BEND AND BREAK*

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June 25, 2007

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*These are notes from a minicourse given at the Summer School "Geometry of complex projective varieties and the minimal model program" held in Grenoble from June 18 to July 6, 2007.

There are few objects canonically attached to a smooth projective algebraic variety X of dimension n. One of them is the invertible sheaf of differential n-forms, or equivalently the corresponding divisor class, for which one chooses a representant K_X, called a *canonical divisor*. It is therefore not surprising that a lot of the geometry of X is reflected in intersection properties of curves on X with this canonical class K_X.

More generally, Mori's theory aims at relating the birational geometry of a variety X to the structure of the cone of numerical equivalences classes of curves on X with respect to the class K_X.

1 Rational curves

Birational geometry has to do with the study of birational maps between varieties. A typical example of a birational morphism is the blow-up $\varepsilon : Y \to X$ of a smooth subvariety Z of a smooth variety X (and a typical example of a birational rational map is the inverse $\varepsilon^{-1} : X \dashrightarrow Y$). The variety Y is again smooth and

$$K_Y = \varepsilon^* K_X + (c-1)E$$

where E is the exceptional divisor of ε and c the codimension of Z in X. Note that E is a $\mathbf{P}^{c-1}$-bundle over Z. In particular, it is *ruled,* and any line ℓ contracted by ε satisfies

$$K_Y \cdot \ell = -(c-1) < 0$$

The principle we want to illustrate here is that the more rational curves a variety contains, the more complicated its birational geometry is. More precisely, on a smooth variety Y, rational curves C satisfying $K_Y \cdot C < 0$ will be of special interest (see Proposition 3).

Proposition 1 *Let X and Y be projective varieties, with X smooth, and let $\pi : Y \to X$ be a birational morphism. Through a general point of each component of* $\operatorname{Exc}(\pi)$,[1] *there exists a rational curve contracted by π.*

The proof shows more precisely that every irreducible component of the hypersurface $\operatorname{Exc}(\pi)$ is *ruled:* it is birational to a product $\mathbf{P}^1 \times Z$.

PROOF. Upon replacing Y with its normalization, we may assume that Y is nonsingular in codimension 1. Each component of $E = \operatorname{Exc}(\pi)$ has codimension 1 hence, by shrinking Y, we may assume that Y and E are smooth and irreducible. Set $U_0 = X \smallsetminus \operatorname{Sing}(\overline{\pi(E)})$, so that the closure in U_0 of the image of $E \cap \pi^{-1}(U_0)$ is smooth, of codimension at least 2. Let $\varepsilon_1 : X_1 \to U_0$ be its blow-up; by the universal property of blow-ups ([H1], II, Proposition 7.14), there exists a factorization

$$\pi|_{V_1} : V_1 \xrightarrow{\pi_1} X_1 \xrightarrow{\varepsilon_1} U_0 \subset X$$

where the complement of $V_1 = \pi^{-1}(U_0)$ in Y has codimension at least 2 and $\overline{\pi_1(E \cap V_1)}$ is contained in the support of the exceptional divisor of ε_1. If the codimension of $\overline{\pi_1(E \cap V_1)}$ in X_1 is at least 2, the divisor $E \cap V_1$ is contained in the exceptional locus of π_1 and, upon replacing V_1 by the complement V_2 of a closed subset of codimension at least 2 and X_1 by an open subset U_1, we may repeat the construction. After i steps, we get a factorization

$$\pi : V_i \xrightarrow{\pi_i} X_i \xrightarrow{\varepsilon_i} U_{i-1} \subset X_{i-1} \xrightarrow{\varepsilon_{i-1}} \cdots \xrightarrow{\varepsilon_2} U_1 \subset X_1 \xrightarrow{\varepsilon_1} U_0 \subset X$$

as long as the codimension of $\overline{\pi_{i-1}(E \cap V_{i-1})}$ in X_{i-1} is at least 2, where V_i is the complement in Y of a closed subset of codimension at least 2. Let $E_j \subset X_j$ be the exceptional divisor of ε_j. We have

$$\begin{aligned} K_{X_i} &= \varepsilon_i^* K_{U_{i-1}} + c_i E_i \\ &= (\varepsilon_1 \circ \cdots \circ \varepsilon_i)^* K_X + c_i E_i + c_{i-1} E_{i,i-1} + \cdots + c_1 E_{i,1} \end{aligned}$$

[1]This is the complement in Y of the largest open set over which π is an isomorphism. Since X is normal, by Zariski's Main Theorem, this is also the complement of the union of positive-dimensional fibers of π; furthermore, $\pi(\operatorname{Exc}(\pi))$ has codimension ≥ 2 in X and $\operatorname{Exc}(\pi) = \pi^{-1}(\pi(\operatorname{Exc}(\pi)))$. Since X is smooth, every irreducible component of $\operatorname{Exc}(\pi)$ has codimension 1 in Y.

where $E_{i,j}$ is the inverse image of E_j in X_i and

$$c_i = \operatorname{codim}_{X_{i-1}}(\overline{\pi_{i-1}(E \cap V_{i-1})}) - 1 > 0$$

Since π_i is birational, $\pi_i^* \mathscr{O}_{X_i}(K_{X_i})$ is a subsheaf of $\mathscr{O}_{V_i}(K_{V_i})$. Moreover, since $\pi_j(E \cap V_j)$ is contained in the support of E_j, the divisor $\pi_j^* E_j - E|_{V_j}$ is effective, hence so is $E_{i,j} - E|_{V_i}$.

It follows that $\mathscr{O}_Y(\pi^* K_X + (c_i + \cdots + c_1)E)|_{V_i}$ is a subsheaf of $\mathscr{O}_{V_i}(K_{V_i}) = \mathscr{O}_Y(K_Y)|_{V_i}$. Since Y is normal and the complement of V_i in Y has codimension at least 2, $\mathscr{O}_Y(\pi^* K_X + (c_i + \cdots + c_1)E)$ is also a subsheaf of $\mathscr{O}_Y(K_Y)$. Since there are no infinite ascending sequences of subsheaves of a coherent sheaf on a Noetherian scheme, the process must terminate at some point: $\overline{\pi_i(E \cap V_i)}$ is a divisor in X_i for some i, hence $E \cap V_i$ is not contained in the exceptional locus of π_i. The morphism π_i then induces a birational isomorphism between $E \cap V_i$ and E_i, and the latter is ruled: more precisely, through every point of E_i there is a rational curve contracted by ε_i. This proves the proposition. □

Corollary 2 *Let Y and X be projective varieties. Assume that X is smooth and that Y contains no rational curves. Any rational map $f : X \dashrightarrow Y$ is defined everywhere.*

PROOF. Let $X' \subset X \times Y$ be the graph of f. The first projection induces a birational morphism $p : X' \to X$. Assume its exceptional locus $\operatorname{Exc}(p)$ is nonempty. By Proposition 1, there exists a rational curve on $\operatorname{Exc}(p)$ which is contracted by p. Since Y contains no rational curves, it must also be contracted by the second projection, which is absurd since it is contained in $X \times Y$. Hence $\operatorname{Exc}(p)$ is empty and π is defined everywhere. □

Under the hypotheses of Proposition 1, one can say more if Y also is smooth.

Proposition 3 *Let X and Y be smooth projective varieties and let $\pi : Y \to X$ be a birational morphism which is not an isomorphism. There exists a rational curve C on Y contracted by π such that $K_Y \cdot C < 0$.*

Proof. The image $\pi(E)$ of the exceptional locus E of π has codimension at least 2 in X (footnote 1). Let x be a point of $\pi(E)$. By Bertini's theorem, a generic hyperplane section of X is smooth and connected, and a generic hyperplane section of X *passing through* x has the same property. It follows that by taking $\dim(X) - 2$ suitable hyperplane sections, we get a smooth surface S in X that meets $\pi(E)$ in a finite set containing x. Moreover, taking one more hyperplane section, we get on S a smooth curve C_0 that meets $\pi(E)$ only at x and a smooth curve C that does not meet $\pi(E)$.

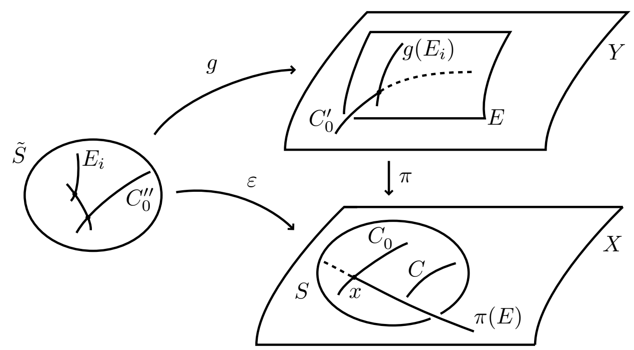

Figure 1: Construction of a rational curve $g(E_i)$ in the exceptional locus E of π

By construction,

$$K_X \cdot C = K_X \cdot C_0$$

We have $K_Y \equiv \pi^* K_X + \mathrm{Ram}(\pi)$, where the support of the divisor $\mathrm{Ram}(\pi)$ is exactly E. Since the curve $C' = \pi^{-1}(C)$ does not meet E, we have

$$K_Y \cdot C' = K_X \cdot C$$

by the projection formula. On the other hand, since the strict transform

$$C_0' = \overline{\pi^{-1}(C_0 - \pi(E))}$$

of C_0 does meet E, we have

$$K_Y \cdot C_0' = (\pi^* K_X + \mathrm{Ram}(\pi)) \cdot C_0' > (\pi^* K_X) \cdot C_0' = K_X \cdot C_0$$

hence

$$K_Y \cdot C_0' > K_Y \cdot C' \tag{1}$$

The indeterminacies of the rational map $\pi^{-1} : S \dashrightarrow Y$ can be resolved by blowing-up a finite number of points of $S \cap \pi(E)$ to get a morphism

$$g : \tilde{S} \xrightarrow{\varepsilon} S \overset{\pi^{-1}}{\dashrightarrow} Y$$

whose image is the strict transform of S in Y. The curve $C'' = \varepsilon^* C$ is irreducible and $g_* C'' = C'$; for C_0, we write

$$\varepsilon^* C_0 = C_0'' + \sum_i m_i E_i$$

where the m_i are nonnegative integers, the E_i are exceptional divisors for ε (hence in particular rational curves), and $g_* C_0'' = C_0'$. Since C and C_0 are linearly equivalent on S, we have

$$C'' \equiv C_0'' + \sum_i m_i E_i$$

on $\tilde{S}$ hence, by applying g_*,

$$C' \equiv C_0' + \sum_i m_i (g_* E_i)$$

Taking intersections with K_Y, we get

$$K_Y \cdot C' \equiv K_Y \cdot C_0' + \sum_i m_i (K_Y \cdot g_* E_i)$$

It follows from (1) that $(K_Y \cdot g_* E_i)$ is negative for some i. In particular, $g(E_i)$ is not a point hence is a rational curve on Y. Moreover, $\pi(g(E_i)) = \varepsilon(E_i)$ hence $g(E_i)$ is contracted by π. □

Let us interpret these results from the point of view of the classification of algebraic varieties. Let $\mathscr{C}$ be a birational equivalence class of smooth projective varieties. One aims at finding a "simplest" member in $\mathscr{C}$. The picture we now have is the following.

Proposition 2 says that if $\mathscr{C}$ contains a (smooth) variety X_0 without rational curves, it is automatically "minimal" in the sense that for each member X of $\mathscr{C}$, there is a birational *morphism* $X \to X_0$. Moreover, by

Proposition 3, X_0 is uniquely determined up to isomorphism (this is the best possible situation!).

If $\mathscr{C}$ contains a variety X_1 such that K_{X_1} has nonnegative degree on rational curves[2] (so this is weaker than containing no rational curves), one may only say by Proposition 3 that any birational morphism from X_1 to another member of $\mathscr{C}$ is an isomorphism.

2 The Cone Theorem

We are now convinced that on a (smooth projective) variety X, rational curves C such that $K_X \cdot C < 0$ play an important role.

Let X be a projective variety. Recall that the real vector space

$$N_1(X)_{\mathbf{R}} = \{\text{1-cycles on } X \text{ with real coefficients}\}/\text{numerical equivalence}$$

is finite dimensional. It contains the convex cone[3] $\mathrm{NE}(X)$ spanned by classes of curves and its closure $\overline{\mathrm{NE}}(X)$.

If S is a subset of $N_1(X)_{\mathbf{R}}$ and h is a Cartier divisor class on X, we write

$$S_{h>0} = \{z \in S \mid h \cdot z > 0\}$$

and similarly for $S_{h\geq 0}$. If h is nef, the cone $\overline{\mathrm{NE}}(X)$ is contained in the half-space $(N_1(X)_{\mathbf{R}})_{h\geq 0}$. If h is ample, we have $\overline{\mathrm{NE}}(X) - \{0\} \subset (N_1(X)_{\mathbf{R}})_{h>0}$ (this is Kleiman's criterion for ampleness).

An extremal ray $\mathbf{R}^+ z$ of $\overline{\mathrm{NE}}(X)$ is K_X*-negative* if $K_X \cdot z < 0$.

Theorem 4 (Cone Theorem) *Let X be a projective manifold. The set $\mathscr{R}$ of all K_X-negative extremal rays of $\overline{\mathrm{NE}}(X)$ is countable and*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X\geq 0} + \sum_{R\in\mathscr{R}} R$$

These rays are locally discrete in the half-space $(N_1(X)_{\mathbf{R}})_{K_X<0}$.

For each $R \in \mathscr{R}$, there exists a rational curve Γ in X such that

$$R = \mathbf{R}^+[\Gamma] \quad , \quad 0 < -K_X \cdot \Gamma \leq \dim(X) + 1.$$

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[2] We will show in Theorem 4 that this is equivalent to K_{X_1} nef.

[3] A cone is a subset V of $\mathbf{R}^m$ stable by multiplication by $\mathbf{R}^+$. A subcone W of a closed cone V is *extremal* if it is closed and convex and if any two elements of V whose sum is in W are both in W. An extremal subcone of dimension 1 is called an *extremal ray.* See §9.1 for more on cones.

About the singularities of X. We assume here that X is smooth, but it is very important for Mori's program to be able to relax this hypothesis. Note that we still need the intersection number of K_X with a curve to be defined, so K_X needs at least to be $\mathbf{Q}$-Cartier. The theorem still holds when X has canonical singularities. It also still holds when X is replaced by a projective klt pair (X, Δ) (here Δ is a "small" effective divisor on the projective variety X). The methods of proof are very different (see §10.2).

If it is three-dimensional, the cone $\overline{\mathrm{NE}}(X)$ looks like Figure 2. However, in general, this cone can be *not locally polyhedral* in the half-space $(N_1(X)_{\mathbf{R}})_{K_X<0}$.

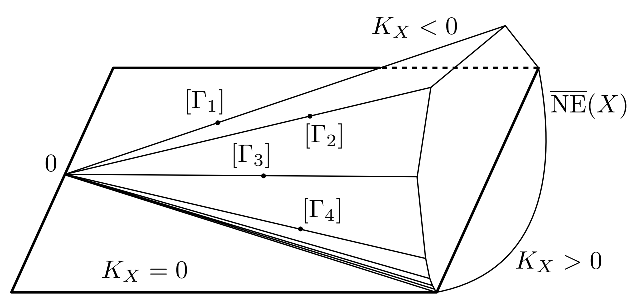


Figure 2: A three-dimensional closed cone of curves

Example 5 If K_X is nef, the effective cone lies entirely in the closed half-space $(N_1(X)_{\mathbf{R}})_{K_X \geq 0}$ and the cone theorem does not give any information. This is the case for example when X is an abelian surface. If we fix an ample class h on X, we have in this case

$$\overline{\mathrm{NE}}(X) = \{z \in N_1(X)_{\mathbf{R}} \mid z^2 \geq 0 \ , \ h \cdot z \geq 0\}$$

By Hodge theory, the intersection form on $N_1(X)_{\mathbf{R}}$ has exactly one positive eigenvalue, so that when this vector space has dimension 3,[4] the closed cone of curves of X looks like Figure 3. In particular, it is not finitely generated. Every boundary point generates an extremal ray, hence there are extremal rays whose only rational point is 0: they cannot be generated by the class of a curve on X.

[4]This is the case fo example when $X = E \times E$, where E is a very geenral elliptic curve ([Ko], Exercise II.4.16).

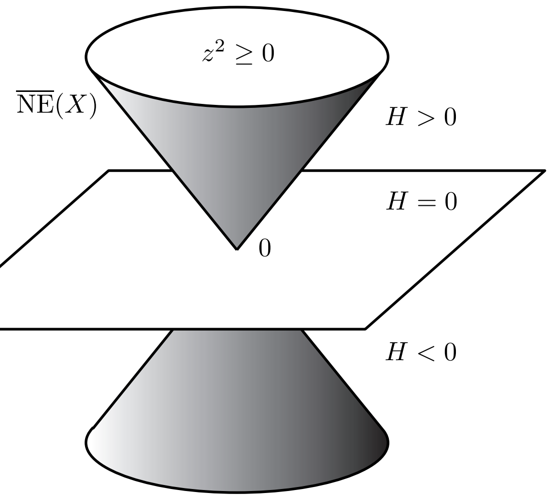


Figure 3: The effective cone of an abelian surface X

Example 6 Let X be a Fano variety, i.e., a projective manifold with $-K_X$ ample; $\overline{\mathrm{NE}}(X) - \{0\}$ is contained in the half-space $(N_1(X)_{\mathbf{R}})_{K_X<0}$, hence the set of extremal rays, being locally discrete and compact, is finite. The Cone Theorem yields

$$\overline{\mathrm{NE}}(X) = \sum_{i=1}^{m} \mathbf{R}^+[\Gamma_i] = \mathrm{NE}(X)$$

Example 7 Let $X \to \mathbf{P}^2$ be the blow-up at the 9 base-points of a general pencil of cubics, let $\pi : X \to \mathbf{P}^1$ be the morphism given by the pencil of cubics and let B be the finite subset of X where π is not smooth. The exceptional divisors $E_0, \ldots, E_8$ are sections of π. Smooth fibers of π are elliptic curves, hence become abelian groups by choosing E_0 as the origin; translations by elements of E_i then generate a subgroup of $\mathrm{Aut}(X - B)$ which can be shown to be isomorphic to $\mathbf{Z}^8$.

Since X is smooth, any automorphism σ of $X - B$ extends to X.[5] For any

[5]Indeed, if $\tilde{\sigma} : \tilde{X} \xrightarrow{\varepsilon} X \overset{\sigma}{\dashrightarrow} X$ is a resolution of its indeterminacies and if E is the last (-1)-curve of the composition of blow-ups ε, its image $\tilde{\sigma}(E)$ in X is a curve, and any point x of $\tilde{\sigma}(E) - B$ has at least two preimages in $\tilde{X}$ (one on E, and $\varepsilon^{-1}(\sigma^{-1}(x))$). By Zariski's heorem, the fibers of $\tilde{\sigma}$ are connected hence positive dimensional above $\tilde{\sigma}(E)$, which is absurd. It follows that there are no such curves E in $\tilde{X}$, hence ε is an isomorphism.

such σ, the curve $E_\sigma = \sigma(E_0)$ is rational with self-intersection -1 and $K_X \cdot E_\sigma = -1$. Each of these curves generates an extremal ray (more generally, on a smooth projective surface, any irreducible curve with negative self-intersection generates an extremal ray).

It follows that $\overline{\mathrm{NE}}(X)$ has infinitely many extremal rays contained in the open half-space $(N_1(X)_\mathbf{R})_{K_X<0}$, which are *not* locally finite in a neighborhood of $K_X^\perp$, because $K_X \cdot E_\sigma = -1$ but $(E_\sigma)_{\sigma\in\mathbf{Z}^8}$ is unbounded since the set of classes of irreducible curves is discrete in $N_1(X)_\mathbf{R}$.

3 Contractions

3.1 Relative cone of curves.

Let X and Y be projective varieties. We define the relative cone of a morphism $\pi : X \to Y$ as the convex subcone $\mathrm{NE}(\pi)$ of $\mathrm{NE}(X)$ generated by the classes of curves contracted by π. Since Y is projective, an irreducible curve C on X is contracted by π if and only if $\pi_*[C] = 0$: being contracted is a numerical property. It follows that $\mathrm{NE}(\pi)$ is the intersection of $\mathrm{NE}(X)$ with the vector space $\mathrm{Ker}(\pi_*)$. It is therefore *closed* in $\mathrm{NE}(X)$ and *the class of an irreducible curve C lies in* $\mathrm{NE}(\pi)$ *if and only if this curve is contracted by* π.

For any proper morphism $\pi : X \to Y$ with Stein factorization $\pi : X \xrightarrow{\pi'} Y' \to Y$, the curves contracted by π and the curves contracted by π' are the same, hence the relative cones of π and π' are the same. So will assume that the following condition is satisfied

$$\pi_*\mathscr{O}_X \simeq \mathscr{O}_Y \tag{2}$$

We show that morphisms π defined on a projective variety X which satisfy (2) are characterized by their relative cone $\mathrm{NE}(\pi)$. Moreover, this closed convex subcone of $\mathrm{NE}(X)$ is extremal (see footnote 3 and §9.1; geometrically, this means that $\mathrm{NE}(X)$ lies on one side of some hyperplane containing $\mathrm{NE}(\pi)$).

One of our aims will be to give sufficient conditions on an extremal subcone (often an extremal ray) of $\mathrm{NE}(X)$ for it to be associated with an actual morphism, thereby converting geometric data on the (relatively) simple object $\mathrm{NE}(X)$ into geometric information about the variety X.

Proposition 8 *Let X, Y and Y' be projective varieties and let $\pi : X \to Y$ be a morphism.*

a) The subcone $\mathrm{NE}(\pi)$ *of* $\mathrm{NE}(X)$ *is* extremal.

b) Assume $\pi_*\mathscr{O}_X \simeq \mathscr{O}_Y$ and let $\pi' : X \to Y'$ be another morphism.

- *If* $\mathrm{NE}(\pi)$ *is contained in* $\mathrm{NE}(\pi')$*, there is a* unique *morphism $f : Y \to Y'$ such that $\pi' = f \circ \pi$.*
- *The morphism π is uniquely determined by* $\mathrm{NE}(\pi)$ *up to isomorphism.*

Proof. Let $a = \sum a_i[C_i]$ and $a' = \sum a'_j[C'_j]$ be elements of $\mathrm{NE}(X)$, where a_i and a'_j are positive real numbers. If $a + a'$ is in $\mathrm{NE}(\pi)$, there exists a decomposition

$$\sum a_i[C_i] + \sum a'_j[C'_j] = \sum a''_k[C''_k]$$

where the C''_k are irreducible curves contracted by π and the a''_k are positive. Applying π_*, we get $\sum a_i\pi_*[C_i] + \sum a'_j\pi_*[C'_j] = 0$ in $N_1(Y)_{\mathbf{R}}$. Since Y is projective, the C_i and C'_j must be contracted by π hence a and a' are in $\mathrm{NE}(\pi)$. This proves a).

To prove b), we need the following rigidity result.

Lemma 9 (Rigidity Lemma) *Let X, Y and Y' be varieties and let $\pi : X \to Y$ and $\pi' : X \to Y'$ be proper morphisms. Assume $\pi_*\mathscr{O}_X \simeq \mathscr{O}_Y$.*

a) If π' contracts one fiber $\pi^{-1}(y_0)$ of π, there is an open neighborhood Y^0 of y_0 in Y and a factorization

$$\pi'|_{\pi^{-1}(Y_0)} : \pi^{-1}(Y_0) \xrightarrow{\pi} Y_0 \longrightarrow Y'$$

b) If π' contracts each *fiber of π, it factors through π.*

Proof. Note that π is surjective. Let Z be the image of

$$g : X \xrightarrow{(\pi,\pi')} Y \times Y'$$

and let $p : Z \to Y$ and $p' : Z \to Y'$ be the two projections. Then $\pi^{-1}(y_0) = g^{-1}(p^{-1}(y_0))$ is contracted by π', hence by g. It follows that

$p^{-1}(y_0) = g(g^{-1}(p^{-1}(y_0)))$ is a point hence the proper surjective morphism p is finite over an open neighborhood Y^0 of y_0 in Y. Set $X^0 = f^{-1}(Y^0)$, $Z^0 = p^{-1}(Y^0)$, and $p_0 = p|_{Z^0} : Z^0 \to Y^0$; we have $\mathscr{O}_{Z^0} \subset g_* \mathscr{O}_{X^0}$ and

$$\mathscr{O}_{Y^0} \subset p_{0*}\mathscr{O}_{Z^0} \subset p_{0*}g_*\mathscr{O}_{X^0} = \pi_*\mathscr{O}_{X^0} = \mathscr{O}_{Y^0}$$

hence $p_{0*}\mathscr{O}_{Z^0} = \mathscr{O}_{Y^0}$. It follows that p_0 is an isomorphism and $\pi' = p' \circ p_0^{-1} \circ \pi|_{X^0}$. This proves a).

If π' contracts *each* fiber of π, the morphism p above is finite, one can take $Y_0 = Y$ and π' factors through π. This proves b). □

Going back to the proof of item b) in the proposition, we assume now $\pi_*\mathscr{O}_X \simeq \mathscr{O}_Y$ and $\mathrm{NE}(\pi) \subset \mathrm{NE}(\pi')$. This means that every irreducible curve contracted by π is contracted by π', hence every (connected) fiber of π is contracted by π'. The existence of f follows from item b) of the lemma. If $f' : Y \to Y'$ satisfies $\pi' = f' \circ \pi$, the composition $Z \xrightarrow{p} Y \xrightarrow{f'} Y'$ must be the second projection, hence $f' \circ p = p'$ and $f' = p' \circ p^{-1}$ is uniquely determined.

The second item in b) follows from the first. □

3.2 Contraction of an extremal subcone

Part of Mori's Minimal Model Program aims at giving conditions under which a given extremal subcone V of $\mathrm{NE}(X)$ can be contracted. Once a contraction exists, its Stein factorization yields a contraction which is unique by the lemma and will be called *the* contraction map of V.

4 Parametrizing morphisms

4.1 The scheme $\mathrm{Mor}(Y, X)$

If X and Y are varieties defined over a field $\mathbf{k}$, with X quasi-projective and Y projective, Grothendieck showed ([G1], 4.c) that morphisms from Y to X are parametrized by a locally Noetherian scheme $\mathrm{Mor}(Y, X)$, which will in general have countably many components. One way to remedy that is to fix an ample divisor H on X and a polynomial P with rational coefficients: the subscheme

$\mathrm{Mor}_P(Y, X)$ of $\mathrm{Mor}(Y, X)$ which parametrizes morphisms $f : Y \to X$ with fixed *Hilbert polynomial*

$$P(m) = \chi(Y, mf^*H)$$

is now quasi-projective over $\mathbf{k}$, and $\mathrm{Mor}(Y, X)$ is the disjoint union of the $\mathrm{Mor}_P(Y, X)$, for all polynomials P.[6] Note that when Y is an irreducible curve, fixing the Hilbert polynomial amounts to fixing the degree of the 1-cycle f_*Y for the embedding of X defined by some multiple of H.

Let us make more precise this notion of *parameter space*. First, there is a universal morphism

$$f^{\mathrm{univ}} : Y \times \mathrm{Mor}(Y, X) \to X$$

such that, for each point t of $\mathrm{Mor}(Y, X)$, the morphism $f^{\mathrm{univ}}_t : Y \to X$ is the morphism corresponding to t (conversely, for any morphism $f : Y \to X$, we will write $[f]$ for the corresponding point of $\mathrm{Mor}(Y, X)$).

The second property, called the *universal* property, says that for any family of morphisms $f_t : Y \to X$ parametrized by a scheme T, the map $T \to \mathrm{Mor}(Y, X)$ obtained by sending t to $[f_t]$ is algebraic. The correct way to express that is to say that there is a one-to-one correspondence between

- morphisms $\varphi : T \to \mathrm{Mor}(Y, X)$ and
- morphisms $f : Y \times T \to X$

obtained by sending φ to

$$f(y, t) = f^{\mathrm{univ}}(y, \varphi(t))$$

Given φ, we will call the morphism f the *evaluation map* and often denote it by ev.

4.2 The tangent space to $\mathrm{Mor}(Y, X)$

We will use the universal property to determine the Zariski tangent space to $\mathrm{Mor}(Y, X)$ at a point $[f]$. This vector space parametrizes by definition

[6] The fact that Y is projective is essential in this construction: the space $\mathrm{Mor}(\mathbf{A}^1, \mathbf{A}^N)$ is *not* a disjoint union of quasi-projective varieties.

morphisms from $\operatorname{Spec}\mathbf{k}[\varepsilon]/(\varepsilon^2)$ to $\operatorname{Mor}(Y,X)$ with image $[f]$, hence extensions of f to morphisms

$$f_\varepsilon : Y\times \operatorname{Spec}\mathbf{k}[\varepsilon]/(\varepsilon^2)\to X$$

which should be thought of as first-order infinitesimal deformations of f.

Proposition 10 *Let X and Y be varieties, with X quasi-projective and Y projective, and let $f : Y\to X$ be a morphism. One has*

$$T_{[f]}\operatorname{Mor}(Y,X)\simeq H^0(Y,\mathscr{H}om(f^*\Omega_X,\mathscr{O}_Y))$$

PROOF. Assume first that Y and X are affine and write $Y=\operatorname{Spec}(B)$ and $X=\operatorname{Spec}(A)$ (where A and B are finitely generated $\mathbf{k}$-algebras). Let $f^\sharp : A\to B$ be the morphism corresponding to f; we are looking for $(\mathbf{k}[\varepsilon]/(\varepsilon^2))$-algebra homomorphisms $f^\sharp_\varepsilon : A[\varepsilon]\to B[\varepsilon]$ of the type

$$f^\sharp_\varepsilon(a)=f(a)+\varepsilon g(a)$$

where a is in A. The equality $f^\sharp_\varepsilon(aa')=f^\sharp_\varepsilon(a)f^\sharp_\varepsilon(a')$ is equivalent to

$$g(aa')=f^\sharp(a)g(a')+f^\sharp(a')g(a)$$

In other words, $g : A\to B$ must be a $\mathbf{k}$-derivation of the A-module B, hence must factor as $g : A\to\Omega_A\to B$ ([H], II, § 8). Such extensions are therefore parametrized by $\operatorname{Hom}_A(\Omega_A,B)=\operatorname{Hom}_B(\Omega_A\otimes_A B,B)$.

In general, cover X by affine open subsets $U_i=\operatorname{Spec}(A_i)$ and Y by affine open subsets $V_i=\operatorname{Spec}(B_i)$ such that $f(V_i)$ is contained in U_i. First-order extensions of $f|_{V_i}$ are parametrized by $h_i\in\operatorname{Hom}_{B_i}(\Omega_{A_i}\otimes_{A_i}B_i,B_i)=H^0(V_i,\mathscr{H}om(f^*\Omega_X,\mathscr{O}_Y))$. To glue these, we need the compatibility condition

$$h_i|_{V_i\cap V_j}=h_j|_{V_i\cap V_j}$$

which is exactly saying that the h_i define a global section on Y. □

In particular, when X is smooth along the image of f,

$$T_{[f]}\operatorname{Mor}(Y,X)\simeq H^0(Y,f^*T_X)$$

4.3 The local structure of $\mathrm{Mor}(Y, X)$

We prove a result on the local structure of the scheme $\mathrm{Mor}(Y, X)$ whose main use will be to provide a lower bound for its dimension at a point $[f]$, thereby allowing us in certain situations to produce many deformations of f. This lower bound is very accessible, via the Riemann-Roch theorem, when Y is an irreducible curve.

Theorem 11 *Let X be a quasi-projective variety, let Y be a projective variety, and let $f : Y \to X$ be a morphism such that X is smooth along $f(Y)$. Locally around $[f]$, the scheme $\mathrm{Mor}(Y, X)$ can be defined by $h^1(Y, f^*T_X)$ equations in a nonsingular scheme of dimension $h^0(Y, f^*T_X)$. In particular, any irreducible component of $\mathrm{Mor}(Y, X)$ through $[f]$ has dimension at least*

$$h^0(Y, f^*T_X) - h^1(Y, f^*T_X)$$

Proof. Locally around the $\mathbf{k}$-point $[f]$, the $\mathbf{k}$-scheme $\mathrm{Mor}(Y, X)$ can be defined by certain polynomial equations $P_1, \dots, P_m$ in an affine space $\mathbf{A}^n_{\mathbf{k}}$. The rank r of the corresponding Jacobian matrix $((\partial P_i/\partial x_j)([f]))$ is the codimension of the Zariski tangent space $T_{[f]} \mathrm{Mor}(Y, X)$ in $\mathbf{k}^n$. The subvariety V of $\mathbf{A}^n_{\mathbf{k}}$ defined by r equations among the P_i for which the corresponding rows have rank r is smooth at $[f]$ with the same Zariski tangent space as $\mathrm{Mor}(Y, X)$.

Letting $h^i = h^i(Y, f^*T_X)$, we are going to show that $\mathrm{Mor}(Y, X)$ can be locally around $[f]$ defined by h^1 equations inside the smooth h^0-dimensional variety V. For that, it is enough to show that in the regular local ring $R = \mathscr{O}_{V,[f]}$, the ideal I of functions vanishing on $\mathrm{Mor}(Y, X)$ can be generated by h^1 elements. Note that since the Zariski tangent spaces are the same, I is contained in the square of the maximal ideal $\mathfrak{m}$ of R. Finally, by Nakayama's Lemma ([M], Theorem 2.3), it is enough to show that the $\mathbf{k}$-vector space $I/\mathfrak{m}I$ has dimension at most h^1.

The canonical morphism $\mathrm{Spec}(R/I) \to \mathrm{Mor}(Y, X)$ corresponds to an extension $f_{R/I} : Y \times \mathrm{Spec}(R/I) \to X$ of f. Since $I^2 \subset \mathfrak{m}I$, the obstruction to extending it to a morphism $f_{R/\mathfrak{m}I} : Y \times \mathrm{Spec}(R/\mathfrak{m}I) \to X$ lies by Lemma 12 below in

$$H^1(Y, f^*T_X) \otimes_{\mathbf{k}} (I/\mathfrak{m}I)$$

Write this obstruction as

$$\sum_{i=1}^{h^1} a_i \otimes \bar{b}_i$$

where $(a_1, \dots, a_{h^1})$ is a basis for $H^1(Y, f^*T_X)$ and $b_1, \dots, b_{h^1}$ are in I. The obstruction vanishes modulo the ideal $(b_1, \dots, b_{h^1})$, which means that the morphism $\operatorname{Spec}(R/I) \to \operatorname{Mor}(Y, X)$ lifts to a morphism $\operatorname{Spec}(R/\big(\mathfrak{m}I+(b_1, \dots, b_{h^1})\big)) \to \operatorname{Mor}(Y, X)$. In other words, the identity $R/I \to R/I$ factors as

$$R/I \to R/\big(\mathfrak{m}I + (b_1, \dots, b_{h^1})\big) \xrightarrow{\pi} R/I$$

where π is the canonical projection. By Lemma 13 below,

$$I = \mathfrak{m}I + (b_1, \dots, b_{h^1})$$

which means that $I/\mathfrak{m}I$ is generated by the classes of $b_1, \dots, b_{h^1}$. $\square$

We now prove the two lemmas used in the proof above.

Lemma 12 *Let R be a Noetherian local $\mathbf{k}$-algebra with maximal ideal $\mathfrak{m}$ and residue field $\mathbf{k}$ and let I be an ideal contained in $\mathfrak{m}$ such that $\mathfrak{m}I = 0$. Let $f : Y \to X$ be a morphism and let $f_{R/I} : Y \times \operatorname{Spec}(R/I) \to X$ be an extension of f. Assume X is smooth along the image of f. The obstruction to extending $f_{R/I}$ to a morphism $f_R : Y \times \operatorname{Spec}(R) \to X$ lies in*

$$H^1(Y, f^*T_X) \otimes_{\mathbf{k}} I$$

Proof. In the case where Y and X are affine, and with the notation of the proof of Proposition 10, we are looking for R-algebra liftings $f_R^\sharp$ fitting into the diagram

$$\begin{array}{ccc} A \otimes_{\mathbf{k}} R & \overset{f_R^\sharp}{\dashrightarrow} & B \otimes_{\mathbf{k}} R \\ \downarrow & & \downarrow \\ A \otimes_{\mathbf{k}} R/I & \xrightarrow{f_{R/I}^\sharp} & B \otimes_{\mathbf{k}} R/I \end{array}$$

Because X is smooth along the image of f and $I^2 = 0$, such a lifting exists ([H], II, Ex. 8.6), and two liftings differ by an R-derivation of $A \otimes_{\mathbf{k}} R$ into

$B\otimes_{\mathbf{k}}I$, that is by an element of $\mathrm{Hom}_{A\otimes_{\mathbf{k}}R}(\Omega_{A\otimes_{\mathbf{k}}R/R}, B\otimes_{\mathbf{k}}I)$. Since $\Omega_{A\otimes_{\mathbf{k}}R/R} \simeq \Omega_A \otimes_{\mathbf{k}} R$ ([B], X, § 7, n° 10, déf. 2), we have

$$\mathrm{Hom}_{A\otimes_{\mathbf{k}}R}(\Omega_{A\otimes_{\mathbf{k}}R/R}, B \otimes_{\mathbf{k}} I) \simeq \mathrm{Hom}_{A\otimes_{\mathbf{k}}R}(\Omega_A \otimes_{\mathbf{k}} R, B \otimes_{\mathbf{k}} I)$$

Since $\mathfrak{m}I = 0$, we further have[7]

$$\begin{aligned} \mathrm{Hom}_{A\otimes_{\mathbf{k}}R}(\Omega_A \otimes_{\mathbf{k}} R, B \otimes_{\mathbf{k}} I) &\simeq \mathrm{Hom}_{A\otimes_{\mathbf{k}}(R/\mathfrak{m})}(\Omega_A \otimes_{\mathbf{k}} (R/\mathfrak{m}), B \otimes_{\mathbf{k}} I) \\ &\simeq \mathrm{Hom}_A(\Omega_A, B \otimes_{\mathbf{k}} I) \end{aligned}$$

because $R/\mathfrak{m} = \mathbf{k}$. In the end, we get

$$\begin{aligned} \mathrm{Hom}_{A\otimes_{\mathbf{k}}R}(\Omega_{A\otimes_{\mathbf{k}}R/R}, B \otimes_{\mathbf{k}} I) &\simeq \mathrm{Hom}_A(\Omega_A, B \otimes_{\mathbf{k}} I) \\ &\simeq \mathrm{Hom}_B(B \otimes_{\mathbf{k}} \Omega_A, B \otimes_{\mathbf{k}} I) \\ &\simeq H^0(Y, \mathscr{H}om(f^*\Omega_X, \mathscr{O}_Y)) \otimes_{\mathbf{k}} I \\ &\simeq H^0(Y, f^*T_X) \otimes_{\mathbf{k}} I \end{aligned}$$

To pass to the global case, one needs to patch up various local extensions to get a global one. There is an obstruction to doing that: on each intersection $V_i \cap V_j$, two extensions differ by an element of $H^0(V_i \cap V_j, f^*T_X) \otimes_{\mathbf{k}} I$; these elements define a 1-cocycle, hence an element in $H^1(Y, f^*T_X) \otimes_{\mathbf{k}} I$ whose vanishing is necessary and sufficient for a global extension to exist (on a separated Noetherian scheme, the cohomology of a coherent sheaf is isomorphic to its Čech cohomology relative to any open affine covering; [H1], III, Theorem 4.5). In case such an extension exists, two extensions differ as above by an element of $H^0(Y, f^*T_X) \otimes_{\mathbf{k}} I$. □

Lemma 13 *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let J be an ideal in A contained in $\mathfrak{m}^2$. If the canonical projection $\pi : A \to A/J$ has a section, $J = 0$.*

---

[7] If A is a ring, J is an ideal in A, and M and N are A-modules such that $JN = 0$, the canonical map

$$\mathrm{Hom}_{A/J}(M/JM, N) \to \mathrm{Hom}_A(M, N)$$

is bijective.

PROOF. Let σ be a section of π: if a and b are in R, we can write $\sigma \circ \pi(a) = a + a'$ and $\sigma \circ \pi(b) = b + b'$, where a' and b' are in I. If a and b are in $\mathfrak{m}$, we have

$$(\sigma \circ \pi)(ab) = (\sigma \circ \pi)(a)\ (\sigma \circ \pi)(b) = (a + a')(b + b') \in ab + \mathfrak{m}J + J^2$$

Since J is contained in $\mathfrak{m}^2$, we get, for any x in J,

$$0 = \sigma \circ \pi(x) \in x + \mathfrak{m}J$$

hence $J \subset \mathfrak{m}J$. Nakayama's Lemma ([M], Theorem 2.2) implies $J = 0$. □

4.4 Morphisms with fixed points, flat families

Assume as usual that X is quasi-projective and Y projective. We will need a slightly more general situation: fix a subscheme B of Y and a morphism $g : B \to X$. Morphisms $f : Y \to X$ which restrict to g on B can be parametrized by a subscheme of $\mathrm{Mor}(Y, X)$ that we denote by $\mathrm{Mor}(Y, X; g)$.

The same techniques as above yield the following extension of Theorem 11 ([Mo], Proposition 2): let $\mathscr{I}_B$ be the ideal sheaf of B in Y; around a point $[f]$ such that X is smooth along $f(Y)$, *the scheme* $\mathrm{Mor}(Y, X; f|_B)$ *can be locally defined by* $h^1(Y, f^*T_X \otimes \mathscr{I}_B)$ *equations in a nonsingular scheme of dimension* $h^0(Y, f^*T_X \otimes \mathscr{I}_B)$. *In particular, its irreducible components are all of dimension at least*

$$h^0(Y, f^*T_X \otimes \mathscr{I}_B) - h^1(Y, f^*T_X \otimes \mathscr{I}_B)$$

All this can be done over a Noetherian base scheme S as in [Mo]: if $Y \to S$ is a *flat* projective S-scheme, $X \to S$ a *flat* quasi-projective S-scheme and B a subscheme of Y *flat* over S with an S-morphism $g : B \to X$, the S-morphisms from Y to X that restrict to g on B can be parametrized by a locally Noetherian S-scheme $\mathrm{Mor}_S(Y, X; g)$. The universal property implies in particular that for any point s of S, one has

$$\mathrm{Mor}_S(Y, X; g)_s \simeq \mathrm{Mor}(Y_s, X_s; g_s)$$

In other words, the schemes $\mathrm{Mor}(Y_s, X_s; g_s)$ fit together to form a scheme over S ([Mo], Proposition 1, and [Ko], Proposition II.1.5).

4.5 Morphisms from a curve

Everything takes a particularly simple form when Y is a curve C (and B is finite): for any $f : C \to X$, one has by Riemann-Roch

$$\begin{aligned} \dim_{[f]} \operatorname{Mor}(C, X) &\geq \chi(C, f^*T_X) \\ &= -K_X \cdot f_*C + (1 - g(C)) \dim(X) \end{aligned}$$

where $g(C) = 1 - \chi(C, \mathscr{O}_C)$, and

$$\begin{aligned} \dim_{[f]} \operatorname{Mor}(C, X; f|_B) &\geq \chi(C, f^*T_X) - \lg(B) \dim(X) \\ &= -K_X \cdot f_*C + (1 - g(C) - \lg(B)) \dim(X) \end{aligned} \tag{3}$$

5 Producing rational curves

The following is the original bend-and-break lemma ([Mo], Theorems 5 and 6). It says that a curve deforming nontrivially while keeping a point fixed must break into an effective 1-cycle with a rational component passing through the fixed point.

Proposition 14 *Let X be a projective variety, let $f : C \to X$ be a smooth curve, and let c be a point on C. If $\dim_{[f]} \operatorname{Mor}(C, X; f|_{\{c\}}) \geq 1$, there exists a rational curve on X through $f(c)$.*

According to (3), *when X is smooth along $f(C)$*, the hypothesis is fulfilled whenever

$$-K_X \cdot f_*C - g(C) \dim(X) \geq 1$$

The proof actually shows that there exists a morphism $f' : C \to X$ and a nonzero effective rational 1-cycle Z on X passing through $f(c)$ such that

$$f_*C \sim f'_*C + Z$$

Proof. Let T be the normalization of a 1-dimensional subvariety of $\operatorname{Mor}(C, X; f|_{\{c\}})$ passing through $[f]$ and let $\overline{T}$ be a smooth compactification of T. The indeterminacies of the rational map

$$\operatorname{ev} : C \times \overline{T} \dashrightarrow X$$

coming from the morphism $T \to \mathrm{Mor}(C, X; f|_{\{c\}})$ can be resolved by blowing up points to get a morphism

$$e : S \xrightarrow{\varepsilon} C \times \overline{T} \overset{\mathrm{ev}}{\dashrightarrow} X$$

where S is a smooth surface. The following picture sums up our constructions

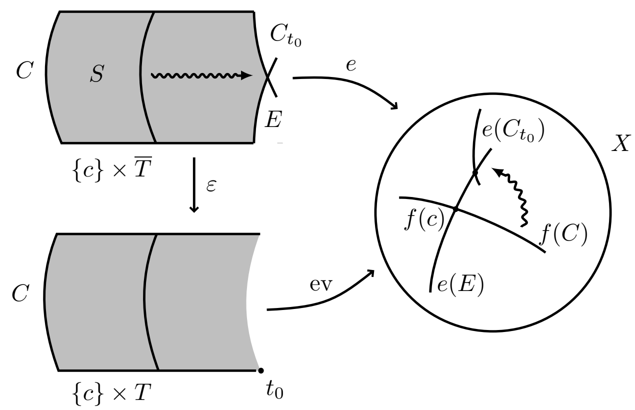


Figure 4: The 1-cycle f_*C degenerates to a 1-cycle with a rational component $e(E)$.

If ev is defined at every point of of $\{c\} \times \overline{T}$, the Rigidity Lemma 9(a) implies that there exist a neighborhood V of c in C and a factorization

$$\mathrm{ev}|_{V \times \overline{T}} : V \times \overline{T} \xrightarrow{p_1} V \xrightarrow{g} X$$

The morphism g must be equal to $f|_V$. It follows that ev and $f \circ p_1$ coincide on $V \times T$, hence on $C \times T$. But this means that the image of T in $\mathrm{Mor}(C, X; f|_{\{c\}})$ is just the point $[f]$, and this is absurd.

Hence there exists a point t_0 in $\overline{T}$ such that ev is not defined at (c, t_0). The fiber of t_0 under the projection $S \to \overline{T}$ is the union of the strict transform of $C \times \{t_0\}$ and a (connected) exceptional rational 1-cycle E which is not entirely contracted by e and meets the strict transform of $\{c\} \times \overline{T}$. Since the latter is contracted by e to the point $f(c)$, the rational 1-cycle e_*E passes through $f(c)$. $\square$

The same proof shows that this result still holds when X is a complex compact *Kähler* manifold. However, it fails for curves on general compact complex manifolds.

Once we know there is a rational curve, it may under certain conditions be broken up into several components. More precisely, if it deforms nontrivially while keeping two points fixed, it must break up (into an effective 1-cycle with rational components).

Proposition 15 *Let X be a projective variety and let $f : \mathbf{P}^1 \to X$ be a rational curve. If $\dim_{[f]}(\mathrm{Mor}(\mathbf{P}^1, X; f|_{\{0,\infty\}})) \geq 2$, the 1-cycle $f_*\mathbf{P}^1$ is numerically equivalent to a connected nonintegral effective rational 1-cycle passing through $f(0)$ and $f(\infty)$.*

According to (3), *when X is smooth along $f(\mathbf{P}^1)$*, the hypothesis is fulfilled whenever

$$-K_X \cdot f_*\mathbf{P}^1 - \dim(X) \geq 2$$

Proof. The group of automorphisms of $\mathbf{P}^1$ fixing two points is the multiplicative group $\mathbf{G}_m$. Let T be the normalization of a 1-dimensional subvariety of $\mathrm{Mor}(\mathbf{P}^1, X; f|_{\{0,\infty\}})$ passing through $[f]$ but not contained in its $\mathbf{G}_m$-orbit. The corresponding map

$$F : \mathbf{P}^1 \times T \to X \times T$$

is finite. Let $\overline{T}$ be a smooth compactification of T. The indeterminacies of the rational map $\mathbf{P}^1 \times \overline{T} \dashrightarrow X \times \overline{T}$ induced by F can be resolved by blowing up points to get a morphism

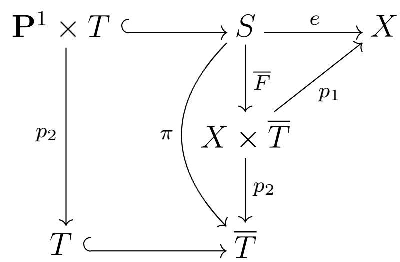
$$\overline{F}' : S' \to \mathbf{P}^1 \times \overline{T} \dashrightarrow X \times \overline{T}$$

whose Stein factorization is

$$\overline{F}' : S' \to S \xrightarrow{\overline{F}} X \times \overline{T}$$

In other words, the surface S is obtained from S' by contracting the components of fibers of $S' \to \overline{T}$ that are contracted by $\overline{F}'$.

Since F is finite, $\overline{F}^{-1}(X \times T) = \mathbf{P}^1 \times T$, so that there is a commutative diagram



This construction is similar to the one we performed in the proof of Proposition 14; however, S might not be smooth but on the other hand, we know that no component of a fiber of π is contracted by e (because it would then be contracted by $\overline{F}$).

Since $\overline{T}$ is a smooth curve and S is integral, π is flat ([H], III, Proposition 9.7), hence each fiber C is a 1-dimensional projective scheme without embedded component, whose genus is constant hence equal to 0 ([H], III, Corollary 9.10). In particular, any component C_1 of C_{red} is smooth rational, because $\mathscr{O}_{C_1}$ is a quotient of $\mathscr{O}_C$ hence $H^1(C_1, \mathscr{O}_{C_1})$ is a quotient of $H^1(C, \mathscr{O}_C)$, hence vanishes. In particular, if C is integral, it is a smooth rational curve.

Assume all fibers of π are integral; then S is a (minimal) ruled surface in the sense of [H], V, § 2 (Hartshorne assumes that S is smooth, but this hypothesis is not used in the proofs hence follows from the others). Let T_0 be the closure of $\{0\} \times T$ in S and let T_∞ be the closure of $\{\infty\} \times T$. These sections of π are contracted by e (to $f(0)$ and $f(\infty)$ respectively).

If H is an ample divisor on $e(S)$, which is a surface by construction, we have $(e^*H)^2 > 0$ and $e^*H \cdot T_0 = e^*H \cdot T_\infty = 0$, hence T_0^2 and T_∞^2 are negative by the Hodge Index Theorem.

However, since T_0 and T_∞ are both sections of π, their difference is linearly equivalent to the pull-back by π of a divisor on $\overline{T}$ ([H], V, Proposition 2.3). In particular,

$$0 = (T_0 - T_\infty)^2 = T_0^2 + T_\infty^2 - 2T_0 \cdot T_\infty < 0$$

which is absurd.

It follows that at least one fiber of π is not integral. Since none of its components is contracted by e, its direct image on X is the required 1-cycle. □

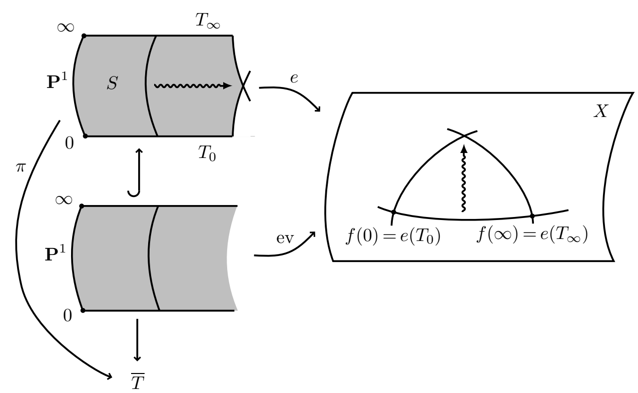


Figure 5: The rational 1-cycle f_*C bends and breaks

6 Rational curves on Fano varieties

A Fano variety is a smooth projective variety with ample anticanonical bundle.

Example 16 The projective space is a Fano variety. Any smooth complete intersection in $\mathbf{P}^n$ defined by equations of degrees $d_1, \ldots, d_s$ with $d_1 + \cdots + d_s \leq n$ is a Fano variety. A finite product of Fano varieties is a Fano variety.

We will apply the bend-and-break lemmas to show that any Fano variety X is covered by rational curves. We start from any curve $f : C \to X$ and want to show, using the estimate (3), that it deforms nontrivially while keeping a point x fixed. We only know how to do that in positive characteristic, where the Frobenius morphism allows to increase the degree of f without changing the genus of C. This gives in that case the required rational curve through x. Using the second bend-and-break lemma, we can bound the degree of this curve by a constant depending only on the dimension of X, and this will be essential for the remaining step: reduction of the characteristic zero case to positive characteristic.

Assume for a moment that X and x are defined over $\mathbf{Z}$; for almost all prime numbers p, the reduction of X modulo p is a Fano variety of the same dimension hence there is a rational curve (defined over the algebraic closure of $\mathbf{Z}/p\mathbf{Z}$) through x. This means that the scheme $\mathrm{Mor}(\mathbf{P}^1, X; 0 \to x)$, which is defined over $\mathbf{Z}$, has a geometric point modulo almost all primes p. Since we can moreover bound the degree of the curve by a constant independent of p, we are in fact dealing with a quasi-projective scheme, and this implies that it has a point over $\bar{\mathbf{Q}}$, hence over $\mathbf{k}$ (this is essentially because a system of polynomials equations with integral coefficients that has a solution modulo almost all primes has a solution). In general, X and x are defined over some finitely generated ring and a similar reasoning yields the existence of a $\mathbf{k}$-point of $\mathrm{Mor}(\mathbf{P}^1, X; 0 \to x)$, i.e., of a rational curve on X through x.

Theorem 17 *Let X be a Fano variety of positive dimension n. Through any point of X there is a rational curve of $(-K_X)$-degree at most $n+1$.*

There is no known proof of this result that uses only transcendental methods.

Proof. Let x be a point of X. To construct a rational curve through x, it is enough by Proposition 14 to produce a curve $f : C \to X$ and a point c on C with $\dim_{[f]} \mathrm{Mor}(C, X; f|_{\{c\}}) \geq 1$ and $f(c) = x$. By the dimension estimate of (3), it is enough to have

$$-K_X \cdot f_*C - ng(C) \geq 1$$

Unfortunately, there is no known way to achieve that, except in positive characteristic. Here is how it works.

Assume that the field $\mathbf{k}$ has characteristic $p > 0$; choose a smooth curve $f : C \to X$ through x and a point c of C such that $f(c) = x$. Consider the ($\mathbf{k}$-linear, degree p) Frobenius morphism $C_1 \to C$.[8] Iterating the construction, we get a morphism $F_m : C_m \to C$ of degree p^m between curves of the same genus. But

$$-K_X \cdot (f \circ F_m)_* C_m - ng(C_m) = -p^m K_X \cdot f_*C - ng(C)$$

is positive for m large enough. By Proposition 14, there exists a rational curve $f' : \mathbf{P}^1 \to X$, with say $f'(0) = x$. If

$$-K_X \cdot f'_*\mathbf{P}^1 - n \geq 2$$

[8]The abstract scheme C_1 is the same as C (so it is a curve with the same genus), but its $\mathbf{k}$-structure is defined as the composition

$$\begin{array}{ccccc} C & \to & \operatorname{Spec} \mathbf{k} & \to & \operatorname{Spec} \mathbf{k} \\ & & x & \mapsto & x^p \end{array}$$

the scheme $\mathrm{Mor}(\mathbf{P}^1, X; f'|_{\{0,1\}})$) has dimension at least 2 at $[f']$. By Proposition 15, one can break up the rational curve $f'(\mathbf{P}^1)$ into at least two (rational) pieces. The component passing through x has smaller $(-K_X)$-degree, and we can repeat the process as long as $-K_X \cdot \mathbf{P}^1 - n \geq 2$, until we get to a rational curve of degree no more than $n+1$.

This proves the theorem in positive characteristic. Assume now that $\mathbf{k}$ has characteristic 0. Embed X in some projective space, where it is defined by a finite set of equations, and let R be the (finitely generated) subring of $\mathbf{k}$ generated by the coefficients of these equations and the coordinates of x. There is a projective scheme $\mathscr{X} \to \mathrm{Spec}(R)$ with an R-point x_R, such that X is obtained from its generic fiber by base change from the quotient field of R to $\mathbf{k}$. The geometric generic fiber is a Fano variety of dimension n. There is a dense open subset U of $\mathrm{Spec}(R)$ over which $\mathscr{X}$ is smooth of dimension n ([G3], th. 12.2.4.(iii)). Since ampleness is an open property ([G3], cor. 9.6.4), we may even, upon shrinking U, assume that the relative dualizing sheaf $\omega_{\mathscr{X}_U/U}$ is ample on all fibers. It follows that over each maximal ideal $\mathfrak{m}$ of R in U, the (geometric) fiber is a Fano variety of dimension n.

The finitely generated ring R has the following properties:[9]

- for each maximal ideal $\mathfrak{m}$ of R, the field $R/\mathfrak{m}$ is finite;
- maximal ideals are dense in $\mathrm{Spec}(R)$.

As explained in §4.4, there is a quasi-projective scheme

$$\rho : \mathrm{Mor}_{\leq n+1}(\mathbf{P}^1_R, \mathscr{X}; 0 \mapsto x_R) \to \mathrm{Spec}(R)$$

[9]The first item is proved as follows. The field $R/\mathfrak{m}$ is a finitely generated $(\mathbf{Z}/\mathbf{Z}\cap\mathfrak{m})$-algebra, hence is finite over the quotient field of $\mathbf{Z}/\mathbf{Z}\cap\mathfrak{m}$ by a theorem of Zariski (which says that if k is a field and K a finitely generated k-algebra which is a field, K is an algebraic hence finite extension of k; see [M], Theorem 5.2). If $\mathbf{Z}\cap\mathfrak{m} = 0$, the field $R/\mathfrak{m}$ is a finite dimensional $\mathbf{Q}$-vector space with basis $e_1, \dots, e_m$. If $x_1, \dots, x_r$ generate the $\mathbf{Z}$-algebra $R/\mathfrak{m}$, there exists an integer q such that qx_j belongs to $\mathbf{Z}e_1 \oplus \cdots \oplus \mathbf{Z}e_m$ for each j. This implies

$$\mathbf{Q}e_1 \oplus \cdots \oplus \mathbf{Q}e_m = R/\mathfrak{m} \subset \mathbf{Z}[1/q]e_1 \oplus \cdots \oplus \mathbf{Z}[1/q]e_m$$

which is absurd; therefore, $\mathbf{Z}/\mathbf{Z}\cap\mathfrak{m}$ is finite and so is $R/\mathfrak{m}$.

For the second item, we need to show that the intersection of all maximal ideals of R is $\{0\}$. Let a be a nonzero element of R and let $\mathfrak{n}$ be a maximal ideal of the localization R_a. The field $R_a/\mathfrak{n}$ is finite by the first item hence its subring $R/R\cap\mathfrak{n}$ is a finite domain hence a field. Therefore $R\cap\mathfrak{n}$ is a maximal ideal of R which is in the open subset $\mathrm{Spec}(R_a)$ of $\mathrm{Spec}(R)$.

which parametrizes morphisms of degree at most $n+1$.

Let $\mathfrak{m}$ be a maximal ideal of R. Since the field $R/\mathfrak{m}$ is finite, hence of positive characteristic, what we just saw implies that the (geometric) fiber over a closed point of the dense open subset U of $\mathrm{Spec}(R)$ is nonempty; it follows that the image of ρ, which is a constructible[10] subset of $\mathrm{Spec}(R)$ by Chevalley's Theorem, contains all closed points of U, therefore is dense by the second item, hence contains the generic point. This implies that the generic fiber is nonempty; it has therefore a geometric point, which corresponds to a rational curve on X through x, of degree at most $n+1$, defined over an algebraic closure of the quotient field of R, hence over $\mathbf{k}$.[11] □

7 A stronger bend-and-break lemma

We will need the following generalization of Proposition 14 which gives some control over the degree of the rational curve that is produced. We start from a curve that deforms nontrivially with any (nonzero) number of fixed points. The more points are fixed, the better the bound on the degree.

Proposition 18 *Let X be a projective variety and let H be an ample divisor*[12] *on X. Let $f : C \to X$ be a smooth curve and let B be a finite nonempty subset of C. Assume*

$$\dim_{[f]} \mathrm{Mor}(C, X; f|_B) \geq 1$$

There exists a rational curve Γ on X which meets $f(B)$ and such that

$$H \cdot \Gamma \leq \frac{2H \cdot f_*C}{\mathrm{Card}(B)}$$

According to (3), *when X is smooth along $f(C)$*, the hypothesis is fulfilled whenever

$$-K_X \cdot f_*C + (1 - g(C) - \mathrm{Card}(B)) \dim(X) \geq 1$$

[10]A constructible subset is a finite union of locally closed subsets.

[11]It is important to remark that the "universal" bound on the degree of the rational curve is essential for the proof. Also, there is no known proof of this result based solely on characteristic 0 methods.

[12]The conclusion still holds if H is only a nef $\mathbf{R}$-divisor.

The proof actually shows that there exist a morphism $f' : C \to X$ and a nonzero effective rational 1-cycle Z on X such that

$$f_*C \sim f'_*C + Z$$

one component of which meets $f(B)$ and satisfies the degree condition above.

Proof. Set $B = \{c_1, \dots, c_b\}$. Let C' be the normalization of the image of f. If C' is rational and f has degree $\geq b/2$ onto its image, just take $\Gamma = C'$. From now on, we will assume that if C' is rational, f has degree $< b/2$ onto its image.

By §4.4, the dimension of the space of maps from C to $f(C)$ that send B to $f(B)$ is at most $h^0(C, f^*T_{C'} \otimes \mathscr{I}_B)$. When C' is irrational, $f^*T_{C'} \otimes \mathscr{I}_B$ has negative degree, and, under our assumption, this remains true when C' is rational. In both cases, the space is therefore 0-dimensional, hence any 1-dimensional subvariety of $\mathrm{Mor}(C, X; f|_B)$ through $[f]$ corresponds to morphisms with varying images. Let $\overline{T}$ be a smooth compactification of the normalization of such a subvariety. Resolve the indeterminacies of the rational map $\mathrm{ev} : C \times \overline{T} \dashrightarrow X$ by blowing up points to get a morphism

$$e : S \xrightarrow{\varepsilon} C \times \overline{T} \overset{\mathrm{ev}}{\dashrightarrow} X$$

whose image is a *surface*.

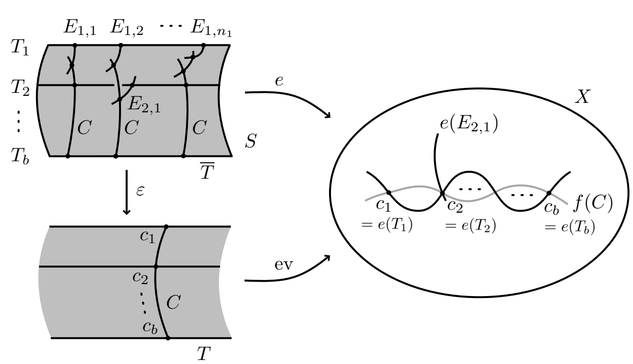


Figure 6: The 1-cycle f_*C bends and breaks, keeping $c_1, \dots, c_b$ fixed

For $i = 1, \dots, b$, we denote by $E_{i,1}, \dots, E_{i,n_i}$ the inverse images on S of the (-1)-exceptional curves that appear every time some point lying over

$\{c_i\} \times \overline{T}$ is blown up. We have

$$E_{i,j} \cdot E_{i',j'} = -\delta_{i,i'}\delta_{j,j'}$$

Write the strict transform T_i of $\{c_i\} \times \overline{T}$ as

$$T_i \sim \varepsilon^*\overline{T} - \sum_{j=1}^{n_i} \varepsilon_{i,j} E_{i,j}$$

where $\varepsilon_{i,j} = T_i \cdot E_{i,j}$ is 1 if the point blown-up is on the (smooth) strict transform of $\{c_i\} \times \overline{T}$, and 0 if it is not. Write also

$$e^*H \sim a\varepsilon^*C + d\varepsilon^*\overline{T} - \sum_{i=1}^{b}\sum_{j=1}^{n_i} a_{i,j}E_{i,j} + G$$

where G is orthogonal to the $\mathbf{R}$-vector subspace of $N^1(S)_{\mathbf{R}}$ generated by ε^*C, $\varepsilon^*\overline{T}$ and the $E_{i,j}$. Note that e^*H is nef, hence

$$a = e^*H \cdot \varepsilon^*\overline{T} \geq 0$$

Since T_i is contracted by e to $f(c_i)$, we have for each i

$$0 = e^*H \cdot T_i = a - \sum_{j=1}^{n_i} \varepsilon_{i,j} a_{i,j}$$

Summing up over i, we get

$$ba = \sum_{i,j} \varepsilon_{i,j} a_{i,j} \tag{4}$$

Moreover, since $\varepsilon^*C \cdot G = 0 = (\varepsilon^*C)^2$ and ε^*C is nonzero, the Hodge Index Theorem implies $G^2 \leq 0$, hence (using (4))

$$\begin{aligned}
(e^*H)^2 &= 2ad - \sum_{i,j} a_{i,j}^2 + G^2 \\
&\leq 2ad - \sum_{i,j} a_{i,j}^2 \\
&= \frac{2d}{b}\sum_{i,j} \varepsilon_{i,j} a_{i,j} - \sum_{i,j} a_{i,j}^2 \\
&\leq \frac{2d}{b}\sum_{i,j} \varepsilon_{i,j} a_{i,j} - \sum_{i,j} \varepsilon_{i,j} a_{i,j}^2 \\
&= \sum_{i,j} \varepsilon_{i,j} a_{i,j}\left(\frac{2d}{b} - a_{i,j}\right)
\end{aligned}$$

Since $e(S)$ is a surface, this number is positive, hence there exist indices i_0 and j_0 such that $\varepsilon_{i_0,j_0} > 0$ and $0 < a_{i_0,j_0} < \frac{2d}{b}$.

But $d = e^*H \cdot \varepsilon^*C = H \cdot C$, and $e^*H \cdot E_{i_0,j_0} = a_{i_0,j_0}$ is the H-degree of the rational 1-cycle $e_*(E_{i_0,j_0})$. The latter is nonzero since $a_{i_0,j_0} > 0$, and it passes through $f(c_{i_0})$ since E_{i_0,j_0} meets T_{i_0} (their intersection number is $\varepsilon_{i_0,j_0} = 1$) and the latter is contracted by e to $f(c_{i_0})$. This proves the proposition: take for Γ a component of $e_*E_{i_0,j_0}$ which passes through $f(c_{i_0})$ but is not contracted by e. □

8 Rational curves on varieties whose canonical bundle is not nef

We proved in §6 that when X is a Fano variety, there is a rational curve through any point of X. The following result considerably weakens the hypothesis: assuming only that K_X has negative degree on *one* curve C, we still prove that there is a rational curve through any point of C.

Note that the proof of Theorem 17 goes through in positive characteristic under this weaker hypothesis and does prove the existence of a rational curve through any point of C. However, to pass to the characteristic 0 case, one needs to bound the degree of this rational curve by some "universal" constant so that we deal only with a quasi-projective part of a morphism space. Apart from that, the ideas are essentially the same as in Theorem 17.

The following theorem will be used to prove the Cone Theorem in the smooth case and Kawamata's theorem on lengths of rational curves (Theorem 26).

Theorem 19 *Let X be a projective variety, let H be an ample divisor on X, and let $f : C \to X$ be a smooth curve such that X is smooth along $f(C)$ and $K_X \cdot C < 0$. Given any point x on $f(C)$, there exists a rational curve Γ on X through x with*

$$H \cdot \Gamma \leq 2\dim(X)\ \frac{H \cdot f_*C}{-K_X \cdot f_*C}$$

When X is smooth, the rational curve can be broken up, using Proposition 15 and (3), into several pieces (of lower H-degree) keeping any two points fixed (one of which being on $f(C)$), until one gets a rational curve Γ which satisfies $-K_X \cdot \Gamma \leq \dim(X)+1$ in addition to the bound on the degree.

It is nevertheless useful to have a more general statement allowing X to be singular. It shows that a normal projective variety X with ample (**Q**-Cartier) anticanonical bundle is covered by rational curves of $(-K_X)$-degree at most $2\dim(X)$, and will be used in this form in §10.4.

Proof. The idea is to take b as big as possible in Proposition 18, in order to get the lowest possible degree for the rational curve. As in the proof of Theorem 17, we first assume that the characteristic of the ground field $\mathbf{k}$ is positive, and use the Frobenius morphism to construct sufficiently many morphisms from C to X.

Assume then that the characteristic is $p > 0$. We compose f with m Frobenius morphisms to get $f_m : C_m \to X$ of degree $p^m \deg(f)$ onto its image. We have by §4.5, for any finite subset B_m of C_m with b_m elements,

$$\dim_{[f_m]} \operatorname{Mor}(C_m, X; f_m|_{B_m}) \geq -p^m K_X \cdot f_*C + (1 - g(C) - b_m)\dim(X)$$

The right-hand-side is positive if we take

$$b_m = \left[\frac{-p^m K_X \cdot f_*C}{\dim(X)} - g(C) \right]$$

which is positive for m sufficiently large. This is what we need to apply Proposition 18. It follows that there exists a rational curve Γ_m on X through some point of $f(B_m)$, such that

$$H \cdot \Gamma_m \leq \frac{2H \cdot (f_m)_* C_m}{b_m} = \frac{2p^m}{b_m} H \cdot f_*C$$

As m goes to infinity, p^m/b_m goes to $\dim(X)/(-K_X \cdot C)$. Since the left-hand side is an integer, we get

$$H \cdot \Gamma_m \leq \frac{2\dim(X)}{-K_X \cdot C} H \cdot C$$

for m sufficiently large. By Lemma 20 below, the set of points of $f(C)$ through which passes a rational curve of degree at most $\frac{2\dim(X)}{-K_X \cdot f_*C} H \cdot f_*C$ is *closed* (it is the intersection of $f(C)$ and the image of the evaluation map); it cannot be finite since we could then take B_m such that $f_m(B_m)$ lies outside of that locus, hence it is equal to $f(C)$. This finishes the proof when the characteristic is positive.

As in the proof of Theorem 17, the characteristic 0 case is done by considering a finitely generated domain R over which X, C, f, H and a point x of $f(C)$ are defined. The family of rational curves mapping 0 to x and of H-degree at most $2\dim(X)\frac{H\cdot f_*C}{-K_X\cdot f_*C}$ (constructed in §4.4) is nonempty modulo any maximal ideal, hence is nonempty over an algebraic closure in $\mathbf{k}$ of the quotient field of R. $\square$

Lemma 20 *Let X be a projective variety and let d be a positive integer. Let M_d be the quasi-projective scheme that parametrizes morphisms $\mathbf{P}^1 \to X$ of degree at most d. The image of the evaluation map*

$$\mathrm{ev}_d : \mathbf{P}^1 \times M_d \to X$$

is closed in X.

The image of ev_d is the set of points of X through which passes a rational curve of degree at most d.

PROOF. The idea is that a rational curve can only degenerate into a union of rational curves of lower degrees.

Let x be a point in $\overline{\mathrm{ev}_d(M_d)} - \mathrm{ev}_d(M_d)$ and let T be the normalization of a 1-dimensional subvariety of $\mathbf{P}^1 \times M$ which dominates a curve in $\overline{\mathrm{ev}_d(M_d)}$ which passes through x and meets $\mathrm{ev}_d(M_d)$. Note that since x is not in the image of ev_d, the variety T is not contracted by the projection $\mathbf{P}^1 \times M_d \to M_d$. Let $\overline{T}$ be a smooth compactification of T.

The image of the rational map

$$\mathrm{ev} : \mathbf{P}^1 \times \overline{T} \dashrightarrow X$$

coming from the nonconstant morphism $T \to M_d$ is a surface and its indeterminacies can be resolved by blowing up a finite number of points to get a morphism

$$e : S \xrightarrow{\varepsilon} \mathbf{P}^1 \times \overline{T} \overset{\mathrm{ev}}{\dashrightarrow} X$$

The surface $e(S)$ contains x; it is covered by the images of the fibers of the projection $S \to \overline{T}$, which are unions of rational curves of degree at most d. This proves the lemma. $\square$

9 Proof of the Cone Theorem

9.1 Elementary properties of cones

Let V be a cone in $\mathbf{R}^m$, i.e., a subset stable by multiplication by $\mathbf{R}^+$; we define its dual cone by

$$V^* = \{\ell \in (\mathbf{R}^m)^* \mid \ell \geq 0 \text{ on } V\}$$

A subcone W of V is *extremal* if it is closed and convex and if any two elements of V whose sum is in W are both in W. An extremal subcone of dimension 1 is called an *extremal ray.* A nonzero linear form ℓ in V^* is a *supporting function* of the extremal subcone W if it vanishes on W.

Lemma 21 *Let V be a closed convex cone in $\mathbf{R}^m$.*

a) We have $V = V^{**}$ and

$$V \text{ contains no lines} \iff V^* \text{ spans } (\mathbf{R}^m)^*$$

The interior of V^* is

$$\{\ell \in (\mathbf{R}^m)^* \mid \ell > 0 \text{ on } V - \{0\}\}$$

b) If V contains no lines, it is the convex hull of its extremal rays.

c) Any proper extremal subcone of V has a supporting function.

d) If V contains no lines and W is a proper closed subcone of V, there exists a linear form in V^* which is positive on $W - \{0\}$ and vanishes on some extremal ray of V.

Proof. Obviously, V is contained in V^{**}. Choose a scalar product on $\mathbf{R}^m$. If $z \notin V$, let $p_V(z)$ be the projection of z on the closed convex set V; since V is a cone, $z - p_V(z)$ is orthogonal to $p_V(z)$. The linear form $\langle p_V(z) - z, \cdot\rangle$ is nonnegative on V and negative at z, hence $z \notin V^{**}$.

If V contains a line L, any element of V^* must be nonnegative, hence must vanish, on L: the cone V^* is contained in $L^\perp$. Conversely, if V^* is contained in a hyperplane H, its dual V contains the line by $H^\perp$ in $\mathbf{R}^m$.

Let ℓ be an interior point of V^*; for any nonzero z in V, there exists a linear form ℓ' with $\ell'(z) > 0$ and small enough so that $\ell - \ell'$ is still in V^*. This implies $(\ell - \ell')(z) \geq 0$, hence $\ell(z) > 0$. Since the set $\{\ell \in (\mathbf{R}^m)^* \mid \ell > 0 \text{ on } V - \{0\}\}$ is open, this proves a).

Assume that V contains no lines; we will prove by induction on m that any point of V is in the linear span of m extremal rays.

Note that for any point v of ∂V, there exists by a) a nonzero element ℓ in V^* that vanishes at v. An extremal ray $\mathbf{R}^+ r$ in $\mathrm{Ker}(\ell) \cap V$ (which exists thanks to the induction hypothesis) is still extremal in V: if $r = x_1 + x_2$ with x_1 and x_2 in V, since $\ell(x_i) \geq 0$ and $\ell(r) = 0$, we get $x_i \in \mathrm{Ker}(\ell) \cap V$ hence they are both proportional to r.

Given $v \in V$, the set $\{\lambda \in \mathbf{R}^+ \mid v - \lambda r \in V\}$ is a closed nonempty interval which is bounded above (otherwise $-r = \lim_{\lambda \to +\infty} \frac{1}{\lambda}(v - \lambda r)$ would be in V). If λ_0 is its maximum, $v - \lambda_0 r$ is in ∂V, hence there exists by a) an element ℓ' of V^* that vanishes at $v - \lambda_0 r$. Since

$$v = \lambda_0 r + (v - \lambda_0 r)$$

item b) follows from the induction hypothesis applied to the closed convex cone $\mathrm{Ker}(\ell') \cap V$ and the fact that any extremal ray in $\mathrm{Ker}(\ell') \cap V$ is still extremal for V.

Let us prove c). We may assume that V spans $\mathbf{R}^m$. Note that an extremal subcone W of V distinct from V is contained in ∂V: if W contains an interior point v, then for any small x, we have $v \pm x \in V$ and $2v = (v + x) + (v - x)$ implies $v \pm x \in W$. Hence W is open in the interior of V; since it is closed, it contains it. In particular, the interior of W is empty, hence its span $\langle W \rangle$ is not $\mathbf{R}^m$. Let w be a point of its interior in $\langle W \rangle$; by a), there exists a nonzero element ℓ of V^* that vanishes at w. By a) again (applied to W^* in its span), ℓ must vanish on $\langle W \rangle$ hence is a supporting function of W.

Let us prove d). Since W contains no lines, there exists by a) a point in the interior of W^* which is not in V^*. The segment connecting it to a point in the interior of V^* crosses the boundary of V^* at a point in the interior of W^*. This point corresponds to a linear form ℓ that is positive on $W - \{0\}$ and vanishes at a nonzero point of V. By b), the closed cone $\mathrm{Ker}(\ell) \cap V$ has an extremal ray, which is still extremal in V. This proves d). $\square$

9.2 Proof of the Cone Theorem

We now prove the Cone Theorem (Theorem 4): if X is a smooth projective variety,

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbf{R}^+[\Gamma_i]$$

where Γ_i are rational curves on X such that $0 < -K_X \cdot \Gamma_i \leq \dim(X) + 1$.

The idea is quite simple: if $\overline{\mathrm{NE}}(X)$ is not equal to the closure of the right-hand side, there exists a divisor M on X which is nonnegative on $\overline{\mathrm{NE}}(X)$ (hence nef), positive on the closure of the right-hand side, and vanishes at some nonzero point z of $\overline{\mathrm{NE}}(X)$, which must therefore satisfy $K_X \cdot z < 0$. We approximate M by an ample divisor, z by an effective 1-cycle and use the bend-and-break Theorem 19 to get a contradiction. In the third and last step, we prove that the right-hand side is closed by a formal argument with no geometric content.

Proof. There are only countably many families of, hence classes of, rational curves on X. Pick a representative Γ_i for each such class z_i that satisfies $0 < -K_X \cdot z_i \leq \dim(X) + 1$.

First step: the rays $\mathbf{R}^+ z_i$ are locally discrete in the half-space $(N_1(X)_{\mathbf{R}})_{K_X < 0}$.

Let H be an ample divisor on X. It is enough to show that for each $\varepsilon > 0$, there are only finitely many classes z_i in the half-space $(N_1(X)_{\mathbf{R}})_{K_X + \varepsilon H < 0}$, since the union of these half-spaces is $(N_1(X)_{\mathbf{R}})_{K_X < 0}$. If $(K_X + \varepsilon H) \cdot \Gamma_i < 0$, we have

$$H \cdot \Gamma_i < -\frac{1}{\varepsilon} K_X \cdot \Gamma_i \leq \frac{1}{\varepsilon}(\dim(X) + 1)$$

and there are finitely many such classes of curves on X.[13]

Second step: $\overline{\mathrm{NE}}(X)$ is equal to the closure of

$$V = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i \mathbf{R}^+ z_i$$

[13] For any ample divisor H and any integer k, the set $\{z \in \overline{\mathrm{NE}}(X) \mid H \cdot z \leq k\}$ contains only finitely many classes of irreducible curves. Indeed, let $D_1, \dots, D_r$ be Cartier divisors on X such that $([D_1], \dots, [D_r])$ is a basis for $N_1(X)_{\mathbf{Q}}$. There exists an integer m such that $mH \pm D_i$ is ample for each i in $\{1, \dots, r\}$. For any z in $\overline{\mathrm{NE}}(X)$, we then have $(mH \pm D_i) \cdot z \geq 0$ hence $|D_i \cdot z| \leq mH \cdot z$. If $H \cdot z \leq k$, this bounds the coordinates of z and defines a compact set. It contains at most finitely many classes of irreducible curves, because the set of this classes is by construction discrete in $N_1(X)_{\mathbf{R}}$.

If this is not the case, there exists by Lemma 21.d) (since $\overline{\mathrm{NE}}(X)$ contains no lines) an $\mathbf{R}$-divisor M on X which is nonnegative on $\overline{\mathrm{NE}}(X)$ (it is in particular nef), positive on $\overline{V} - \{0\}$ and which vanishes at some nonzero point z of $\overline{\mathrm{NE}}(X)$. This point cannot be in V, hence $K_X \cdot z < 0$.

Choose a norm on $N_1(X)_{\mathbf{R}}$ such that $\|[C]\| \geq 1$ for each irreducible curve C (this is possible since the set of classes of irreducible curves is discrete). We may assume, upon replacing M with a multiple, that $M \cdot v \geq 2\|v\|$ for all v in $\overline{V}$. Since the class $[M]$ is a limit of classes of ample $\mathbf{Q}$-divisors, and z is a limit of classes of effective rational 1-cycles, there exist an ample $\mathbf{Q}$-divisor H and an effective 1-cycle Z such that

$$2\dim(X)(H \cdot Z) < -K_X \cdot Z \qquad \text{and} \qquad H \cdot v \geq \|v\| \tag{5}$$

for all v in $\overline{V}$. We may further assume, by throwing away the other components, that each component C of Z satisfies $-K_X \cdot C > 0$.

Since X is smooth, the bend-and-break Theorem 19 implies that, for every component C of Z, there exists a rational curve Γ on X such that

$$2\dim(X)\ \frac{H \cdot C}{-K_X \cdot C} \geq H \cdot \Gamma$$

and $-K_X \cdot \Gamma \leq \dim(X) + 1$. But $[\Gamma]$ is then in $\overline{V}$, hence $H \cdot [\Gamma] \geq \|[\Gamma]\| \geq 1$ by (5). This contradicts the first inequality in (5) and finishes the proof of the second step.

Third step: for any set J of indices, the cone

$$V_J = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_{j \in J} \mathbf{R}^+ z_j$$

is closed.

By Lemma 21.b), it is enough to show that any extremal ray $\mathbf{R}^+ r$ in $\overline{V_J}$ satisfying $K_X \cdot r < 0$ is in V_J. Let H be an ample divisor on X and let ε be a positive number such that $(K_X + \varepsilon H) \cdot r < 0$. By the first step, there are only finitely many classes $z_{j_1}, \ldots, z_{j_q}$, with $j_\alpha \in J$, such that $(K_X + \varepsilon H) \cdot z_{j_\alpha} < 0$.

Write r as the limit of a sequence $(r_m + s_m)$, where $r_m \in \overline{\mathrm{NE}}(X)_{K_X + \varepsilon H \geq 0}$ and $s_m = \sum_{\alpha=1}^{q} \lambda_{\alpha,m} z_{j_\alpha}$. Since $H \cdot r_m$ and $H \cdot z_{j_\alpha}$ are positive, the sequences $(H \cdot r_m)$ and $(\lambda_{\alpha,m})$ are bounded, hence we may assume, after taking subsequences, that all sequences (r_m) and $(\lambda_{\alpha,m})$ have limits.[14] Because r spans

[14]See footnote 13.

an extremal ray in $\overline{V_J}$, the limits must be nonnegative multiples of r, and since $(K_X + \varepsilon H) \cdot r < 0$, the limit of (r_m) must vanish. Moreover, r is a multiple of one the z_{j_α}, hence is in V_J.

If we choose a set I of indices such that $(\mathbf{R}^+ z_j)_{j\in I}$ is the set of all (distinct) extremal rays among all $\mathbf{R}^+ z_i$, the proof shows that any extremal ray of $\overline{\mathrm{NE}}(X)_{K_X<0}$ is spanned by a z_i, with $i \in I$. This finishes the proof of the Cone Theorem. □

Corollary 22 *Let X be a smooth projective variety and let R be a K_X-negative extremal ray. There exists a nef divisor M_R on X such that*

a) $R = \{z \in \overline{\mathrm{NE}}(X) \mid M_R \cdot z = 0\}$;

b) the divisor $mM_R - K_X$ is ample for all integers $m \gg 0$.

The divisor M_R will be called a *supporting divisor* for R. Property b) is useful in conjunction with Theorem 24: it implies that in characteristic zero, the linear system $|mM_R|$ is base-point-free for all integers m sufficiently large, and defines the contraction of R.

PROOF. With the notation of the proof of the Cone Theorem, there exists i_0 in I such that $R = \mathbf{R}^+ z_{i_0}$. By the third step of the proof, the subcone

$$V = V_{I - \{i_0\}} = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_{i\in I,\, i\neq i_0} \mathbf{R}^+ z_i$$

of $\overline{\mathrm{NE}}(X)$ is closed and proper since it does not contain R. By Lemma 21.d), there exists a linear form which is nonnegative on $\overline{\mathrm{NE}}(X)$, positive on $V - \{0\}$ and which vanishes at some nonzero point of $\overline{\mathrm{NE}}(X)$, hence on R since $\overline{\mathrm{NE}}(X) = V + R$. The intersection of the interior of V^* and the *rational* hyperplane $R^\perp$ is therefore nonempty, hence contains an integral point: there exists a divisor M_R on X which is positive on $V - \{0\}$ and vanishes on R. It is in particular nef and a) holds.

Choose a norm on $N_1(X)_\mathbf{R}$ and let a be the (positive) minimum of M_R on the set of elements of V with norm 1. If b is the maximum of K_X on the same compact, the divisor $mM_R - K_X$ is positive on $V - \{0\}$ for m rational greater than b/a, and positive on $R - \{0\}$ for $m \geq 0$, hence ample for $m > \max(b/a, 0)$ by Kleiman's criterion. This proves b). □

10 The Cone Theorem for projective klt pairs

Let (X, Δ) be a complex projective klt pair.[15] The Cone Theorem takes the following form: *the set* $\mathscr{R}$ *of all* $(K_X + \Delta)$*-negative extremal rays of* $\overline{\mathrm{NE}}(X)$ *is countable and*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\Delta \geq 0} + \sum_{R \in \mathscr{R}} R$$

These rays are locally discrete in the half-space $(N_1(X)_{\mathbf{R}})_{K_X+\Delta<0}$. *For each* $R \in \mathscr{R}$, *there exists a rational curve* Γ *in* X *such that*

$$R = \mathbf{R}^+[\Gamma] \qquad\qquad 0 < -(K_X + \Delta) \cdot \Gamma \leq 2 \dim(X)$$

As the following exercise shows, even when X is smooth, this theorem has applications that do not follow from Theorem 4. However, the method of proof is totally different, since most bend-and-break results break down on singular varieties.

Exercise 1 Let X be a smooth projective variety of general type.[16]

1) Prove that there exist an effective divisor D such that $K_X - D$ is ample and $\varepsilon \in [0, 1)$ such that the pair $(X, \varepsilon D)$ is klt.

2) Prove that there are only finitely many K_X-negative extremal rays in $\overline{\mathrm{NE}}(X)$.

3) If X contains no rational curves, prove that K_X is ample. (*Hint:* show that K_X is nef and that, if it is not ample, X contains a curve C and an effective divisor D such that $K_X \cdot C = 0$ (use Theorem 24).)

---

[15] For us, this means the following: X is a projective normal variety, Δ is a $\mathbf{Q}$-divisor on X with coefficients in $[0, 1)$ such that some integral mutliple of $K_X + \Delta$ is a Cartier divisor, and for some (hence all) log resolution $\pi : Y \to X$ (i.e., a projective birational morphism where Y is smooth and $\mathrm{Exc}(\pi) + \pi^*\Delta$ is a divisor with simple normal crossings), if one writes

$$K_Y + \Delta' \sim \pi^*(K_X + \Delta) + E$$

where Δ' is the strict tranform of Δ and E is π-exceptional, then the coefficients of E are all > -1.

[16] This means that K_X is big.

10.1 The Rationality Theorem

This is the main result that is needed for the proof of the Cone Theorem in the singular case. We will not prove it here. It can be "deduced" from the Base-Point-Free Theorem 24 and the Kawamata–Viehweg Vanishing Theorem (but this is not easy; see [D], §7.8).

Theorem 23 (Rationality Theorem) *Let (X,Δ) be a complex projective klt pair such that the $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor $K_X+\Delta$ is not nef. Let H be an ample Cartier divisor on X.*

The number $r=\sup\{t\in\mathbf{R}^+ \mid H+t(K_X+\Delta)\ \mathrm{nef}\}$ is rational *and can be written as $r=\frac{ju}{v}$, where j is the smallest positive integer such that $j(K_X+\Delta)$ is a Cartier divisor, and u and v are positive integers with $v\le j(\dim(X)+1)$.*

Exercise 2 Deduce the Rationality Theorem from the Cone Theorem for klt pairs, assuming only that H is nef.

10.2 Proof of the Cone Theorem in the singular case

We may assume that $K_X+\Delta$ is not nef. Let j be a positive integer such that $j(K_X+\Delta)$ is a divisor. For any nef divisor M on X, the set $V_M=\{z\in\overline{\mathrm{NE}}(X)\mid M\cdot z=0\}$ is a extremal subcone of $\overline{\mathrm{NE}}(X)$ which is nonzero if M is not ample by Kleiman's criterion. We say that V_M is $(K_X+\Delta)$-*negative* if $V_M-\{0\}\subset(N_1(X)_{\mathbf{R}})_{K_X+\Delta<0}$.

First step: if M is a nef and nonample divisor such that V_M is $(K_X+\Delta)$-negative, there exists a nef divisor L such that V_L is an extremal ray contained in V_M.

Let $v=(j(\dim(X)+1))!$ and let H be an ample divisor on X. Let m be a positive integer; since $mM+H$ is ample, the positive number u_m defined by

$$\frac{ju_m}{v}=\sup\{t\in\mathbf{R}\mid mM+H+t(K_X+\Delta)\ \mathrm{nef}\}$$

is an integer by the Rationality Theorem 23. Since M is nef, we have $u_m\le u_{m+1}$. Moreover, if z_0 is some nonzero element of V_M, the inequalities $(mM+H+\frac{ju_m}{v}(K_X+\Delta))\cdot z_0\ge 0$ and $(K_X+\Delta)\cdot z_0<0$ imply

$$u_m\le v\ \frac{(mM+H)\cdot z_0}{-j(K_X+\Delta)\cdot z_0}=v\ \frac{H\cdot z_0}{-j(K_X+\Delta)\cdot z_0}$$

It follows that the sequence (u_m) becomes stationary, equal to some integer u_H for $m \geq m_0$. The divisor

$$L_{m,H} = v(mM + H + \frac{ju_H}{v}(K_X + \Delta))$$

is by construction nef and not ample for $m \geq m_0$. If $z \in V_{L_{m_0+1,H}}$, we have

$$0 = L_{m_0+1,H} \cdot z = L_{m_0,H} \cdot z + vM \cdot z \geq vM \cdot z \geq 0$$

hence $V_{L_{m_0+1,H}} \subset V_M$.

For each ample divisor H, we get a nef and nonample divisor $L_H = L_{m_0+1,H}$ such that $0 \neq V_{L_H} \subset V_M$. Ample classes span $N_1(X)^*_{\mathbf{R}}$, hence their restrictions span $\langle V_M \rangle^*$. If $\langle V_M \rangle$ has dimension at least 2, there exists an ample divisor H on X such that

$$(L_H)|_{V_M} = (H + \frac{ju_H}{v}(K_X + \Delta))|_{V_M}$$

does not vanish, and for this divisor, V_{L_H} is strictly contained in V_M. This proves the first step by induction on the dimension of $\langle V_M \rangle$.

Second step: $\overline{\mathrm{NE}}(X)$ *is the closure of*

$$V = \overline{\mathrm{NE}}(X)_{K_X+\Delta \geq 0} + \sum_{V_L \in \mathscr{L}} V_L$$

where $\mathscr{L}$ *is the set of* $(K_X + \Delta)$*-negative extremal rays of the type* V_L*, with* L *nef nonample divisor.*

If this is not the case, there exists by Lemma 21.d) (since $\overline{\mathrm{NE}}(X)$ contains no lines) an $\mathbf{R}$-divisor M on X which is nonnegative on $\overline{\mathrm{NE}}(X)$ but vanishes on some extremal ray $\mathbf{R}^+ z_0$ (it is in particular nef and nonample), and is positive on $\overline{V} - \{0\}$ (i.e., it is in the interior of $\overline{V}^*$). We want to show that we can find such a divisor with integral coefficients. The class of M is the limit of a sequence $([H_m])$ of classes of ample $\mathbf{Q}$-divisors. By the Rationality Theorem, there exists a sequence (r_m) of positive *rational* numbers such that $M_m = H_m + r_m(K_X + \Delta)$ is nef but not ample.

The point z_0 cannot be in V, hence $(K_X + \Delta) \cdot z_0 < 0$. This implies

$$r_m \leq \frac{H_m \cdot z_0}{-(K_X + \Delta) \cdot z_0}$$

hence the sequence (r_m) converges to 0. It follows that the sequence $([M_m])$ of classes of $\mathbf{Q}$-divisors converges to $[M]$, hence its terms are eventually in the interior of $\overline{V}^*$.

So we may assume that M has integral coefficients; since M is not ample, V_M is nonzero. Since M is positive on $V - \{0\}$, the cone V_M is $(K_X + \Delta)$-negative. By the first step, the cone V_M contains an extremal ray V_L, and we get a contradiction since M is positive on $V_L - \{0\}$.

Third step: $\mathscr{L}$ is countable and locally discrete in the half-space $(N_1(X)_{\mathbf{R}})_{K_X+\Delta<0}$.

As in the proof of the Cone Theorem in the smooth case (Theorem 4), it is enough to show that given an ample divisor H_0 and $\varepsilon > 0$, there are finitely many such rays in the half-space $(N_1(X)_{\mathbf{R}})_{K_X+\Delta+\varepsilon H_0<0}$. For any $V_L \in \mathscr{L}$, let z_L be the unique point on V_L such that $-(K_X + \Delta) \cdot z_L = 1$.

Let H be any ample divisor on X. The first step yields a nef divisor $L_H = v((m_0 + 1)L + H + \frac{u_H}{v}(K_X + \Delta))$ with $V_L = V_{L_H}$, and $vH \cdot z_L = u_H$ is a positive integer.

If V_L is in the half-space $(N_1(X)_{\mathbf{R}})_{K_X+\Delta+\varepsilon H_0<0}$, one has $H_0 \cdot z_L \le 1/\eta$, hence the z_L are in a compact. If there are infinitely many rays V_L in that half-space, the z_L must accumulate. This is impossible since $H \cdot z_L$ can only take discrete values for all ample H (the integer v is independent of H).

Fourth step: for any subset $\mathscr{L}'$ of $\mathscr{L}$, the cone

$$\overline{\mathrm{NE}}(X)_{K_X+\Delta\ge 0} + \sum_{V_L \in \mathscr{L}'} V_L$$

is closed.

This is proved exactly as the third step of the proof of the Cone Theorem in the smooth case (Theorem 4).

10.3 Existence of contractions

We use the following result (for the proof, see [D], §7.7).

Theorem 24 (Base-point-free theorem) *Let (X, Δ) be a complex projective klt pair and let D be a nef Cartier divisor on X such that $aD - (K_X + \Delta)$ is nef and big for some positive rational a. The linear system $|mD|$ is base-point-free for all integers m sufficiently large.*

The fact that extremal rays can be contracted is essential to the realization of Mori's minimal model program, but proved unattainable by the bend-and-break methods. It comes now for free as an immediate consequence of Theorem 24.

Theorem 25 *Let (X,Δ) be a projective complex klt pair and let R be a $(K_X+\Delta)$-negative extremal ray.*

a) There exists an irreducible curve C on X whose class generates R.

b) The contraction $c_R : X \twoheadrightarrow Y$ of R exists.

c) There is an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Pic}(Y) & \xrightarrow{c_R^*} & \mathrm{Pic}(X) & \longrightarrow & \mathbf{Z} \\ & & & & [L] & \longmapsto & L\cdot C \end{array}$$

Proof. By the fourth step of the proof of Theorem 4, the subcone

$$V = \overline{\mathrm{NE}}(X)_{K_X+\Delta\geq 0} + \sum_{R'\in\mathscr{R}-\{R\}} R'$$

of $\overline{\mathrm{NE}}(X)$ is closed and distinct from $\overline{\mathrm{NE}}(X)$, hence there exists a nef $\mathbf{R}$-divisor M which is positive on $V-\{0\}$ but vanishes at some nonzero point of $\overline{\mathrm{NE}}(X)$, which must be in R. By the same procedure as in the second step of the proof of Theorem 4, we may assume that M has integral coefficients. As in the proof of Corollary 22.b), $mM-(K_X+\Delta)$ is ample for all m sufficiently large. By Theorem 24, any sufficiently large multiple of M defines a morphism whose Stein factorization $c : X \to Y$ is the contraction of R. This proves b).

Since the contraction is unique (it is determined by the classes of the curves that it contracts), any two successive multiples of M yield isomorphic c hence M is linearly equivalent to the pull-back of an (ample) divisor class on Y.

The morphism c^* is injective: since $c_*\mathscr{O}_X\simeq\mathscr{O}_Y$, we have $c_*(c^*\mathscr{L})\simeq\mathscr{L}$ for any line bundle $\mathscr{L}$ on Y by the projection formula. Let now D be a Cartier divisor on X which vanishes on R; since M is positive on $V-\{0\}$, the divisor $mM+D$ has the same property for all m sufficiently large, hence is a supporting divisor for R. By what we have just seen, $\mathscr{O}_X(mM+D)$ and $\mathscr{O}_X(mM)$ are therefore in $c^*\operatorname{Pic}(Y)$, hence so is $\mathscr{O}_X(D)$. It follows that we have an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Pic}(Y) & \xrightarrow{c^*} & \operatorname{Pic}(X) & \longrightarrow & \mathbf{R} \\ & & & & [L] & \longmapsto & L\cdot z \end{array}$$

where z is any nonzero element of R.

This shows in particular that c is not an isomorphism; Y being normal, Zariski's Main Theorem implies that c contracts at least one irreducible curve C, which must satisfy $M\cdot C=0$. It follows that $[C]$ generates R and proves a) and c). □

For any fiber F of c_R, we saw in the proof above that for $m\gg 0$, the restrictions

$$(mM-(K_X+\Delta))|_F=-(K_X+\Delta)|_F$$

are ample. We say that $-(K_X+\Delta)$ *is c_R-ample.*

10.4 Kawamata's theorem

We now prove that $K_X+\Delta$-negative extremal rays are generated by classes of rtional curves on X. Note the difference with the smooth case, where rational curves entered the picture right from the beginning. Here, we know that the contraction c_R of a $K_X+\Delta$-negative extremal ray R exists (Theorem 25). If c_R is birational and $c_R(X)$ *is smooth,* it follows from Proposition 1 that there is a rational curve on X contracted by c_R; however, this is rarely the case. We need a different argument taking into account the important fact that $-(K_X+\Delta)$ is c_R-ample.

Theorem 26 *Let X and Y be complex normal varieties and let $c:X\to Y$ be a projective morphism. Assume that Δ is an effective divisor on X such that (X,Δ) is a klt pair and that $-(K_X+\Delta)$ is c-ample. Every irreducible component E of* Exc(c) *is covered by rational curves Γ contracted by c and such that*

$$0<-(K_X+\Delta)\cdot\Gamma\le 2(\dim(E)-\dim(c(E)))$$

Proof. The idea of the proof is to apply the bend-and-break theorem 19 to E, but we have to be careful with the singularities. Replacing c with its Stein factorization, we may assume that it has connected fibers. Let e be the dimension of E.

Since Y is normal, $E = c^{-1}(c(E))$. We may replace Y with the intersection of codim$(c(E))$ general hyperplane sections and assume that $c(E)$ is a point: the normality of Y is preserved, so is the c-ampleness of $-(K_X + \Delta)$, and one checks that the pair (X, Δ) remains klt.

Let now H be a very ample divisor on X, let e be the dimension of E, and let $\nu : \tilde{E} \to E$ be its normalization. The intersection C in $\tilde{E}$ of $e-1$ general elements of $|\nu^* H|$ is a smooth curve contained in the smooth locus of $\tilde{E}$.

Lemma 27 *In the above situation, we have*

$$(K_X + \Delta) \cdot H^{e-1} \cdot E \geq K_{\tilde{E}} \cdot (\nu^* H)^{e-1} \tag{6}$$

In terms of the curve C, the lemma reads

$$\nu^*(K_X + \Delta) \cdot C = (K_X + \Delta)|_E \cdot \nu_* C \geq K_{\tilde{E}} \cdot C \tag{7}$$

The proof shows that the inequality is strict if $E \neq X$.

Proof. If $E = X$, there is equality in (6). We may therefore assume that c is birational.

Assume $e \geq 2$. If we take a general (normal) member X_H of $|H|$ and let Y_H be the normalization of $c(X_H)$, the hypersurface $E_H = E \cap X_H$ of X_H is a component of dimension $e-1$ of the exceptional locus of $c_H : X_H \to Y_H$ and $\tilde{E}_H = \nu^{-1}(H)$ is normal hence is the normalization of E_H. Set $\Delta_H = \Delta|_H$; since

$$\begin{aligned}(K_{X_H} + \Delta_H) \cdot H^{e-2} \cdot E_H &= (K_X + \Delta + H)|_H \cdot H^{e-2} \cdot E|_H \\ &= (K_X + \Delta) \cdot H^{e-1} \cdot E + H^e \cdot E\end{aligned}$$

and

$$\begin{aligned}K_{\tilde{E}_H} \cdot (\nu^* H)^{e-2} &= (K_{\tilde{E}} + \nu^* H)|_{\tilde{E}_H} \cdot (\nu^* H)^{e-2} \\ &= K_{\tilde{E}} \cdot (\nu^* H)^{e-1} + H^e \cdot E\end{aligned}$$

it is the same to prove the lemma for c or for c_H. We may therefore assume $e=1$ (but we lose the c-ampleness of $-(K_X+\Delta)$, of course) and prove

$$(K_X+\Delta)\cdot E > \deg(K_{\tilde{E}})$$

Assume to the contrary

$$(K_X+\Delta)\cdot E \leq \deg(K_{\tilde{E}})$$

(note that the left-hand side is a rational number, whereas the right-hand side is an integer). Let D_E be a divisor on E_{reg} such that $\nu^* D_E \equiv K_{\tilde{E}}$. There is an injective trace map $\nu_*\omega_{\tilde{E}} \to \omega_E$. The projection formula implies

$$h^0(E,\omega_E(-D_E)) \geq h^0(E,\nu_*\omega_{\tilde{E}}(-D_E)) = h^0(\tilde{E},\omega_{\tilde{E}}(-\nu^*D_E)) = 1$$

hence

$$H^1(E,D_E)\neq 0 \tag{8}$$

by duality. Note for future reference that

$$D_E-(K_X+\Delta)|_E \text{ has nonnegative degree.} \tag{9}$$

After shrinking Y, we may assume that it is contractible and Stein[17] and that c induces an isomorphism over the complement of the point $0=c(E)$. The curve E is a component of the fiber $X_0=c^{-1}(0)$. The sheaves $R^ic_*\mathscr{O}_X$ and $R^ic_*\mathbf{Z}$, for $i>0$, are supported at 0 and the corresponding Leray exact sequences yield isomorphisms

$$H^i(X,\mathbf{Z})\simeq H^0(Y,R^ic_*\mathbf{Z})\simeq H^i(X_0,\mathbf{Z})$$

and

$$H^i(X,\mathscr{F})\simeq H^0(Y,R^ic_*\mathscr{F}) \tag{10}$$

for any coherent sheaf $\mathscr{F}$ on X and integer i.

We will make the simplifying assumption[18] *that X_0 has dimension* 1. Using (10), this implies $H^2(X,\mathscr{F})=0$ for any coherent sheaf $\mathscr{F}$ on X. The

[17] So that the higher cohomology of $\mathbf{Z}$ and of any coherent sheaf vanishes.

[18] This holds for example when E is a top-dimensional component of the exceptional locus of c, and this is enough for us. In general, one may use a neat trick of Kawamata's to reduce to the case where X_0 is actually irreducible. This goes roughly as follows (see [K], (2.3.3)): take a very ample divisor on X that meets X_0 transversely and shrink X to an analytic neighborhood of E so that this divisor has a component which meets only the other components of X_0, not E. The holomorphic map to a projective space associated with the sections of (a multiple of) this divisor contracts only E. Its image might not be algebraic, but the vanishing theorem that we use below is still valid in this context (see [N], 3.6) and the proof therefore proceeds in the same way.

exponential exact sequences for X and X_0 induce a commutative diagram

$$\begin{array}{ccccccc}
H^1(X,\mathscr{O}_X) & \to & \mathrm{Pic}(X) & \to & H^2(X,\mathbf{Z}) & \to & H^2(X,\mathscr{O}_X)=0 \\
\downarrow & & \downarrow & & \downarrow\wr & & \downarrow \\
H^1(X_0,\mathscr{O}_{X_0}) & \to & \mathrm{Pic}(X_0) & \to & H^2(X_0,\mathbf{Z}) & \to & H^2(X_0,\mathscr{O}_{X_0})=0 \\
\downarrow & & & & & & \\
H^2(X,\mathscr{I}_{X_0/X})=0 & & & & & &
\end{array}$$

hence a surjection $\mathrm{Pic}(X) \twoheadrightarrow \mathrm{Pic}(X_0)$. Since E meets the rest of the components of X_0 in a finite set, there is a Cartier divisor D on X which restricts to D_E on E and is as ample as we wish on $\overline{X_0 - E}$. In particular, the $\mathbf{Q}$-Cartier divisor $D-(K_X+\Delta)$ is c-nef by (9), and c-big because c is birational. Since X has canonical singularities, by the relative logarithmic Kawamata-Viehweg Vanishing Theorem,[19] the sheaves $R^ic_*(\mathscr{O}_X(D))$ vanish for $i>0$, hence also $H^i(X,D)$ by (10).

Part of the long exact sequence in cohomology associated with the exact sequence

$$0 \to \mathscr{I}_{E/X}(D) \to \mathscr{O}_X(D) \to \mathscr{O}_E(D_E) \to 0$$

reads

$$0 = H^1(X,D) \to H^1(E,D_E) \to H^2(X,\mathscr{I}_{E/X}(D)) = 0$$

It contradicts (8) and proves the lemma. □

We now go back to the proof of the theorem. Since the left-hand side of (7) is negative, Theorem 19 implies that $\tilde{E}$ contains a rational curve through any given point of C; in particular, E is covered by rational curves. More precisely, since $-\nu^*(K_X+\Delta)$ is ample on $\tilde{E}$, there exists through any point x of $\tilde{E}$ a rational curve Γ on $\tilde{E}$ with

$$-\nu^*(K_X+\Delta)\cdot\Gamma \le 2e\,\frac{-\nu^*(K_X+\Delta)\cdot C}{-K_{\tilde{E}}\cdot C} \le 2e$$

by (7). This finishes the proof of the theorem. □

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[19]This is the following statement: let X and Y be complex projective varieties, with X smooth, and let $\pi : X \to Y$ be a morphism. Let D be a π-nef and π-big $\mathbf{Q}$-divisor on X whose fractional part has simple normal crossings. Then

$$R^i\pi_*(\mathscr{O}_X(K_X+\lceil D\rceil)) = 0$$

for all $i>0$.

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