Fake projective spaces and fake tori

Olivier DEBARRE

École normale supérieure, Paris

Institute for Basic Science, Pohang, October 18, 2016

Hirzebruch and Kodaira's theorem

Theorem (Hirzebruch–Kodaira)

If a compact Kähler manifold is homeomorphic to $\mathbf{CP}^n$*, it is biholomorphic to* $\mathbf{CP}^n$*.*

Sketch of proof

Since X is compact Kähler, the Hodge numbers $h^{p,q}(X)$ can be computed from the Betti numbers $b_r(X)$:

$$\sum_{p+q=r} h^{p,q}(X) = b_r(X) = b_r(\mathbf{CP}^n) = \begin{cases} 1 & \text{if } r \text{ even and } 0 \leq r \leq 2n; \\ 0 & \text{otherwise.} \end{cases}$$

We obtain

- $H^{p,q}(X) = 0$ for $p \neq q$, hence $\chi(X, \mathscr{O}_X) = 1$;
- $H^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X) = 0$, hence $H^2(X, \mathbf{C}) \simeq \mathbf{C}$ is generated by the class of a Kähler metric, $\mathrm{Pic}(X) = \mathbf{Z}[L]$, with L ample, and X is projective (Kodaira).

Sketch of proof

If $f\colon X \xrightarrow{\sim} \mathbf{CP}^n$ homeomorphism, $f^*H = \pm L$ (both generate $H^2(X,\mathbf{Z})$).

Pontryagin classes are topological invariants (Novikov), hence

$$p_i(X) = f^*p_i(\mathbf{CP}^n) = f^*\left(\binom{n+1}{i}H^{2i}\right) = \binom{n+1}{i}L^{2i}.$$

Write $c_1(X) = (n+1+2s)L$ (the number $2s$ is even because $c_1(X)$ mod 2, the first Stiefel–Whitney class, is invariant under homeomorphism).
Hirzebruch–Riemann–Roch theorem and values of the Pontryagin classes yield

$$\chi(X,\mathscr{O}_X) = \binom{n+s}{n}.$$

Sketch of proof

Since this number is 1, we have

$$\binom{n+s}{n} = 1$$

and

- either $s = 0$, $c_1(X) = (n+1)L$, and X is a Fano variety;
- or $s = -n-1$, n is even, $c_1(X) = -(n+1)L$, and X is of general type.

In the first case, X is biholomorphic to $\mathbf{CP}^n$ (Kobayashi–Ochiai).

Sketch of proof

In the second case, X carries a Kähler–Einstein metric ω with $\mathrm{Ric}(\omega) = -\omega$ (Aubin–Yau). Moreover (Yau),

$$(-1)^n\Big(\frac{2(n+1)}{n}c_2(X) - c_1^2(X)\Big) \cdot c_1^{n-2}(X) \geq 0.$$

Using the known values for $c_1(X)$ and $p_1(X)$, the left side vanishes.

This implies that (X, ω) has constant negative holomorphic curvature, hence is covered by the unit ball in $\mathbf{C}^n$.

Since X is simply connected (and compact), this is impossible.

What if we only assume that X has the same integral cohomology ring as $\mathbf{CP}^n$?

Theorem (Fujita, Libgober–Wood)

A compact Kähler manifold with same integral cohomology ring as $\mathbf{CP}^n$ ($n \leq 6$) is

- *either isomorphic to $\mathbf{CP}^n$;*
- *or a quotient of the unit balls $\mathbf{B}^4$ or $\mathbf{B}^6$.*

No quotients of $\mathbf{B}^{2m}$ with the same integral cohomology rings as $\mathbf{CP}^{2m}$ are known (all examples have torsion in H^2).

Question

Is a compact Kähler manifold with same integral cohomology ring as $\mathbf{CP}^n$ isomorphic to $\mathbf{CP}^n$?

What if we drop the Kähler assumption?

$n = 1$: a compact complex manifold with same integral cohomology groups as $\mathbf{CP}^1$ is isomorphic to $\mathbf{CP}^1$.

$n = 2$: a compact complex manifold with same integral cohomology groups as $\mathbf{CP}^2$ is isomorphic to $\mathbf{CP}^2$ (even first Betti number implies Kähler).

$n = 3$: answer unknown. Same Betti numbers and simply connected is not enough (quadrics in $\mathbf{CP}^4$).

Complex structure on $\mathbf{S}^6$

If $\mathbf{S}^6$ has a complex structure (non-Kähler since $b_2(\mathbf{S}^6) = 0$), its blow-up X at a point satisfies

$$X \underset{\text{diff}}{\sim} \mathbf{S}^6 \sharp \overline{\mathbf{CP}^3} \underset{\text{diff}}{\sim} \overline{\mathbf{CP}^3} \underset{\text{diff}}{\sim} \mathbf{CP}^3.$$

However, $c_1(\mathbf{S}^6) = 0$ (because $b_2(\mathbf{S}^6) = 0$), hence $c_1(X) = -2E$ and $c_1(X)^3 = -8$, whereas $c_1(\mathbf{CP}^3)^3 = 4^3$, hence X is not biholomorphic to $\mathbf{CP}^3$.

Conclusion: if the theorem holds without the Kähler assumption, there is no complex structure on $\mathbf{S}^6$.

Catanese's theorem

If a compact Kähler manifold is homeomorphic to a complex torus, is it biholomorphic to a complex torus? We have more!

Theorem (Catanese)

Let X be a compact Kähler manifold such that there is a ring isomorphism

$$\bigwedge^{\bullet} H^1(X, \mathbf{Z}) \xrightarrow{\sim} H^{\bullet}(X, \mathbf{Z}).$$

Then X is biholomorphic to a complex torus.

Sketch of proof

Since X is Kähler, the Albanese map $a_X\colon X \to A_X$ induces an isomorphism

$$a_X^{*1}\colon H^1(A_X, \mathbf{Z}) \xrightarrow{\sim} H^1(X, \mathbf{Z}).$$

The hypothesis then implies that $a_X^*\colon H^\bullet(A_X, \mathbf{Z}) \to H^\bullet(X, \mathbf{Z})$ is also an isomorphism.
Set $n := \dim(X)$; since $a_X^{*2n}\colon H^{2n}(A_X, \mathbf{Z}) \xrightarrow{\sim} H^{2n}(X, \mathbf{Z})$ is an isomorphism, a_X is birational.
Since we have an isomorphism of the whole cohomology rings and X is Kähler, a_X contracts no subvarieties of X and is therefore finite. Thus, a_X is an isomorphism.

Note: in dimensions ≥ 3, the hypothesis "X Kähler" is necessary for the conclusion to hold.

Catanese's question

Assume that the compact Kähler manifold X is a *rational cohomology torus,* i.e., there is an isomorphism

$$\bigwedge^\bullet H^1(X, \mathbf{Q}) \xrightarrow{\sim} H^\bullet(X, \mathbf{Q}).$$

of graded $\mathbf{Q}$-algebras. Is X biholomorphic to a complex torus?

The answer is NO! But one can still describe the structure of rational cohomology tori and produce "exotic" examples.

Note: the Albanese map $a_X : X \to A_X$ is still surjective and finite.

The rest of this presentation is joint work with Zhi Jiang and Martì Lahoz.

A rational cohomology torus which is not a torus

C curve of genus ≥ 2 with an involution τ such that $g(C/\tau) = 1$.

E elliptic curve with an involution σ such that $g(E/\sigma) = 1$.

$X = (C \times E)/(\tau \times \sigma)$, smooth surface, is a rational cohomology torus, but not a torus.

Its Albanese map $a_X \colon X \to (C/\tau) \times (E/\sigma)$ has degree 2.

Equivalently, the image of $\bigwedge^4 H^1(X, \mathbf{Z}) \to H^4(X, \mathbf{Z}) \simeq \mathbf{Z}$ has index 2.

One checks that X is a rational cohomology torus if and only if there is a finite morphism $f\colon X \to A$ to a torus such that

$$f^*\colon H^\bullet(A,\mathbf{Q}) \xrightarrow{\sim} H^\bullet(X,\mathbf{Q}).$$

Kawamata's theorem

Given $f : X \to A$ finite, there are (Kawamata)

- a subtorus K of A,
- a normal projective variety Y,

and a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{I_X \text{ Iitaka fibration}} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g \text{ finite}} \\ A & \longrightarrow & A/K, \end{array}$$

so that $\dim(Y) = \kappa(X)$, and $X = Y$ if and only if X is of general type. General fibers of I_X are tori which are étale covers of K.

Then, X *is a rational cohomology torus if and only if* Y *is a (possibly singular) rational cohomology torus.*

Iitaka torus towers

One may then repeat the construction for a rational cohomology torus X and obtain

$$X \xrightarrow{I_X} X_1 \xrightarrow{I_{X_1}} X_2 \longrightarrow \cdots \longrightarrow X_{k-1} \xrightarrow{I_{X_{k-1}}} X_k,$$

with finite morphisms $f_i \colon X_i \to A_i$ to quotient tori of A, and

- either X_k is a rational cohomology torus of general type of positive dimension;
- or X_k is a point, in which case we call X an *Iitaka torus tower.*

New question: *are there rational cohomology tori of general type of positive dimension?* or equivalently, *are all rational cohomology tori Iitaka torus towers?*

Rational cohomology tori of general type

The answer is (unfortunately) YES! There rational cohomology tori of general type of any dimension ≥ 3, but they must be singular.

Theorem (Sawin)

Let $X \to A$ be a finite morphism from a smooth complex projective variety of general type to an abelian variety. Then

$$(-1)^{\dim(X)} \chi_{\text{top}}(X) > 0.$$

In particular, X is not a rational cohomology torus.

Sketch of proof

The pull-back to X of a general 1-form on A vanishes along a finite scheme, whose length is therefore $c_n(\Omega_X) = (-1)^n \chi_{\mathrm{top}}(X)$, where $n := \dim(X)$.

Since X is of general type, any 1-form on X vanishes at some point (Popa–Schnell), hence the result.

A singular rational cohomology torus of general type of dimension 3

For each $j \in \{1, 2, 3\}$, consider Cartesian diagrams of double covers

$$\begin{array}{ccc} C'_j & \xrightarrow{/\sigma_j} & C_j \\ {\scriptstyle /\tau_j}\big\downarrow & & \big\downarrow \\ E'_j & \xrightarrow{\text{étale}} & E_j \end{array}$$

where E_j is an elliptic curve. Then,

$$C'_1 \times C'_2 \times C'_3 / \langle \mathrm{id}_1 \times \tau_2 \times \sigma_3, \sigma_1 \times \mathrm{id}_2 \times \tau_3, \tau_1 \times \sigma_2 \times \mathrm{id}_3, \tau_1 \times \tau_2 \times \tau_3 \rangle$$

is of general type, has rational isolated singularities, and is a rational cohomology torus.

Building on this example, we can produce, in all dimensions ≥ 4, smooth rational cohomology tori that are not Iitaka torus towers.

Conclusion

We "reduced" the classification of rational cohomology tori to the classification of possibly singular rational cohomology tori of general type.

This seems a hard task. Here are a couple of facts that we can prove:

- the degree of the Albanese map a_X is divisible by the square of a prime number;
- the number of simple factors of A_X is greater than the smallest prime number that divides $\deg(a_X)$.

In the example, $\deg(a_X) = 4$ and A_X is the product of 3 elliptic curves.