COHOMOLOGICAL CHARACTERIZATIONS OF THE COMPLEX PROJECTIVE SPACE

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Abstract. In this survey, we discuss whether the complex projective space can be characterized by its integral cohomology ring among compact complex manifolds.

1. Introduction

Our starting point is the following 1957 result from [HK].

Theorem 1 (Hirzebruch–Kodaira, Yau). *Any compact Kähler manifold which is homeomorphic to* $\mathbf{CP}^n$ *is biholomorphic to* $\mathbf{CP}^n$.

Actually, Hirzebruch–Kodaira proved this result under the additional assumptions that the Kähler manifold is *diffeomorphic* to $\mathbf{CP}^n$ and that, when n is even, $c_1(T_X)$ is not $n+1$ times a negative generator of $H^2(X, \mathbf{Z})$. The first assumption was dropped (see [M]) when Novikov proved in [N] that Pontryagin classes are invariant under homeomorphisms; the second assumption was also dropped later thanks to work of Yau ([Y]). If one does not assume that X is Kähler, the conclusion still holds for $n \leq 2$ (complex surfaces with even first Betti number are Kähler by [B, L]), but nothing is known for $n \geq 3$; if the Kähler assumption can be dropped when $n = 3$, the sphere $\mathbf{S}^6$ has no complex structure.

Stronger characterizations were proved in dimensions $n \leq 6$ by Fujita ([F1]) and Libgober–Wood ([LW]) assuming only that the Kähler manifold has the homotopy type of $\mathbf{CP}^n$. Looking carefully through their arguments, it is not too difficult to extract a proof of the following stronger result.

Theorem 2. *Let n be an integer with $n \leq 6$. Any compact Kähler manifold with the same integral cohomology ring as* $\mathbf{CP}^n$ *is*

- *either isomorphic to* $\mathbf{CP}^n$*;*
- *or a quotient of the unit balls* $\mathbf{B}^4$ *or* $\mathbf{B}^6$.

No quotients of even dimensional unit balls $\mathbf{B}^{2m}$ with the same integral cohomology rings as $\mathbf{P}^{2m}$ are known (all known examples have torsion in H^2). It is therefore legitimate to ask the following question.

Question. Is any compact Kähler manifold with same integral cohomology ring as $\mathbf{CP}^n$ isomorphic to $\mathbf{CP}^n$?

The methods used in the proof of the theorem above are completely computational and it seems unlikely that they can be generalized to higher dimensions (we obtain only partial results in dimension 7 in Theorem 10). Using geometrical arguments would perhaps be a good idea to make further progress.

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2. Preliminaries

From now on, we will write $\mathbf{P}^n$ instead of $\mathbf{CP}^n$ for the complex projective space of dimension n.

2.1. **Hirzebruch–Riemann–Roch.** Let X be a projective complex manifold of dimension n. Following [H], we set

$$\chi^p(X) := \sum_{q=0}^{n}(-1)^q h^{p,q}(X) = \chi(X,\Omega_X^p)$$

and we define the χ_y-genus

$$\chi_y(X) := \sum_{p=0}^{n}\chi^p(X)y^p = \sum_{p,q=0}^{n}(-1)^q h^{p,q}(X)y^p \in \mathbf{Z}[y].$$

For instance, $\chi_0(X)=\chi(X,\mathscr{O}_X)$ and $\chi_{-1}(X)=\chi_{\mathrm{top}}(X)$. One consequence of the Hirzebruch–Riemann–Roch theorem is that the coefficients $\chi^p(X)$ of the polynomial $\chi_y(X)$ can be expressed in terms of the Chern classes of X ([H, Section IV.21.3, (10)])

$$\chi^p(X) = T_n^p(c_1(X),\dots,c_n(X)) \quad \text{or} \quad \chi_y(X) = T_n(y;c_1(X),\dots,c_n(X)),$$

where $T_n(y;c_1,\dots,c_n) := \sum_{p=0}^{n} T_n^p(c_1,\dots,c_n)y^p$. These polynomials satisfy $T_n^p = (-1)^n T_n^{n-p}$ and they can be explicitly determined ([H, Section I.1.8, (10)]). For example, the constant terms $T_n^0(c_1,\dots,c_n)$ (which are also $(-1)^n$ times the leading term) are the Todd polynomials $\mathrm{td}_n(c_1,\dots,c_n)$ ([H, Section I.1.7, (10)]) and $T_n(-1;c_1,\dots,c_n)=c_n$, so that

$$c_n(X) = \chi_{\mathrm{top}}(X). \tag{1}$$

Libgober–Wood also introduce the polynomials

$$t_n(z;c_1,\dots,c_n) := T_n(z-1;c_1,\dots,c_n). \tag{2}$$

They show ([LW, Lemma 2.2])

$$t_n(z;c_1,\dots,c_n) = c_n - \tfrac{1}{2}nc_nz + \tfrac{1}{12}\big(\tfrac{1}{2}n(3n-5)c_n + c_1c_{n-1}\big)z^2 + \cdots \tag{3}$$

and they compute these polynomials for $n\le 6$ ([LW, p. 145]). We extend their computations to all $n\le 9$ in Section 9.

2.2. **Compact Kähler manifolds with same Betti numbers as $\mathbf{P}^n$.** Let X be a compact Kähler manifold with the same Betti numbers as $\mathbf{P}^n$.

Since X is Kähler, one can compute the numbers $h^{p,q}(X)$ from its Betti numbers, and we see that $h^{p,q}(X)=h^{p,q}(\mathbf{P}^n)=1$ if $p=q\in\{0,\dots,n\}$, and $h^{p,q}(X)=0$ otherwise. In particular, X is projective (Kodaira). Setting $c_i(X):=c_i(T_X)\in H^{2i}(X,\mathbf{Z})$, we deduce from the Hirzebruch–Riemann–Roch theorem (Section 2.1) the equalities

$$\begin{aligned} t_n(z;c_1(X),\dots,c_n(X)) &= t_n(z;c_1(\mathbf{P}^n),\dots,c_n(\mathbf{P}^n)) \\ &= t_n\big(z;\tbinom{n+1}{1},\dots,\tbinom{n+1}{n}\big) = \sum_{i=0}^{n}\tbinom{n+1}{i+1}(-1)^i z^i. \end{aligned} \tag{4}$$

In particular, (3) implies

$$c_n(X) = c_n(\mathbf{P}^n) = n+1\ ,\ \ c_1(X)c_{n-1}(X) = c_1(\mathbf{P}^n)c_{n-1}(\mathbf{P}^n) = \tfrac{1}{2}n(n+1)^2. \tag{5}$$

Assume now $c_1(X) < 0$,[1] so that X is of general type. We have ([Y, Remark (iii)])

$$\begin{array}{c}\big(\tfrac{2(n+1)}{n}c_2(X) - c_1(X)^2\big)\cdot(-c_1(X))^{n-2} \geq 0 \\ \text{with equality if and only if } X \text{ is covered by the unit ball in } \mathbf{C}^n,\end{array} \tag{6}$$

in which case, by the Hirzebruch proportionality principle, all the Chern numbers of X are the same as those of $\mathbf{P}^n$ and n is even.

If on the other hand $c_1(X) > 0$, so that X is a Fano manifold, the group $\mathrm{Pic}(X) \simeq H^2(X, \mathbf{Z})$ is torsion-free ([IP, Proposition 2.1.2]) and we write $K_X = -c_1 L$, where L is an ample generator of $\mathrm{Pic}(X)$. We have ([KO])

$$c_1 \leq n+1 \quad \text{with equality if and only if } X \simeq \mathbf{P}^n. \tag{7}$$

2.3. **Compact Kähler manifolds with same integral cohomology ring as $\mathbf{P}^n$.** Assume now that X has the same integral cohomology ring as $\mathbf{P}^n$. We have $\mathrm{Pic}(X) = \mathbf{Z}L$, where L is ample with $L^n = 1$, and $\ell := c_1(L)$ generates $H^2(X, \mathbf{Z})$. We define integers $c_1, \dots, c_n$ by setting $c_i(X) = c_i \ell^i$ and we compute Euler characteristics using the Hirzebruch–Riemann–Roch theorem ([H, Theorem 20.3.2])

$$\begin{aligned}\chi(X, L^m) &= \big[e^{m\ell}\cdot \mathrm{td}(X)\big]_n = \mathrm{td}_n(X) + \cdots + \frac{m^{n-2}}{(n-2)!}\,\mathrm{td}_2(X) \\ &\qquad + \frac{m^{n-1}}{(n-1)!}\,\mathrm{td}_1(X) + \frac{m^n}{n!},\end{aligned} \tag{8}$$

$$\chi(X, T_X \otimes L^m) = \big[\mathrm{ch}(T_X)\cdot e^{m\ell}\cdot \mathrm{td}(X)\big]_n. \tag{9}$$

Lemma 3. *We have*

$$c_1 - (n+1) \equiv 0 \pmod 2, \tag{10}$$

$$c_1^2 + c_2 - 3nc_1 + \tfrac{1}{2}(n+1)(3n-2) \equiv 0 \pmod{12}. \tag{11}$$

Proof. Since the polynomial in m which appears in (8) takes integral values at all integers m, it decomposes as

$$\binom{m+n}{n} + a_1\binom{m+n-1}{n-1} + a_2\binom{m+n-2}{n-2} + \cdots$$

where $a_1, a_2, \dots$ are integers. The coefficient of m^{n-1} is

$$\frac{1}{n!}\sum_{i=1}^{n} i + \frac{1}{(n-1)!}a_1 = \frac{1}{(n-1)!}\,\mathrm{td}_1(X) = \frac{1}{(n-1)!}\frac{1}{2}c_1,$$

hence $\frac{n+1}{2} + a_1 = \frac{1}{2}c_1$. This proves the congruence (10).

The coefficient of m^{n-2} is

$$\frac{1}{n!}\sum_{1\leq i<j\leq n} ij + \frac{1}{(n-1)!}\sum_{i=1}^{n-1} ia_1 + \frac{1}{(n-2)!}a_2 = \frac{1}{(n-2)!}\,\mathrm{td}_2(X) = \frac{1}{(n-2)!}\frac{1}{12}(c_1^2 + c_2).$$

The first sum is

$$\sum_{1\leq i<j\leq n} ij = \sum_{1\leq j\leq n}\frac{j(j-1)}{2}j = \frac{1}{2}\Big[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\Big] = \frac{n(n-1)(n+1)(3n+2)}{24}.$$

We obtain

$$\frac{1}{12}(c_1^2 + c_2) = \frac{(n+1)(3n+2)}{24} + \frac{n}{2}\Big(\frac{1}{2}c_1 - \frac{n+1}{2}\Big) + a_2$$

[1]In the sense that the image of $c_1(X)$ in $H^2(X, \mathbf{R})$ is a negative multiple of the class of a Kähler metric.

and the congruence (11) follows. □

3. Surfaces

Theorem 4 ([Y]). *Any compact complex manifold with the same Betti numbers as $\mathbf{P}^2$ is*

- *either isomorphic to $\mathbf{P}^2$;*
- *or a quotient of the unit ball $\mathbf{B}^2$.*

Proof. A compact complex surface with even first Betti number is Kähler ([B, L]). Equations (5) then give $c_1(X)^2 = 9$ and $c_2(X) = 3$.

If $c_1(X) > 0$, the surface X is isomorphic to $\mathbf{P}^2$ by (7).

If $c_1(X) < 0$, there is equality in (6) and X is a quotient of $\mathbf{B}^2$. □

Corollary 5. *Any compact complex manifold with the same integral cohomology groups as $\mathbf{P}^2$ is isomorphic to $\mathbf{P}^2$.*

Proof. Compact quotients X of the unit ball $\mathbf{B}^2$ are called fake projective planes. They are all classified and it was proved in [PY, Theorem 10.1] that $H_1(X, \mathbf{Z})$ is always nonzero (and torsion). It follows that $H^2(X, \mathbf{Z})_{\mathrm{tors}} \simeq H_1(X, \mathbf{Z})_{\mathrm{tors}}$ is nonzero, so the integral cohomology groups of fake projective planes are different from those of $\mathbf{P}^2$. □

4. Threefolds

In odd dimensions $2m-1$, it is definitely not enough to assume that the Betti numbers of X and $\mathbf{P}^{2m-1}$ are the same: a smooth odd-dimensional quadric $X \subset \mathbf{P}^{2m}$ has this property, but, if $m \geq 2$, a positive generator L of $H^2(X, \mathbf{Z})$ satisfies $L^{2m-1} = 2$, hence X is not even homeomorphic to $\mathbf{P}^{2m-1}$. In dimension 3, there are two other examples of Fano threefolds with the same Betti numbers as $\mathbf{P}^3$ (this is equivalent in that case to $b_2 = 1$ and $h^{1,2} = 0$): one with $L^3 = 5$ and one with $L^3 = 22$ ([IP, Table 12.2]).

Theorem 6 ([F1, LW]). *Any compact Kähler manifold with the same integral cohomology ring as $\mathbf{P}^3$ is isomorphic to $\mathbf{P}^3$.*

Proof. If ℓ is a positive generator of $H^2(X, \mathbf{Z})$, we write as before $c_i(X) = c_i \ell^i$. Equations (5) give $c_3 = 4$ and $c_1 c_2 = 24$. If $c_1 < 0$, we get $c_2 < 0$, but this contradicts (6). Therefore, c_1 is a positive divisor of 24 which we can assume, by (7), to be 1, 2, 3, or 4. By Lemma 3, c_1 is even, so we need only exclude $c_1 = 2$. In that case, (X, L) is a so-called del Pezzo variety (coindex 2) with $L^3 = 1$. It is therefore isomorphic to a hypersurface of degree 6 in the weighted projective space $\mathbf{P}(3, 2, 1, 1, 1)$ ([F2], [IP, Theorem 3.2.5]). But such a variety has $h^{1,2} = 21$, so this is a contradiction. □

5. Fourfolds

Theorem 7 ([F1, LW]). *Any compact Kähler manifold with the same integral cohomology ring as $\mathbf{P}^4$ is*

- *either isomorphic to $\mathbf{P}^4$;*
- *or a quotient of the unit ball $\mathbf{B}^4$.*

Four examples of compact quotients X of $\mathbf{B}^4$ with the same Betti numbers as $\mathbf{P}^4$ are known, but the groups $H_1(X, \mathbf{Z})$ are never zero ([PY2, Theorem 4]) hence they do not have the same integral cohomology ring as $\mathbf{P}^4$. It is therefore possible that the second case of the theorem never occurs.

Proof. If ℓ is a positive generator of $H^2(X,\mathbf{Z})$, we write as before $c_i(X)=c_i\ell^i$. Equations (5) give $c_4=5$ and $c_1c_3=50$. By Lemma 3, c_1 is odd, so by (7), we are reduced to $c_1\in\{\pm1,\pm5,-25\}$.

Equation (5) reads $c_4=5$. Equation (8) with $m=0$ then gives (using the values for the Todd polynomials given in Section 9)

$$1=\chi(X,\mathscr{O}_X)=\tfrac{1}{720}\big(-c_1^4+4c_1^2c_2+3c_2^2+c_1c_3-c_4\big)=\tfrac{1}{720}\big(-c_1^4+4c_1^2c_2+3c_2^2+50-5\big)$$

so c_2 is an integral root of the quadratic equation

$$3c_2^2+4c_1^2c_2-c_1^4-675=0.$$

Its (reduced) discriminant $7c_1^4+2025$ is therefore a square, which leaves only (among our possible values) the cases $c_1=\pm5$, $c_2=10$.

If $c_1=5$, the fourfold X is isomorphic to $\mathbf{P}^4$ by (7). If $c_1=-5$, there is equality in (6) and X is a quotient of $\mathbf{B}^4$. □

6. Fivefolds

Theorem 8 ([F1, LW])**.** *Any compact Kähler manifold with the same integral cohomology ring as $\mathbf{P}^5$ is isomorphic to $\mathbf{P}^5$.*

Proof. If ℓ is a positive generator of $H^2(X,\mathbf{Z})$, we write as before $c_i(X)=c_i\ell^i$. Equations (5) give $c_5=6$ and $c_1c_4=90$ and, by (10), c_1 is even. The relation $\chi(X,\mathscr{O}_X)=T_5^0(c_1,\dots,c_5)$ (Section 2.1) gives

$$1=\chi(X,\mathscr{O}_X)=-\tfrac{1}{1440}\big(c_1^3c_2-3c_1c_2^2-c_1^2c_3+c_1c_4\big)=-\tfrac{1}{1440}\big(c_1^3c_2-3c_1c_2^2-c_1^2c_3+90\big)$$

so c_2 is an integral root of the quadratic equation

$$3c_1c_2^2-c_1^3c_2+c_1^2c_3-1530=0. \tag{12}$$

Moreover, the congruence (11) reads

$$c_1^2-3c_1+c_2+3\equiv 0 \pmod{12}. \tag{13}$$

If $c_1\equiv 0 \pmod 9$, we obtain, reducing (12) modulo 27, the contradiction $1530\equiv 0 \pmod{27}$. Using (7), it follows that the possible values for c_1 are $\pm2,\pm6,-10,-30$.

We rules these cases out one by one.

Case $c_1=-30$. Reducing (12) modulo 25, we obtain $3\cdot(-30)c_2^2\equiv 1530\equiv 5 \pmod{25}$, hence $1\equiv 3\cdot(-6)c_2^2\equiv 2c_2^2 \pmod 5$, which is impossible.

Case $c_1=-10$. Reducing (12) modulo 25, we obtain $3\cdot(-10)c_2^2\equiv 1530\equiv 5 \pmod{25}$, hence $c_2^2\equiv -1 \pmod 5$. We have $c_4=-9$. Equation (8) reads, for $m=1$,

$$\begin{aligned}720\chi(X,L) &= 720+(-c_1^4+4c_1^2c_2+3c_2^2+c_1c_3-c_4)+15c_1c_2+10(c_1^2+c_2)+15c_1+6\\ &\equiv 9+3\cdot(-1)+6 \pmod 5,\end{aligned}$$

which is absurd.

Case $c_1=\pm2$. By (13), we can write $c_2=12d_2-1$.

Assume $c_1=2$. Substituting $c_5=6$, $c_4=45$, and $c_3=(1530-6c_2^2+8c_2)/4$ (obtained from (12)) in (9) with $m=-1$, we compute, following [F1],

$$24\chi(X,T_X\otimes L^{-1})=-3c_2^2+2c_2+731=-3(12d_2-1)^2+2(12d_2-1)+731\equiv 6 \pmod{24},$$

which is absurd. When $c_1=-2$, we obtain similarly the contradiction

$$24\chi(X,T_X\otimes L)=3c_2^2-2c_2+727\equiv 12 \pmod{24}.$$

Case $c_1 = -6$. By (13), we can write $c_2 = 12d_2 + 3$. We compute, using (12) and (9) with $m = -1$ again,

$$\begin{aligned} 24\chi(X, T_X \otimes L) &= 45c_2^2 - \tfrac{520}{3}c_2 - 171 = 45(12d_2+3)^2 - \tfrac{520}{3}(12d_2+3) - 171 \\ &\equiv 45 \cdot 9 - 520 \cdot 4d_2 - 520 - 171 \pmod{24}, \end{aligned}$$

but this is absurd since this last number is $\equiv 2 \pmod 8$. □

7. Sixfolds

Theorem 9 ([LW]). *Any compact Kähler manifold with the same integral cohomology ring as $\mathbf{P}^6$ is*

- *either isomorphic to $\mathbf{P}^6$;*
- *or a quotient of the unit ball $\mathbf{B}^6$.*

Proof. If ℓ is a positive generator of $H^2(X, \mathbf{Z})$, we write as before $c_i(X) = c_i\ell^i$. Equations (5) gives $c_6 = 7$ and $c_1c_5 = 3 \cdot 7^2 = 147$. From the fact that the polynomial $t_6(y; c_1, \dots, c_6)$ is the same for X and $\mathbf{P}^6$, we obtain

$$\begin{aligned} 720\tbinom{7}{5} &= -c_1^3c_3 + 3c_1c_2c_3 + c_1^2c_4 - 3c_3^2 + 3c_2c_4 + 69c_1c_5 + 186c_6, \\ 60480\tbinom{7}{7} &= 2c_1^6 - 12c_1^4c_2 + 11c_1^2c_2^2 + 5c_1^3c_3 + 10c_2^3 + 11c_1c_2c_3 - 5c_1^2c_4 - c_3^2 - 9c_2c_4 - 2c_1c_5 + 2c_6. \end{aligned}$$

Plugging in the values $c_6 = 7$ and $c_1c_5 = 147$, we obtain ([LW, p. 150])

$$-c_1^3c_3 + 3c_1c_2c_3 + c_1^2c_4 - 3c_3^2 + 3c_2c_4 = 3675, \tag{14}$$

$$2c_1^6 - 12c_1^4c_2 + 11c_1^2c_2^2 + 5c_1^3c_3 + 10c_2^3 + 11c_1c_2c_3 - 5c_1^2c_4 - c_3^2 - 9c_2c_4 = 60760. \tag{15}$$

Eliminating c_4 between these two equations, we see that c_3 is a solution of the quadratic equation

$$a_2c_3^2 + a_1c_3 + a_0 = 0, \tag{16}$$

where

$$\begin{aligned} a_2 &= 2(15 + 8c_1^2), \\ a_1 &= -4c_1c_2(15c_2 + 8c_1^2), \\ a_0 &= -30c_2^2 - 43c_1^2c_2^3 + 25c_1^4c_2^2 + 6c_1^6c_2 + 215355c_2 - 2c_1^8 + 79135c_1^2. \end{aligned}$$

In particular, $15c_2 + 8c_1^2$ divides a_0, hence also the remainder of the division of $1125a_0$ by $15c_2 + 8c_1^2$, which is

$$R(c_1) := c_1^2(6758c_1^6 - 40186125).$$

Moreover, Equation (8) gives that the polynomial

$$\begin{aligned} P(m) = {} & m\frac{1}{1440}\left(-c_1c_4 + c_1^2c_3 + 3c_1c_2^2 - c_1^3c_2\right) + \frac{m^2}{2}\frac{1}{720}\left(-c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4\right) \\ & + \frac{m^3}{6} \cdot \frac{1}{24}c_1c_2 + \frac{t^4}{24} \cdot \frac{1}{12}\left(c_1^2 + c_2\right) + \frac{m^5}{120} \cdot \frac{1}{2}c_1 + \frac{m^6}{720} \end{aligned}$$

takes integral values for all integers m. In particular,

$$720(P(1) + P(-1)) = -c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4 + 5(c_1^2 + c_2) + 2 \equiv 0 \pmod{720}. \tag{17}$$

Finally, by (11), we have the congruence

$$c_1^2 + c_2 + 6c_1 \equiv 4 \pmod{12}. \tag{18}$$

Case $c_1 \equiv 0 \pmod 3$. We get $c_2 \equiv 1 \pmod 3$ (from (18)), $c_4 \equiv 1 \pmod 3$ (from (17)), and $c_3 \equiv 0 \pmod 3$ (from (14)).

We write $c_1 = 3d_1$, $c_2 = 3d_2+1$, and $c_4 = 3d_4+1$. We have $-c_4+3+5c_2+2 \equiv 0 \pmod 9$ (from (17) again), i.e., $d_2 + d_4 \equiv 0 \pmod 3$. Moreover, using (14) modulo 27, we obtain $9d_1^2(3d_4+1) + 3(3d_2+1)(3d_4+1) \equiv 3 \pmod{27}$ (from (14) again), i.e., $d_4 + d_2 + d_1^2 \equiv 0 \pmod 3$. This gives $d_1 \equiv 0 \pmod 3$, which is impossible, since d_1 is a power of 7.

The case $c_1 = 49$ being excluded by (7), there remains to consider the cases $c_1 = -49$, $c_1 = \pm 1$, and $c_1 = \pm 7$. In all these cases, we have $c_2 \equiv -3 \pmod{12}$ by (18) and we write $c_2 = 12e_2 - 3$. We saw above that $R(c_1)$ must be divisible by $15c_2 + 8c_1^2 = 180e_2 + 8c_1^2 - 45$.

Case $c_1 = -49$**.** The integer $d = 15c_2 + 8c_1^2 = 180e_2 - 45 + 8 \cdot 49^2$ is positive by (6) and divides $R(c_1) = 7^6 \cdot 37 \cdot 251 \cdot 1559 \cdot 131849$. A computer check gives us all the positive divisors d of $R(c_1)$ for which e_2 is an integer; we then compute the discriminant D of the quadratic equation (16). We find

- $d = 37 \cdot 1559$, for which $e_2 = 214$ and $D = 2^7 \cdot 37 \cdot 1559 \cdot 7087681 \cdot 21780337$;
- $d = 7 \cdot 131849$, for which $e_2 = 5021$ and $D = 2^8 \cdot 7^3 \cdot 13 \cdot 193 \cdot 131849 \cdot 9694436995073$;
- $d = 7^3 \cdot 251 \cdot 1559 \cdot 131849$, for which $e_2 = 98314662210$ and $D = 2^7 \cdot 7^8 \cdot 17 \cdot 251 \cdot 317 \cdot 1559 \cdot 131849 \cdot 165057229 \cdot 120326342604749673066085 9$.

In each case, D is not a perfect square hence the system of equations (14) and (15) has no integral solutions.

Case $c_1 = \pm 1$**.** We have $R(c_1) = -23 \cdot 1746929$ and there are no divisors of $R(c_1)$ for which e_2 is an integer.

Case $c_1 = 7$**.** There is then equality in (7) and X is isomorphic to $\mathbf{P}^6$.

Case $c_1 = -7$**.** We have $R(c_1) = 7^4 \cdot 101 \cdot 152533$. The only divisor of $R(c_1)$ for which e_2 is an integer is $7 \cdot 101$, for which $e_2 = 2$ and $c_2 = 21$. There is then equality in (6) and X is a quotient of $\mathbf{B}^6$. □

8. Sevenfolds

Theorem 10. *Any compact Kähler manifold X with the same integral cohomology ring as $\mathbf{P}^7$ is isomorphic to $\mathbf{P}^7$, unless $c_1(X)^7 \in \{\pm 2^7, \pm 4^7\}$.*

Proof. If ℓ is a positive generator of $H^2(X, \mathbf{Z})$, we write as before $c_i(X) = c_i \ell^i$. Equations (5) give $c_7 = 8$ and $c_1 c_6 = 2^5 \cdot 7 = 224$ and, by (10), c_1 is even. From the fact that the polynomial $t_7(y; c_1, \ldots, c_7)$ is the same for X and $\mathbf{P}^7$, we obtain, comparing the coefficients of y^5 and y^7 and plugging in the values $c_7 = 8$ and $c_1 c_6 = 224$, the equations

$$0 = c_1^3 c_4 - 3c_1 c_2 c_4 - c_1^2 c_5 + 3c_3 c_4 - 3c_2 c_5 + 7728, \tag{19}$$

$$0 = -2c_1^5 c_2 + 10c_1^3 c_2^2 + 2c_1^4 c_3 - 10c_1 c_2^3 - 11c_1^2 c_2 c_3 - 2c_1^3 c_4 + c_1 c_3^2 + 9c_1 c_2 c_4 + 2c_1^2 c_5 + 120512, \tag{20}$$

with $7728 = 2^4 \cdot 3 \cdot 7 \cdot 23$ and $120512 = 2^6 \cdot 7 \cdot 269$.

By (11), we also have the congruence

$$c_1^2 + c_2 + 3c_1 \equiv 8 \pmod{12}. \tag{21}$$

We also compute, for all integers m,

$$
\begin{aligned}
60480\chi(X,L^m) \;=\;& 12m^7+42m^6c_1+42m^5(c_1^2+c_2)+105m^4c_1c_2\\
&+2m^3(-7c_1^4+28c_1^2c_2+21c_2^2+7c_1c_3-7c_4)\\
&+m^2(-21c_1^3c_2+63c_1c_2^2+21c_1^2c_3-21c_1c_4)\\
&+m(2c_1^6-12c_1^4c_2+11c_1^2c_2^2+5c_1^3c_3+10c_2^3+11c_1c_2c_3\\
&\qquad -5c_1^2c_4-c_3^2-9c_2c_4-2c_1c_5+2c_6)\\
&+60480,
\end{aligned}
\tag{22}
$$

with $60480=2^6\cdot 3^3\cdot 5\cdot 7$.

Case $7\mid c_1$. We write $c_1=7d_1$. Equation (20) divided by 7 gives

$$-10d_1c_2^3+d_1c_3^2+9d_1c_2c_4+2^6\cdot 269\equiv 0\pmod 7,\text{ hence } d_1(3c_2^3-c_3^2-2c_2c_4)\equiv 3\pmod 7,$$

and, from (22) with $m=1$, we get

$$24+2(10c_2^3-c_3^2-9c_2c_4+2c_6)\equiv 0\pmod 7,\text{ hence } 5+3c_2^3-c_3^2-2c_2c_4+2c_6\equiv 0\pmod 7.$$

Together, these two congruences imply $5d_1+3+2d_1c_6\equiv 0\pmod 7$. Since $d_1c_6=224/7=32$, we finally get $d_1\equiv 2\pmod 7$. But $d_1\le 1$ by (7) and $d_1\mid 2^5$, and all these conditions are incompatible.

Since c_1 must be even, we now would like to exclude the cases $c_1\in\{\pm 2,\pm 4,-8,-16,-32\}$. Unfortunately, playing around with congruences is not enough when $c_1\in\{\pm 2,\pm 4\}$ (even with (9)) and we were unable to exclude these cases.

We therefore assume $c_1\in\{-8,-16,-32\}$ and write $c_1=4d_1$.

Congruence (21) implies $c_2\equiv 0\pmod 4$ and we write $c_2=4d_2$. Equation (22) with $m=1$, taken modulo 32 and divided by 2, gives

$$
\begin{aligned}
0\;\equiv\;& -2d_1c_3^2-8d_1d_2c_4+8d_1c_3-c_3^2-20d_1c_4-4d_2c_4-8d_1c_5+8d_1+8d_2\\
&-14c_4+2c_6+12\pmod{16},
\end{aligned}
\tag{23}
$$

hence c_3 is even; we write $c_3=2d_3$. When $c_1=\pm 32$, we already get the contradiction $0\equiv 2^6\pmod{2^7}$ by reducing equation (20) modulo 2^7.

So we assume $(d_1,c_6)\in\{(-2,-28),(-4,-14)\}$ and we write $c_6=2d_6$. We obtain from (23) that c_4 is even, we write $c_4=2d_4$, and, after dividing by 4, we get

$$0\;\equiv\;-2d_1d_3^2-d_3^2-2d_1d_4-2d_2d_4-2d_1c_5+2d_1+2d_2-7d_4+d_6+3\pmod 4$$

hence $d_3\equiv d_4+d_6+1\pmod 2$.

Case $c_1=-16$. We have $d_3\equiv d_4\pmod 2$. Substituting the integers d_2, d_3, and d_4 into (20) and dividing by 64, we find

$$160d_2^3-10240d_2^2-352d_2d_3-d_3^2-18d_2d_4+131072d_2+4096d_3+256d_4+8c_5+1883=0,$$

hence $d_3\equiv 1\pmod 2$. Doing the same with (19), we obtain, after dividing by 4,

$$96d_2d_4+3d_3d_4-3d_2c_5-2048d_4-64c_5+1932=0,$$

hence $d_2\equiv c_5\equiv 1\pmod 2$. So we write $d_2=2e_2+1$, $d_3=2e_3+1$, $d_4=2e_4+1$, $c_5=2d_5+1$, and, substituting these variables into (22) with $m=-2$, we obtain the contradiction

$$756\chi(X,L^{-2})=-640e_2^3+7424e_2^2+704e_2e_3+2e_3^2+36e_2e_4+80818e_2+290e_3+14e_4-8d_5-713529$$

(the right side is odd, but the left side is even).

Case $c_1 = -8$. We have $d_3 \equiv d_4 + 1 \pmod 2$. Substituting the integers d_2, d_3, and d_4 into (20) and dividing by 32, we find

$$160d_2^3 - 2560d_2^2 - 176d_2d_3 - d_3^2 - 18d_2d_4 + 8192d_2 + 512d_3 + 64d_4 + 4c_5 + 3766 = 0,$$

hence $d_3 \equiv 0 \pmod 2$ and $d_4 \equiv 1 \pmod 2$. We write $d_3 = 2e_3$ and $d_4 = 2e_4 + 1$. Doing the same with (22) with $m = -1$, we get

$$\begin{aligned} 60480\chi(X, L^{-1}) &= -50d_2^3 + 1310d_2^2 + 880d_2e_3 + 20e_3^2 + 90d_2e_4 + 45d_2 + 22670d_2 \\ &\qquad - 160e_3 - 20e_4 - 80c_5 - 469710, \end{aligned}$$

hence $d_2 \equiv 0 \pmod 2$. We write $d_2 = 2e_2$ and, substituting into (20) and dividing by 64, we obtain the contradiction

$$640e_2^3 - 5120e_2^2 - 352e_2e_3 - 2e_3^2 - 36e_2e_4 + 8174e_2 + 512e_3 + 64e_4 + 2c_5 + 1915 = 0.$$

This finishes the proof of the theorem. □

9. Computations

We list here the polynomials $t_n(z; c_1, \dots, c_n)$ (defined in (2)) for $n \le 9$ (they were given for $n \le 6$ in [LW, p. 145])

$$\begin{aligned}
t_1 &= c_1 - \tfrac{1}{2}c_1 z \\
t_2 &= c_2 - c_2 z + \tfrac{1}{12}(c_1^2 + c_2)z^2 \\
t_3 &= c_3 - \tfrac{3}{2}c_3 z + \tfrac{1}{12}(c_1c_2 + 6c_3)z^2 - \tfrac{1}{24}c_1c_2z^3 \\
t_4 &= c_4 - 2c_4 z + \tfrac{1}{12}(c_1c_3 + 14c_4)z^2 + \tfrac{1}{12}(c_1c_3 + 2c_4)z^3 + \tfrac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4)z^4 \\
t_5 &= c_5 - \tfrac{5}{2}c_5 z + \tfrac{1}{12}(c_1c_4 + 25c_5)z^2 - \tfrac{1}{8}(c_1c_4 + 5c_5)z^3 \\
&\quad + \tfrac{1}{720}(-c_1^3c_2 + 3c_1c_2^2 + c_1^2c_3 + 29c_1c_4 + 30c_5)z^4 + \tfrac{1}{1440}(c_1^3c_2 - 3c_1c_2^2 - c_1^2c_3 + c_1c_4)z^5 \\
t_6 &= c_6 - 3c_6 z + \tfrac{1}{12}(c_1c_5 + 39c_6)z^2 - \tfrac{1}{6}(c_1c_5 + 9c_6)z^3 \\
&\quad + \tfrac{1}{720}(-c_1^3c_3 + 3c_1c_2c_3 + c_1^2c_4 - 3c_3^2 + 3c_2c_4 + 69c_1c_5 + 186c_6)z^4 \\
&\quad + \tfrac{1}{720}(c_1^3c_3 - 3c_1c_2c_3 - c_1^2c_4 + 3c_3^2 - 3c_2c_4 - 9c_1c_5 - 6c_6)z^5 \\
&\quad + \tfrac{1}{60480}(2c_1^6 - 12c_1^4c_2 + 11c_1^2c_2^2 + 5c_1^3c_3 + c_2^3 + 11c_1c_2c_3 - 5c_1^2c_4 \\
&\qquad - c_3^2 - 9c_2c_4 - 2c_1c_5 + 2c_6)z^6 \\
t_7 &= c_7 - \tfrac{7}{2}c_7 z + \tfrac{1}{12}(c_1c_6 + 56c_7)z^2 - \tfrac{5}{24}(c_1c_6 + 14c_7)z^3 \\
&\quad + \tfrac{1}{720}(-c_1^3c_4 + 3c_1c_2c_4 + c_1^2c_5 - 3c_3c_4 + 3c_2c_5 + 124c_1c_6 + 602c_7)z^4 \\
&\quad + \tfrac{1}{480}(c_1^3c_4 - 3c_1c_2c_4 - c_1^2c_5 + 3c_3c_4 - 3c_2c_5 - 24c_1c_6 - 42c_7)z^5 \\
&\quad + \tfrac{1}{60480}(2c_1^5c_2 - 10c_1^3c_2^2 - 2c_1^4c_3 + 10c_1c_2^3 + 11c_1^2c_2c_3 - 40c_1^3c_4 - c_1c_3^2 \\
&\qquad + 117c_1c_2c_4 + 40c_1^2c_5 - 126c_3c_4 + 126c_2c_5 + 170c_1c_6 + 84c_7)z^6 \\
&\quad + \tfrac{1}{120960}(-2c_1^5c_2 + 10c_1^3c_2^2 + 2c_1^4c_3 - 10c_1c_2^3 - 11c_1^2c_2c_3 - 2c_1^3c_4 \\
&\qquad + c_1c_3^2 + 9c_1c_2c_4 + 2c_1^2c_5 - 2c_1c_6)z^7 \\
t_8 &= c_8 - 4c_8 z + \tfrac{1}{12}(c_1c_7 + 76c_8)z^2 - \tfrac{1}{4}(c_1c_7 + 20c_8)z^3 \\
&\quad + \tfrac{1}{720}(-c_1^3c_5 + 3c_1c_2c_5 + c_1^2c_6 - 3c_3c_5 + 3c_2c_6 + 194c_1c_7 + 1458c_8)z^4 \\
&\quad + \tfrac{1}{360}(c_1^3c_5 - 3c_1c_2c_5 - c_1^2c_6 + 3c_3c_5 - 3c_2c_6 - 44c_1c_7 - 138c_8)z^5 \\
&\quad + \tfrac{1}{60480}(2c_1^5c_3 - 10c_1^3c_2c_3 - 2c_1^4c_4 + 10c_1c_2^2c_3 + 10c_1^2c_3^2 + c_1^2c_2c_4 - 96c_1^3c_5 - 10c_2c_3^2 \\
&\quad + 10c_2^2c_4 - 11c_1c_3c_4 + 295c_1c_2c_5 + 96c_1^2c_6 - 20c_4^2 - 264c_3c_5 + 284c_2c_6
\end{aligned}$$

$$
\begin{aligned}
&\quad + 1206c_1c_7 + 1524c_8)z^6 \\
&+ \tfrac{1}{60480}(-2c_1^5c_3 + 10c_1^3c_2c_3 + 2c_1^4c_4 - 10c_1c_2^2c_3 - 10c_1^2c_3^2 - c_1^2c_2c_4 + 12c_1^3c_5 + 10c_2c_3^2 \\
&- 10c_2^2c_4 + 11c_1c_3c_4 - 43c_1c_2c_5 - 12c_1^2c_6 + 20c_4^2 + 12c_3c_5 - 32c_2c_6 \\
&\quad - 30c_1c_7 - 12c_8)z^7 \\
&+ \tfrac{1}{3628800}(-3c_1^8 + 24c_1^6c_2 - 50c_1^4c_2^2 - 14c_1^5c_3 + 8c_1^2c_2^3 + 26c_1^3c_2c_3 + 14c_1^4c_4 + 21c_2^4 \\
&\quad + 50c_1c_2^2c_3 + 3c_1^2c_3^2 - 19c_1^2c_2c_4 - 7c_1^3c_5 - 8c_2c_3^2 - 34c_2^2c_4 - 13c_1c_3c_4 \\
&\quad - 16c_1c_2c_5 + 7c_1^2c_6 + 5c_4^2 + 3c_3c_5 + 13c_2c_6 + 3c_1c_7 - 3c_8)z^8
\end{aligned}
$$

$$
\begin{aligned}
t_9 = {}& c_9 - \tfrac{9}{2}c_9z + \tfrac{1}{12}(c_1c_8 + 99c_9)z^2 + \tfrac{1}{24}(-7c_1c_8 - 189c_9)z^3 \\
&+ \tfrac{1}{720}(-c_1^3c_6 + 3c_1c_2c_6 + c_1^2c_7 - 3c_3c_6 + 3c_2c_7 + 279c_1c_8 + 2979c_9)z^4 \\
&+ \tfrac{1}{288}(c_1^3c_6 - 3c_1c_2c_6 - c_1^2c_7 + 3c_3c_6 - 3c_2c_7 - 69c_1c_8 - 333c_9)z^5 \\
&+ \tfrac{1}{60480}(2c_1^5c_4 - 10c_1^3c_2c_4 - 2c_1^4c_5 + 10c_1c_2^2c_4 + 10c_1^2c_3c_4 + c_1^2c_2c_5 - 173c_1^3c_6 - 10c_2c_3c_4 \\
&\quad - 10c_1c_4^2 + 10c_2^2c_5 - c_1c_3c_5 + 526c_1c_2c_6 \\
&\quad + 173c_1^2c_7 - 10c_4c_5 - 505c_3c_6 + 515c_2c_7 + 4041c_1c_8 + 9075c_9)z^6 \\
&+ \tfrac{1}{40320}(-2c_1^5c_4 + 10c_1^3c_2c_4 + 2c_1^4c_5 - 10c_1c_2^2c_4 - 10c_1^2c_3c_4 - c_1^2c_2c_5 + 33c_1^3c_6 + 10c_2c_3c_4 \\
&\quad + 10c_1c_4^2 - 10c_2^2c_5 + c_1c_3c_5 - 106c_1c_2c_6 - 33c_1^2c_7 + 10c_4c_5 + 85c_3c_6 - 95c_2c_7 \\
&\quad - 261c_1c_8 - 255c_9)z^7 \\
&+ \tfrac{1}{3628800}(-3c_1^7c_2 + 21c_1^5c_2^2 + 3c_1^6c_3 - 42c_1^3c_2^3 - 29c_1^4c_2c_3 \\
&\quad + 57c_1^5c_4 + 21c_1c_2^4 + 50c_1^2c_2^2c_3 + 8c_1^3c_3^2 - 274c_1^3c_2c_4 - 57c_1^4c_5 - 8c_1c_2c_3^2 \\
&\quad + 266c_1c_2^2c_4 + 287c_1^2c_3c_4 + 14c_1^2c_2c_5 - 153c_1^3c_6 - 300c_2c_3c_4 - 295c_1c_4^2 \\
&\quad + 300c_2^2c_5 - 27c_1c_3c_5 + 673c_1c_2c_6 + 153c_1^2c_7 - 300c_4c_5 - 30c_3c_6 + 330c_2c_7 \\
&\quad + 267c_1c_8 + 90c_9)z^8 \\
&+ \tfrac{1}{7257600}(3c_1^7c_2 - 21c_1^5c_2^2 - 3c_1^6c_3 + 42c_1^3c_2^3 + 29c_1^4c_2c_3 + 3c_1^5c_4 - 21c_1c_2^4 \\
&\quad - 50c_1^2c_2^2c_3 - 8c_1^3c_3^2 - 26c_1^3c_2c_4 - 3c_1^4c_5 + 8c_1c_2c_3^2 + 34c_1c_2^2c_4 \\
&\quad + 13c_1^2c_3c_4 + 16c_1^2c_2c_5 + 3c_1^3c_6 - 5c_1c_4^2 - 3c_1c_3c_5 - 13c_1c_2c_6 \\
&\quad - 3c_1^2c_7 + 3c_1c_8)z^9
\end{aligned}
$$

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