_i} & \longrightarrow & 0 \\ & & & & & \searrow & \downarrow\!\!\!\downarrow & & \\ & & & & & & T_{C_i,q_i} & & \end{array}$$

For $m \gg 0$, we have $H^1(C_0, (N_{C/X})|_{C_0}(-q_i)) = 0$ by the lemma below. All in all, we obtain that the composition

$$H^0(C, N_{C/X}) \to H^0(C_0, (N_{C/X})|_{C_0}) \to T_{C_i,q_i}$$

is surjective: a general section of $(N_{C/X})|_{C_0}$ has nonzero image in each T_{C_i,q_i}, hence a general first-order deformation of C in X is smooth.

Lemma 2.44 *Let m_0 be an integer. For $m \gg 0$, we have*

$$H^1(C_0, (N_{C/X})|_{C_0}(-D)) = 0$$

for all divisors D on C_0 of degree $\leq m_0$.

Granting the lemma for a moment, and using the bijectivity of (2.5), we obtain $H^1(C, N_{C/X}) = 0$, so that deformations of C in X are unobstructed, and a general deformation C' is therefore smooth (of same genus g as C).

Apply again the lemma with a divisor of degree $g + m_0$ on $C_0 - \{q_1, \ldots, q_m\}$. We have by semicontinuity

$$H^1(C', N_{C'/X}(-D'_0)) = 0$$

for some divisor D'_0 on C' of degree $g + m_0$. Let D' be a divisor on C' of degree $\leq m_0$. By Riemann-Roch, we can write $D'_0 \sim_{\text{lin}} D' + E'$, where E' is effective, and the exact sequence

$$0 \to N_{C'/X}(-D'_0) \to N_{C'/X}(-D') \to N_{C'/X}(-D')|_{E'} \to 0$$

implies $H^1(C', N_{C'/X}(-D')) = 0$, which proves the proposition. $\square$

Proof of the Lemma. There is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{C_0/X}(-D) & \longrightarrow & (N_{C/X})|_{C_0}(-D) & \longrightarrow & \bigoplus_{i=1}^{m} T_{C_i,q_i} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_{C_0/X}(-D) & \longrightarrow & N_{C_0/X}(Q - D) & \longrightarrow & \bigoplus_{i=1}^{m} N_{C_0/X,q_i} & \longrightarrow & 0 \end{array}$$

For m not smaller than some integer m_1, we have

$$H^1(C_0, N_{C_0/X}(Q - D)) = 0$$

for all divisors D on C_0 of degree $\leq m_0$, so that

$$\delta_m : \bigoplus_{i=1}^{m} N_{C_0/X,q_i} \to H^1(C_0, N_{C_0/X}(-D))$$

is surjective. Consider preimages $(t_{1,j}, \ldots, t_{m_1,j})_{1\leq j\leq h}$ by δ_{m_1} of a basis $(\omega_j)_{1\leq j\leq h}$ of $H^1(C_0, N_{C_0/X}(-D))$. The restriction of $\delta_{m_1 h}$ to $\bigoplus_{i,j} \mathbf{C}t_{i,j}$ is surjective, hence also, for $m \geq m_1 h$, the restriction of δ_m to $\bigoplus_{i=1}^m T_{C_i,q_i}$ since the points q_i and the directions T_{C_i,q_i} are general. We obtain $H^1(C_0, (N_{C/X})|_{C_0}(-D)) = 0$, hence the lemma. □

2.8 Rationally chain connected varieties

We know study varieties for which two general points can be connected by a chain of rational curves (so this is a property weaker than rational connectedness). For the same reasons as in §2.3, we have to modify slightly this geometric definition. We will eventually show that rational chain connectedness implies rational connectedness for *smooth* varieties in *characteristic zero* (this will be proved in Theorem 2.49).

Definition 2.45 Let $\mathbf{k}$ be a field and let $\mathbf{K}$ be an algebraically closed extension of $\mathbf{k}$. A $\mathbf{k}$-variety X is *rationally chain connected* if it is *proper* and if there exist a $\mathbf{K}$-variety M and a closed subscheme $\mathscr{C}$ of $M \times X_{\mathbf{K}}$ such that:

- the fibers of the projection $\mathscr{C} \to M$ are connected proper curves with only rational components;
- the projection $\mathscr{C} \times_M \mathscr{C} \to X_{\mathbf{K}} \times X_{\mathbf{K}}$ is dominant.

This definition does not depend on the choice of the algebraically closed extension $\mathbf{K}$.

Remark 2.46 Rational chain connectedness is *not* a birational property: the projective cone over an elliptic curve E is rationally chain connected (pass through the vertex to connect any two points by a rational chain of length 2), but its canonical desingularization (a $\mathbf{P}^1_{\mathbf{k}}$-bundle over E) is not. However, it is a birational property among *smooth* projective varieties in characteristic zero, because it is then equivalent to rational connectedness (Theorem 2.49).

Remark 2.47 If X is a rationally chain connected variety, two general points of $X_{\mathbf{K}}$ can be connected by a chain of rational curves (and the converse is true when $\mathbf{K}$ is uncountable); actually *any* two points of $X_{\mathbf{K}}$ can be connected by a chain of rational curves (this follows from "general principles"; see [K1], Corollary 3.5.1).

Remark 2.48 Let $X \to T$ be a proper and equidimensional morphism with normal fibers defined over a field of characteristic zero. The set

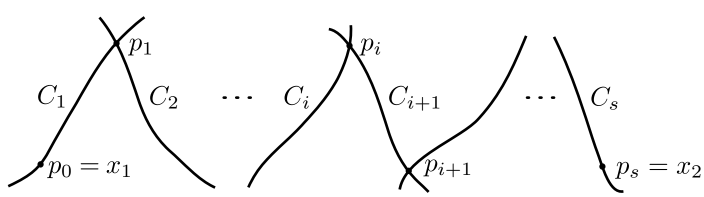
$$\{t \in T \mid X_t \text{ is rationally chain connected}\}$$

is closed (this is difficult; see [K1], Theorem 3.5.3). If the morphism is moreover smooth, this set is also open (Theorem 2.49 and Corollary 2.29).

We now prove that in characteristic zero, a *smooth* rationally chain connected variety is rationally connected (recall that this is false for singular varieties by Remark 2.46). The basic idea of the proof is to use Proposition 2.37 to smooth a rational chain connecting two points. The problem is to make *each* link free; this is achieved by adding lots of free teeth to each link and by deforming the resulting comb into a free rational curve, keeping the two endpoints fixed, in order not to lose connectedness of the chain.

Theorem 2.49 *A smooth rationally chain connected projective variety defined over a field of characteristic zero is rationally connected.*

PROOF. Let X be a smooth rationally chain connected projective variety defined over a field $\mathbf{k}$ of characteristic zero. We may assume that $\mathbf{k}$ is algebraically closed and uncountable. We will prove that X contains a very free rational curve. Let x_1 and x_2 be points of X. There exists a rational chain connecting x_1 and x_2, which can be described as the union of rational curves $f_i : \mathbf{P}^1_{\mathbf{k}} \to C_i \subset X$, for $i \in \{1, \ldots, s\}$, with $f_1(0) = x_1$, $f_i(\infty) = f_{i+1}(0)$, $f_s(\infty) = x_2$.



The rational chain connecting x_1 *and* x_2

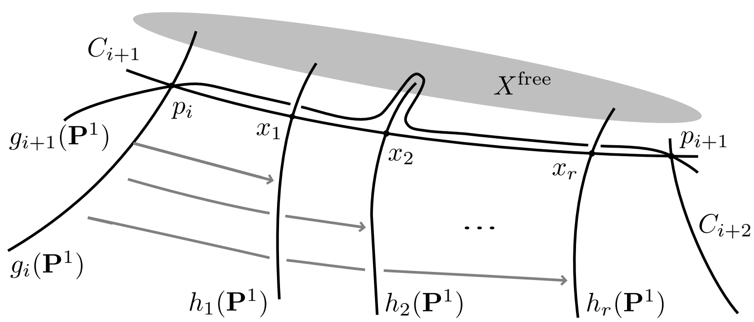
Assume that x_1 is in the subset X^{free} of X defined in Proposition 2.17, so that f_1 is free. We will construct by induction on i rational curves $g_i : \mathbf{P}^1_{\mathbf{k}} \to X$ with $g_i(0) = f_i(0)$ and $g_i(\infty) = f_i(\infty)$, whose image meets X^{free}.

When $i = 1$, take $g_1 = f_1$. Assume that g_i is constructed with the required properties; it is free, so the evaluation map

$$\begin{array}{rccc} \mathrm{ev}: & \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X) & \longrightarrow & X \\ & g & \longmapsto & g(\infty) \end{array}$$

is smooth at $[g_i]$ (this is not exactly Proposition 2.12, but follows from its proof). Let T be an irreducible component of $\mathrm{ev}^{-1}(C_{i+1})$ that passes through $[g_i]$; it dominates C_{i+1}.

We want to apply principle 2.19 to the family of rational curves on X parametrized by T: since the curve g_i meets X^{free}, so do very general members of the family T. Since they also meet C_{i+1} by construction, it follows that given a very general point q of C_{i+1}, there exists a deformation $h_q : \mathbf{P}^1_{\mathbf{k}} \to X$ of g_i which meets X^{free} and x.



Replacing a link with a free link

Picking distinct very general points $q_1, \dots, q_m$ in $C_{i+1} - \{p_i, p_{i+1}\}$, we get free rational curves $h_{q_1}, \dots, h_{q_m}$ which, together with the handle C_{i+1}, form a rational comb C with m teeth (as defined in Definition 2.38) with a morphism $f : C \to X$ whose restriction to the teeth is free. By Theorem 2.39.a), for m large enough, there exists a subcomb $C' \subset C$ with at least one tooth such that $f|_{C'}$ can be smoothed leaving p_i and p_{i+1} fixed. Since C' meets X^{free}, so does a very general smooth deformation by principle 2.19 again. So we managed to construct a rational curve $g_{i+1} : \mathbf{P}^1_{\mathbf{k}} \to X$ through $f_{i+1}(0)$ and $f_{i+1}(\infty)$ which meets X^{free}.

In the end, we get a chain of free rational curves connecting x_1 and x_2. By Proposition 2.37, this chain can be smoothed leaving x_2 fixed. This means that x_1 is in the closure of the image of the evaluation map $\mathrm{ev} : \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto x_2) \to X$. Since x_1 is any point in X^{free}, and the latter is dense in X because the ground field is uncountable, ev is dominant. In particular, its image meets the dense subset $X^{\mathrm{vfree}}_{x_2}$ defined in Proposition 2.34, hence there is a very free rational curve on X, which is therefore rationally connected (Corollary 2.28.a)). □

Corollary 2.50 *A smooth projective rationally chain connected complex variety is simply connected.*

Proof. A smooth projective rationally chain connected complex variety is rationally connected by the theorem, hence simply connected by Corollary 2.32.b). □

Chapter 3

Sections of families of separably rationally connected varieties

3.1 (C_1) fields

Definition 3.1 (Artin-Lang) A field $\mathbf{k}$ is (C_1) if any hypersurface of $\mathbf{P}^n_{\mathbf{k}}$ of degree $\leq n$ has a $\mathbf{k}$-rational point.

It is a classical theorem of Chevalley-Warning that any finite field is (C_1). The following is also classical ([T]).

Theorem 3.2 (Tsen) *The function field of a curve defined over an algebraically closed field is* (C_1).

Proof. Let B be a smooth projective curve defined over an algebraically closed field $\mathbf{k}$, with function field $\mathbf{K}$, and let $X \subset \mathbf{P}^n_{\mathbf{K}}$ be a hypersurface defined by a homogeneous polynomial $F(\mathbf{x}) = \sum_{|\mathbf{m}|=d} a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$ of degree d. The elements $a_{\mathbf{m}}$ of $\mathbf{K}$ can be viewed as sections of the same $\mathscr{O}_B(D)$ for some effective nonzero divisor D on B. We consider, for any positive integer q, the map

$$\begin{aligned} f_q : H^0(B, qD)^{n+1} &\longrightarrow H^0(B, D + dqD) \\ \mathbf{x} &\longmapsto \sum_{|\mathbf{m}|=d} a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}. \end{aligned}$$

For $q \gg 0$, the dimension of the $\mathbf{k}$-vector space on the left-hand-side is

$$a_q = (n+1)(q \deg(D) + 1 - g(B)),$$

whereas the dimension of the $\mathbf{k}$-vector space on the right-hand-side is

$$b_q = (dq+1)\deg(D) + 1 - g(B).$$

If $d \leq n$, we have $a_q > b_q$ for $q \gg 0$. Since $\mathbf{k}$ is algebraically closed, there exists then a point $\mathbf{x} \in \mathbf{P}(H^0(B, qD)^{n+1})(\mathbf{k})$ in the intersection of the b_q hypersurfaces given by the components of f_q. This $\mathbf{x}$ defines a $\mathbf{K}$-point in X. □

Corollary 3.3 *Let X be a projective surface with a morphism $\pi : X \to B$ onto a smooth projective curve B, defined over an algebraically closed field $\mathbf{k}$. Assume that the generic fiber is a geometrically integral curve of genus 0. Then π has a section and X is birational over B to $B \times \mathbf{P}^1_{\mathbf{k}}$.*

Proof. Any geometrically integral projective curve of genus 0 over any field $\mathbf{K}$ is isomorphic to a nondegenerate conic in $\mathbf{P}^2_{\mathbf{K}}$. By Tsen's theorem, this conic has a $\mathbf{K}$-point when $\mathbf{K} = k(B)$. □

Corollary 3.4 *Let X be a projective surface with a morphism $\pi : X \to B$ onto a smooth projective curve B, defined over an algebraically closed field $\mathbf{k}$. Assume that fibers over closed points are all isomorphic to $\mathbf{P}^1_{\mathbf{k}}$. There exists a locally free rank-2 sheaf $\mathscr{E}$ on B such that X is isomorphic over B to $\mathbf{P}(\mathscr{E})$.*

Proof. The sheaf $\pi_*\mathscr{O}_X$ is a locally free on B. For all closed points $b \in B$, we have $H^0(X_b, \mathscr{O}_{X_b}) = 1$, where X_b is the fiber of b. Since π is flat, the base change theorem ([H1], Theorem III.12.11) implies that $\pi_*\mathscr{O}_X$ has rank 1 hence is isomorphic to $\mathscr{O}_B$. In particular, the generic fiber of π is geometrically integral.

Similarly, since $H^1(X_b, \mathscr{O}_{X_b}) = 0$ for all closed points $c \in B$, the base change theorem again implies that the sheaf $R^1\pi_*\mathscr{O}_X$ is zero and that the generic fiber also has genus 0. By Corollary 3.3, π has a section $B \to X$ whose image we denote by C. We then have $(C \cdot X_c) = 1$ for all $c \in B$, hence, by the base change theorem again, $\mathscr{E} = \pi_*(\mathscr{O}_X(B))$ is a locally free rank-2 sheaf on B. Furthermore, the canonical morphism

$$\pi^*(\pi_*(\mathscr{O}_X(B))) \to \mathscr{O}_X(B)$$

is surjective, hence there exists, by the universal property of $\mathbf{P}(\mathscr{E})$ ([H1], Proposition II.7.12), a morphism $f : X \to \mathbf{P}(\mathscr{E})$ over B such that $f^*\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1) = \mathscr{O}_X(B)$. Since $\mathscr{O}_X(B)$ is very ample on each fiber, one checks that f is an isomorphism. □

3.2 Sections of families of separably rationally connected varieties

Since smooth hypersurfaces of degree $\leq n$ in $\mathbf{P}^n_{\mathbf{k}}$ are rationally connected varieties, the following question seems a natural extension of Tsen's theorem.

Question 3.5 Let X be a smooth proper separably rationally connected variety defined over a (C_1) field $\mathbf{k}$. Is $X(\mathbf{k})$ nonempty?

The answer is affirmative when $\mathbf{k}$ is a finite field ([Es]) and, by the following theorem, when $\mathbf{k}$ is the function field of a curve defined over an algebraically closed field.

Theorem 3.6 (de Jong-Graber-Harris-Starr) *A proper morphism from a variety onto a smooth curve, defined over an algebraically closed field, whose generic fiber is smooth and separably rationally connected, has a section.*

This theorem generalizes Corollary 3.3. It was first proved over $\mathbf{C}$ in [GHS], then in general in [dJS]. Here is one consequence of this fundamental theorem. It complements Corollary 2.32.

Corollary 3.7 (Kollár) *A smooth proper separably rationally connected variety defined over an algebraically closed field is algebraically simply connected.*

Proof. Let X be such a variety, defined over an algebraically closed field $\mathbf{k}$ which we can assume, by Corollary 2.32.b), to be of positive characteristic p. We need to show that any connected Galois covering $Y \to X$ is trivial. If this is not the case, there exists a factorization $Y \xrightarrow{u} X' \to X$, where u is Galois with Galois group of prime order ℓ. In this situation, X' is also proper, smooth, and separably rationally connected (Remark 2.24). The classification of cyclic coverings tells us (Kummer and Artin–Schreier theories) that there exist a covering of X' by affine open subsets U_i and

- if $\ell \neq p$, regular maps $h_i : U_i \to \mathbf{k}^*$ and $e_{ij} : U_i \cap U_j \to \mathbf{k}^*$ such that $e_{ij}e_{jk}e_{ki} = 1$ on $U_i \cap U_j \cap U_k$ and $e_{ij}^{\ell} = h_i h_j^{-1}$ on $U_i \cap U_j$, the affine open subset $u^{-1}(U_i)$ being defined by the equation $h_i = t_i^{\ell}$ in $U_i \times \mathbf{A}^1_{\mathbf{k}}$, and the glueings by $t_i = e_{ij}t_j$;
- if $\ell = p$, regular maps $h_i : U_i \to \mathbf{k}$ and $e_{ij} : U_i \cap U_j \to \mathbf{k}$ such that $e_{ij} + e_{jk} + e_{ki} = 0$ on $U_i \cap U_j \cap U_k$ and $e_{ij}^p - e_{ij} = h_i - h_j$ on $U_i \cap U_j$, the affine open subset $u^{-1}(U_i)$ being defined by the equation $h_i = t_i^p - t_i$ in $U_i \times \mathbf{A}^1_{\mathbf{k}}$, and the glueings by $t_i = e_{ij} + t_j$.

Letting $A = \operatorname{Spec} \mathbf{k}[t]$, we construct a subvariety $\mathscr{Y}$ of $X' \times A$ as the union of the affine varieties defined in $U_i \times \mathbf{A}^1_{\mathbf{k}} \times A$ as follows:

- if $\ell \neq p$, by the equation $th_i = t_i^{\ell}$, and the glueings $t_i = e_{ij}t_j$;
- if $\ell = p$, by the equation $h_i = t_i^p - t^{p-1}t_i$, and the glueings $t_i = te_{ij} + t_j$.

The fibers $\mathscr{Y} \to A$ outside of 0 are isomorphic to Y, hence are separably rationally connected. The fiber at 0 is isomorphic to X', with multiplicity ℓ. This morphism has a section by Theorem 3.6, hence $\ell = 1$. □

Corollary 3.8 *Let $u : X \to Y$ be a surjective morphism of proper varieties defined over a field of characteristic zero. Assume that Y is rationally connected and that a general fiber of u is rationally connected. Then X is rationally connected.*

PROOF. We may assume that the base field $\mathbf{k}$ is algebraically closed and uncountable, and that X and Y are smooth. Let x and x' be closed general points of X which are on smooth fibers of u, and let $\mathbf{P}^1_{\mathbf{k}} \to Y$ be a rational curve joining $u(x)$ and $u(x')$. Let $X' = X \times_Y \mathbf{P}^1_{\mathbf{k}}$. A general fiber of $u' : X' \to \mathbf{P}^1_{\mathbf{k}}$ is smooth and rationally connected (Corollary 2.29), hence there is a section $\sigma : \mathbf{P}^1_{\mathbf{k}} \to X'$ of u' (Theorem 3.6). The fibers $X'_x = {u'}^{-1}(x)$ and $X'_{x'} = {u'}^{-1}(x')$ are smooth and rationally connected, hence there are by Theorem 2.40 rational curves, one in X'_x joining x and $\sigma(u'(x))$, and one in $X'_{x'}$ joining x' and $\sigma(u'(x'))$. Together with the section σ, we obtain a chain of three rational curves joining x and x' in X'. It follows that X is rationally chain connected, hence rationally connected by Theorem 2.49. □

3.2.1 Strategy of the proof of the theorem

Let X be a proper separably rationally connected variety and let $u : X \to B$ be a surjective morphism onto a smooth curve. Assume that the generic fiber is smooth and separably rationally connected. We want to prove that u has a section. If all fibers of u are reduced, and if X is projective of dimension ≥ 3, our strategy is the following:

- construct a multisection $C \subset X$ of u, where C is a smooth projective curve contained in the smooth locus of u;
- in this case, by attaching to C sufficiently many general vertical rational curves and deforming the resulting comb as a *subscheme* of X, we may assume that the normal bundle to C in X, even after twisting by any divisor of degree bounded by some fixed constant, has vanishing higher cohomology;
- upon attaching to C sufficiently many vertical bisecant rational curves and deforming the resulting subscheme of X, we obtain a section.

More details are given below (§3.2.2). One reduces the general case to this special by a clever argument (§3.2.3).

3.2.2 Case where all fibers are reduced

Upon replacing X with $X \times \mathbf{P}^3_{\mathbf{k}}$, we can always assume $\dim(X) \geq 3$. The first step of the strategy outlined above follows from a count of parameters and the Bertini theorem. Let g be the genus of C and let b be the genus of B. If d is the degree of $u|_C$, its ramification R has degree $m_0 = 2g-2-d(2b-2)$. By Theorem 2.42, we can attach to C many vertical rational curves at general points with general tangents, and the resulting comb can be deformed as a subscheme of X by Theorem 2.43 to a smooth multisection $C' \subset X$ such that

$$H^1(C', N_{C'/X}(-D')) = 0$$

for all divisors D' on C' of degree $\leq m_0$. The genus of C' is still g, the morphism $u|_{C'}$ still has degree d hence the degree of its ramification R' is still m_0. In particular, $H^1(C', N_{C'/X}(-R')) = 0$ and can then conclude with the following theorem.

Theorem 3.9 *Let X be a proper variety, let B be a smooth projective curve, let $u : X \to B$ be a morphism with smooth rationally connected general fibers, and let $C \subset X$ be a smooth projective curve contained in the smooth locus of u. Let R be the ramification divisor of $u|_C : C \to B$. If $H^1(C, N_{C/X}(-R)) = 0$, the morphism u has a section.*

PROOF. We start from a finite separable morphism $f : C \to B$ of degree d between smooth projective curves and a finite subset $S \subset B$ which contains the branching locus of f. The following lemma shows that, upon identifying in C sufficiently many suitable pairs of points in fibers outside of $f^{-1}(S)$, one can deform the resulting (singular) covering $\mathscr{C}_0 \to B$ to a (reducible) covering $\mathscr{C}_\infty \to B$ with a section.

Lemma 3.10 *There exist a projective irreducible surface $\mathscr{C}$ and surjective morphisms*

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & B \\ {\scriptstyle p}\downarrow & & \\ \mathbf{P}^1_{\mathbf{k}} & & \end{array}$$

such that,

(1) for $t \in \mathbf{P}^1_{\mathbf{k}}$ general, the curve $\mathscr{C}_t := p^{-1}(t)$ is smooth;

(2) $\mathscr{C}_0$ is an integral nodal curve with normalization C, on which F restricts to f;

(3) $\mathscr{C}_\infty$ has an irreducible component B' which F sends isomorphically onto B;

(4) *$\mathscr{C}$ is smooth along $\mathscr{C}_0$ and at a general point of B', and $F(\mathrm{Sing}(\mathscr{C}_0)) \cap S = \varnothing$;*

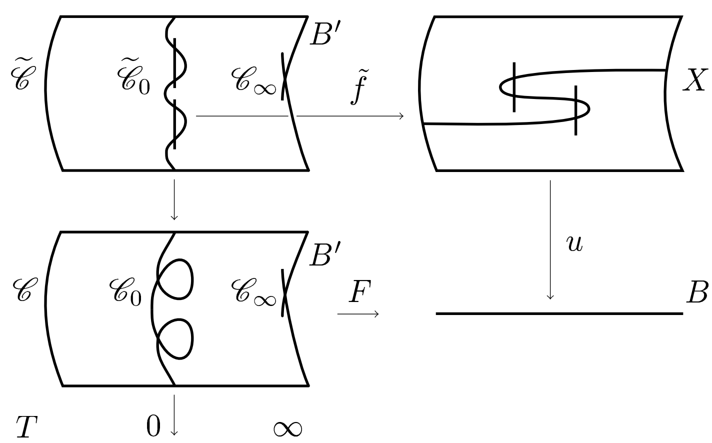
(5) *for each $s \in S$ and all $t \neq \infty$, the local formal structure of the coverings $F_t : \mathscr{C}_t \to B$ and $f : C \to B$ in a neighborhood of the fiber of s are the same: there are isomorphisms*

$$\mathscr{C}_t \times_B \operatorname{Spec} \widehat{\mathscr{O}}_{B,s} \simeq C \times_B \operatorname{Spec} \widehat{\mathscr{O}}_{B,s}$$

of $\operatorname{Spec} \widehat{\mathscr{O}}_{B,s}$*-schemes.*

Only item (5) is not elementary and it will not be used until the next section. Granting the lemma for the time being, we continue with the proof of Theorem 3.9.

Let $T \to \mathbf{P}^1_{\mathbf{k}}$ be a double cover ramified at 0 and let $\tilde{\mathscr{C}} \to \mathscr{C} \times_{\mathbf{P}^1_{\mathbf{k}}} T$ be the blow-up of the singular points that appear above the nodes of $\mathscr{C}_0$. We denote again by $0 \in T$ the only point above 0. The fiber of $\tilde{p} : \tilde{\mathscr{C}} \to T$ at 0 is reduced and is the union of C and of rational curves meeting each C transversely in two points. Let $\tilde{F} : \tilde{\mathscr{C}} \to \mathscr{C} \to B$ be the composed morphism. If we join in X the pairs of points in C which are identified in $\mathscr{C}_0$, by very free rational curves contained in fibers of u (Theorem 2.42) and avoiding its singular locus, we may define a morphism $\tilde{f} : \tilde{\mathscr{C}}_0 \to X$. We then have the following diagram:



Consider, in $\mathrm{Mor}_T(\tilde{\mathscr{C}}, X \times T)$, the subscheme $\mathrm{Mor}_T(\tilde{\mathscr{C}}, X \times T; B)$ that consists of T-morphisms $\tilde{\mathscr{C}} \to X \times T$ whose composition with $u \circ p_1$ is $\tilde{F}$. In other words, we are looking at the fiber at $[\tilde{F}]$ of the morphism

$$\mathrm{Mor}_T(\tilde{\mathscr{C}}, X \times T) \to \mathrm{Mor}_T(\tilde{\mathscr{C}}, B \times T)$$

induced by composition by u. One checks that $\mathrm{Mor}_T(\tilde{\mathscr{C}}, X \times T; B)$ is smooth at $[\tilde{f}]$ if $H^1(\tilde{\mathscr{C}}_0, \tilde{f}^* T_{X/B})$ vanishes, where $T_{X/B}$ is the relative tangent bundle, defined as the kernel of the tangent map $Tu : T_X \to f^*T_B$ (it is locally free on $\tilde{f}(\tilde{\mathscr{C}}_0)$). By construction, the restriction of this sheaf to the rational components of $\tilde{\mathscr{C}}_0$ is ample. On the component C, it restricts to $N_{C/X}(-R)$. One then shows as in Lemma 2.36 that the hypothesis $H^1(C, N_{C/X}(-R)) = 0$ implies $H^1(\tilde{\mathscr{C}}_0, \tilde{f}^*T_{X/B}) = 0$.

The morphism $\mathrm{Mor}_T(\tilde{\mathscr{C}}, X \times T; B) \to T$ is then smooth at $[\tilde{f}]$, hence contains a curve smooth at $[\tilde{f}]$ which dominates T. Let T' be a smooth compactification of that curve. There is a rational map

$$G : \tilde{\mathscr{C}} \times_T T' \dashrightarrow X$$

which coincides with $\tilde{f}$ on $\tilde{\mathscr{C}}_0$ and such that $u \circ G = \tilde{F}$. It is defined in the complement of a finite subset in the normal locus of $\tilde{\mathscr{C}} \times_T T'$, hence at the generic point of B'. The resulting rational map $G|_{B'} : B' \dashrightarrow X$ gives the desired section of u, and this proves Theorem 3.9. □

Let us go back now to the proof of Lemma 3.10.

Proof of Lemma 3.10. Let m be a large integer and let $g : C \to \mathbf{P}^1_{\mathbf{k}}$ be the morphism defined by two general sections of $\mathscr{O}_C(mf^*S)$. One checks that

- the morphism $(f, g) : C \to B \times \mathbf{P}^1_{\mathbf{k}}$ is birational onto its image C_0;
- the singularities of the curve C_0 are nodes which are not in $S \times \mathbf{P}^1_{\mathbf{k}}$;
- the curve C_0 is in the linear system $|p_1^*\mathscr{O}_C(dmS) \otimes p_2^*\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(d)|$.

This linear system also contains the divisor

$$C_\infty = (dmS \times \mathbf{P}^1_{\mathbf{k}}) + (B \times \{u_1, \dots, u_d\}),$$

where $u_1, \dots, u_d$ are general points of $\mathbf{P}^1_{\mathbf{k}}$. Consider the rational map

$$B \times \mathbf{P}^1_{\mathbf{k}} \dashrightarrow \mathbf{P}^1_{\mathbf{k}}$$

defined by the pencil $\langle C_0, C_\infty \rangle$. If $\mathscr{C} \to B \times \mathbf{P}^1_{\mathbf{k}}$ is the blow-up of the scheme $C_0 \cap C_\infty$, and $p : \mathscr{C} \to B \times \mathbf{P}^1_{\mathbf{k}} \dashrightarrow \mathbf{P}^1_{\mathbf{k}}$ and $F : \mathscr{C} \to B \times \mathbf{P}^1_{\mathbf{k}} \to B$ are the composed morphisms, one easily checks items (1), (2), (3), and (4) of the lemma.

Let s be a point of S and let π be a uniformizing parameter in the discrete valuation ring $\mathscr{O}_{B,s}$. In $\mathrm{Spec}\, \mathscr{O}_{B,s} \times \mathbf{P}^1_{\mathbf{k}}$, the curve C_0 is a defined by one equation σ_0 (with $\sigma_0(s, u_i) \neq 0$ for each i), the curve C_∞ by the equation $\pi^{md} \prod(u - u_i)$, hence the curve $\mathscr{C}_t$ by $\sigma_0 + t\pi^{md} \prod(u - u_i)$. Modulo π^{md}, the fibers at s of the morphisms $\mathscr{C}_t \to B$ induced by F are therefore isomorphic for $t \neq \infty$. Since these morphisms are finite (this follows from the explicit equation of $\mathscr{C}_t$ in $B \times \mathbf{P}^1_{\mathbf{k}}$), the following lemma implies item (5) of Lemma 3.10 and this finishes its proof. □

Lemma 3.11 *Let A be a complete discrete valuation ring with uniformizing parameter π and fraction field K. Let K' be a separable extension of K and let A' be the integral closure of A in K'. There exists a positive integer m_0 such that, for any discrete valuation ring A'' finite over A, the following conditions are equivalent:*

(i) the A-algebras A' and A'' are isomorphic;

(ii) there exists an integer $m > m_0$ such that the A-algebras $A'/\pi^m A'$ and $A''/\pi^m A''$ are isomorphic.

Proof. Since K' is a separable extension of K, there exists $a' \in A'$ such that $K' = K(a')$. Its minimal polynomial P has coefficients in A and satisfies $P'(a') \neq 0$. There exists a positive integer m_1 such that $P'(a') \in \pi^{m_1} A' - \pi^{m_1+1} A'$. Let $m_0 = 2m_1$.

Let us show (ii) $\Rightarrow$ (i). Since A'' is the integral closure of A in the fraction field K'' of A'', it is enough to show that the K-extensions K' and K'' are isomorphic. The A-modules A' and A'' are free, of same rank since $A'/\pi A' \simeq A''/\pi A''$. The K-extensions K' and K'' therefore have same degree and it is enough to show that there exists a K-homomorphism from K' to K''.

Let a'' be an element of A'' representing the image of a' by the projection $A' \to A'/\pi^m A' \simeq A''/\pi^m A''$. We have

$$P(a'') \in \pi^m A'' \quad \text{and} \quad P'(a'') \in \pi^{m_1} A'' - \pi^{m_1+1} A''.$$

The discrete valuation ring A'' is complete for the π-adic topology. By Hensel's lemma ([Bo], chap. III, §4, n°5, cor. 1), there exists $b'' \in A''$ such that $P(b'') = 0$ and $b'' \equiv a'' \pmod{\pi^{m-m_1}}$. This defines a morphism $A[X]/(P) \to A''$ which induces the K-homomorphism $K' \to K''$ we were looking for. □

3.2.3 Reduction to the case where the fibers are reduced

Following de Jong and Starr, we now prove that, in order to find a section of $u : X \to B$, it is enough to find one after any base change on B such that the fibers of the induced morphism are all reduced.

Proposition 3.12 *Let X be a proper normal variety, let B be a smooth projective curve, and let $u : X \to B$ be a morphism with reduced general fibers, defined over an uncountable algebraically closed field. Then,*

1) *there exist a smooth projective curve C and a separable finite morphism $C \to B$ such that the fibers of the induced morphism*

$$u_C : (C \times_B X)^{\mathrm{norm}} \to C$$

are reduced;

2) *if, for all separable finite morphisms $C \to B$ of smooth projective curves such that the fibers of u_C are reduced, u_C has a section, u has a section.*

Proof. Item 1) can be found in [E] and [BLR], th. 2.1′, p. 368.

Consider a morphism $f : C \to B$ satisfying the properties in item 1) and apply Lemma 3.10, taking for S the set of branching points of f and of points of B whose u-fiber is not reduced. It yields a diagram

$$\begin{array}{ccc} \mathscr{C} \times_B X & \longrightarrow & X \\ \downarrow & & \downarrow u \\ \mathscr{C} & \xrightarrow{F} & B \\ p\downarrow & & \\ T := \mathbf{P}^1_{\mathbf{k}}. & & \end{array}$$

Consider the fibres of $u_{\mathscr{C}_t} : (\mathscr{C}_t \times_B X)^{\mathrm{norm}} \to \mathscr{C}_t$ for $t \neq \infty$. Outside of S, they are reduced because the u-fiber is reduced. At a point of S, the formal fibers are reduced because they are isomorphic to the u_C-fiber. Hence all fibers of $u_{\mathscr{C}_t}$ are reduced. For t general, $\mathscr{C}_t$ is smooth hence there is by hypothesis a section $\sigma_t : \mathscr{C}_t \to (\mathscr{C}_t \times_B X)^{\mathrm{norm}}$ of $u_{\mathscr{C}_t}$. Since the base field is uncountable and the scheme $\mathrm{Mor}_T(\mathscr{C}, X \times T; B)$ defined on page 49 only has countably many irreducible components, at least one of them contains almost all the points corresponding to the morphisms $\mathscr{C}_t \to X$ induced by σ_t, hence contains a smooth irreducible curve that dominates $\mathbf{P}^1_{\mathbf{k}}$. Considering a smooth compactification T' of that curve, we obtain as on page 50 a rational map $\mathscr{C} \times_T T' \dashrightarrow X$ whose composition with $F \circ p_1$ is u and again, this gives the desired section of u. □

Chapter 4

Separably rationally connected varieties over non algebraically closed fields

4.1 Very free rational curves on separably rationally connected varieties over large fields

Let $\mathbf{k}$ be a field, let $\bar{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$, and let X be a smooth projective separably rationally connected $\mathbf{k}$-variety. Given any point of the $\bar{\mathbf{k}}$-variety $X_{\bar{\mathbf{k}}}$, there is a very free rational curve $f : \mathbf{P}^1_{\bar{\mathbf{k}}} \to X_{\bar{\mathbf{k}}}$ passing through that point (Theorem 2.40). One can ask about the existence of such a curve defined over $\mathbf{k}$, passing through a given $\mathbf{k}$-point of X. The answer is unknown in general, but Kollár proved that such a curve does exist over certain fields ([K3]).

Definition 4.1 A field $\mathbf{k}$ is *large* if for all smooth connected $\mathbf{k}$-variety X such that $X(\mathbf{k}) \neq \varnothing$, the set $X(\mathbf{k})$ is Zariski-dense in X.

The field $\mathbf{k}$ is large if and only if, for all smooth $\mathbf{k}$-curve C such that $C(\mathbf{k}) \neq \varnothing$, the set $C(\mathbf{k})$ is infinite.

Examples 4.2 1) Local fields such as $\mathbf{Q}_p$, $\mathbf{F}_p((t))$, $\mathbf{R}$, and their finite extensions, are large (because the implicit function theorem holds for analytic varieties over these fields).

2) For any field $\mathbf{k}$, the field $\mathbf{k}((x_1, \ldots, x_n))$ is large for $n \geq 1$.

Theorem 4.3 (Kollár) *Let $\mathbf{k}$ be a large field, let X be a smooth projective separably rationally connected $\mathbf{k}$-variety, and let $x \in X(\mathbf{k})$. There exists a very free $\mathbf{k}$-rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$ such that $f(0) = x$.*

PROOF. The $\mathbf{k}$-scheme $\mathrm{Mor}^{\mathrm{vfree}}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto x)$ is smooth and nonempty (because, by Corollary 2.28, it has a point in $\bar{\mathbf{k}}$). It therefore has a point in a finite separable extension ℓ of $\mathbf{k}$, which corresponds to a very free ℓ-rational curve $f_\ell : \mathbf{P}^1_\ell \to X_\ell$. Let $M \in A^1_{\mathbf{k}}$ be a closed point with residual field ℓ. The curve

$$C = (0 \times \mathbf{P}^1_{\mathbf{k}}) \cup (\mathbf{P}^1_{\mathbf{k}} \times M) \subset \mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}}$$

is a comb over $\mathbf{k}$ with handle $C_0 = 0 \times \mathbf{P}^1_{\mathbf{k}}$, and $\mathrm{Gal}(\ell/\mathbf{k})$ acts simply transitively on the set of teeth of $C_{\bar{\mathbf{k}}}$.

The constant morphism $0 \times \mathbf{P}^1_{\mathbf{k}} \to x$ and $f_\ell : \mathbf{P}^1_{\mathbf{k}} \times M \to X$ coincide on $0 \times M$ hence define a $\mathbf{k}$-morphism $f : C \to X$.

As in §2.7.1, let $T = \mathbf{P}^1_{\mathbf{k}}$, let $\mathscr{C}$ be the smooth $\mathbf{k}$-surface obtained by blowing-up the closed point $M \times 0$ in $\mathbf{P}^1_{\mathbf{k}} \times T$, and let $\pi : \mathscr{C} \to T$ be the first projection, so that the curve $\mathscr{C}_0 = \pi^{-1}(0)$ is isomorphic to C. We let $\mathscr{X} = X \times T$ and $x_T = x \times T \subset \mathscr{X}$, and we consider the inverse image ∞_T in $\mathscr{C}$ of the curve $\infty \times T$. The morphism f then defines $f_0 : \mathscr{C}_0 \to \mathscr{X}_0$, hence a $\mathbf{k}$-point of the T-scheme $\mathrm{Mor}_T(\mathscr{C}, \mathscr{X}; \infty_T \mapsto x_T)$ above $0 \in T(\mathbf{k})$.

Lemma 4.4 *The T-scheme $\mathrm{Mor}_T(\mathscr{C}, \mathscr{X}; \infty_T \mapsto x_T)$ is smooth at $[f_0]$.*

PROOF. It is enough to check $H^1(C, (f^*T_X)(-\infty)) = 0$. The restriction of $(f^*T_X)(-\infty)$ to the handle C_0 is isomorphic to $\mathscr{O}_{C_0}(-1)^{\oplus \dim(X)}$, and its restriction to the teeth is $f_\ell^* T_{X_\ell}$, hence is ample. We conclude with Lemma 2.36. □

This lemma already implies, since $\mathbf{k}$ is large, that $\mathrm{Mor}_T(\mathscr{C}, \mathscr{X}; \infty_T \mapsto x_T)$ has a $\mathbf{k}$-point whose image in T is not 0. It corresponds to a $\mathbf{k}$-rational curve $\mathbf{P}^1_{\mathbf{k}} \to X$ sending ∞ to x. However, there is no reason why this curve should be very free, and we will need to work a little bit more for that. By Lemma 4.4, there exists a smooth connected $\mathbf{k}$-curve

$$T' \subset \mathrm{Mor}_T(\mathscr{C}, \mathscr{X}; \infty_T \mapsto x_T)$$

passing through $[f_0]$ and dominating T. It induces a $\mathbf{k}$-morphism

$$F : \mathscr{C} \times_T T' \to X$$

such that $F(\infty_T \times_T T') = \{x\}$. Since $T'(\mathbf{k})$ is nonempty (it contains $[f_0]$), it is dense in T' because $\mathbf{k}$ is large. Let $T'_0 = (T - \{0\}) \times_T T'$ and let $t \in T'_0(\mathbf{k})$. The restriction of F to $C \times_T t$ is a $\mathbf{k}$-rational curve $F_t : \mathbf{P}^1_{\mathbf{k}} \to X$ sending ∞ to x.

For F_t to be very free, we need to check $H^1(\mathbf{P}^1_{\mathbf{k}}, (F_t^* T_X)(-2)) = 0$. By semi-continuity and density of $T'_0(\mathbf{k})$, it is enough to find an effective relative divisor $D \subset \mathscr{C}$, of degree ≥ 2 on the fibers of π, such that

$$H^1(\mathscr{C} \times_T [f_0], (F^* T_X)(-D')|_{\mathscr{C} \times_T [f_0]}) = 0,$$

where $D' = D \times_T T'$. Take for $D \subset \mathscr{C}$ the union of ∞_T and of the strict transform of $M \times T$ in $\mathscr{C}$. The divisor $(D_0)_{\bar{\mathbf{k}}}$ on the comb $(\mathscr{C} \times_T [f_0])_{\bar{\mathbf{k}}}$ has degree 1 on the handle and 1 on each tooth. We conclude with Lemma 2.36 again. $\square$

4.2 R-equivalence

Definition 4.5 Let X be a proper variety defined over a field $\mathbf{k}$. Two points x and y in $X(\mathbf{k})$ are *directly R-equivalent* if there exists a morphism $f : \mathbf{P}^1_{\mathbf{k}} \to X$ such that $f(0) = x$ and $f(\infty) = y$.

They are *R-equivalent* if there are points $x_0, \ldots, x_m \in X(\mathbf{k})$ such that $x_0 = x$ and $x_m = y$, and x_i and x_{i+1} are directly R-equivalent for all $i \in \{0, \ldots, m-1\}$. This is an equivalence relation on $X(\mathbf{k})$ called *R-equivalence.*

The following result is [K3], Corollary 1.8 ([K3], Corollary 1.5 is a more general analog for all local fields).

Theorem 4.6 (Kollár) *Let X be a smooth projective rationally connected real variety. The R-equivalence classes are the connected components of $X(\mathbf{R})$.*

Proof. Let $x \in X(\mathbf{R})$ and let $f : \mathbf{P}^1_{\mathbf{R}} \to X$ be a very free $\mathbf{R}$-rational curve such that $f(0) = x$ (Theorem 4.3). The $\mathbf{R}$-scheme $M = \mathrm{Mor}^{\mathrm{vfree}}(\mathbf{P}^1_{\mathbf{R}}, X; \infty \mapsto f(\infty))$ is locally of finite type and the evaluation morphism $M \times \mathbf{P}^1_{\mathbf{R}} \to X$ is smooth on $M \times A^1_{\mathbf{R}}$ (Proposition 2.27.a)). By the local inversion theorem, the induced map $M(\mathbf{R}) \times A^1(\mathbf{R}) \to X(\mathbf{R})$ is therefore open. Its image contains x, hence a neighborhood of x, which is contained in the R-equivalence class of x (any point in the image is directly R-equivalent to $f(\infty)$, hence R-equivalent to x).

It follows that R-equivalence classes are open and connected in $X(\mathbf{R})$. Since they form a partition of this topological space, they are its connected components. $\square$

Let X be a smooth projective separably rationally connected $\mathbf{k}$-variety. When $\mathbf{k}$ is large, there is a very free curve through any point of $X(\mathbf{k})$ (Theorem 4.3). When $\mathbf{k}$ is algebraically closed, there is such a curve through any finite subset of $X(\mathbf{k})$ (Theorem 2.40). This cannot hold in general, even when $\mathbf{k}$ is large (when $\mathbf{k} = \mathbf{R}$, two points belonging to different connected components of

$X(\mathbf{R})$ cannot be on the same rational curve defined over $\mathbf{R}$). Kollár proved that over a large field, this is the only obstruction (see [K4]).

Lemma 4.7 (Kollár) *Let X be a smooth projective separably rationally connected variety over a large field $\mathbf{k}$ and let x and $y \in X(\mathbf{k})$ be directly R-equivalent points. There exists a very free $\mathbf{k}$-rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$ such that $f(0) = x$ and $f(\infty) = y$.*

Sketch of proof. Since x and y are directly R-equivalent, there exists a $\mathbf{k}$-rational curve $g : \mathbf{P}^1_{\mathbf{k}} \to X$ such that $g(0) = x$ and $g(\infty) = y$. We want to make this rational curve very free. Using Theorem 4.3, we can form a comb C defined over $\mathbf{k}$, with handle $C_0 = g(\mathbf{P}^1_{\mathbf{k}})$, by adding as many very free $\mathbf{k}$-rational teeth as we want.

However, the argument of the proof of Theorem 2.49 does not work in our situation: it is based on the smoothing result Theorem 2.39, which uses a dimension count to prove that a nontrivial subcomb deforms (over an algebraically closed field). In our case, there is no way to make sure that this subcomb, or its deformation, will still be defined over $\mathbf{k}$.

Kollár's idea is to use instead (an analog of) Theorem 2.43 to deform the full comb C as a $\mathbf{k}$-subscheme of X. There are several conditions that need to be met at the same time:

- C needs to be embedded in X so its teeth need to meet C_0 in distinct points, and they should be mutually disjoint;
- we need a family of combs such that the tangents to the teeth at the points where they meet the handle C_0 span the normal bundle to C_0 in X at that point (this replaces the hypothesis that the tangent directions to the teeth have to be general in Theorem 2.43).

Kollár constructs a cone over $\bar{\mathbf{k}}$ in such a way that it satisfies these two conditions and, upon adding all the conjugates of its teeth, the resulting comb still satisfies them ([K4], Lemma 14 and Theorem 15). All in all, we obtain a $\mathbf{k}$-comb $C \subset X$ such that x and y are smooth points of C and

$$H^1(C, N_{C/X}(-x-y)) = 0.$$

We can then follow the arguments of the proof of Theorem 4.3: if $\mathscr{C} \to T = \mathbf{P}^1_{\mathbf{k}}$ is a smoothing of the comb C and $\mathscr{X} = X \times T$, and we denote by $\mathrm{Hilb}_T(\mathscr{X}; x, y)$ the subscheme of $\mathrm{Hilb}_T(\mathscr{X})$ that parametrizes subschemes of $\mathscr{X}$ that contain $x \times T$ and $y \times T$, we obtain that the morphism $\mathrm{Hilb}_T(\mathscr{X}; x, y) \to T$ is smooth at $[C]$ hence there exists ($\mathbf{k}$ being large) a very free $\mathbf{k}$-rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$ such that $f(0) = x$ and $f(\infty) = y$. $\square$

Theorem 4.8 (Kollár) *Let X be a smooth projective separably rationally connected variety defined over a large field $\mathbf{k}$. Let $x_1, \dots, x_r \in X(\mathbf{k})$ be R-equivalent points. There exists a very free $\mathbf{k}$-rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$ and distinct points $t_1, \dots, t_r \in \mathbf{k}$ such that $f(t_i) = x_i$ for all i.*

In particular, $x_1, \dots, x_r$ are all mutually directly R-equivalent. This theorem generalizes both Theorem 4.3 and Theorem 2.40.

Sketch of proof. First, we prove that direct R-equivalence is the same as R-equivalence on $X(\mathbf{k})$. Fix x, y, $z \in X(\mathbf{k})$ such that x is directly R-equivalent to y and y is directly R-equivalent to z. By Lemma 4.7, there exist very free rational $\mathbf{k}$-rational curves $f : \mathbf{P}^1_{\mathbf{k}} \to X$ and $g : \mathbf{P}^1_{\mathbf{k}} \to X$ such that $f(0) = x$, $f(\infty) = y$, $g(0) = y$, and $g(\infty) = z$. Using the arguments of the proof of Proposition 2.37, and the fact that $\mathbf{k}$ is large, one shows that the one-toothed comb obtained by glueing f and g deforms, fixing x and z, to a very free $\mathbf{k}$-rational curve $h : \mathbf{P}^1_{\mathbf{k}} \to X$ joining x and z.

Let now $x_1, \dots, x_r \in X(\mathbf{k})$ be R-equivalent points. Fix a point $y \in X(\mathbf{k})$ belonging to the R-equivalence class of the x_i. By what we just saw, the exist very free rational $\mathbf{k}$-rational curves $f_i : \mathbf{P}^1_{\mathbf{k}} \to X$ such that $f_i(0) = y$ and $f_i(\infty) = x_i$. One can form a $\mathbf{k}$-comb C with r teeth and define a morphism $C \to X$, using the f_i on the teeth and contracting the handle to y. Reasoning as in the proof of Theorem 4.3, one shows that this morphism can be smoothed, fixing the points $x_1, \dots, x_r$, to a very free $\mathbf{k}$-rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$ that satisfies the conditions in the theorem. □

Bibliography

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, Formal and rigid geometry. IV. The reduced fibre theorem, *Invent. Math.* **119** (1995), 361–398.

[Bo] N. Bourbaki, *Éléments de mathématique. Algèbre commutative.* Chapitres 1 à 4. Masson, Paris, 1985.

[C] F. Campana, Remarques sur le revêtement universel des variety kählériennes compactes, *Bull. Soc. Math. Fr.* **122** (1994), 255–284.

[CL] A. Chambert-Loir, À propos du groupe fondamental des variétés rationnellement connexes, eprint `math.AG/0303051.`

[D] O. Debarre, *Higher-dimensional algebraic geometry,* Universitext. Springer-Verlag, New York, 2001.

[dJS] A.J. de Jong, J. Starr, Every rationally connected variety over the function field of a curve has a rational point, *Amer. J. Math.* **125** (2003), 567–580.

[De] P. Deligne, Théorie de Hodge II, *Publ. Math. Inst. Hautes Études Sci.* **40** (1972), 5–57.

[E] H. Epp, Eliminating wild ramification, *Invent. Math.* **19** (1973), 235–249.

[Es] H. Esnault, Varieties over a finite field with trivial Chow group of 0-cycles have a rational point, *Invent. Math.* **151** (2003), 187–191.

[GHS] T. Graber, J. Harris, J. Starr, Families of Rationally Connected Varieties, *J. Amer. Math. Soc.* **16** (2003), 57–67.

[G1] A. Grothendieck, Techniques de construction et théorèmes d'existence en géométrie algébrique IV : les schemes de Hilbert, Séminaire Bourbaki, Exp. **221**, 1960/61. Astérisque hors série **6**, Soc. Math. Fr., 1997.

[G2] A. Grothendieck, Eléments de géométrie algébrique IV, 2, Inst. Hautes Études Sci. Publ. Math. **24**, 1965.

[H1] R. Hartshorne, *Algebraic Geometry,* Graduate Texts in Mathematics **52**, Springer Verlag, New York, 1977.

[K1] J. Kollár, *Rational Curves on Algebraic Varieties.* Ergebnisse der Mathematik und ihrer Grenzgebiete **32**, Springer Verlag, Berlin, 1996.

[K2] J. Kollár, Nonrational hypersurfaces, *J. Amer. Math. Soc.* **8** (1995), 241–249.

[K3] J. Kollár, Rationally connected varieties over local fields, *Ann. of Math.* **150** (1999), 357–367.

[K4] J. Kollár, Specialization of zero cycles, *Publ. Res. Inst. Math. Sci.* **40** (2004), 689–708.

[KMM] J. Kollár, Y. Miyaoka, S. Mori, Rationally Connected Varieties, *J. Alg. Geom.* **1** (1992), 429–448.

[L] R. Lazarsfeld, *Positivity in Algebraic Geometry I & II*, Ergebnisse der Mathematik und ihrer Grenzgebiete **48** & **49**, Springer Verlag, Berlin, 2004.

[M] S. Mori, Projective manifolds with ample tangent bundles, *Ann. of Math.* **110** (1979), 593–606.

[S] E. Sernesi, *Deformations of algebraic schemes,* Grundlehren der Mathematischen Wissenschaften **334**. Springer-Verlag, Berlin, 2006.

[T] C. Tsen, Quasi-algebraisch-abgeschlossene Funktionenkörper, *J. Chinese Math.* **1** (1936), 81–92.