MODULI AND PERIODS OF HYPERKÄHLER MANIFOLDS

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Abstract. This is an introduction to hyperkähler manifolds, their moduli spaces, and their period maps. Few proofs are provided, so that these notes should be seen as an introductory guide to the theory more than a complete reference text.

Contents

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1. Hyperkähler manifolds: general theory

We work over the complex numbers.

In dimension 1, compact kähler manifolds with trivial first (real) Chern class are (projective) elliptic curves. In dimension 2, they are either bielliptic surfaces (they are projective, their integral first Chern class has order 2, 3, 4, or 6, and their universal cover is a complex torus), Enriques surfaces (they are projective, their integral first Chern class has order 2 and their universal cover is a K3 surface), K3 surfaces, or complex tori.

In higher dimensions, the situation gets more complicated and there is no complete classification. Even the finiteness of the number of deformation types is unknown.

1.1. The decomposition theorem for compact kähler manifolds with trivial first Chern class.

The first result that we present, without proof, shows that hyperkähler manifolds are one of the building blocks for all compact kähler manifolds with trivial real first Chern class.

Theorem 1.1 (Beauville, Bogomolov)**.** *Let X be a compact kähler manifold with $c_1(X)_{\mathbf{R}} = 0$. There exists a finite étale cover $\widetilde{X} \to X$ such that*

$$\widetilde{X} \simeq T \times \prod_i CY_i \times \prod_j HK_j,$$

where T is a complex torus, CY_i is a Calabi–Yau manifold and HK_j is a hyperkähler manifold.

The proof of this theorem depends on a difficult result of S.T. Yau: the existence on X of a Ricci-flat kähler metric.

Let us define the Calabi–Yau and hyperkähler manifolds that appear in the theorem..

Definition 1.2. A *Calabi–Yau manifold* is a simply connected compact kähler manifold X of dimension $n \geq 3$ with K_X trivial and $H^0(X, \Omega_X^p) = 0$ for all $0 < p < n$.

One has in particular $\chi(X, \mathscr{O}_X) = 1+(-1)^n$. Examples are easy to find: any smooth complete intersection of multidegree $(d_1, \dots, d_r)$ in $\mathbf{P}^{n+r}$, with $d_1 + \cdots + d_r = n + r + 1$ and $n \geq 3$, is a Calabi–Yau manifold of dimension n.

Definition 1.3. A *hyperkähler manifold* is a simply connected compact kähler manifold X with an everywhere nondegenerate holomorphic 2-form σ_X such that $H^0(X, \Omega_X^2) = \mathbf{C}\sigma_X$.

A hyperkähler manifold X has even dimension $n = 2m$. One has $H^0(X, \Omega_X^p) = \mathbf{C}\sigma_X^{\wedge p/2}$ if p is even and $H^0(X, \Omega_X^p) = 0$ if p is odd; in particular, $\chi(X, \mathscr{O}_X) = m + 1$.

In dimension 2, hyperkähler manifolds are K3 surfaces. They exist in any even dimension but are harder to construct (see Section 1.2 below).

It follows from the decomposition theorem that the integral Chern class $c_1(X) \in H^2(X, \mathbf{Z})$ of a compact kähler manifold X with $c_1(X)_{\mathbf{R}} = 0$ is torsion (one has $c_1(\widetilde{X}) = 0$).[1]

Example 1.4. A compact kähler fourfold X with $c_1(X)_{\mathbf{R}} = 0$ and $\chi(X, \mathscr{O}_X) = 3$ is a hyperkähler manifold.

Indeed, we have $\chi(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = 3d$, where d is the degree of the étale cover $\widetilde{X} \to X$. Since this is nonzero, there is no torus factor, hence either $\widetilde{X}$ is a Calabi–Yau fourfold, but then $\chi(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = 2$,

[1] In dimension 3, Beauville proved that it has order ≤ 66 and that this bound is optimal ([Be, Proposition 8]).

which is impossible, or it is a product of two K3 surfaces, but then $\chi(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = 4$, which is impossible, or it is a hyperkähler manifold, but then $\chi(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = 3$, hence $d = 1$.

1.2. **Examples of hyperkähler manifolds.**

Two infinite series of examples were constructed by Beauville.

1.2.1. *Punctual Douady spaces of K3 surfaces.* If S is a compact analytic surface, the Douady space $S^{[m]}$ parametrizes analytic subspaces of S of length m; we will call it the mth Douady space of S. It is a smooth (Fogarty) compact complex manifold of dimension $2m$.

Theorem 1.5 (Beauville, Fujiki)**.** *Let S be a K3 surface. The Douady space $S^{[m]}$ is a hyperkähler manifold of dimension $2m$.*

All K3 surfaces are kähler (Siu) and simply connected; this implies that $S^{[m]}$ is also kähler (Varouchas) and simply connected.

The structure of $S^{[m]}$ is quite complicated, but it is enough to construct the symplectic form on the big open subset parametrizing subspaces of length m where at most two points coincide. It is obtained from the corresponding open subset in S^m by blowing up the diagonals and taking the quotient by the action of the symmetric group $\mathfrak{S}_m$, simply ramified along the exceptional divisor (in particular, this describes $S^{[2]}$ completely).

When S is projective, so is $S^{[m]}$ (it is then called a Hilbert scheme).

1.2.2. *Punctual Douady spaces of complex tori of dimension 2.* One can do the same construction starting from a complex torus T of dimension 2. One gets again a manifold $T^{[m+1]}$, but it is not simply connected.

Theorem 1.6 (Beauville)**.** *Let T be a complex torus of dimension 2. The fiber $K_m(T)$ of 0 of the sum map $T^{[m+1]} \to T$ is a hyperkähler manifold of dimension $2m$.*

The manifold $K_m(T)$ is called the mth Kummer variety of T.[2]

1.2.3. *Other examples.* We will see soon (Examples 1.8 and 1.9) that smooth deformations of the hyperkähler manifolds constructed above are still hyperkähler manifolds which are not in general of that type.

O'Grady constructed two other families of hyperkähler manifolds (of respective dimensions 6 and 10) which are not deformations of the examples above (they have different topological types).

These are the only examples known.

1.3. **Deformations of hyperkähler manifolds.**

[2]When $m = 1$, it is isomorphic to the minimal desingularization of the quotient $T/\{\pm \mathrm{Id}_T\}$, usually called the Kummer variety of T, hence the name.

1.3.1. *Small deformations.* Any smooth kähler deformation of a hyperkähler manifold is again a hyperkähler manifold (any small deformation is kähler, but there are large deformations of compact kähler manifolds that cease to be kähler).

Let X be a hyperkähler manifold. Then $T_X \simeq \Omega_X$, hence $H^0(X, T_X) \simeq H^0(X, \Omega_X) \subset H^1(X, \mathbf{C})$ vanishes. By Kuranishi's theorem, there is a universal local deformation. Its base is (a germ of) an analytic subspace of $H^1(X, T_X)$ defined by $h^2(X, T_X)$ equations. Note that $h^2(X, T_X) = h^2(X, \Omega_X) = h^{1,2}(X) = \frac{1}{2}b_3(X)$ can very well be nonzero. However, we have the following result.

Theorem 1.7 (Bogomolov, Tian, Todorov)**.** *Small deformations of a hyperkähler manifold X are unobstructed. The base of the universal local deformation of X is therefore smooth of dimension $h^1(X, T_X) = b_2(X) - 2$.*

Example 1.8. Let S be a K3 surface. Beauville showed that for any $m \geq 2$, one has

$$H^2(S^{[m]}, \mathbf{Z}) \simeq H^2(S, \mathbf{Z}) \oplus \mathbf{Z}\delta$$

for a certain class δ. In particular, $b_2(S^{[m]}) = b_2(S) + 1 = 23$: the universal local deformation space of $S^{[m]}$ has dimension 21 and a very general deformation of $S^{[m]}$ is not of that type.

Example 1.9. If T is a complex torus of dimension 2, one has again, for $m \geq 2$,

$$H^2(K_m(T), \mathbf{Z}) \simeq H^2(T, \mathbf{Z}) \oplus \mathbf{Z}\delta$$

for a certain class δ, so that $b_2(T^{[m]}) = b_2(T) + 1 = 7$: the universal local deformation space of $K_m(T)$ has dimension 5 and a very general deformation of $K_m(T)$ is not of that type.

1.3.2. *The local period map.* Let X be a hyperkähler manifold and let $\pi\colon \mathscr{X} \to B$ be its universal local deformation, with $X = \mathscr{X}_0$, the fiber at $0 \in B$. Upon shrinking B, we may assume that it is smooth and simply connected. The family $\pi\colon \mathscr{X} \to B$ is then differentially trivial and we may uniquely identify each $H^2(\mathscr{X}_b, \mathbf{Z})$ with $H^2(X, \mathbf{Z})$ (the local system $R^2\pi_*\mathbf{Z}$ is trivial because B is simply connected). The (local) *period map* is the map

$$\begin{aligned} \wp\colon B &\longrightarrow \mathbf{P}(H^2(X, \mathbf{C})) \\ b &\longmapsto [H^{2,0}(\mathscr{X}_b)]. \end{aligned}$$

Griffiths showed that $\wp$ is holomorphic and that its differential at the point $0 \in B$ is the composition

$$\begin{aligned} T_{B,0} \xrightarrow{\sim} H^1(X, T_X) &\xrightarrow{g} \operatorname{Hom}(H^{2,0}(X), H^1(X, \Omega_X)) \\ &\subset \operatorname{Hom}(H^{2,0}(X), H^2(X, \mathbf{C})/H^{2,0}(X)) = T_{\mathbf{P}(H^2(X,\mathbf{C})),[H^{2,0}(X)]}, \end{aligned}$$

where g is induced by interior product.

Theorem 1.10 (Local Torelli theorem)**.** *The local period map of a hyperkähler manifold is an embedding and its image is (a germ of) a smooth analytic hypersurface in $\mathbf{P}(H^2(X, \mathbf{C}))$.*

Proof. The symplectic form σ_X induces an isomorphism $T_X \xrightarrow{\sim} \Omega_X$. The map g above is therefore an isomorphism, because it factors as

$$g\colon H^1(X, T_X) \xrightarrow[\sim]{\lrcorner\sigma_X} H^1(X, \Omega_X) \xrightarrow{\sim} \operatorname{Hom}(H^{2,0}(X), H^1(X, \Omega_X))$$

$$\alpha \longmapsto (\sigma_X \longmapsto \alpha).$$

This finishes the proof of the theorem. ∎

1.4. **The Beauville–Bogomolov quadratic form.** One can be much more precise: we are going to show that the image of the local period map is an open subset of a smooth quadric in $\mathbf{P}(H^2(X,\mathbf{C}))$.

Theorem 1.11 (Beauville, Bogomolov, Fujiki)**.** *Let X be a hyperkähler manifold of dimension $n=2m$. There exist a unique (integral) quadratic form q_X on $H^2(X,\mathbf{C})$, which takes integral values on $H^2(X,\mathbf{Z})$, and a unique constant $c_X\in\mathbf{Q}_{>0}$ such that*

(a) *for all $\alpha\in H^2(X,\mathbf{C})$, one has*

$$\int_X \alpha^n = c_X q_X(\alpha)^m, \tag{1}$$

(b) *the form q_X is nondivisible on $H^2(X,\mathbf{Z})$ and takes positive values on kähler classes in $H^2(X,\mathbf{C})$.*

The form q_X is called the Beauville–Bogomolov form and the constant c_X the Fujiki constant.

Proof. As we saw above, the image of the local period map is a smooth analytic hypersurface $D\subset\mathbf{P}(H^2(X,\mathbf{C}))$ and

$$\forall\sigma\in D\qquad \sigma^{m+1}=0 \tag{2}$$

(because it has type $(n+2,0)$ in $H^{n+2}(\mathscr{X}_b,\mathbf{C})$).

The vanishing of the polynomial

$$f(\alpha):=\int_X\alpha^n$$

defines an algebraic hypersurface of degree $n=2m$ in $\mathbf{P}(H^2(X,\mathbf{C}))$. By taking derivatives in (2), one sees that f vanishes along D (hence along its Zariski closure $\overline{D}$) with multiplicity $\geq m$. Moreover, D is not contained in a hyperplane (otherwise, the hyperplane $H^{2,0}(\mathscr{X}_b)\oplus H^{1,1}(\mathscr{X}_b)=\operatorname{Im}(T_{\wp,b})$ would be the same for all $b\in B$), hence $\overline{D}$ is a quadric with equation $q=0$ and $f=q^m$. Since f has rational coefficients, one can choose q_X proportional to q integral and nondivisible, and c_X as in (1). This formula determines q_X up to sign. We choose this sign so that q_X is positive on the convex cone of kähler forms. ☐

We now prove some properties of the Beauville–Bogomolov form. Recall that if ω is a kähler form on a compact manifold X of dimension n, one defines the primitive cohomology by

$$H^2_{\mathrm{prim}}(X,\mathbf{R}):=\operatorname{Ker}\bigl(H^2(X,\mathbf{R})\xrightarrow{\cdot\,\omega^{n-1}}H^{2n}(X,\mathbf{R})\xrightarrow[\sim]{\int_X}\mathbf{R}\bigr).$$

Lefschetz theory tells us that the quadratic form

$$q_\omega(\alpha):=\int_X\alpha^2\omega^{n-2}$$

is nondegenerate of signature $(2,b_2(X)-3)$ on $H^2_{\mathrm{prim}}(X,\mathbf{R})$.

Theorem 1.12. *Let X be a hyperkähler manifold with symplectic form σ_X. Its Beauville–Bogomolov form q_X satisfies the following properties.*

(a) *One has*

$$q_X(\sigma_X)=0\quad,\quad q_X(\sigma_X,\bar\sigma_X)>0.$$

(b) *One has*

$$H^{1,1}(X)=(H^{2,0}(X)\oplus H^{0,2}(X))^{\perp_{q_X}}.$$

(c) *If ω is a kähler form on X, the restriction of q_X to $H^2_{\rm prim}(X,\mathbf{R})$ is a positive multiple of the form q_ω.*
(d) *The signature of q_X on $H^2(X,\mathbf{R})$ is $(3, b_2(X)-3)$.*

By (d), the form q_X is nondegenerate. We denote by (Λ_X, q_X) the lattice $(H^2(X,\mathbf{Z}), q_X)$.

Proof. Taking the derivative of (1) at a kähler form ω, we get

$$\forall \beta \in H^2(X,\mathbf{C}) \qquad n\int_X \omega^{n-1}\beta = mc_X q_X(\omega)^{m-1} q_X(\omega,\beta). \tag{3}$$

Since $q_X(\omega)>0$, this implies that $H^2_{\rm prim}(X,\mathbf{R})$ is the same as $\omega^{\perp q_X}$. Taking another derivative, we get

$$n(n-1)\int_X \omega^{n-2}\beta\gamma = m(m-1)c_X q_X(\omega)^{m-2} q_X(\omega,\beta) q_X(\omega,\gamma) + mc_X q_X(\omega)^{m-1} q_X(\beta,\gamma)$$

for all $\beta,\gamma \in H^2(X,\mathbf{C})$. When $\beta,\gamma\in H^2_{\rm prim}(X,\mathbf{R}) = \omega^{\perp q_X}$, this shows (c), which in turn implies (a) and (b) (since $\sigma_X, \bar\sigma_X \in H^2_{\rm prim}(X,\mathbf{C})$) and (d) (since $q_X(\omega)>0$). □

Remark 1.13. Let X be a hyperkähler manifold. Fujiki proved that q_X is proportional to the quadratic form $\alpha \longmapsto \int_X \sqrt{\mathrm{td}(X)}\,\alpha^2$ (where $\mathrm{td}(X)$ is the *Todd class* of X). In particular, it only depends, up to multiplication by a positive rational constant (see footnote 8), on the topological structure of X, not on its complex structure.

Example 1.14. Let S be a K3 surface. Recall from Example 1.8 that for any $m\ge 2$, one has

$$H^2(S^{[m]},\mathbf{Z}) \simeq H^2(S,\mathbf{Z}) \oplus \mathbf{Z}\delta.$$

This decomposition is orthogonal for the form $q_{S^{[m]}}$, the restriction of $q_{S^{[m]}}$ to $H^2(S,\mathbf{Z})$ is the usual intersection product (giving rise to the K3-lattice $\Lambda_{\rm K3}$), and $q_{S^{[m]}}(\delta) = -2(m-1)$. One has $c_{S^{[m]}} = \frac{(2m)!}{m!2^m}$.

Example 1.15. If T is a complex torus of dimension 2, one has, for $m\ge 2$ (Example 1.9),

$$H^2(K_m(T),\mathbf{Z}) \simeq H^2(T,\mathbf{Z}) \oplus \mathbf{Z}\delta.$$

This decomposition is orthogonal for the form $q_{K_m(T)}$, the restriction of $q_{K_m(T)}$ to $H^2(T,\mathbf{Z})$ is the usual intersection product (giving rise to the lattice $U^{\oplus 3}$, where U is the hyperbolic plane $\big(\mathbf{Z}^2, \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\big)$), and $q_{K_m(T)}(\delta) = -2(m+1)$. One has $c_{K_m(T)} = \frac{(2m)!(m+1)}{m!2^m}$.

To end this section, I will mention without proof an important result of Huybrechts. Recall that if ω is a kähler class on a hyperkähler manifold X, one has $q_X(\omega)>0$. In particular, if L is an ample line bundle on X, one has $q_X(c_1(L))>0$. The following theorem states a sort of converse (note that an L with $q_X(c_1(L))>0$ is not necessarily ample, but it becomes plus or minus ample on a small deformation).

Theorem 1.16 (Huybrechts, Demailly–Boucksom)**.** *A hyperkähler manifold is projective if and only if there is a line bundle L on X such that $q_X(c_1(L))>0$.*

1.5. **The Hirzebruch–Riemann–Roch formula.** The classical Hirzebruch–Riemann–Roch formula says that the Euler characteristic of a line bundle L on a compact complex manifold X of dimension n is given by the formula

$$\chi(X,L) = \sum_{i=0}^n \frac{1}{i!}\int_X c_1(L)^i\, \mathrm{td}_{n-i}(X),$$

where $\mathrm{td}(X)$ is as above the Todd class of X. On a hyperkähler manifold, it takes a particularly simple form.

Theorem 1.17 (Huybrechts)**.** *Let X be a hyperkähler manifold of dimension $2m$. There exist rational constants $a_0, a_2, \dots, a_{2m}$ such that, for every line bundle L on X, one has*

$$\chi(X, L) = \sum_{i=0}^{m} a_{2i} q_X(c_1(L))^i \tag{4}$$

The proof is based on the fact that, for any class $\beta \in H^{4j}(X, \mathbf{R})$ which remains of type $(2j, 2j)$ on every small deformation of X, the form $\alpha \mapsto \int_X \beta\alpha^{2(m-j)}$ on $H^2(X, \mathbf{R})$ is proportional to the form $\alpha \mapsto q_X(\alpha)^{m-j}$ (formula (1) is the particular case where $\beta = 1 \in H^0(X, \mathbf{R})$). This applies to each $\mathrm{td}_{2m-2j}(X)$ and gives (since $T_X \simeq T_X^\vee$, the odd Chern classes of X vanish, hence so do its odd Todd classes)

$$\chi(X, L) = \sum_{j=0}^{m} \frac{1}{(2j)!} a_j q_X(c_1(L))^j.$$

When L is nef and big, its higher cohomology groups vanish (Kawamata–Viehweg), hence $\chi(X, L) = h^0(X, L)$ and $q_X(L) > 0$. Moreover, some positive tensor power of L is base-point free (Kawamata base-point free theorem).

The relevant constants in (4) have been computed for the two main series of examples:

- when X is $S^{[m]}$ (or a deformation), we have (Ellingsrud–Göttsche–M. Lehn)

$$\chi(X, L) = \binom{\frac{1}{2}q_X(L) + m + 1}{m}; \tag{5}$$

- when X is $K_m(T)$ (or a deformation), we have (Britze)

$$\chi(X, L) = (m+1)\binom{\frac{1}{2}q_X(L) + m}{m}.$$

It is conjectured that in general, the polynomial function $x \mapsto \sum_{i=0}^{m} a_{2i}x^i$ in (4) is increasing on $\mathbf{R}^{\geq 0}$ (as it is in the two examples above).

1.6. **Cohomology of hyperkähler manifolds.** Let X be a hyperkähler manifold of dimension $n = 2m$. The formula

$$\int_X \alpha^n = c_X q_X(\alpha)^m$$

seen in (1) has very strong consequences on the structure of the cohomology algebra $H^{2\bullet}(X, \mathbf{C})$.

Theorem 1.18 (Verbitsky)**.** *Let X be a hyperkähler manifold of dimension $2m$. The kernel of the canonical morphism*

$$\mu\colon \mathrm{Sym}^\bullet H^2(X, \mathbf{C}) \longrightarrow H^{2\bullet}(X, \mathbf{C})$$

is generated by the relations α^{m+1} for all $\alpha \in H^2(X, \mathbf{C})$ such that $q_X(\alpha) = 0$. In particular, μ is injective in degrees $\leq m$.

Proof. We have already seen during the proof of Theorem 1.11 that

$$\forall \alpha \in H^2(X, \mathbf{C}) \qquad q_X(\alpha) = 0 \iff \big(\alpha^{m+1} = 0 \text{ in } H^{2m+2}(X, \mathbf{C})\big).$$

(The implication $\Longleftarrow$ follows from (1) and the implication $\Longrightarrow$ from the fact that $\alpha \mapsto \alpha^{m+1}$ vanishes on D, hence on its Zariski closure $(q_X = 0)$.)

It is a general fact from algebra that if V is a complex vector space with a nondegenerate quadratic form q, the graded algebra $A^\bullet := \mathrm{Sym}^\bullet V / \langle \alpha^{m+1} \mid q(\alpha) = 0\rangle$ is Gorenstein with socle in degree $2m$: one has $A^{2m} = \mathbf{C}$ and the pairings $A^i \times A^{2m-i} \to A^{2m} = \mathbf{C}$ are nondegenerate for $0 \le i \le 2m$.

In our case, we take $V = H^2(X, \mathbf{C})$. The morphism μ factors through a morphism $\bar\mu \colon A^\bullet \to H^{2\bullet}(X, \mathbf{C})$ which is an isomorphism in degree $2m$, since $A^{2m} = \mathbf{C}$ and $\mu(\omega^m) \ne 0$ for every kähler class ω. We want to show that $\bar\mu$ is injective; if $\alpha \in A^k$ is nonzero, there exists $\beta \in A^{2m-k}$ such that $\alpha\beta \ne 0$, hence $\mu(\alpha\beta) \ne 0$, hence $\mu(\alpha) \ne 0$. □

Example 1.19. When S is a K3 surface, the (injective) canonical morphism $\mathrm{Sym}^2 H^2(S^{[m]}, \mathbf{C}) \to H^4(S^{[m]}, \mathbf{C})$ is an isomorphism for $m = 2$, but not when $m \ge 3$ (the dimensions of all these spaces are known by work of Göttsche).

The proof says nothing about the cohomology in odd degrees (it vanishes in degree 1, but can be nontrivial in degrees ≥ 3; for example, $b_3(K_2(T)) = 8$).

Corollary 1.20 (Beauville)**.** *Let X be a hyperkähler fourfold. One has $b_2(X) \le 23$.*

Proof. The proof is based on the relation (Libgober–Wood)

$$\int_X c_1(T_X) c_{n-1}(T_X) = \sum_{p=0}^{n} (-1)^p \Big(6p^2 - \frac{1}{2} n(3n+1)\Big) \chi(X, \Omega_X^p),$$

valid for all compact complex manifolds X of dimension n. One has $\chi(X, \Omega_X^p) = \sum_q (-1)^q h^{p,q}(X)$, and when X is hyperkähler, the isomorphisms $\Omega_X^p \simeq \Omega_X^{n-p}$ give $h^{p,q} = h^{n-p,q}$. All these identities can be combined to prove the equality (Salamon)

$$\sum_{p=0}^{4m} (-1)^p (6p^2 - m(6m+1)) b_p(X) = 0$$

fo all hyperkähler manifolds X of dimension $2m$. In dimension 4, this gives

$$b_3(X) + b_4(X) = 46 + 10 b_2(X).$$

Theorem 1.18 also gives $b_4(X) \ge b_2(X)(b_2(X) + 1)/2$, hence $46 + 10b_2(X) \ge b_2(X)(b_2(X) + 1)/2$. This gives the desired inequality (and $b_3(X) = 0$ if there is equality). □

For hyperkähler fourfolds, one can prove that either $b_2(X) = 23$ or $3 \le b_2(X) \le 8$ (Guan). Note that for $m \ge 2$, one has $b_2(S^{[m]}) = 23$ for any K3 surface S and $b_2(K_m(T)) = 7$ for any complex torus T of dimension 2.

2. Cones in the cohomology of hyperkähler manifolds

2.1. **Monodromy groups.**

The proofs of some of the results in this section depend on the constructions presented in the next section, but we chose to group them together here for clarity.

We are interested in the bimeromorphic geometry of hyperkähler manifolds. The whole theory rests on the following elementary result. If X is irreducible, we say that an open subset $U \subset X$ is *big* if $\mathrm{codim}_X(X \smallsetminus U) \ge 2$.

Proposition 2.1 (Huybrechts)**.** *Let X and X' be hyperkähler manifolds with a bimeromorphic isomorphism $u \colon X \overset{\sim}{\dashrightarrow} X'$. There exist big open subsets $U \subset X$ and $U' \subset X'$ such that u induces an isomorphism $U \xrightarrow{\sim} U'$ and a Hodge isometry $u^* \colon (H^2(X', \mathbf{Z}), q_{X'}) \xrightarrow{\sim} (H^2(X, \mathbf{Z}), q_X)$.*

Sketch of proof. Let $U \subset X$ (resp. $U' \subset X'$) be the largest open subset on which u (resp. u^{-1}) is defined. Since X is normal and Ω_X^2 is locally free, U is big and restriction induces an isomorphism $H^0(X, \Omega_X^2) \xrightarrow{\sim} H^0(U, \Omega_U^2)$. These vector spaces are spanned by the symplectic form σ_X and, since $(u|_U)^*\sigma_{X'}$ is nonzero, it is a nonzero multiple of $\sigma_X|_U$. Since this 2-form is nowhere degenerate, $u|_U$ is quasi-finite and, being bimeromorphic, it is an open embedding by Zariski's Main Theorem. This implies $u(U) \subset U'$ and, applying the same reasoning with u^{-1}, we deduce that u induces an isomorphism between U and U'. Since U is big, the restriction $H^2(X, \mathbf{Z}) \to H^2(U, \mathbf{Z})$ is an isomorphism[3] and we get an isomorphism $u^*\colon H^2(X', \mathbf{Z}) \xrightarrow{\sim} H^2(X, \mathbf{Z})$ such that $u^*(H^{2,0}(X')) = H^{2,0}(X)$.

For the proof that u^* is a Hodge isometry, we refer to [Hu1, Section 27.1]. □

In the situation of the proposition, if one of the manifolds X or X' is projective, so is the other: this is because a compact kähler manifold which is bimeromorphically isomorphic to a projective manifold is projective.

Let X be a hyperkähler manifold. Since the quadratic form q_X has signature $(1, b_2(X) - 3)$ on $H^{1,1}(X)_{\mathbf{R}}$, the set $\{\alpha \in H^{1,1}(X)_{\mathbf{R}} \mid q_X(\alpha) > 0\}$ has two components (when $b_2(X) > 3$). We let

$$O^+(\Lambda_X) \subseteq O(\Lambda_X) \tag{6}$$

be the subgroup of the full isometry group that preserves each of these components. We let

$$\mathrm{Mon}(X) \subseteq O^+(\Lambda_X) \tag{7}$$

be the subgroup generated by the images of the monodromy representations $\pi_1(B, b) \to O(\Lambda_X)$ for all smooth proper families $\mathscr{X} \to B$ with B connected and $\mathscr{X}_b = X$.

We let

$$\mathrm{Mon}_{\mathrm{Hdg}}(X) \subset \mathrm{Mon}(X) \tag{8}$$

be the subgroup of elements of $\mathrm{Mon}(X)$ that preserve the Hodge structure of $H^2(X)$. In other words, their complexifications preserve the line $H^{2,0}(X) \subset H^2(X, \mathbf{C})$.

Any bimeromorphic automorphism $u\colon X \overset{\sim}{\dashrightarrow} X$ induces, by Proposition 2.1, an isometry u^* of Λ_X which is in fact in $\mathrm{Mon}_{\mathrm{Hdg}}(X)$ ([M, Theorem 3.1]). We let

$$\mathrm{Mon}_{\mathrm{Bir}}(X) \subset \mathrm{Mon}_{\mathrm{Hdg}}(X)$$

be the subgroup consisting of all these isometries.

We will say that an irreducible hypersurface $Y \subset X$ is *negative* if $q_X(Y) < 0$. The reflection R_Y about the hyperplane $Y^\perp$ is then integral and belongs to $\mathrm{Mon}_{\mathrm{Hdg}}(X)$.[4] We let

$$W_{\mathrm{Exc}}(X) \subset \mathrm{Mon}_{\mathrm{Hdg}}(X) \tag{9}$$

be the subgroup generated by all these reflections ([M, Definition 6.8 and Theorem 6.18(3)]).

The following is [M, Corollary 6.9 and Theorem 6.18(5)] (this is actually a consequence of Theorem 2.5).

---

[3]If $n := \dim_{\mathbf{C}}(X)$ and $Y := X \smallsetminus U$, this follows from example from the long exact sequence

$$H_{2n-2}(Y, \mathbf{Z}) \to H_{2n-2}(X, \mathbf{Z}) \to H_{2n-2}(X, Y, \mathbf{Z}) \to H_{2n-3}(Y, \mathbf{Z}),$$

the fact that $H_j(Y, \mathbf{Z}) = 0$ for $j > 2(n-2) \geq \dim_{\mathbf{R}}(Y)$, and the duality isomorphisms $H_{2n-i}(X, \mathbf{Z}) \simeq H^i(X, \mathbf{Z})$ and $H_{2n-i}(X, Y, \mathbf{Z}) \simeq H^i(X \smallsetminus Y, \mathbf{Z})$.

[4]When X is projective, this is [M, Proposition 6.2]; this result extends to the general case as explained in [AV1, Theorem 3.24].

Proposition 2.2 (Markman). *Let X be a* projective *hyperkähler manifold. The group* $\mathrm{Mon}_{\mathrm{Hdg}}(X)$ *is the semi-direct product of its normal subgroup* $W_{\mathrm{Exc}}(X)$ *and its subgroup* $\mathrm{Mon}_{\mathrm{Bir}}(X)$.

2.2. **Cones in $H^{1,1}(X)_{\mathbf{R}}$.** Let X be a hyperkähler manifold. The set of kähler classes on X form an open convex cone

$$\mathrm{Kah}(X) \subset H^{1,1}(X)_{\mathbf{R}}$$

called the *kähler cone.* By Theorem 1.11(b), it is contained in one component

$$\mathrm{Pos}(X) \subset H^{1,1}(X)_{\mathbf{R}}$$

of the set $\{\alpha \in H^{1,1}(X)_{\mathbf{R}} \mid q_X(\alpha) > 0\}$ which we call the *positive cone.*

By Proposition 2.1, it makes sense to define the (possibly nonconvex) open *bimeromorphic kähler cone*

$$\mathrm{BirKah}(X) := \bigcup_{u\colon X \overset{\sim}{\dashrightarrow} X'} u^*(\mathrm{Kah}(X')) \subseteq \mathrm{Pos}(X). \tag{10}$$

There might be infinitely many different cones $u^*(\mathrm{Kah}(X'))$. The following result shows that these cones are either disjoint or equal.

Proposition 2.3 (Fujiki). *Let X and X' be hyperkähler manifolds with a bimeromorphic isomorphism $u\colon X \overset{\sim}{\dashrightarrow} X'$. There are equivalences*

$$u \text{ is an isomorphism} \iff u^*\big(\mathrm{Kah}(X')\big) = \mathrm{Kah}(X) \iff u^*\big(\mathrm{Kah}(X')\big) \cap \mathrm{Kah}(X) \neq \varnothing.$$

There is only one nontrivial implication (that if the pullback of one kähler class is a kähler class, u is an isomorphism). For the proof of a stronger result, see [Hu1, Proposition 27.7].

The closed cone $\overline{\mathrm{BirKah}}(X)$ is convex: this is a consequence of the following result.

Theorem 2.4 (Boucksom, Huybrechts). *Let X be a hyperkähler manifold. We have the following characterizations*

$$\begin{aligned}
\mathrm{Kah}(X) &= \{\alpha \in \mathrm{Pos}(X) \mid \alpha \cdot C > 0 \text{ for all rational curves } C \subset X\},\\
\overline{\mathrm{Kah}}(X) &= \{\alpha \in \overline{\mathrm{Pos}}(X) \mid \alpha \cdot C \geq 0 \text{ for all rational curves } C \subset X\},\\
\overline{\mathrm{BirKah}}(X) &= \{\alpha \in \overline{\mathrm{Pos}}(X) \mid q_X(\alpha, Y) \geq 0 \text{ for all irreducible hypersurfaces } Y \subset X\}\\
&= \{\alpha \in \overline{\mathrm{Pos}}(X) \mid q_X(\alpha, Y) \geq 0 \text{ for all uniruled irreducible hypersurfaces } Y \subset X\}\\
&= \{\alpha \in \overline{\mathrm{Pos}}(X) \mid q_X(\alpha, Y) \geq 0 \text{ for all negative irreducible hypersurfaces } Y \subset X\}.
\end{aligned}$$

This is [Hu1, Propositions 28.2, 28.5, and 28.7]. Regarding the description of $\overline{\mathrm{BirKah}}(X)$, the inequality $q_X(\omega, Y) > 0$ for any kähler class ω and any hypersurface $Y \subset X$ follows from (3); this gives one inclusion. Also, it follows from the signature property of the quadratic form q_X that $q_X(\alpha, \beta) \geq 0$ for all $\alpha, \beta \in \overline{\mathrm{Pos}}(X)$. It is therefore enough to check $q_X(\alpha, Y) \geq 0$ when Y is negative. Boucksom proved that this implies that Y is uniruled ([Hu1, Corollary 28.9]).

We also consider the interior

$$\overset{\circ}{\overline{\mathrm{BirKah}}}(X) := \{\alpha \in \mathrm{Pos}(X) \mid q_X(\alpha, Y) > 0 \text{ for all negative irr. hypersurfaces } Y \subset X\} \tag{11}$$

of the closed birational kähler cone.[5] It contains $\mathrm{BirKah}(X)$.

[5] Markman denotes this cone by $\mathscr{F}\mathscr{E}_X$ and calls it the *fundamental exceptional chamber* ([M, Definition 5.2]).

Let $u\colon X \overset{\sim}{\dashrightarrow} X'$ be a bimeromorphic isomorphism between hyperkähler manifolds. Since u induces an isomorphism between big open subsets and u^* is an isometry (Proposition 2.1), u sends negative irreducible hypersurfaces to negative irreducible hypersurfaces, hence

$$u^*\big(\overset{\circ}{\overline{\mathrm{BirKah}}}(X')\big) = \overset{\circ}{\overline{\mathrm{BirKah}}}(X). \tag{12}$$

We will see later a kind of converse to that statement (Section 3.1.3).

The importance of this cone is demonstrated by the following result ([M, Theorems 6.17 and 6.18]).

Theorem 2.5 (Markman)**.** *Let X be a* projective *hyperkähler manifold. There is a subset $\mathscr{S}_X \subset \mathrm{Pic}(X) \smallsetminus \{0\}$, which is stable under the action of $\mathrm{Mon}_{\mathrm{Hdg}}(X)$ and consists of classes of negative q_X-square,*[6] *such that $\overset{\circ}{\overline{\mathrm{BirKah}}}(X)$ is the connected component of*

$$\mathrm{Pos}(X) \smallsetminus \bigcup_{W\in\mathscr{S}_X} W^\perp \tag{13}$$

that contains the kähler cone. The group $\mathrm{Mon}_{\mathrm{Hdg}}(X)$ acts transitively on the set of connected components and its subgroup $W_{\mathrm{Exc}}(X)$ acts faithfully and transitively.

The groups $\mathrm{Mon}_{\mathrm{Hdg}}(X)$ and $W_{\mathrm{Exc}}(X)$ were defined in (8) and (9). The connected components of (13) are called *exceptional chambers* in [M, Definition 5.10(1)]. There is a further decomposition for which the kähler cone is a connected component ([AV1, Theorem 1.19]).

Theorem 2.6 (Amerik–Verbitsky)**.** *Let X be a hyperkähler manifold. There is a subset $\mathscr{W}_X \subset \mathrm{Pic}(X)\smallsetminus\{0\}$ containing $\mathscr{S}_X$, which is stable under the action of $\mathrm{Mon}_{\mathrm{Hdg}}(X)$ and consists of classes of negative q_X-square,*[7] *such that the connected components of*

$$\mathrm{Pos}(X) \smallsetminus \bigcup_{W\in\mathscr{W}_X} W^\perp \tag{14}$$

are the $g(u^*(\mathrm{Kah}(X')))$, for all $g \in \mathrm{Mon}_{\mathrm{Hdg}}(X)$ and all bimeromorphic isomorphisms $u\colon X \overset{\sim}{\dashrightarrow} X'$.

The cone $\mathrm{Kah}(X)$ is therefore a component of (14). The connected components of (14) are called *kähler-type chambers* in [M, Definition 5.10(2)], where they were defined, and *kähler-Weyl chambers* in [AV1, Definition 6.1].

Corollary 2.7. *With the notation of the theorem, the connected components of*

$$\overset{\circ}{\overline{\mathrm{BirKah}}}(X) \smallsetminus \bigcup_{W\in\mathscr{W}_X} W^\perp \tag{15}$$

are the $u^*(\mathrm{Kah}(X'))$, for all bimeromorphic isomorphisms $u\colon X \overset{\sim}{\dashrightarrow} X'$.

Proof. Theorem 2.6 says that the components of (14) are the $g(u^*(\mathrm{Kah}(X')))$, for $g \in \mathrm{Mon}_{\mathrm{Hdg}}(X)$ and $u\colon X \overset{\sim}{\dashrightarrow} X'$ bimeromorphic isomorphism. By Proposition 2.2, we can write $g = wv^*$, where

[6] Elements of *Spe* are defined right before [M, Definition 6.6]; our set $\mathscr{S}_X$ is $\mathit{Spe} \cup (-\mathit{Spe})$. The set *Spe* contains $c_1(\mathscr{O}_X(Y))$ for all negative irreducible hypersurfaces $Y \subset X$ (see [M, Proposition 6.6 and Corollary 6.9]).

[7] The set $\mathscr{W}_X$ is defined in [AV1, Definition 1.13], where its elements are called *MBM classes* (for monodromy birationally minimal), and in [Mo1, Definition 1.2], where they are called *wall divisors.* The fact that the two definitions are equivalent (in the sense that MBM classes and wall divisors are rational multiples one of another) is explained in [KLM, Remark 2.4].

$w \in W_{\mathrm{Exc}}(X)$ and $v\colon X \overset{\sim}{\dashrightarrow} X$. Then, $g(u^*(\mathrm{Kah}(X'))) = w((uv)^*(\mathrm{Kah}(X')))$, and, by Theorem 2.5, this component does not meet $\overset{\circ}{\overline{\mathrm{BirKah}}}(X)$ unless $w = \mathrm{Id}$. This finishes the proof. □

2.3. **Cones in** $\mathrm{Pic}(X)_{\mathbf{R}}$**.** Assuming that the hyperkähler manifold X is projective, we now look at the intersections of the cones in $H^{1,1}(X)_{\mathbf{R}}$ that we have constructed with the real subspace $\mathrm{Pic}(X)_{\mathbf{R}} := \mathrm{Pic}(X) \otimes \mathbf{R} = \langle H^{1,1}(X)_{\mathbf{R}} \cap H^2(X, \mathbf{Q})\rangle$. The open cone

$$\mathrm{Pos}^{\mathrm{alg}}(X) := \mathrm{Pos}(X) \cap \mathrm{Pic}(X)_{\mathbf{R}}$$

is again the component of $\{L \in \mathrm{Pic}(X)_{\mathbf{R}} \mid L^2 > 0\}$ that contains the ample classes.

The intersection

$$\mathrm{Kah}(X) \cap \mathrm{Pic}(X)_{\mathbf{R}} = \mathrm{Amp}(X)$$

is the *ample cone,* generated by classes of ample line bundles, hence the intersection

$$\overline{\mathrm{Kah}}(X) \cap \mathrm{Pic}(X)_{\mathbf{R}} = \overline{\mathrm{Amp}}(X) = \mathrm{Nef}(X)$$

is the *nef cone,* generated by classes of nef line bundles. Theorem 2.4 implies

$$\begin{aligned} \mathrm{Amp}(X) &= \{\alpha \in \mathrm{Pos}^{\mathrm{alg}}(X) \mid \alpha \cdot C > 0 \text{ for all rational curves } C \subset X\}, \\ \mathrm{Nef}(X) &= \{\alpha \in \overline{\mathrm{Pos}}^{\mathrm{alg}}(X) \mid \alpha \cdot C \geq 0 \text{ for all rational curves } C \subset X\}. \end{aligned}$$

The *movable cone*

$$\mathrm{Mov}(X) \subset \mathrm{Pic}(X)_{\mathbf{R}}$$

is the convex cone generated by classes of line bundles on X whose base locus has codimension ≥ 2 (no fixed divisors). Integral points that are in the interior $\overset{\circ}{\mathrm{Mov}}(X)$ of $\mathrm{Mov}(X)$ do correspond to line bundles, some positive multiple of which has no fixed divisors (Kawamata).

The following result ([Ma, Proposition 2.1]) characterizes movable line bundles on hyperkähler manifolds. Its proof uses the Minimal Model Program.

Proposition 2.8 (Matsushita)**.** *Let L be a movable line bundle on a projective hyperkähler manifold X. There exists a projective hyperkähler manifold X' with a birational isomorphism $u\colon X \overset{\sim}{\dashrightarrow} X'$ and a* nef *line bundle L' on X' such that $L = u^*L'$.*

Corollary 2.9. *Let X be a projective hyperkähler manifold. We have*

$$\overline{\mathrm{Mov}}(X) = \overline{\bigcup_{u\colon X \overset{\sim}{\dashrightarrow} X'} u^*(\mathrm{Nef}(X'))} = \overline{\mathrm{BirKah}}(X) \cap \mathrm{Pic}(X)_{\mathbf{R}}. \tag{16}$$

In particular,

$$\begin{aligned} \overline{\mathrm{Mov}}(X) &= \{\alpha \in \overline{\mathrm{Pos}}^{\mathrm{alg}}(X) \mid q_X(\alpha, Y) \geq 0 \textit{ for all irreducible hypersurfaces } Y \subset X\} \\ &= \{\alpha \in \overline{\mathrm{Pos}}^{\mathrm{alg}}(X) \mid q_X(\alpha, Y) \geq 0 \textit{ for all uniruled irreducible hypersurfaces } Y \subset X\} \\ &= \{\alpha \in \overline{\mathrm{Pos}}^{\mathrm{alg}}(X) \mid q_X(\alpha, Y) \geq 0 \textit{ for all negative irreducible hypersurfaces } Y \subset X\}. \end{aligned}$$

Proof. Theorem 2.8 implies that $\mathrm{Mov}(X)$ is contained in $\bigcup_u u^*(\mathrm{Nef}(X'))$. Conversely, each cone $u^*(\mathrm{Amp}(X'))$ is obviously contained in $\mathrm{Mov}(X)$. This gives the first equality in (16) and the second equality follows easily. □

Theorem 2.10. *Let X be a projective hyperkähler manifold. We have*

$$\mathring{\mathrm{Mov}}(X) = \overline{\mathring{\mathrm{BirKah}}} \cap \mathrm{Pic}(X)_{\mathbf{R}} \tag{17}$$

and this cone is the connected component of

$$\mathrm{Pos}^{\mathrm{alg}}(X) \smallsetminus \bigcup_{H \in \mathscr{S}_X} H^\perp \tag{18}$$

that contains the ample cone.

The image of the restriction map $\mathrm{Mon}_{\mathrm{Hdg}}(X) \to O(\mathrm{Pic}(X))$ has finite index in $O(\mathrm{Pic}(X))$ and contains $W_{\mathrm{Exc}}(X)$ as a normal subgroup which acts faithfully and transitively on the set of connected components of (18)*.*

Proof. The equality (17) is [M, Lemmas 6.22]. The statement that the cone $\mathring{\mathrm{Mov}}(X)$ is a connected component of (18) is obtained from the analogous statement in Theorem 2.5 by restricting to $\mathrm{Pic}(X)_{\mathbf{R}}$. The rest is [M, Lemma 6.23]. ☐

We get a similar statement by restricting the decomposition in Corollary 2.7 to algebraic classes ([MY, Proposition 2.1]).

Theorem 2.11. *Let X be a projective hyperkähler manifold. The connected components of*

$$\mathring{\mathrm{Mov}}(X) \smallsetminus \bigcup_{H \in \mathscr{W}_X} H^\perp \tag{19}$$

are the $u^*(\mathrm{Amp}(X'))$, for all birational isomorphisms $u \colon X \overset{\sim}{\dashrightarrow} X'$.

Example 2.12. Let S be a projective K3 surface and let $X = S^{[m]}$, with $m \geq 2$. We have (Example 1.14)

$$\begin{aligned} H^2(X, \mathbf{Z}) &\simeq H^2(S, \mathbf{Z}) \oplus \mathbf{Z}\delta, \\ \mathrm{Pic}(X) &\simeq \mathrm{Pic}(S) \oplus \mathbf{Z}\delta, \end{aligned}$$

with $q_X(\delta) = -2(m-1)$. The irreducible hypersurface in X that parametrizes nonreduced subschemes has class 2δ; it is therefore negative. The whole nef cone $\mathrm{Nef}(S) \subset \mathrm{Pic}(S)_{\mathbf{R}}$ corresponds to classes that are nef but not ample on X: it is contained in the boundary of $\mathrm{Nef}(X) \subset \mathrm{Pic}(X)_{\mathbf{R}}$; by Corollary 2.9, it is also contained in the boundary of the movable cone $\overline{\mathrm{Mov}}(X) \subset \mathrm{Pic}(X)_{\mathbf{R}}$. The precise determination of the cones $\mathrm{Nef}(X) \subset \overline{\mathrm{Mov}}(X)$ is a difficult problem which was solved only recently (at least theoretically) by Bayer–Macrì.

When $m = 2$, they proved in particular that the set $\mathscr{S}_X$ of Theorem 2.10 is the set of all classes $W \in \mathrm{Pic}(X)$ such that $q_X(W) = -2$ and they also determine the set $\mathscr{W}_X$. If (S, L) is a very general polarized K3 surface, with L is ample of degree $2e$, so that $\mathrm{Pic}(S) = \mathbf{Z}L$ and $\mathrm{Pic}(X) = \mathbf{Z}L \oplus \mathbf{Z}\delta$, we have $q_X(aL + b\delta) = 2ea^2 - 2b^2$, hence

$$\mathrm{Pos}^{\mathrm{alg}}(X) = \mathbf{R}_{>0}(L - \sqrt{e}\delta) + \mathbf{R}_{>0}(L + \sqrt{e}\delta).$$

To find the elements of $\mathscr{S}_X$, one needs to find the integral solutions of the equation

$$ea^2 - b^2 = -1.$$

The solutions $(a, b) = (0, \pm 1)$ gives the elements $\pm\delta$ of $\mathscr{S}_X$ and the "wall" $\mathbf{R}L$. Beside that,

- either $e = f^2$ is a perfect square and there are no other solutions, so that

$$\overline{\mathrm{Mov}}(X) = \mathbf{R}_{\geq 0}L + \mathbf{R}_{\geq 0}(L - f\delta) \subset \mathrm{Pic}(X)_{\mathbf{R}},$$

- or e is *not* a perfect square, they are infinitely elements in $\mathscr{S}_X$, and
$$\overline{\mathrm{Mov}}(X) = \mathbf{R}_{\geq 0} L + \mathbf{R}_{\geq 0}(x_1 L - e y_1 \delta) \subset \mathrm{Pic}(X)_{\mathbf{R}},$$
where (x_1, y_1) is the minimal solution of the Pell equation $x^2 - ey^2 = 1$. There are infinitely many chambers in the decomposition (18): they are bounded by "successive" solutions to the Pell equation.

Bayer–Macrì also proved that the set $\mathscr{W}_X \smallsetminus \mathscr{S}_X$ consists of the classes $W = 2aL + b\delta$ such that $q_X(W) = -10$. One therefore needs to find the integral solutions of the equation $4ea^2 - b^2 = -5$.

If this equation has no solutions (for example, if $e \equiv \pm 2 \pmod 5$), we have $\overline{\mathrm{Mov}}(X) = \mathrm{Nef}(X)$: any birational isomorphism $X \overset{\sim}{\dashrightarrow} X'$ to another hyperkähler manifold is an isomorphism.

If this equation does have solutions, one checks that the set (19) may have up to 3 connected components. This is the case for $e = 29$: there is a nonregular involution $\tau\colon X \overset{\sim}{\dashrightarrow} X$ that exchanges the two "extreme" chambers and a nontrivial birational model $X \overset{\sim}{\dashrightarrow} X'$ on which τ induces a biregular involution.

Example 2.13. Let us present another example originally worked out geometrically by Hassett–Tschinkel. There is a projective hyperkähler fourfold X (deformation of the Hilbert square of a K3 surface) for which $(\mathrm{Pic}(X), q_X) = \big(\mathbf{Z}L \oplus \mathbf{Z}M, \big(\begin{smallmatrix} 6 & 0 \\ 0 & -4 \end{smallmatrix}\big)\big)$. We have
$$\mathrm{Pos}^{\mathrm{alg}}(X) = \mathbf{R}_{>0}(L - \sqrt{3/2}\, M) + \mathbf{R}_{>0}(L + \sqrt{3/2}\, M).$$
To determine $\mathscr{S}_X$, one needs to solve the equation $3a^2 - 2b^2 = -1$, which has no integral solutions (reduce modulo 3). Therefore,
$$\overline{\mathrm{Mov}}(X) = \overline{\mathrm{Pos}}^{\mathrm{alg}}(X).$$
The set $\mathscr{W}_X \smallsetminus \mathscr{S}_X$ consists of the classes $W = aL + bM$ such that $q_X(W) = -10$. One therefore needs to find the integral solutions of the equation $3a^2 - 2b^2 = -5$. There are infinitely many of them (they are all given by $a\sqrt{3} + 2b\sqrt{2} = (\pm 3\sqrt{3} \pm 8\sqrt{2})(5 + 2\sqrt{6})^r$, for $r \in \mathbf{Z}$) hence the set (19) has infinitely many chambers, hence infinitely many "different" birational isomorphisms $u_r\colon X \overset{\sim}{\dashrightarrow} X_r$. But half of these are explained by the fact that the group of birational automorphisms of X is the infinite dihedral group $\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$: the fourfold X_r is either isomorphic to X or to X_1.

Example 2.14. Here is a similar example. There is a projective hyperkähler fourfold X (deformation of the Hilbert square of a K3 surface) for which $(\mathrm{Pic}(X), q_X) = \big(\mathbf{Z}L \oplus \mathbf{Z}M, \big(\begin{smallmatrix} 6 & 0 \\ 0 & -22 \end{smallmatrix}\big)\big)$ (we will explain in Example 3.9 why such an X exists). To determine $\mathscr{S}_X$ and $\mathscr{W}_X$, one needs to solve the equations $3a^2 - 11b^2 = -1$ or -5, which have no integral solutions (reduce modulo 11). Therefore,
$$\mathrm{Nef}(X) = \overline{\mathrm{Mov}}(X) = \overline{\mathrm{Pos}}^{\mathrm{alg}}(X) = \mathbf{R}_{>0}(L - \sqrt{3/11}\, M) + \mathbf{R}_{>0}(L + \sqrt{3/11}\, M).$$

3. The Torelli theorem for hyperkähler manifolds

3.1. The global period map and the Torelli theorem.

3.1.1. *The moduli space and the period map.* Our aim is to construct a global period map for hyperkähler manifolds of a certain topological type. It should be defined on a "moduli space" for these hyperkähler manifolds and should locally look like the local period map that we defined in Section 1.3.2. In particular, its target space should be a fixed projective space $\mathbf{P}(V)$. In order to define the period of any given X of the type under consideration, we need to choose an isomorphism $H^2(X, \mathbf{C}) \simeq V$. There are too many such isomorphisms so we rigidify the situation: we fix a lattice (Λ, q_Λ) and we consider pairs (X, φ), where X is a hyperkähler manifold and

$\varphi\colon (H^2(X,\mathbf{Z}),q_X) \xrightarrow{\sim} (\Lambda,q_\Lambda)$ an isometry. Such pairs are called Λ-*marked hyperkähler manifolds.* One defines isomorphisms (isomorphisms between the varieties that are compatible with the markings) and families $\mathscr{X} \to B$ of marked hyperkähler manifolds (with a trivialization $\varphi\colon R^2\pi_*\mathbf{Z} \xrightarrow{\sim} \Lambda \times B$). Given such a family, there is a holomorphic period map

$$B \longrightarrow \mathbf{P}(\Lambda \otimes \mathbf{C})$$

that sends a point $b \in B$ to $\varphi_{b,\mathbf{C}}([H^{2,0}(\mathscr{X}_b])$. It takes its values in

$$Q_\Lambda := \{[x] \in \mathbf{P}(\Lambda \otimes \mathbf{C}) \mid q_\Lambda(x) = 0,\ q_\Lambda(x,\bar{x}) > 0\}.$$

If Λ has signature $(3, r-3)$, this is an open subset of a smooth quadric hypersurface of dimension $r-2$. It is connected, simply connected, and isomorphic to $O(3,r-3,\mathbf{R})/SO(2,\mathbf{R}) \times O(1,r-3,\mathbf{R})$.

The (possibly empty) moduli space $M_{\Lambda,n}$ for Λ-marked hyperkähler manifolds of dimension n and the holomorphic period map

$$\wp_{\Lambda,n}\colon M_{\Lambda,n} \longrightarrow Q_\Lambda$$

are constructed by glueing together the local constructions of Sections 1.3.1 and 1.3.2. If nonempty, the moduli space $M_{\Lambda,n}$ is smooth of pure dimension $\mathrm{rank}(\Lambda) - 2$ but *nonseparated,* and there is in general no universal family over $M_{\Lambda,n}$.

It is known that $M_{\Lambda,n}$ has finitely many components if one fixes in addition the Fujiki constant c_X.[8] Let $M'_{\Lambda,n}$ be the union of all components of $M_{\Lambda,n}$ that correspond to a fixed deformation type, that is, to pairs (X,φ), where X is a deformation of a fixed hyperkähler manifold X_0 (for example, X is a deformation of mth Hilbert schemes of K3 surfaces). By definition of the monodromy group (see (7)), the number of connected components of $M'_{\Lambda,n}$ is the (finite) index

$$[O(\Lambda_{X_0}) : \mathrm{Mon}(X_0)]. \tag{20}$$

Since $\mathrm{Mon}(X_0) \subseteq O^+(\Lambda_{X_0})$, the number of components is even (if $b_2(X) > 3$): if (X,φ) is in one component, $(X,-\varphi)$ is in another.

Example 3.1. When $n = 2$, there is only one lattice Λ for which $M_{\Lambda,2}$ is nonempty: the K3 lattice Λ_{K3}. The space $M_{\Lambda_{\mathrm{K3}},2}$ has exactly two components.

Example 3.2. Let $m \geq 2$ and consider the lattice $\Lambda_{\mathrm{K3}^{[m]}} := \Lambda_{\mathrm{K3}} \oplus \mathbf{Z}(-2(m-1))$ (see Example 1.14). Let $M'_{\Lambda_{\mathrm{K3}^{[m]}},2m}$ be the union of the components of $M_{\Lambda_{\mathrm{K3}^{[m]}},2m}$ that correspond to marked pairs (X,φ), where X is a deformation of the mth Hilbert scheme of a K3 surface.

For any such X, Markman proved in [M, Lemma 9.2] that $\mathrm{Mon}(X)$ is a *normal subgroup* of $O(\Lambda_X)$ of index $2^{\max\{\rho(m-1),1\}}$, where $\rho(m-1)$ is the number of distinct prime factors of $m-1$. This index is also the number of components of $M'_{\Lambda_{\mathrm{K3}^{[m]}},2m}$. When $m = 2$ or m is a prime power, it follows that $M'_{\Lambda_{\mathrm{K3}^{[m]}},2m}$ has exactly two components.

Example 3.3. Let $m \geq 2$ and consider the lattice $\Lambda_{K_m} := U^{\oplus 3} \oplus \mathbf{Z}(-2(m+1))$ (see Example 1.15). Let $M'_{\Lambda_{K_m},2m}$ be the union of the components of $M_{\Lambda_{K_m},2m}$ that correspond to marked pairs (X,φ), where X is a deformation of the mth Kummer variety of an abelian surface.

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[8] See the discussion of [Hu1, Section 26.5], and in particular [Hu1, Proposition 26.20], for various finiteness results. It turns out that the normalization we chose for the form q_X (integral and nondivisible) may not be the right one. For each dimension n, there is a universal positive rational constant c_n such that the quadratic form $\tilde{q}_X\colon \alpha \longmapsto c_n \int_X \sqrt{\mathrm{td}(X)}\,\alpha^2$ is integral (see Remark 1.13; just take for c_n a positive integer divisible enough to kill all denominators in $\sqrt{\mathrm{td}(X)}$). Then $M_{\Lambda,n}$ has finitely many components if one considers markings $(H^2(X,\mathbf{Z}),\tilde{q}_X) \xrightarrow{\sim} (\Lambda,q_\Lambda)$.

In particular, they are only finitely many deformation types in a given topological type.

For such an X, Mongardi proved in [Mo2, Theorem 4.3] that $\mathrm{Mon}(X)$ is a *normal subgroup* of $O(\Lambda_X)$ of index $2^{\rho(m+1)+1}$, where $\rho(m+1)$ is the number of distinct prime factors of $m+1$. In particular, $\mathrm{Mon}(X)$ is always a proper subgroup of $O^+(\Lambda_X)$ and $M'_{\Lambda_{\mathrm{K3}^{[m]}},2m}$ has at least 4 components, and exactly 4 if and only if $m+1$ is a prime power.

This explains Namikawa's example in [N]: if T is a very general complex torus of dimension 2, with dual $T^\vee$, there is a Hodge isometry $H^2(K_2(T),\mathbf{Z}) \simeq H^2(K_2(T^\vee),\mathbf{Z})$ (so that, for suitable markings φ and $\varphi^\vee$, the periods of $(K_2(T),\varphi)$, $(K_2(T),-\varphi)$, $(K_2(T^\vee),\varphi^\vee)$, and $(K_2(T^\vee),-\varphi^\vee)$ are the same) but T and $T^\vee$ are not bimeromorphically isomorphic (there is a similar example with T projective; see [N, Remark 2]).

3.1.2. *The Torelli theorem.* We first express the Torelli theorem in terms of the surjectivity and injectivity properties of the period map for hyperkähler manifolds. The following statement combines fundamental results of Huybrechts and Verbitsky ([V]). We fix as before a lattice Λ and an even integer n, and we let $M_{\Lambda,n}$ be the (possibly empty) moduli space for Λ-marked hyperkähler manifolds of dimension n.

Theorem 3.4 (Torelli theorem, first version)**.** *Let Λ be a lattice and let n be a positive integer. The period map*

$$\wp_{\Lambda,n}\colon M_{\Lambda,n} \longrightarrow Q_\Lambda$$

is a local isomorphism. Its restriction to any connected component of $M_{\Lambda,n}$ is surjective and it is injective over the complement of the union of countably many hypersurfaces of Q_Λ.

The proof of this important result is difficult and uses constructions that we have not seen (see [Hu3] for a very nice account).

Let M be a component of $M_{\Lambda,n}$. The period map cannot be injective on M because M is not separated, while Q_Λ is. This accounts entirely for the noninjectivity of $\wp_{\Lambda,n}|_M$: points (X,φ) and (X',φ') of M have the same image by $\wp_{\Lambda,n}$ if and only if they are inseparable, meaning that any neighborhood of one meets any neighborhood of the other ([Hu3, Corollaries 4.10 and 5.9]). In that case, the isometry $\varphi'^{-1}\circ\varphi\colon (H^2(X,\mathbf{Z}),q_X) \xrightarrow{\sim} (H^2(X',\mathbf{Z}),q_{X'})$ maps the subsets (13) and (14) of $\mathrm{Pos}(X)$ to the corresponding subsets of $\mathrm{Pos}(X')$ ([M, Corollary 5.13]) and there exists a bimeromorphic isomorphism $X \overset{\sim}{\dashrightarrow} X'$ ([Hu3, Proposition 4.7]).

Remark 3.5. There is a bijection between the fiber of $\wp_{\Lambda,n}|_M$ at the point (X,φ) (which consists entirely of inseparable points) and the set of connected components of (14): it maps a point $(X',\varphi') \in M$ in the fiber of (X,φ) to the component $\varphi^{-1}(\varphi'(\mathrm{Kah}(X')))$ of (14) ([M, Proposition 5.14]).

In particular, this fiber is a single point if and only if $\mathrm{Kah}(X) = \mathrm{Pos}(X)$. This certainly happens when the period is in $Q_\Lambda \smallsetminus \bigcup_{\alpha\in\Lambda\smallsetminus\{0\}} \alpha^\perp$: for any (X,φ) with period $[x]$ in that set, we have

$$\mathrm{Pic}(X) = H^{1,1}(X)\cap H^2(X,\mathbf{Z}) \xrightarrow{\sim} x^\perp \cap \bar{x}^\perp \cap \Lambda = \{0\},$$

hence the set $\mathscr{W}_X$ in (14) is empty. This explain the very general injectivity statement in Theorem 3.4.[9]

Remark 3.6. The fiber of $\wp_{\Lambda,n}|_M$ at the point (X,φ) might very well be infinite: if $\mathrm{Bir}(X)$ is the group of bimeromorphic automorphisms of X and $\mathrm{Aut}(X)$ the subgroup of (biregular) automorphisms, *the number of points that are inseparable from (X,φ) is at least the index of* $\mathrm{Aut}(X)$

[9] Note that the fiber of the period map at such a point $[x]$ has a single point on *every* component; in general, it could possibly happen that the cardinality of the fiber of a period point depends on the component.

in $\mathrm{Bir}(X)$. Indeed, by Remark 3.5, this number is the number of connected components of (14), which is at least the number of connected components of (15), which is at least the index $[\mathrm{Bir}(X) : \mathrm{Aut}(X)]$ (this is because, by Proposition 2.3, $u, v\colon X \overset{\sim}{\dashrightarrow} X$ satisfy $u^*(\mathrm{Kah}(X)) = v^*(\mathrm{Kah}(X))$ if and only if $vu^{-1} \in \mathrm{Aut}(X)$).

In the situation of Example 2.13, all these numbers are infinite.

Remark 3.7. Points in the same fiber of $\wp_{\Lambda,n}$ but on different components of $M_{\Lambda,n}$ correspond to hyperkähler manifolds that may not be bimeromorphically isomorphic (see Example 3.3).

3.1.3. *Another form of the Torelli theorem.* We now use the Torelli theorem to try to answer the following basic questions: given hyperkähler manifolds X and X' of the same dimension, and a Hodge isometry

$$\psi\colon (H^2(X',\mathbf{Z}), q_{X'}) \xrightarrow{\sim} (H^2(X,\mathbf{Z}), q_X),$$

are X and X' bimeromorphically isomorphic, or isomorphic? More precisely, is ψ induced by a bimeromorphic isomorphism, or by an isomorphism, between X and X'?

A necessary condition for a positive answer to any of these questions is that X and X' are *of the same deformation type* (that is, they are deformation one of another).[10] However, as noted in Remark 3.7, even under this assumption, the existence of the isometry ψ does not in general imply that X and X' are bimeromorphically isomorphic. As stated in [M, Theorem 1.3], only specific isometries (which Markman calls *parallel transport operators*) are induced by bimeromorphic isomorphisms and they are in practice hard to characterize.

We will therefore make the additional assumption that *the group* $\mathrm{Mon}(X)$ *has index* 2 *in* $O(H^2(X,\mathbf{Z}), q_X)$*;* equivalently, by the discussion preceding Example 3.1, only two components of $M_{\Lambda,\dim(X)}$ correspond to deformations of X: if (X', φ') is in one component, $(X', -\varphi')$ is in the other. This assumption is satisfied in some important cases (see Examples 3.1 and 3.2).

Theorem 3.8 (Torelli theorem, second version)**.** *Let X and X' be hyperkähler manifolds of the same deformation type. Assume that the group* $\mathrm{Mon}(X)$ *has index* 2 *in* $O(H^2(X,\mathbf{Z}), q_X)$.

If there is a Hodge isometry $\psi\colon (H^2(X',\mathbf{Z}), q_{X'}) \xrightarrow{\sim} (H^2(X,\mathbf{Z}), q_X)$*, the manifolds X and X' are bimeromorphically isomorphic. Moreover, we have the following equivalences.*

(a) *The following properties are equivalent:*
  (i) *there exists a bimeromorphic isomorphism* $u\colon X \overset{\sim}{\dashrightarrow} X'$ *such that* $\psi = u^*$*;*
  (ii) *we have* $\psi\big(\overset{\circ}{\overline{\mathrm{BirKah}}}(X')\big) = \overset{\circ}{\overline{\mathrm{BirKah}}}(X)$;
  (iii) *we have* $\psi\big(\overset{\circ}{\overline{\mathrm{BirKah}}}(X')\big) \cap \overset{\circ}{\overline{\mathrm{BirKah}}}(X) \neq \varnothing$.
  If X is projective, these properties are also equivalent to
  (iv) $\psi\big(\overset{\circ}{\mathrm{Mov}}(X')\big) \cap \overset{\circ}{\mathrm{Mov}}(X) \neq \varnothing$.

(b) *The following properties are equivalent:*
  (i) *there exists an isomorphism* $u\colon X \xrightarrow{\sim} X'$ *such that* $\psi = u^*$*;*
  (ii) *we have* $\psi\big(\mathrm{Kah}(X')\big) = \mathrm{Kah}(X)$;
  (iii) *we have* $\psi\big(\mathrm{Kah}(X')\big) \cap \mathrm{Kah}(X) \neq \varnothing$.

[10] This is because bimeromorphically isomorphic hyperkähler manifolds are deformations one of another ([Hu1, Proposition 27.8]). More precisely, if (X, φ) is a Λ-marked hyperkähler manifold of dimension n and $u\colon X \overset{\sim}{\dashrightarrow} X'$ is a bimeromorphic isomorphism with another hyperkähler manifold, the pair $(X', \varphi \circ u^*)$ is a Λ-marked pair that belongs to the same component of $M_{\Lambda,n}$ as (X, φ) (this is because, by [M, Theorem 3.1], the isometry u^* is a *parallel transport operator).*

If X is projective, these properties are also equivalent to
(iv) $\psi\big(\mathrm{Amp}(X')\big)\cap \mathrm{Amp}(X)\neq\varnothing$.

Proof. Any marking $\varphi\colon (H^2(X,\mathbf{Z}),q_X)\simeq(\Lambda,q_\Lambda)$ of X induces a marking $\varphi':=\varphi\circ\psi$ of X', and (X,φ) and (X',φ') have the same period. As explained above, (X,φ) and $(X',\varepsilon\varphi')$, for some $\varepsilon\in\{-1,1\}$, are in the same component of $M_{\Lambda,\dim(X)}$ and have the same period. They are therefore inseparable hence bimeromorphically isomorphic.

We already saw in (12) that (a)(i) implies (a)(ii), which trivially implies (a)(iii). Assume (a)(iii). As explained above, either ψ or $-\psi$ is a parallel transport operator; since ψ satisfies (iii), it is a parallel transport operator. By [M, Theorem 1.6], there exist $w\in W_{\mathrm{Exc}}(X')$ (see (9)) and $u\colon X \overset{\sim}{\dashrightarrow} X'$ such that $\psi=w\circ u^*$. We then have $w\big(\overline{\mathrm{BirKah}}^{\circ}(X')\big)=\overline{\mathrm{BirKah}}^{\circ}(X')$ and by Theorem 2.5, $w=\mathrm{Id}$. Finally, if X is projective, so is X', and by (17), (a)(iv) implies (a)(iii), and (a)(ii) implies (a)(iv).

As for (b), the implications (b)(i) $\Rightarrow$ (b)(ii) $\Rightarrow$ (b)(iii) are obvious. Assume (b)(iii). By (1), there exists $u\colon X\overset{\sim}{\dashrightarrow} X'$ such that $\psi=u^*$. Since u^* maps a kähler class to a kähler class, u is biregular by Proposition 2.3. Finally, if X is projective, so is X', (b)(iv) implies (b)(iii), and (b)(ii) implies (b)(iv). □

3.1.4. *Construction of hyperkähler manifolds with given Picard lattice.* The surjectivity of the period map can be used to prove that hyperkähler manifolds with given Picard lattice exist.

Example 3.9. Let Λ be a lattice for which the moduli space $M_{\Lambda,n}$ is non empty (that is, there exists a hyperkähler manifold X of dimension n such that the lattice $(H^2(X,\mathbf{Z}),q_X)$ is isometric to Λ). In particular, the signature of Λ is $(3,r-3)$, with $r\geq 3$. Let Λ_0 be a lattice with signature $(1,s)$ and assume that Λ_0 embeds into Λ as a primitive sublattice (when Λ is even, this is always possible when $2s\leq r-3$ by a theorem of Nikulin).

Using that Λ_0 is primitive in Λ, one easily checks that for a very general element $[x]\in Q_\Lambda\cap\Lambda_0^\perp$, one has $x^\perp\cap\Lambda=\Lambda_0$. By the surjectivity statement in Theorem 3.4, there is a marked hyperkähler manifold (X,φ) whose period is $[x]$. Its Picard group is precisely

$$H^{1,1}(X)\cap H^2(X,\mathbf{Z})\simeq x^\perp\cap\bar{x}^\perp\cap\Lambda=x^\perp\cap\Lambda=\Lambda_0.$$

Note that X is projective by Theorem 1.16. This proves for example that a hyperkähler fourfold X as in Example 2.14 exists (in that case, since $\mathrm{Nef}(X)=\overline{\mathrm{Pos}}^{\mathrm{alg}}(X)$, every class in Λ_0 with positive square will be plus or minus ample).

Example 3.10. The case where Λ has signature $(3,0)$ is interesting (although it is expected to lead to empty moduli spaces!): in that case, Q_Λ is a smooth conic (this is the only case where it is compact). It is a twistor line, as defined in [Hu3, Definition 3.3]. If X is a hyperkähler manifold with $b_2(X)=3$, it has a universal deformation $\mathscr{X}\to\mathbf{P}^1$ which is its twistor space. Projective deformations of X correspond to countably many points in the moduli space $\mathbf{P}^1$; they are rigid, hence defined over $\overline{\mathbf{Q}}$.

3.1.5. *Automorphisms of hyperkähler manifolds.* Let X be a hyperkähler manifold, let $\mathrm{Bir}(X)$ be the group of its bimeromorphic automorphisms, and let $\mathrm{Aut}(X)$ be the subgroup of its (biholomorphic) automorphisms. By Proposition 2.1, there are representations

$$\Psi_X^B\colon \mathrm{Bir}(X)\longrightarrow O(H^2(X,\mathbf{Z}),q_X)\quad\text{and}\quad \Psi_X^A\colon \mathrm{Aut}(X)\longrightarrow O(H^2(X,\mathbf{Z}),q_X). \tag{21}$$

Proposition 3.11 (Huybrechts, Hassett–Tschinkel)**.** *Let X be a hyperkähler manifold. The kernels of Ψ_X^A and Ψ_X^B are equal and finite, and they are invariant by deformation of X.*

Sketch of proof. The first part is [Hu2, Proposition 9.1]: any bimeromorphic automorphism in the kernel of Ψ_X^B preserves a kähler class, hence is biholomorphic (Proposition 2.3). If an automorphism of X fixes a kähler class, it fixes the unique Calabi–Yau metric in that class hence is an isometry of the underlying riemannian manifold. Since the group of isometries of a compact riemanniann manifold is compact and the group $\mathrm{Aut}(X)$ is discrete (its Lie algebra is $H^0(X, T_X) = 0$), it is finite.

The deformation invariance of the kernel is [HT2, Theorem 2.1]. □

In particular, the index of $\Psi_X^A(\mathrm{Aut}(X))$ in $\Psi_X^B(\mathrm{Bir}(X))$ is equal to the index of $\mathrm{Aut}(X)$ in $\mathrm{Bir}(X)$ (this may be used to apply Remark 3.6).

Example 3.12. If X is a deformation of the mth Hilbert scheme of a K3 surface, it follows from [Be, Proposition 10] and Proposition 3.11 that Ψ_X^A and Ψ_X^B are both injective.

If X is a deformation of the mth Kummer variety of an abelian surface, it follows from [BNS, Corollary 5(2)] and Proposition 3.11 that the common kernels of Ψ_X^A and Ψ_X^B are isomorphic to $(\mathbf{Z}/(m+1)\mathbf{Z})^2 \rtimes \mathbf{Z}/2\mathbf{Z}$.[11]

The images of the groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ in the group $O(H^2(X, \mathbf{Z}), q_X)$ can then be studied using the Torelli theorem, for example in its version presented in Theorem 3.8. This has proved to be a very powerful tool to compute these groups and to produce hyperkähler manifolds with interesting automorphism groups (see Example 3.18).

3.2. **The Torelli theorem for polarized hyperkähler manifolds.** We now consider pairs (X, H) consisting of a hyperkähler manifold X and an ample line bundle H on X. The idea is that since any birational isomorphism between two such pairs is automatically biregular (because an ample class pulls back to an ample class), this should allow us to avoid the nonseparation problems described earlier.

3.2.1. *Marked polarized hyperkähler manifolds and their period map.* Let (Λ, q_Λ) be a lattice and let $h \in \Lambda$ be an element with $q_\Lambda(h) > 0$. We consider triples (X, H, φ), where (X, φ) is a Λ-marked hyperkähler manifold such that H is an ample line bundle on X and $\varphi(c_1(H)) = h$; we called them (Λ, h)-*marked hyperkähler manifolds,* or simply *marked polarized hyperkähler manifolds.* Since $c_1(H)$ is orthogonal to $H^{2,0}(X)$, the period of (X, φ) lands in

$$Q_{\Lambda,h} := \{[x] \in \mathbf{P}(\Lambda \otimes \mathbf{C}) \mid q_\Lambda(x) = q_\Lambda(x, h) = 0,\ q_\Lambda(x, \bar{x}) > 0\} = Q_\Lambda \cap h^\perp.$$

If Λ has signature $(3, r-3)$, this variety has dimension $r-3$ and two connected components (one if $r = 3$), each isomorphic to the homogeneous space $O(2, r-3, \mathbf{R})/SO(2, \mathbf{R}) \times O(r-3, \mathbf{R})$ and interchanged by complex conjugation. The manifold $Q_{\Lambda,h}$ is acted on by the group

$$O(\Lambda, h) = \{g \in O(\Lambda, q_\Lambda) \mid g(h) = h\}. \tag{22}$$

The Torelli theorem (Theorem 3.4) now takes the following form.

[11]The action of the latter group on $K_m(T)$ is given as follows: $(\mathbf{Z}/(m+1)\mathbf{Z})^2$ is the group of $(m+1)$-torsion points $t \in T$ and the action is induced by the action $(t, \varepsilon) \cdot (x_0, \dots, x_m) = (t + \varepsilon x_0, \dots, t + \varepsilon x_m)$ on T^{m+1}, where $\varepsilon \in \{-1, 1\}$. In particular, any smooth deformation X of $K_m(T)$ carries several involutions which are deformations of involutions of $K_m(T)$, for example the one induced by the involution $(x_0, \dots, x_m) \mapsto (-x_0, \dots, -x_m)$ of T^{m+1}. Its fixed locus has dimension $m' := \lfloor \frac{1}{2}(m+1) \rfloor$ (one component is the subvariety consisting of all $(x_0, -x_0, \dots, x_{m'-1}, -x_{m'-1})$ when m is odd, and of all $(x_0, -x_0, \dots, x_{m'-1}, -x_{m'-1}, 0)$ when m is even). It follows that X has nontrivial compact analytic subvarieties (Kaledin–Verbitsky). In contrast, a very general deformation of the mth Hilbert scheme of a K3 surface contains no nontrivial compact analytic subvarieties (Verbitsky).

Theorem 3.13. *Let (Λ, q_Λ) be a lattice, let $h \in \Lambda$ be such that $q_\Lambda(h) > 0$, and let n be a positive integer. There is a smooth and separated moduli space $M_{\Lambda,h,n} \subset M_{\Lambda,n}$ for (Λ, h)-marked hyperkähler manifolds. The restriction of the period map*

$$\wp_{\Lambda,h,n} \colon M_{\Lambda,h,n} \longrightarrow Q_{\Lambda,h} \tag{23}$$

to any connected component of $M_{\Lambda,h,n}$ is an open embedding.

Proof. Inside the (closed) inverse image $\wp_{\Lambda,n}^{-1}(Q_{\Lambda,h}) \subset M_{\Lambda,n}$, the locus of pairs (X, φ) where the class $\varphi^{-1}(h)$ is ample is open. We denote it by $M_{\Lambda,h,n}$. One then only needs to prove that inseparable points (X, φ) and (X', φ') in $M_{\Lambda,h,n}$ are equal. This follows from the description of the fiber of the period map given in Remark 3.5: since $\varphi^{-1} \circ \varphi'$ maps the kähler class $c_1(H')$ to the kähler class $c_1(H)$, it maps $\mathrm{Kah}(X')$ to $\mathrm{Kah}(X)$ and the points (X, φ) and (X', φ') are the same. □

3.2.2. *Polarized hyperkähler manifolds and their period map.* The holomorphic period map (23) is equivariant for the obvious actions of the group $O(\Lambda, h)$, defined in (22), on its source and target spaces. This group acts properly discontinuously on $Q_{\Lambda,h}$ and the Baily–Borel theory tells us that the quotient $O(\Lambda, h)\backslash Q_{\Lambda,h}$ has a natural structure of a quasi-projective variety (with finite quotient singularities).

The quotient $\mathscr{M}_{\Lambda,h,n} := O(\Lambda, h)\backslash M_{\Lambda,h,n}$ is a moduli space for polarized hyperkähler manifolds. It can be given an analytic structure via the period map; it can also be given an algebraic structure as follows.

Matsusaka's Big Theorem implies that there is a positive integer $k(n)$ such that, for any hyperkähler manifold X of dimension n and any ample line bundle H on X, the line bundle $H^{\otimes k}$ is very ample for all $k \geq k(n)$. Since smooth subvarieties of a fixed $\mathbf{P}^N$ of bounded degrees form a quasi-projective subscheme of the Hilbert scheme, we obtain a quasi-projective scheme $\mathscr{H}$ parametrizing all hyperkähler manifolds with fixed dimension, Fujiki constant, and Beauville–Bogomolov form. One would then like to take the quotient of $\mathscr{H}$ by the canonical action of $SL(N+1)$. The usual technique for taking this quotient, Geometric Invariant Theory, is difficult to apply directly in that case (one would need to show that points corresponding to our hyperkähler manifolds are semistable) but Viehweg managed to go around this difficulty to avoid a direct check of GIT stability and still construct a quasi-projective coarse moduli space.

Theorem 3.14. *Let n and d be positive integers. There exists a quasi-projective coarse moduli space for polarized complex hyperkähler manifolds (X, H) with $\dim(X) = n$ and $H^n = d$.*

On each connected component of $\mathscr{M}_{\Lambda,h,n}$, the degree H^n is fixed,[12] hence we are in the situation of Theorem 3.14 and this puts an algebraic structure on this component. For this algebraic structure on $\mathscr{M}_{\Lambda,h,n}$, the induced period map

$$\mathscr{M}_{\Lambda,h,n} \longrightarrow O(\Lambda, h)\backslash Q_{\Lambda,h}$$

is algebraic (this is again by Baily–Borel's theory). However, it is not in general the "right" map to consider if one wants it to embed the components of $\mathscr{M}_{\Lambda,h,n}$: one needs instead to take, on the right side, the quotient by a subgroup of $O(\Lambda, h)$ of finite index.

To define this subgroup, we follow [M, Sections 7 and 8]. *We fix a component M of $M_{\Lambda,n}$* (hence in particular a deformation type) and set

$$\mathscr{M}^M_{\Lambda,h,n} := \{(X, H) \in \mathscr{M}_{\Lambda,h,n} \mid \text{there exists a } (\Lambda, h)\text{-marking } \varphi \text{ such that } (X, \varphi) \in M\}. \tag{24}$$

[12] Actually, as explained in [Hu1, Proposition 26.20], H^n is bounded once the dimension n and the lattice (Λ, q_Λ) are fixed, if one uses markings for the modified form $\tilde{q}_X$ (see footnote 8).

By [M, Corollary 7.3], this is an irreducible component of $\mathscr{M}_{\Lambda,h,n}$. Let $(X,\varphi) \in M$; in (7), we defined the monodromy group $\mathrm{Mon}(X) \subset O^+(\Lambda_X)$. The finite-index subgroups

$$\mathrm{Mon}^M_\Lambda := \varphi \circ \mathrm{Mon}(X) \circ \varphi^{-1} \subset O^+(\Lambda) \tag{25}$$

$$\mathrm{Mon}^M_{\Lambda,h} := \mathrm{Mon}^M_\Lambda \cap O^+(\Lambda,h) \subset O^+(\Lambda,h) \tag{26}$$

are independent of the choice of the point (X,φ) of M ([M, Definition 7.2]) and different components M give rise to conjugate subroups. The polarized Torelli theorem takes the following form ([M, Theorem 8.4]).

Theorem 3.15 (Polarized Torelli theorem, first version)**.** *Let (Λ, q_Λ) be a lattice, let h be an element of Λ with $q_\Lambda(h) > 0$, and let n be a positive integer. Let M be a component of the moduli space $M_{\Lambda,n}$. The period map*

$$\mathscr{M}^M_{\Lambda,h,n} \longrightarrow \mathrm{Mon}^M_{\Lambda,h} \backslash Q_{\Lambda,h}$$

is an open (algebraic) embedding between quasi-projective algebraic varieties.

When Λ has signature $(3, r-3)$ with $r > 3$, the variety $Q_{\Lambda,h}$ has two connected components (each stable under the $O^+(\Lambda,h)$-action) and one usually chooses one, $Q^+_{\Lambda,h}$, into which the period map lands; the image of the period map is then an open dense subset of $\mathrm{Mon}^M_{\Lambda,h} \backslash Q^+_{\Lambda,h}$.

Remark 3.16 (Song)**.** We saw in (20) that the number of components of $M_{\Lambda,n}$ in a given deformation type is the index $[O(\Lambda) : \mathrm{Mon}^M_\Lambda]$ (which does depend on the choice of the component M). The number of components of $\mathscr{M}_{\Lambda,h,n}$ (still within the given deformation type) is then $[O(\Lambda) : \mathrm{Mon}^M_\Lambda]/[O(\Lambda,h) : \mathrm{Mon}^M_{\Lambda,h}]$.

When $\mathrm{Mon}(X)$ is a normal subgroup of $O(\Lambda_X)$, these subgroups are independent of M and the number of components is

$$[O(\Lambda) : O(\Lambda,h) \cdot \mathrm{Mon}_\Lambda].$$

This is the case when X is a deformation of either the mth Hilbert scheme of a K3 surface (Example 3.2) or the mth Kummer variety of an abelian surface (Example 3.3) and Jieao Song computed that this number is $2^{\max\{\tilde\rho(\gamma)-1,0\}}$, where γ is the *divisibility* of h (taken to be primitive), which is the positive generator of the subgroup $h \cdot \Lambda$ of $\mathbf{Z}$, and, for every $r > 0$, we denote by $\rho(r)$ the number of distinct prime factors of r and set $\tilde\rho(r) = \rho(r)$ if r is odd and $\tilde\rho(r) = \rho(r/2)$ if r is even.

This remark has the following consequence (note that the divisibility of H divides $2m-2$ and $q_X(H)$; in particular, the theorem applies when $m-1$ is the power of a prime number).

Theorem 3.17 (Polarized Torelli theorem, second version)**.** *Let m be an integer such that $m \ge 2$. Let (X,H) and (X',H') be polarized hyperkähler manifolds that are deformations of the mth Hilbert scheme of a K3 surface or of the mth Kummer variety of an abelian surface. Assume that there is a prime number p and an integer s such that the divisibility of H is either p^s or $2p^s$. If there is a Hodge isometry*

$$\psi\colon (H^2(X',\mathbf{Z}), q_{X'}) \xrightarrow{\sim} (H^2(X,\mathbf{Z}), q_X)$$

such that $\psi(H') = H$, there exists an isomorphism $u\colon X \xrightarrow{\sim} X'$ such that $\psi = u^*$.

This theorem can be used to construct automorphisms of a hyperkähler manifold whose ample cone is known, as shown in the next example.

Example 3.18. Let X be a hyperkähler fourfold as in Example 2.14. We saw in that example that every class in the positive cone of X is ample. Let $O^+(\mathrm{Pic}(X), q_X) \subset O(\mathrm{Pic}(X), q_X)$ be the index-2 subgroup of isometries that preserve the positive cone. A little bit of lattice theory shows that every element of $O^+(\mathrm{Pic}(X), q_X)$, extended by $\varepsilon\,\mathrm{Id}$ on $\mathrm{Pic}(X)^\perp$ for some $\varepsilon \in \{-1, 1\}$, lifts to an isometry of $(H^2(X, \mathbf{Z}), q_X)$. This implies that $\mathrm{Aut}(X)$ contains (and is in fact equal to) the group $O^+(\mathrm{Pic}(X), q_X)$, which is the infinite dihedral group.

3.3. **Image of the period map for polarized hyperkähler manifolds.** The period map in Theorem 3.15 is not surjective and one may ask what its image is. In the next theorem, $M_{\Lambda,n}$ is the moduli space of Λ-marked hyperkähler manifolds of dimension n and $\mathscr{M}^M_{\Lambda,h,n}$ a component—defined in (24)—of the moduli space $\mathscr{M}_{\Lambda,h,n}$ of h-polarized hyperkähler manifolds.

Theorem 3.19. *Let (Λ, q_Λ) be a lattice, let h be an element of Λ with $q_\Lambda(h) > 0$, and let n be a positive integer. Let M be a component of $M_{\Lambda,n}$. The image of the period map*

$$\mathscr{M}^M_{\Lambda,h,n} \longrightarrow \mathrm{Mon}^M_{\Lambda,h} \backslash Q_{\Lambda,h}$$

is the complement of a finite union of Heegner divisors.

A Heegner divisor is defined as follows. Let $\alpha \in \Lambda$ with negative square; the Heegner divisor D_α defined by α is the image in the quotient of the hyperplane section $Q_{\Lambda,h} \cap \alpha^\perp$. It is an irreducible algebraic hypersurface.

Proof. For each hyperkähler manifold X, we defined (Theorem 2.6 and footnote 7) a subset $\mathscr{W}_X \subset \mathrm{Pic}(X)$ of classes of negative squares called wall divisors (or MBM classes). If φ is a Λ-marking on X, we obtain a set $\varphi(\mathscr{W}_X)$ of negative classes in Λ. Given a component M as in the theorem, define (following [AV3, Definition 3.15]) the set

$$\mathscr{W}^M := \bigcup_{(X,\varphi)\in M} \varphi(\mathscr{W}_X) \subset \Lambda.$$

of *wall classes.* For any $(X, \varphi) \in M$, one then has ([Mo1, Theorem 1.3], [AV1, Corollary 5.13], [AV3, Theorem 3.14])

$$\mathscr{W}_X = \mathrm{Pic}(X) \cap \varphi^{-1}(\mathscr{W}^M). \tag{27}$$

We will show that the image of the period map is the complement of the union

$$\bigcup_{\alpha\in\mathscr{W}^M\cap h^\perp} D_\alpha \tag{28}$$

of Heegner divisors.

Let $(X, H) \in \mathscr{M}^M_{\Lambda,h,n}$, with a (Λ, h)-marking such that $(X, \varphi) \in M$. If the period is in D_α for some $\alpha \in \mathscr{W}^M \cap h^\perp$, the class $\varphi^{-1}(\alpha)$ is of type $(1, 1)$ on X hence is in $\mathscr{W}^M$ by (27). Since H is orthogonal to this class, this contradicts Theorem 2.6 for X.[13] It follows that the image of the period map is contained in the complement of (28). There are a priori countably many Heegner divisors involved, but since the image of the period map is Zariski open, there are in fact only finitely many.

We will now show that the image is exactly equal to this complement. Let $[x]$ be any point in $Q_{\Lambda,h}$. By the surjectivity of the period map (Theorem 3.4), there exists (X, φ) in M whose period is $[x]$. Since $q_\Lambda(x, h) = 0$, the class $\varphi^{-1}(h)$ is the class of a line bundle H on X with $q_X(c_1(H)) = q_\Lambda(h) > 0$. By Theorem 1.16, X is projective (but H is not necessarily ample). After possibly changing H into its opposite, we may assume $c_1(H) \in \mathrm{Pos}(X)$.

[13] It is not necessary here to use the full force of Theorem 2.6: by the very definition of wall divisors given in [Mo1, Definition 1.2], the ample class H cannot be orthogonal to a wall divisor.

Assume now that x is not in the union (28) and consider the decomposition (14). If $c_1(H) \in W^\perp$ for some $W \in \mathscr{W}_X$, we get $h \in \varphi(W)^\perp$. But $\alpha := \varphi(W)$ is then in $\mathscr{W}^M \cap h^\perp$ and is orthogonal to x, which contradicts our assumption.

Therefore, by Theorem 2.6, there exist $g \in \mathrm{Mon}_{\mathrm{Hdg}}(X)$ and $u\colon X' \overset{\sim}{\dashrightarrow} X$ such that $u^*(g(c_1(H)))$ is ample on X'. The pair $(X', \varphi \circ g^{-1} \circ (u^*)^{-1})$ still has period $[x]$ and, by footnote 10, belongs to M. We have therefore expressed $[x]$ as the period point of an element of $\mathscr{M}^M_{\Lambda,h,n}$. This proves the theorem. □

3.4. **The period map for polarized deformations of Hilbert schemes of K3 surfaces.** As explained in the proof of Theorem 3.19, in order to describe explicitly the Heegner divisors in the complement of the image of the period map, one needs to know the wall classes. These wall classes were described in [BM, Mo1] for mth Hilbert schemes of K3 surfaces and in [Y] for mth Kummer varieties of abelian surfaces (see also [BNS, Theorem 2.9] for a summary). For deformations of mth Hilbert schemes of K3 surfaces, the explicit description of the image of the period map was worked out in [DM] when $m = 2$ (see below) and in [D, Appendix B] in general.

As explained in Remark 3.16, given an element $h \in \Lambda_{\mathrm{K3}^{[m]}}$ with $q(h) > 0$, the moduli space $\mathscr{M}'_{\Lambda_{\mathrm{K3}^{[m]}},h,2m}$ of hyperkähler manifolds that are deformations of mth Hilbert schemes of K3 surfaces with a polarization "of type h" may have several components. It is not clear whether the image of each component under the period map is the same.

Assume now $m = 2$, take $\Lambda = \Lambda_{\mathrm{K3}^{[2]}}$ (see Example 3.2), and fix integers $e > 0$ and $\gamma \in \{1, 2\}$. All primitive $h \in \Lambda$ with $q_\Lambda(h) = 2e$ and divisibility γ (see Remark 3.16 for the definition) are in the same $O(\Lambda)$-orbit, so that the moduli space $\mathscr{M}_{\Lambda,h,4}$ only depends on e and γ.[14] It then follows from Remark 3.16 that a unique component of $\mathscr{M}_{\Lambda,h,4}$ parametrizes pairs (X, H) such that X is a deformation of the second Hilbert scheme of a K3 surface (when $\gamma = 2$, one needs the extra condition $e \equiv -1 \pmod 4$ to insure that such a component exists). We denote it by $\mathscr{M}_{e,\gamma}$. Theorem 3.19 says that the image of the period map

$$\mathscr{M}_{e,\gamma} \longrightarrow \mathrm{Mon}_{\Lambda,e,\gamma} \backslash Q_{\Lambda,e,\gamma}$$

is the complement of a finite union of Heegner divisors. As mentioned above, these divisors were worked out explicitly in [DM]. Let us mention one case of [DM, Theorem 6.1].

Proposition 3.20. *Let e be a positive integer such that $e \equiv -1 \pmod 4$. The image of the period map*

$$\mathscr{M}_{e,2} \longrightarrow \mathrm{Mon}_{\Lambda,e,2} \backslash Q_{\Lambda,e,2}$$

is the complement of a single (irreducible) Heegner divisor.

Example 3.21. Let $e = 3$. A general element of $\mathscr{M}_{3,2}$ is the variety of lines contained in a smooth cubic hypersurface in $\mathbf{P}^5$. More precisely, let $\mathscr{M}_{\mathrm{cubic}}$ be the (projective) GIT moduli space of cubic hypersurfaces in $\mathbf{P}^5$ and let $\mathscr{M}^{\mathrm{smooth}}_{\mathrm{cubic}} \subset \mathscr{M}_{\mathrm{cubic}}$ be the (affine) open subset corresponding to smooth cubic hypersurfaces. The map that associates with such a cubic the variety of lines that it contains defines ([BD]) an open embedding

$$F\colon \mathscr{M}^{\mathrm{smooth}}_{\mathrm{cubic}} \lhook\joinrel\longrightarrow \mathscr{M}_{3,2}.$$

Proposition 3.20 says that the image of the period map

$$\mathscr{M}_{3,2} \lhook\joinrel\longrightarrow \mathrm{Mon}_{\Lambda,3,2} \backslash Q_{\Lambda,3,2}$$

[14] It follows from [GHS, Proposition 3.6] that this is more generally the case if and only if $\gcd(\frac{2e}{\gamma}, \frac{2m-2}{\gamma}, \gamma) = 1$ or $\gamma = 2$.

is the complement of a single (irreducible) Heegner divisor denoted by D_6 in the literature. The image of the open subset $F(\mathscr{M}_{\rm cubic}^{\rm smooth})$ is the complement of $D_2 \cup D_6$, where D_2 is another Heegner divisor ([L]). The general element of the complement $\mathscr{M}_{3,2} \smallsetminus F(\mathscr{M}_{\rm cubic}^{\rm smooth})$ (which maps to D_2) is a pair $(S^{[2]}, 2L - \delta)$, where (S, L) is a general polarized K3 surface of degree 2 (see Example 1.8 for the notation).

The rational map $F\colon \mathscr{M}_{\rm cubic} \dashrightarrow \mathscr{M}_{3,2}$ is not defined at the (semistable) point corresponding to the (singular) *chordal cubic:* it blows up that point and maps it onto the divisor $\mathscr{M}_{3,2} \smallsetminus F(\mathscr{M}_{\rm cubic}^{\rm smooth})$.

Example 3.22. Let $e = 11$. A general element of $\mathscr{M}_{11,2}$ is a so-called *Debarre–Voisin variety* (see [DV]). More precisely, let $\mathscr{M}_{\rm DV}$ be the (projective) GIT moduli space of Debarre–Voisin varieties. There is a rational map

$$F\colon \mathscr{M}_{\rm DV} \dashrightarrow \mathscr{M}_{11,2}.$$

Proposition 3.20 says that the image of the period map

$$\mathscr{M}_{11,2} \hookrightarrow \mathrm{Mon}_{\Lambda,11,2} \backslash Q_{\Lambda,11,2}$$

is the complement of a single (irreducible) Heegner divisor denoted by D_{22} in the literature. The domain of definition of the map F is not known precisely, but in a work in progress with Frédéric Han, Kieran O'Grady, and Claire Voisin, we identify several points of $\mathscr{M}_{\rm DV}$ at which it is not defined, and which are blown up to the following irreducible divisors in $\mathscr{M}_{11,2}$ (again, see Example 1.8 for the notation):

- the divisor of pairs $(X, 6L - 5\delta)$, where (S, L) is a general polarized K3 surface of degree 2 and X is the only nontrivial birational model of $S^{[2]}$, which is mapped by the period map to the Heegner divisor D_2;
- the divisor of pairs $(S^{[2]}, 2L - \delta)$, where (S, L) is a general polarized K3 surface of degree 6, which is mapped by the period map to the Heegner divisor D_6;
- the divisor of pairs $(S^{[2]}, 2L - 3\delta)$, where (S, L) is a general polarized K3 surface of degree 10, which is mapped by the period map to the Heegner divisor D_{10};
- the divisor of pairs $(S^{[2]}, 2L - 5\delta)$, where (S, L) is a general polarized K3 surface of degree 18, which is mapped by the period map to the Heegner divisor D_{18};
- the divisor of pairs $(S^{[2]}, 2L - 7\delta)$, where (S, L) is a general polarized K3 surface of degree 30, which is mapped by the period map to the Heegner divisor D_{30}.

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