GUSHEL-MUKAI VARIETIES AND THEIR PERIODS

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Abstract. Gushel-Mukai varieties are defined as the intersection of the Grassmannian $\mathsf{Gr}(2,5)$ in its Plücker embedding, with a quadric and a linear space. They occur in dimension 6 (with a slightly modified construction), 5, 4, 3, 2 (where they are just K3 surfaces of degree 10), and 1 (where they are just genus 6 curves). Their theory parallels that of another important class of Fano varieties, cubic fourfolds, with many common features such as the presence of a canonically attached hyperkähler fourfold: the variety of lines for a cubic is replaced here with a double EPW sextic.

There is a big difference though: in dimension at least 3, GM varieties attached to a given EPW sextic form a family of positive dimension. However, we prove that the Hodge structure of any of these GM varieties can be reconstructed from that of the EPW sextic or of an associated surface of general type, depending on the parity of the dimension (for cubic fourfolds, the corresponding statement was proved in 1985 by Beauville and Donagi).

This is joint work with Alexander Kuznetsov.

1. Definition of Gushel–Mukai varieties

Let X be a (smooth complex) Fano variety of dimension n and Picard number 1, so that $\mathrm{Pic}(X) = \mathbf{Z}H$, with H ample. The positive integer such that $-K_X \underset{\mathrm{lin}}{\equiv} rH$ is called the *index* of X and the integer $d := H^n$ its *degree*. For example, a smooth hypersurface $X \subseteq \mathbf{P}^{n+1}$ of degree $d \le n+1$ is a Fano variety of index $r = n+2-d$ and degree d.

The index satisfies the following properties:

- if $r \ge n+1$, then $X \simeq \mathbf{P}^n$ (Kobayashi–Ochiai, 1973);
- if $r = n$, then X is a quadric in $\mathbf{P}^{n+1}$ (Kobayashi–Ochiai, 1973);
- if $r = n-1$ (del Pezzo varieties), classification by Fujita and Iskovskikh (1977–1988);
- if $r = n-2$, classification by Mukai (1989–1992), modulo a result later proved by Mella (1997).

Mukai used the *vector bundle method* (initiated by Gushel in 1982) to prove that in the last case, $d \in \{2,4,6,8,10,12,14,16,18,22\}$.

Theorem 1.1 (Mukai)**.** *Any smooth Fano n-fold with Picard number 1, index $n-2$, and degree* 10 *has dimension $n \in \{3,4,5,6\}$ and can be obtained as follows:*

- $X_6 \to \mathsf{Gr}(2, V_5)$ *double cover branched along* X_5*;*
- $X_5 = \mathsf{Gr}(2, V_5) \cap$ *(quadric)* $\subseteq \mathbf{P}^9$ *or Gushel;*
- $X_4 = X_5 \cap \mathbf{P}^8$ *or Gushel;*
- $X_3 = X_5 \cap \mathbf{P}^7$ *or Gushel.*

If we take further linear sections, we obtain general K3 surfaces of degree 10 and general genus-6 curves.

Why GM varieties?

- They have interesting period maps.

- They have intriguing rationality properties in dimension 4 similar to that of cubic fourfolds (most X_3 are irrational, all X_5 and X_6 are rational).
- They have interesting derived categories.

2. Periods of Gushel–Mukai varieties

Let X be a GM n-fold. Only its middle cohomology $H^n(X;\mathbf{Z})$ is interesting and it is torsion-free. There is a canonical map $\gamma\colon X \to \mathsf{Gr}(2,V_5)$ called the *Gushel map* and we define the vanishing cohomology as

$$H^n(X;\mathbf{Z})_{\mathrm{van}} := \gamma^* H^n(\mathsf{Gr}(2,V_5);\mathbf{Z})^{\perp} \subseteq H^n(X;\mathbf{Z}).$$

Its Hodge numbers are

- when $n=3$, 0 10 10 0;
- when $n=4$, 0 1 20 1 0;
- when $n=5$, 0 0 10 10 0 0;
- when $n=6$, 0 0 1 20 1 0 0.

In other words,

- when n is odd, the Hodge structure has weight 1 and there is a 10-dimensional principally polarized intermediate Jacobian $J(X)$;
- when n is even, the Hodge structure is of K3 type and the period is in *the same* 20-dimensional quasiprojective period domain $\mathscr{D}$.

This defines period maps (here $\mathscr{X}_n$ is the moduli space for GM n-folds)

dim.			dim.
22	$\mathscr{X}_3 \to \mathscr{A}_{10}$		
24	$\mathscr{X}_4 \to \mathscr{D}$	$\mathscr{A}_{10}$	55
25	$\mathscr{X}_5 \to \mathscr{A}_{10}$	$\mathscr{D}$	20
25	$\mathscr{X}_6 \to \mathscr{D}$		

none of which are injective. They are dominant when n is even. To go further, we need to introduce another geometric object.

3. (Double) EPW sextics

Nobody wants to hear the definition of an EPW sextic $Y \subseteq \mathbf{P}(V_6)$ associated with a Lagrangian subspace $A \subseteq \bigwedge^3 V_6$. Under some explicit generality assumptions on A (which we will always assume to hold), there is a canonical double cover

$$f\colon \widetilde{Y} \longrightarrow Y$$

branched over the singular locus Y^2 of Y (a smooth surface of irregularity 0) and $\widetilde{Y}$ is a (general) polarized smooth HK fourfold of BB degree 2 (O'Grady). There is another canonical double étale covering

$$g\colon \widetilde{Y^2} \longrightarrow Y^2,$$

where $\widetilde{Y^2}$ is a smooth connected surface of general type and irregularity 10.

Given a GM n-fold X with Gushel map $X \to \mathsf{Gr}(2,V_5)$, one can associate an EPW sextic $Y \subseteq \mathbf{P}(V_6)$, where V_5 naturally embeds into V_6 as a hyperplane (Iliev–Manivel).

Theorem 3.1 (D–Kuznetsov)**.** (1) *When $n \in \{3,5\}$, there is an isomorphism*

$$J(X) \xrightarrow{\sim} \mathrm{Alb}(\widetilde{Y^2}).$$

(2) *When $n \in \{4,6\}$, there is an isomorphism of Hodge structures*

$$(H^n(X;\mathbf{Z})_{\mathrm{van}}, \smile) \xrightarrow{\sim} (H^2(\widetilde{Y};\mathbf{Z})_0, (-1)^{n/2-1} q_{BB}).$$

Let $\mathscr{E}$ be the quasiprojective GIT moduli space for good Lagrangians A (or for good EPW sextics). There is a map $\mathscr{X}_n \to \mathscr{E}$ and the fiber of $[A] \in \mathscr{E}$ is a subvariety of $\mathbf{P}(V_6^\vee)$ that corresponds to the choice of the hyperplane $[V_5] \in \mathbf{P}(V_6^\vee)$. It is

- the surface $\check{Y}^2$ when $n = 3$;
- the 4-fold $\check{Y}$ when $n = 4$;
- the 5-fold $\mathbf{P}(V_6^\vee) \smallsetminus \check{Y}^2$ when $n = 5$;
- the 5-fold $\mathbf{P}(V_6^\vee) \smallsetminus \check{Y}$ when $n = 6$.

(Here $\check{Y} \subseteq \mathbf{P}(V_6^\vee)$ is the projective dual of $Y \subseteq \mathbf{P}(V_6)$; it is also an EPW sextic, with singular locus a smooth surface $\check{Y}^2$.)

Corollary 3.2. *The period map for GM n-folds factors through the map $\mathscr{X}_n \to \mathscr{E}$ and its image has dimension* 20.

When n is even, the period maps factor as

$$\begin{array}{cccccc}
\text{dim.} & & & & & \text{dim.} \\
24 & \mathscr{X}_4 & \searrow & & & \\
 & & & \mathscr{E} \overset{p}{\hookrightarrow} \mathscr{D} & & 20 \\
25 & \mathscr{X}_6 & \nearrow & & &
\end{array}$$

where the period map p for double EPW sextics is an open embedding by Verbitsky's Torelli theorem (with known image). When n is odd, there are factorizations

$$\begin{array}{cccccc}
\text{dim.} & & & & & \text{dim.} \\
22 & \mathscr{X}_3 & \searrow & & & \\
 & & & \mathscr{E} \longrightarrow \mathscr{E}/\tau \xrightarrow{q} \mathscr{A}_{10} & & 55 \\
25 & \mathscr{X}_5 & \nearrow & & &
\end{array}$$

where τ is the (nontrivial) duality involution of $\mathscr{E}$. The morphism q is known to be unramified (D–Iliev–Manivel 2012) and it is expected to be (generically) injective.

Conjecture 3.3 (Iliev)**.** Let X be a (general) GM 3-fold with intermediate Jacobian $(J(X), \Theta)$ and associated EPW sextic Y. There is a unique component of codimension ≤ 6 of the singular locus of Θ and it is "isomorphic" to $\widetilde{Y^2} \times \widetilde{\check{Y}^2}$.

Iliev has a sketch of proof that follows Voisin's 1988 construction of a component of codimension 5 of the singular locus of the theta divisor of the 10-dimensional intermediate Jacobian $J(W)$ of a quartic double solid W (whereby proving the irrationality of this solid; I proved in 1990 that this is the unique component of codimension ≤ 5). The surface $F(W)$ of lines contained in W embeds into $J(W)$ by the Abel–Jacobi map and Voisin studies those translates $F(W)_u$ that are contained in Θ and the linear systems $|\Theta||_{F(W)_u}$: their base-points correspond to singular points of $\mathrm{Sing}(\Theta)$. In our case, the surface $F(X)$ of conics contained in X maps into $J(X)$ but each of the steps of Voisin's proof becomes much more difficult.

4. Ideas of proofs

The standard argument for this kind of results goes back to Beauville–Donagi (1985), who treated the case of a smooth cubic fourfold $W \subseteq \mathbf{P}(V_6)$ and its (smooth) HK fourfold of lines $F(W) \subseteq \mathsf{Gr}(2, V_6)$. There is an incidence diagram

$$\begin{array}{ccc} & I & \\ {}^{q}\swarrow & & \searrow^{p} \\ W & & F(W), \end{array}$$

where p is the universal line, a $\mathbf{P}^1$-bundle, and q is dominant and generically finite. Beauville and Donagi prove that the Abel–Jacobi map $p_*q^*\colon H^4(W, \mathbf{Z}) \to H^2(F(W), \mathbf{Z})$ is an isomorphism of Hodge structures which induces an isometry between the primitive cohomologies $(H^4(W, \mathbf{Z})_0, \smile)$ and $(H^2(F(W), \mathbf{Z})_0, -q_{BB})$. Two ingredients that one uses here are:

- $F(W)$ parametrizes curves on W, so there is a correspondence between W and $F(W)$;
- $F(W)$ and I are smooth, so one can define p_* in singular cohomology.

Dimension 4. Let X be a GM fourfold satisfying some mild explicit generality assumptions. The scheme $F(X)$ of lines contained in X is harder to relate to the double EPW sextic $\widetilde{Y}$ (which I haven't defined anyway!). It goes as follows.

Any line contained in $\mathsf{Gr}(2, V_5)$ is of the type $L_{V_1,V_3} = \{[V_2] \in \mathsf{Gr}(2, V_5) \mid V_1 \subseteq V_2 \subseteq V_3\}$, so there is a map $\sigma\colon F(X) \to \mathbf{P}(V_5)$ given by $[L_{V_1,V_3}] \mapsto [V_1]$. The steps are (recall that V_5 is naturally a hyperplane in V_6):

- σ factors as

$$\begin{array}{ccccccc} \sigma\colon F(X) & \xrightarrow{\tilde\sigma} & f^{-1}(Y \cap \mathbf{P}(V_5)) & \xrightarrow{f} & Y \cap \mathbf{P}(V_5) & \hookrightarrow & \mathbf{P}(V_5) \\ & & \cap & & \cap & & \cap \\ & & \widetilde{Y} & \xrightarrow{\quad f \quad} & Y & \hookrightarrow & \mathbf{P}(V_6); \end{array}$$

- $F(X)$ is smooth irreducible of dimension 3 and $\tilde\sigma$ is a small resolution;
- the induced composition

$$a\colon H^2(\widetilde{Y}, \mathbf{Z}) \xrightarrow{\sim} H^2(f^{-1}(Y \cap \mathbf{P}(V_5)), \mathbf{Z}) \xrightarrow{\tilde\sigma^*} H^2(F(X), \mathbf{Z})$$

  (where the first map is an isomorphism by the Lefschetz theorem) is injective;
- the Abel–Jacobi map $p_*q^*\colon H^4(X, \mathbf{Z})_{\mathrm{van}} \to H^2(F(X), \mathbf{Z})$ is injective (as in Beauville–Donagi) and induces an antiisometry between $H^4(X, \mathbf{Z})_{\mathrm{van}}$ and $H^2(\widetilde{Y}, \mathbf{Z})_0 \subseteq H^2(\widetilde{Y}, \mathbf{Z}) \overset{a}{\hookrightarrow} H^2(F(X), \mathbf{Z})$

Dimension 6. For a general GM 6-fold X, the proof is similar: one uses instead the smooth fourfold parametrizing σ-planes contained in X (that is, of the form $P_{V_1,V_4} = \{[V_2] \in \mathsf{Gr}(2, V_5) \mid V_1 \subseteq V_2 \subseteq V_4\}$).

Dimension 5. The 5-dimensional case is the easiest: σ-planes contained in a (general) GM fivefold X are parametrized by a smooth connected curve $P(X)$ of genus 161 with, as above, a map $\sigma\colon P(X) \to \mathbf{P}(V_5)$. This map factors as

$$\begin{array}{ccccccc} \sigma\colon P(X) & = & g^{-1}(Y^2 \cap \mathbf{P}(V_5)) & \xrightarrow{g} & Y^2 \cap \mathbf{P}(V_5) & \hookrightarrow & \mathbf{P}(V_5) \\ & & \cap & & \cap & & \cap \\ & & \widetilde{Y^2} & \xrightarrow{g} & Y^2 & \hookrightarrow & \mathbf{P}(V_6). \end{array}$$

In other words, $P(X)$ is a connected double étale cover of a general genus-81 hyperplane section of the smooth surface $Y^2 \subseteq \mathbf{P}(V_6)$. A generalization of an old argument of Clemens (written by Tjurin in his famous 1972 article) shows that the corresponding Abel–Jacobi map $q_*p^*\colon H_1(P(X), \mathbf{Z}) \to H_5(X, \mathbf{Z})$ in homology is surjective. It induces a surjective morphism

$$a\colon J(P(X)) \longrightarrow J(X)$$

with connected kernel. By the Lefschetz theorem, there is another surjective morphism

$$b\colon J(P(X)) \longrightarrow \mathrm{Alb}(\widetilde{Y^2})$$

with connected kernel. We want to show that the morphisms a and b are the same.

One can then use a cheap trick: it follows from Deligne–Picard–Lefschetz theory that since the surface Y^2 is regular, for a very general choice of hyperplane V_5, the Jacobian $J(Y^2 \cap \mathbf{P}(V_5))$ is simple (of dimension 81). It is therefore contracted by both a and b , which induce surjective morphisms

$$a'\colon \text{Prym} \longrightarrow J(X) \qquad \text{and} \qquad b'\colon \text{Prym} \longrightarrow \mathrm{Alb}(\widetilde{Y^2})$$

with connected kernels. Using again monodromy arguments, one shows that the kernel of b' is simple (of dimension 70). It is therefore contracted by a', which induces an isomorphism

$$a''\colon \mathrm{Alb}(\widetilde{Y^2}) \xrightarrow{\sim} J(X).$$

Dimension 3. Lines on a (general) GM threefold X are parametrized by a smooth connected curve $F(X)$ of genus 71 which is the normalization of the singular, arithmetic genus-81, curve $Y^2 \cap \mathbf{P}(V_5)$ (the hyperplane V_5 is not general anymore), but it is hard to relate this curve with the surface $\widetilde{Y^2}$.

However, it was proved by Logachev and Iliev–Manivel that the Hilbert scheme of conics contained in X is the blow up of the smooth surface $\widetilde{\check{Y}^2}$ at a point. This gives an Abel–Jacobi map $\mathrm{Alb}(\widetilde{\check{Y}^2}) \to J(X)$ which should be an isomophism.

We actually proceed differently and prove instead that the Abel–Jacobi map associated with a family of rational quartic curves parametrized by the surface $\widetilde{Y^2}$ gives the desired isomorphism $\mathrm{Alb}(\widetilde{Y^2}) \xrightarrow{\sim} J(X)$ (a similar method also works for fivefolds).

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