Fano varieties and polytopes

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The Fano Conference

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Torino, September 29 – October 5, 2002

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Smooth Fano varieties

A smooth Fano variety (over a fixed algebraically closed field $\mathbf{k}$) *is a smooth projective variety whose anticanonical divisor is ample.*

Examples (1) A Fano curve is $\mathbf{P}^1$.

(2) Smooth Fano (del Pezzo) surfaces are $\mathbf{P}^1 \times \mathbf{P}^1$ and the blow-up of $\mathbf{P}^2$ in at most 8 points, with no three colinear.

(3) Smooth Fano threefolds have been classified in characteristic zero.

(4) A finite product of Fano varieties is a Fano variety.

(5) A smooth complete intersection in $\mathbf{P}^{n+s}$ defined by equations of degrees $d_1 \geq \cdots \geq d_s > 1$ is a Fano variety if and only if $d_1 + \cdots + d_s \leq n + s$. In any given dimension n, there are only finitely many choices for the degrees (because $s \leq \sum(d_i - 1) \leq n$).

(6) A cyclic covering $X \to \mathbf{P}^n$ of degree $d > 1$ branched along a smooth hypersurface of degree de is a Fano variety if and only if $(d-1)e \le n$ (again, there are only finitely many choices for d and e when n is fixed).

(7) In characteristic zero, a projective variety acted on transitively by a connected linear algebraic group (such as a flag variety) is a smooth Fano variety.

(8) If X is a smooth (Fano) variety and $D_1, \dots, D_r$ are nef divisors on X such that $-K_X - D_1 - \cdots - D_r$ is ample, the (Grothendieck) projective bundle $\mathbf{P}(\bigoplus_{i=1}^r \mathscr{O}_X(D_i))$ is a Fano variety.

Toric varieties

The algebraic torus of dimension n over $\mathbf{k}$ is

$$\mathbf{T}^n = \mathrm{Spec}(\mathbf{k}[U_1, U_1^{-1}, \ldots, U_n, U_n^{-1}])$$

A *toric variety* is an algebraic variety defined over $\mathbf{k}$ with an action of $\mathbf{T}^n$ and a dense open orbit isomorphic to $\mathbf{T}^n$. They can all be constructed from very explicit combinatorial data.

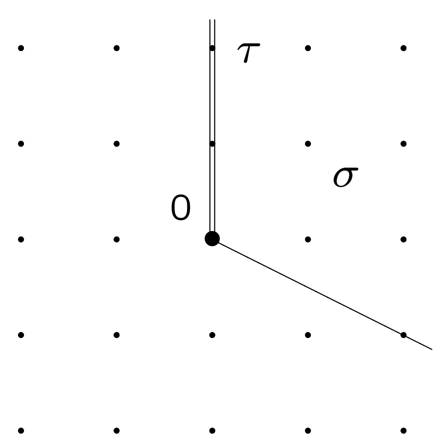
Affine toric varieties

1) Given a pair of dual lattices

$$M \simeq \mathbf{Z}^n \subset M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$$

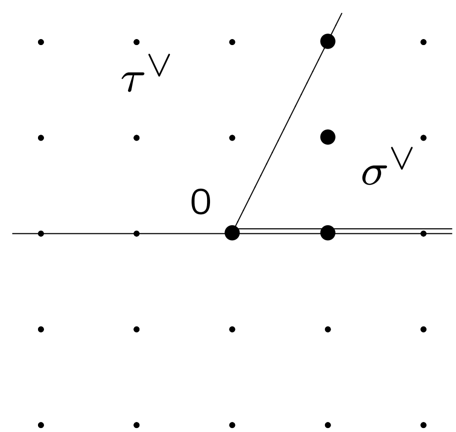
$$N = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Z}) \subset N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$$

2) a cone σ in $N_{\mathbf{R}}$ generated by a finite number of vectors in N and containing no lines, such as



3) and the dual cone

$$\sigma^\vee = \{u \in M_{\mathbf{R}} \mid \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}$$



the semigroup $\sigma^\vee \cap M$ is finitely generated (by the dots).

4) Define an affine variety by

$$X_\sigma = \mathrm{Spec}(\mathbf{k}[\sigma^\vee \cap M])$$

In our example,

$$\begin{aligned} X_\sigma &= \mathrm{Spec}(\mathbf{k}[U, UV, UV^2]) \\ &\simeq \mathrm{Spec}(\mathbf{k}[X, Y, Z]/(XZ - Y^2)) \end{aligned}$$

5) If τ is a face of σ,

$$\sigma^\vee \cap M \subset \tau^\vee \cap M$$

The corresponding map $X_\tau \to X_\sigma$ makes X_τ into a principal open subset of X_σ. (If $u \in M$ is such that $\tau = \sigma \cap u^\perp$, $X_\tau = (X_\sigma)_u$.)

In our example,

$$X_\tau = \mathrm{Spec}(\mathbf{k}[U, U^{-1}, V]) = (X_\sigma)_U$$
$$X_{\{0\}} = \mathrm{Spec}(\mathbf{k}[U, U^{-1}, V, V^{-1}]) = (X_\sigma)_{UV}$$

6) The torus $\mathbf{T}^n$ acts on X_σ and X_τ and the inclusion $X_\tau \to X_\sigma$ is equivariant.

General toric varieties

A *fan* Δ is a finite set of cones in $N_{\mathbf{R}}$ as above such that:

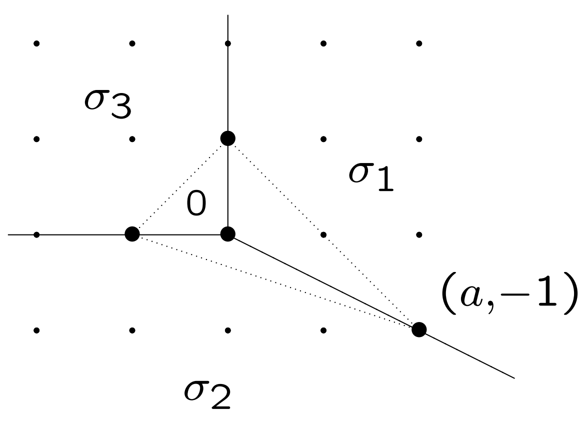
- every face of a cone in Δ is in Δ;
- the intersection of any two cones in Δ is in Δ.

Construct an irreducible n-dimensional $\mathbf{k}$-scheme X_Δ with an action of $\mathbf{T}^n$ by gluing the affine varieties X_σ and $X_{\sigma'}$ along the open set $X_{\sigma\cap\sigma'}$, for all σ and σ' in Δ.

The scheme X_Δ is *irreducible, separated and normal*. Each X_σ, hence also X_Δ, contains the open set $X_{\{0\}} = \mathbf{T}^n$.

All toric varieties are obtained via this construction.

Example The toric surface associated with the fan that consists of $\sigma_1, \sigma_2, \sigma_3$ and their faces:



is the projective cone in $\mathbf{P}^{a+1}$ over the rational normal curve in $\mathbf{P}^a$. For $a = 1$, this is $\mathbf{P}^2$.

Projectivity criterion

Δ is the set of cones spanned by the faces of a polytope Q with vertices in N and the origin in its interior	$\Longleftrightarrow$	X_Δ (or X_Q) projective

Given a polytope Q, we denote by $\mathscr{V}(Q)$ the set of its vertices.

Smooth toric Fano varieties

Let Q be a polytope in $N_{\mathbf{R}}$.

Each facet of Q is the convex hull of a basis of N	$\Rightarrow$	X_Q smooth Fano

In this case, the anticanonical divisor $-K_{X_Q}$ is *very ample.* We call such a polytope a *smooth Fano polytope*. Its only integral interior point is the origin. Its *dual*

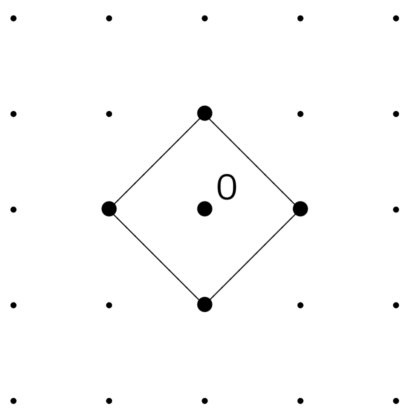
$$P = \{u \in M_{\mathbf{R}} \mid \langle u, v\rangle \leq 1 \text{ for all } v \in Q\}$$

is a polytope with vertices in M. It is usually not a Fano polytope, although its only integral interior point is the origin. We have

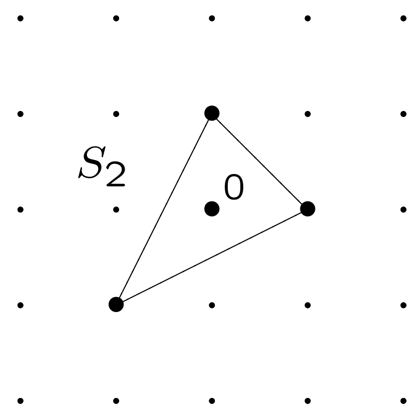
$$\boxed{\begin{array}{ccc} \mathrm{Pic}(X_Q) & \simeq & \mathbf{Z}^{\#\mathscr{V}(Q)-n} \\ h^0(X,-K_X) & = & \#(M \cap P) \\ (-K_X)^n & = & n!\,\mathrm{vol}(P) \end{array}}$$

Examples (1) There is only 1 smooth Fano polytope in $\mathbf{R}$, the interval $S_1 = [-1, 1]$.

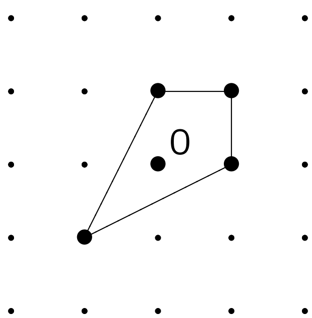
(2) Up to the action of $\mathrm{GL}(2, \mathbf{Z})$, there are only 5 smooth Fano polytopes in $\mathbf{R}^2$:



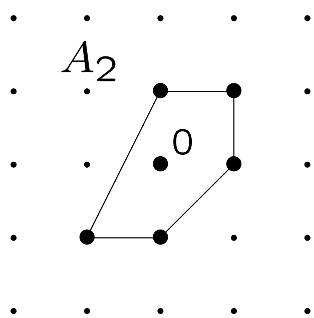
$\mathbf{P}^1 \times \mathbf{P}^1$



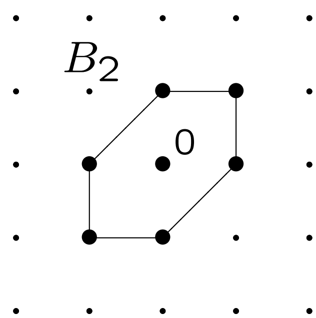
$\mathbf{P}^2$



$\mathbf{P}^2$ blown up at a point

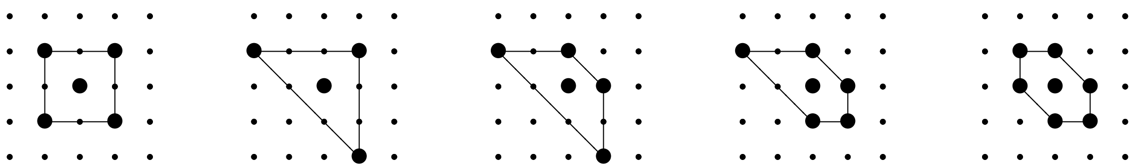


$\mathbf{P}^2$ blown up at 2 points



$\mathbf{P}^2$ blown up at 3 points

The dual polytopes are



and the formulas above are easily checked in these cases.

(3) The product of smooth Fano polytopes Q and Q' is a smooth Fano polytope and $X_{Q\times Q'} = X_Q \times X_{Q'}$.

(4) Let $(e_1,\ldots,e_n)$ be a basis for N and set $e_0 = -e_1 \cdots - e_n$. The simplex

$$S_n = \text{Conv}(\{e_0, e_1, \ldots, e_n\})$$

is a smooth Fano polytope. When n is even,

$$\begin{aligned} A_n &= \text{Conv}(\{e_0, \pm e_1, \ldots, \pm e_n\}) \\ B_n &= \text{Conv}(\{\pm e_0, \pm e_1, \ldots, \pm e_n\}) \end{aligned}$$

are smooth Fano polytopes.

When $n = 2m+1$, the $n+1$ vertices $e_0, -e_1, e_2, \ldots, -e_{2m+1}$ are on the same facet $-x_1 + x_2 - x_3 + \cdots - x_{2m+1} = 1$.

One has $X_{S_n} \simeq \mathbf{P}^n$, whereas $X_{A_{2m}}$ and $X_{B_{2m}}$ are obtained from $\mathbf{P}^{2m}$ by a series of blow ups (b.u.) and blow downs (b.d.), such as

$$
\begin{tikzcd}
& \mathbf{P}^1 \times \mathbf{P}^{2m-2} \arrow[d, hook] & & \mathbf{P}^2 \times \mathbf{P}^{2m-3} \arrow[d, hook] & & \cdots & & \mathbf{P}^{m-1} \times \mathbf{P}^{m} \arrow[d, hook] & \\
& Y_1 \arrow[dl, swap, "\substack{\text{b.u.}\\ \text{inv.}\\ \mathbf{P}^1}"] \arrow[dr, "\substack{\text{b.d.}\\ \text{exc.}\\ \text{div.}}"] & & Y_2 \arrow[dl, swap, "\substack{\text{b.u.}\\ \text{inv.}\\ \mathbf{P}^2}"] \arrow[dr, "\substack{\text{b.d.}\\ \text{exc.}\\ \text{div.}}"] & & \cdots & & Y_{m-1} \arrow[dl, swap, "\substack{\text{b.u.}\\ \text{inv.}\\ \mathbf{P}^{m-1}}"] \arrow[dr, "\substack{\text{b.d.}\\ \text{exc.}\\ \text{div.}}"] & \\
X_1 \arrow[d, swap, "\substack{\text{b.u.}\\ \text{fixed}\\ \text{points}}"] & & X_2 & & X_3 & \cdots & X_{m-1} & & X_{A_{2m}} = X_m \\
\mathbf{P}^{2m} & & & & & & & &
\end{tikzcd}
$$

The following result, which improves on an earlier result of Ewald, is one of the few classification results that are valid in all dimensions.

Theorem (Casagrande) *Any smooth Fano polytope in $N_{\mathbf{R}}$ with a set of vertices symmetric about the origin that spans $N_{\mathbf{R}}$ is isomorphic to a product of S_1, A_{2r} and B_{2s}.*

Kähler–Einstein metrics Compact complex manifolds with a positive Kähler–Einstein metric (i.e., whose Ricci form is a positive (constant) multiple of the Kähler form) are Fano varieties, but it is a difficult problem to find conditions on a complex Fano manifold so that a Kähler–Einstein metric exists.

However, there is a criterion for toric Fano manifolds (Tian, Batyrev & Selivanova): if $Q \subset N_{\mathbf{R}}$ is a smooth Fano polytope and *the only point of N fixed by the finite group*

$$G_Q = \{g \in \mathsf{GL}(N) \mid g(Q) = Q\}$$

is the origin, the Fano variety X_Q has a Kähler–Einstein metric. This condition implies that the barycenter of Q (resp. of $Q^\vee$) is the origin. We have the following implications:

The only point of N fixed by G_Q is the origin

$\Downarrow$

X_Q has a K.E. metric

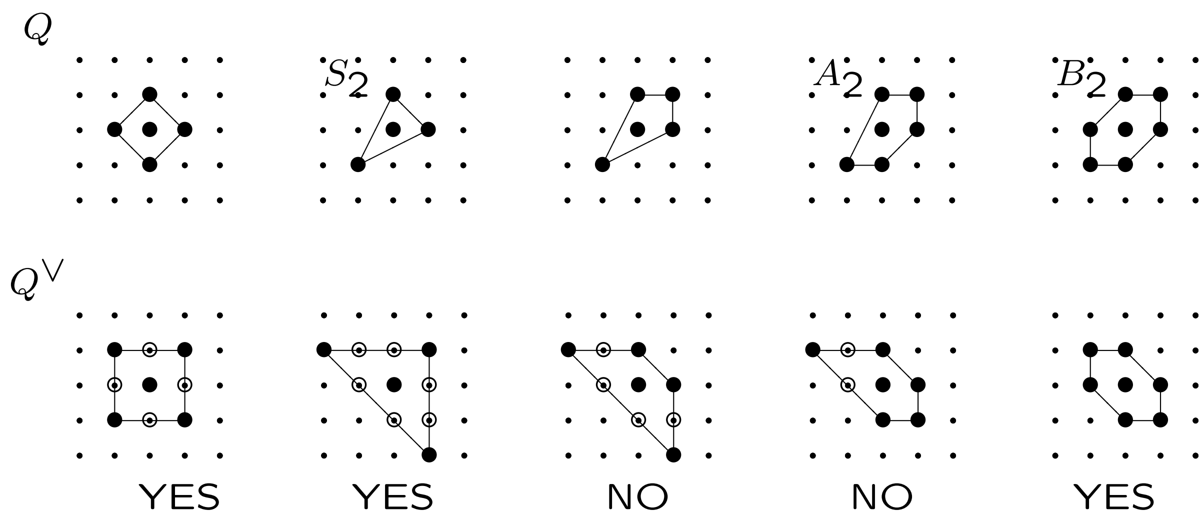
$\Downarrow$

Aut(X_Q) reductive and the Futaki character Lie(Aut(X_Q)) $\to \mathbf{C}$ vanishes

$\Longleftrightarrow$

The set of points of M in the relative interiors of facets of $Q^\vee$ is symmetric about the origin and the barycenter of $Q^\vee$ is the origin

Examples (1) In dimension two, the answer to the question "does X_Q have a Kähler–Einstein metric?" is:



The polytopes S_n and B_n all satisfy the Batyrev–Selivanova criterion. Using the criterion above, one checks that the automorphism group of $X_{A_{2m}}$ is not reductive.

(2) Fix positive integers $a_1, \ldots, a_m, r_1, \ldots, r_m$ with $a_i \le r_i$. The projective bundle

$$X = \mathbf{P}(\mathscr{O}(a_1, 0, \ldots, 0) \oplus \cdots \oplus \mathscr{O}(0, \ldots, a_m))$$

over $\mathbf{P}^{r_1} \times \cdots \times \mathbf{P}^{r_m}$ is a smooth Fano variety (this is a particular case of Example (8)) whose automorphism group is reductive *(by direct computation)*. It is the toric variety associated with the smooth Fano polytope Q with vertices:

$$\begin{aligned}
&g_1, \ldots, g_m \quad \text{with } g_1 + \cdots + g_m = 0 \\
&e_{i1}, \ldots, e_{ir_i}, a_i g_i - (e_{i1} + \cdots + e_{ir_i}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } i = 1, \ldots, m
\end{aligned}$$

If all the r_i are equal and all the a_i are equal, it satisfies the Batyrev–Selivanova criterion, hence X has a Kähler-Einstein metric.

The barycenter of $Q^\vee$ is the origin if and only if the integrals

$$\int_{\substack{g_1,\ldots,g_m \leq 1 \\ g_1,\ldots,g_m = 0}} g_i \prod_{j=1}^{m} (r_j + 1 - a_j g_j)^{r_j} \, d\mu$$

all vanish for $1 \leq i \leq m$. I have checked that for $r_1, \ldots, r_m \leq 10$, they do only when all the r_i are equal and all the a_i are equal.

Number of vertices Let $Q \subset N_{\mathbf{R}}$ be a smooth Fano polytope with dual P. There is a one-to-one correspondence

$$\boxed{\begin{array}{ccc} \left\{\begin{array}{c}\text{vertices of } P \\ \text{(resp. } Q\text{)}\end{array}\right\} & \leftrightarrow & \left\{\begin{array}{c}\text{facets of } Q \\ \text{(resp. } P\text{)}\end{array}\right\} \\ u & \leftrightarrow & F_u \end{array}}$$

For any $u \in \mathscr{V}(P)$ and $v \in \mathscr{V}(Q)$, we have

$$\langle u, v\rangle \in \mathbf{Z} \quad , \quad \langle u, v\rangle \leq 1$$

with equality if and only if $v \in F_u$.

Theorem (Voskresenskiĭ, Klyachko) *A smooth Fano polytope of dimension n has most $2n^2$ vertices.*

Proof. Let Q be a smooth Fano polytope, with dual P. The origin is in the interior of P, hence in the interior of the convex hull of at most $2n$ vertices by Steinitz's theorem: we have

$$0 = \lambda_1 u_1 + \cdots + \lambda_d u_d \qquad \lambda_1, \ldots, \lambda_d > 0$$

where the vertices $u_1, \ldots, u_d$ of P generate $M_{\mathbf{R}}$ and $n < d \leq 2n$.

Let v be a vertex of Q. We have $\langle u_i, v\rangle \neq 0$ for at least one i, hence $\langle u_i, v\rangle > 0$ for at least one i. This implies $v \in F_{u_i}$, hence the vertices of Q are all on $F_{u_1} \cup \cdots \cup F_{u_d}$. Since each facet has n vertices, Q has at most $nd \leq 2n^2$ vertices. $\square$

Batyrev conjectures that a smooth Fano polytope of dimension n has at most

$$\begin{cases} 3n & \text{if } n \text{ is even} \\ 3n-1 & \text{if } n \text{ is odd} \end{cases}$$

vertices. This holds for $n \leq 5$ (Batyrev, Casagrande).

There is equality for n even for $A_2^{n/2}$ and for n odd for $A_2^{(n-1)/2} \times S_1$.

The (asymptotically) best known bound is

$$\begin{aligned} \#\mathscr{V}(Q) &\leq n+2+2\sqrt{(n^2-1)(2n-1)} \\ &\leq 2\sqrt{2n^3} \end{aligned}$$

Corollary *Up to the action of* $\mathrm{GL}(n,\mathbf{Z})$*, there are only finitely many smooth Fano polytopes in* $\mathbf{R}^n$*.*

Proof. The number of vertices of such a polytope Q is at most $2n^2$. Since each facet of Q has n vertices, their number is at most $\binom{2n^2}{n}$, hence the volume of Q is at most $\frac{1}{n!}\binom{2n^2}{n}$. Fix the vertices $e_1, \ldots, e_n$ of a facet of Q and set $S = \text{Conv}(\{0, e_1, \ldots, e_n\})$. For any vertex $v = v_1 e_1 + \cdots + v_n e_n$ of Q, the volume of Conv(v, S) is at most $\frac{1}{n!}\binom{2n^2}{n}$, hence $|v_i| \leq \binom{2n^2}{n}$. The integral vector v may therefore take only finitely many values. $\square$

Volume of the dual polytope Let Q be a smooth Fano polytope. The integer

$$(-K_{X_Q})^n = n!\,\mathrm{vol}(Q^\vee)$$

is often called the *degree* of the smooth Fano variety X_Q.

Theorem *Let Q be a smooth Fano polytope of dimension n with $\rho + n$ vertices, $\rho > 1$. We have*

$$\mathrm{vol}(Q^\vee) \leq \left(\frac{(n-1)^{\rho+1}-1}{n-2}\right)^n \leq n^{\rho n}$$

Equivalently, a smooth toric Fano variety X of dimension n and Picard number $\rho \geq 2$ satisfies

$$(-K_X)^n \leq n!n^{\rho n} \tag{1}$$

Remarks (1) When $\rho = 1$, we have $X \simeq \mathbf{P}^n$, hence

$$(-K_X)^n = (n+1)^n$$

When $\rho = 2$, we have by classification

$$(-K_X)^n \leq n^{2n}$$

For any integers $\rho \geq 2$ and $n \geq 4$ such that $\frac{n}{\log n} \geq 2^{\rho-2}$, there are examples of smooth toric Fano varieties X of dimension n and Picard number ρ with

$$(-K_X)^n \geq \left(\frac{n^\rho}{2^{\rho^2-1}(\log n)^{\rho-1}}\right)^n$$

so the bound (1) is not too far off.

(2) The pseudo-index

$$\iota_X = \min_{C \subset X \text{ rational curve}} (-K_X \cdot C)$$

of X can be computed in a purely combinatorial way from Q. One can get the more precise bound (for $\rho > 1$)

$$\mathrm{vol}(Q^\vee) \leq \Big(n(n+1-\iota_X)^{\rho-1}\Big)^n$$

(3) If X_Q is a compact complex manifold of dimension n with a positive Kähler-Einstein metric, methods from differential geometry give

$$\mathrm{vol}(Q^\vee) \leq (2n-1)^n \, \frac{2^{n+1} n!}{(2n)!} \sim 2(2e)^{n-1/2}$$

(4) Taking $a_1 = r_1$ and $a_2 = \cdots = a_m = r_2 = \cdots = r_m = 1$ in Example (2) above, one gets

a smooth toric Fano variety X of dimension $n = r_1 + 2m - 2$ and Picard number $m + 1$ with *reductive automorphism group* and

$$(-K_X)^n \geq m^n r_1^{r_1}$$

Taking $m = \left[\frac{n}{2\log n}\right]$, we get $(-K_X)^n \geq \left(\frac{an^2}{\log n}\right)^n$ for some positive constant a. In particular, X does not have a Kähler-Einstein metric.

(5) Ewald conjectures that *given a smooth Fano polytope Q, there is a basis of N in which the vertices of Q have all their coordinates in $\{0, \pm 1\}$.* It holds for all the examples presented here.

Let $(e_1, \ldots, e_n)$ be such a basis, let u be a vertex of $Q^\vee$, and let $v_1, \ldots, v_n$ be the vertices of the face F_u of Q. Write

$$v_i = \sum_j \alpha_{ij} e_j,\ \ \alpha_{ij} \in \{0, \pm 1\},\ \ \det(\alpha_{ij}) = \pm 1$$

The coordinates $\langle u, e_1\rangle, \ldots, \langle u, e_n\rangle$ of u, solutions of the linear system

$$1 = \langle u, v_i\rangle = \sum_{j=1}^{n} \alpha_{ij}\langle u, e_j\rangle \qquad i = 1, \ldots, n$$

are $n \times n$ determinants whose entries are in $\{0, \pm 1\}$. By Hadamard's inequality, their absolute value is bounded by $n^{n/2}$, hence

$$\text{vol}(Q^\vee) \leq 2^n n^{n^2/2}$$

The theorem on the volume of the dual of a smooth Fano polytope is a consequence of the following two lemmas.

Lemma *Let P be a polytope in $\mathbf{R}^n$ whose only integral interior point is the origin. Let b be such that for each vertex u of P and each vertex v of $P^\vee$, we have $-b \leq \langle u, v\rangle \leq 1$. Then*

$$\mathrm{vol}(P) \leq (b+1)^n$$

Proof. The hypothesis implies $-\frac{1}{b}P \subset P$. Assume $\mathrm{vol}(P) > (b+1)^n$ and set

$$P' = \frac{1-\varepsilon}{b+1}P$$

We have $\mathrm{vol}(P') > 1$ for ε positive small enough hence, by a theorem of Blichfeldt, P' contains distinct points p and p' such that $p - p' \in \mathbf{Z}^n$. Since P' is convex and $-\frac{1}{b}P' \subset P'$, the point

$$\frac{p-p'}{b+1} = \frac{p + b\left(-\frac{1}{b}p'\right)}{b+1}$$

is in P', hence $p - p'$ is in $(b+1)P'$, which is contained in the interior of P. This contradicts the fact that the origin is the only integral interior point of P. $\square$

Lemma *Let Q be a smooth Fano polytope of dimension n with $\rho+n$ vertices, $\rho>1$. For each vertex u of $Q^\vee$ and each vertex v of Q, we have*

$$-\frac{n-1}{n-2}((n-1)^\rho-1) \leq \langle u,v\rangle \leq 1$$

Singular Fano varieties

In many cases, such as in the Minimal Model Program, it is necessary to allow some kind of singularities for Fano varieties. We assume the characteristic of $\mathbf{k}$ is 0.

Let X be a normal variety such that K_X is a $\mathbf{Q}$-Cartier divisor. For any desingularization $f : Y \to X$, write

$$K_Y \sim f^* K_X + \sum_i a_i E_i$$

where the E_i are f-exceptional divisors, $a_i \in \mathbf{Q}$, and $\sim$ denotes numerical equivalence.

The discrepancy of X is the minimum of the a_i for all possible desingularizations f.

It can be computed from any desingularization f whose exceptional locus is a divisor with normal crossings by the formula

$$\boxed{\mathrm{discr}(X) = \begin{cases} \min(1, a_i) & \text{if all } a_i \text{ are } \geq -1 \\ -\infty & \text{otherwise} \end{cases}}$$

It is therefore either $-\infty$ or a rational number in $[-1, 1]$, and is 1 if X is smooth. We say that X has *log terminal singularities* if $\mathrm{discr}(X) > -1$.

Definition *A Fano variety X is a normal projective variety with log terminal singularities such that $-K_X$ is an ample $\mathbf{Q}$-Cartier divisor.*

Example Let Y be a projectively normal smooth subvariety of $\mathbf{P}^n$, with hyperplane divisor H, such that $K_Y \sim qH$ for some rational number q (e.g., Y a smooth curve).

Set $\pi : \tilde{X} = \mathbf{P}(\mathscr{O}_Y \oplus \mathscr{O}_Y(H)) \to Y$ and let Y_0 be the section such that $Y_0|_{Y_0} \equiv -H$.

The cone X in $\mathbf{P}^{n+1}$ over Y is normal; it is the contraction $f : \tilde{X} \to X$ of Y_0, and

$$\begin{aligned} K_{\tilde{X}} &\sim -2Y_0 + \pi^*(K_Y - H) \\ &\sim -2Y_0 + \pi^*((q-1)H) \end{aligned}$$

hence

$$K_X \sim (q-1)H$$

(These two Weil divisors coincide outside of the vertex of X.)

and

$$\boxed{-K_X \text{ ample} \iff q < 1}$$

Writing $K_{\tilde{X}} \sim f^*K_X + aY_0$, restricting to Y_0 yields $a = -1 - q$, hence

$$\boxed{X \text{ has log terminal singularities} \iff q < 0}$$

It follows that X *is a Fano variety if and only if* Y *is a Fano variety*.

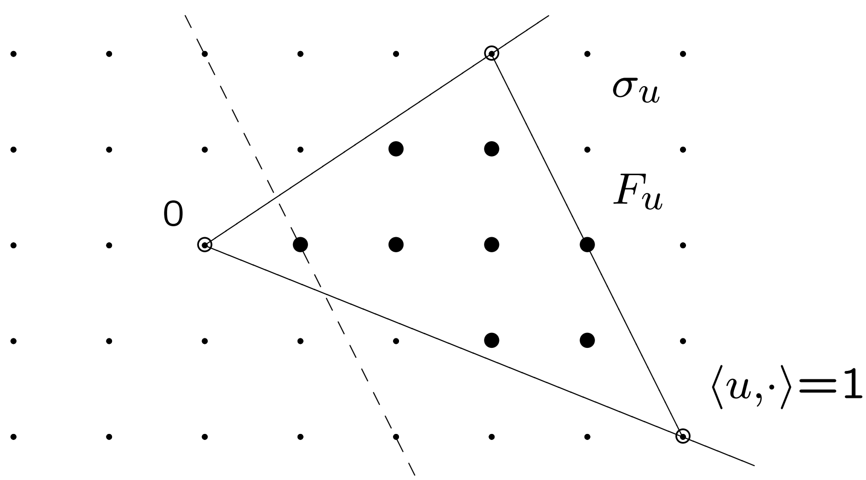
Toric Fano varieties

We have

Vertices of Q are primitive in N	$\Rightarrow$	X_Q Fano

Moreover, if $P = Q^\vee$ and if, for each vertex u of P, we let σ_u be the cone in $N_{\mathbf{R}}$ generated by the facet F_u of Q, we have

$$\mathrm{discr}(X_Q) = -1 + \min_{\substack{u \in \mathscr{V}(P) \\ v \in \sigma_u \cap N - \mathscr{V}(Q) - \{0\}}} \langle u, v \rangle$$

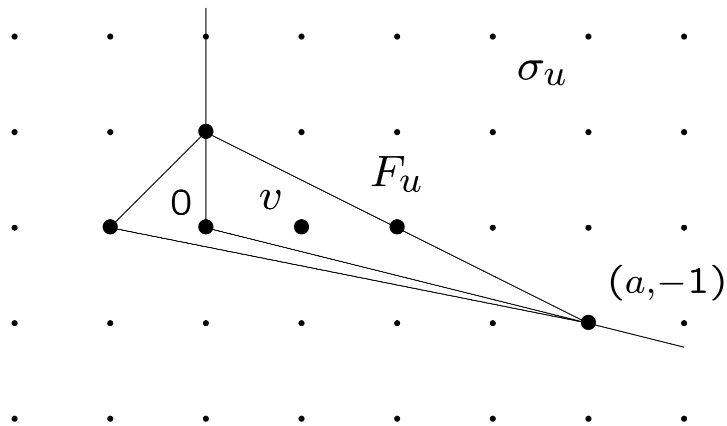


The polytope P has rational vertices and

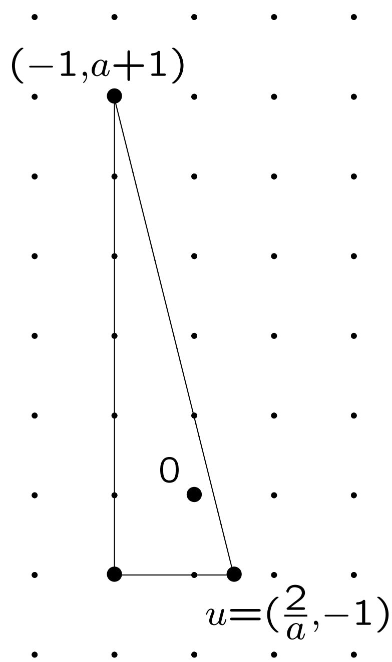
Vertices of mP are in M	$\Longleftrightarrow$	mK_X Cartier

In particular, X is a Gorenstein variety if and only if P has integral vertices (a polytope Q with this property is called *reflexive*; its dual has the same property)

Example The projective cone in $\mathbf{P}^{a+1}$ over the rational normal curve in $\mathbf{P}^a$ is the toric surface associated with the polytope



with dual



It is a Fano surface (this is a particular case of the example above, with $q = -2/a$). We recover

$$\mathrm{discr}(X) = -1 + \langle u, v \rangle = -1 + \frac{2}{a}$$

Also, the smallest positive integer m such that mK_X is a Cartier divisor is a if a is odd, $a/2$ otherwise.

This shows that there are, in each dimension at least 2, infinitely many isomorphism classes of toric Fano varieties. However, by bounding the discrepancy away from -1, we get finitely many isomorphism types: since the vertices of a Fano polytope are primitive, we have for any $r > 0$

$$\mathrm{discr}(X_Q) \geq -1 + \frac{1}{r} \iff \langle u, v \rangle \geq \frac{1}{r} \quad \text{for all } v \in \sigma_u \cap N$$

This implies

$$\mathrm{Int}(Q) \cap rN = \{0\}$$

Corollary *Given positive integers n and r, there are only finitely many isomorphism types of toric Fano varieties of dimension n and discrepancy $\geq -1 + \frac{1}{r}$ defined over $\mathbf{k}$.*

This is due to Borisov & Borisov (1993), who appear to have been unaware of an earlier article of Hensley (1983), which gives the finiteness of the number of isomorphism classes of polytopes $Q \subset \mathbf{R}^n$ such that $\mathrm{Int}(Q) \cap r\mathbf{Z}^n = \{0\}$ under the action of $\mathrm{GL}(\mathbf{Z}^n)$. It also provides explicit (large) bounds in terms of n and r for the smallest integer m such that mK_X is Cartier and for the degree $(-K_X)^n$.

It is conjectured that for each positive integer n and r, there are only finitely many deformation types of Fano varieties of dimension n with discrepancy $\geq -1 + 1/r$. The point is to bound $(-K_X)^n$ *and* an integer m such that mK_X is a Cartier divisor. M^c^Kernan claims in the preprint *Boundedness of log terminal Fano pairs of bounded index* (math.AG/0205214) that given integers n and m, there are only finitely many deformation types of complex Fano varieties X of dimension n such that mK_X is a Cartier divisor.