# PERIODS OF ALGEBRAIC VARIETIES

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Abstract. The *periods* of a compact complex algebraic manifold $X$ are the integrals of its holomorphic 1-forms over paths. These integrals are in general not well-defined, but they can still be used to associate with the variety $X$ a *period point* in a suitable analytic (or sometimes quasi-projective) variety called a *period domain.* As $X$ varies in a smooth family, this period point varies holomorphically in the period domain, defining the *period map* from the parameter space of the family to the period domain. This is a very old construction when $X$ is a complex curve. Griffiths generalized it to higher dimensions in 1968 using Hodge theory, and it became a very useful tool to study moduli spaces of varieties, in particular for K3 surfaces. I will discuss "classical" examples and some more recent results (cubic fourfolds, holomorphic symplectic varieties, Fano fourfolds).

## 1. Elliptic curves

A (complex) elliptic curve is a smooth plane cubic curve $E$, that is, the set of points in the complex projective plane $\mathbf{P}^2$ where a homogeneous polynomial of degree 3 in 3 variables vanishes (smoothness is then equivalent to the fact that the partial derivatives of this polynomial have no common zero). After a change of variables, the equation of $E$ can be written, in affine coordinates, as

$$y^2 = x(x-1)(x-\lambda),$$

where $\lambda \in \mathbf{C} - \{0,1\}$.

The differential 1-form

$$\omega := \frac{dx}{y}$$

is holomorphic everywhere on $E$.[1] *Elliptic integrals* of the type

$$\int_{p_0}^{p} \omega = \int_{p_0}^{p} \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

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[1] At points where $q(x) = x(x-1)(x-\lambda)$ (hence also $y$) vanishes, we may write $\omega = 2dy/q'(x)$, hence $\omega$ is regular everywhere.

were considered classically. This integral is not well-defined because it depends on the choice of a path on $E$ from the point $p_0$ to the point $p$. More precisely, if $(\delta,\gamma)$ is a symplectic basis for the free abelian group $H_1(E,\mathbf{Z})$ endowed with the (skew-symmetric) intersection form, it is only well-defined up to the subgroup of $\mathbf{C}$ generated by the *periods*

$$A := \int_\delta \omega \quad \text{and} \quad B := \int_\gamma \omega.$$

One can always choose the basis $(\delta,\gamma)$ so that $A = 1$ and $\mathrm{Im}(B) > 0$. The point $\tau := B$ of the Siegel upper half-plane $\mathscr{H}$ is then called the *period point* of $E$. It is still well-defined only up to transformations of the type

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),$$

corresponding to a change of the symplectic basis $(\delta,\gamma)$. All in all, we have defined a *single-valued period map*

$$\begin{array}{ccc} \mathbf{C} - \{0,1\} & \longrightarrow & \mathrm{SL}_2(\mathbf{Z})\backslash\mathscr{H} \\ \lambda & \longmapsto & \tau. \end{array}$$

One can prove that it is *holomorphic and surjective.*

In fact, the period map induces an isomorphism between $\mathbf{C} - \{0,1\}$ and the quotient of $\mathscr{H}$ by a subgroup of $\mathrm{SL}_2(\mathbf{Z})$ of index 6 classically denoted by $\Gamma(2)$. Its inverse defines a function

$$\begin{array}{ccc} \mathscr{H} & \longrightarrow & \mathbf{C} - \{0,1\} \\ \tau & \longmapsto & \lambda(\tau) \end{array}$$

which realizes $\mathscr{H}$ as the universal cover of $\mathbf{C} - \{0,1\}$. It is given explicitly by

$$\lambda(\tau) = \left( \sum_{n=-\infty}^{+\infty} q^{\left(n+\frac{1}{2}\right)^2} \Big/ \sum_{n=-\infty}^{+\infty} q^{n^2} \right)^4,$$

where $q = \exp(i\pi\tau)$ ($|q| < 1$). This function was used by Picard to prove his “little theorem” that an entire function on the complex plane which omits more than one value is constant.[2]

[2] We may assume that the function misses the values 0 and 1. Composing it with the $\lambda$ function and the isomorphism $\mathscr{H} \to B(0,1)$ given by $z \mapsto \frac{1+iz}{z+i}$, we obtain a bounded entire function which is therefore constant.

Going back to the period of $E$, we can interpret our constructions in term of the *Hodge decomposition*

$$\begin{array}{ccccccc} H^1(E,\mathbf{C}) = & H^1(E,\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{C} & = & H^{0,1}(E) & \oplus & H^{1,0}(E). \\ & \| & & \| & & \| \\ & \langle\delta,\gamma\rangle^* & & \mathbf{C}\overline{\omega} & & \mathbf{C}\omega \end{array}$$

We have then $\omega = \delta^* + \tau\gamma^*$. The period $\tau$ therefore describes where the line $H^{1,0}(E)$ sits in the 2-dimensional vector space $H^1(E,\mathbf{C})$.

## 2. Quartic surfaces

A quartic surface $X \subset \mathbf{P}^3$ is defined as the set of zeroes of a homogeneous polynomial of degree 4 in 4 variables (this is a particular case of a *K3 surface*). We ask that it be smooth (which is equivalent to saying that the partial derivatives of this polynomial have no common zeroes). This is the case for example for the Fermat quartic, defined by the equation $x_0^4 + \cdots + x_3^4 = 0$.

One can show that the vector space of holomorphic differential 2-forms on $X$ is 1-dimensional. Let $\omega$ be a generator. As in the case of elliptic curves, we choose a basis $(\gamma_1,\ldots,\gamma_{22})$ for the free abelian group $H_2(X,\mathbf{Z})$ and we want to consider the *periods* $\int_{\gamma_i}\omega$. We will present this in a slightly more conceptual fashion by considering as above the Hodge decomposition

$$\begin{array}{ccccccccc} H^2(X,\mathbf{C}) = & H^2(X,\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{C} & = & H^{0,2}(X) & \oplus & H^{1,1}(X) & \oplus & H^{2,0}(X) \\ & \| & & \| & & & & \| \\ & \langle\gamma_1,\ldots,\gamma_{22}\rangle^* & & \mathbf{C}\overline{\omega} & & & & \mathbf{C}\omega \end{array}$$

into complex vector subspaces. We will view the line $H^{2,0}(X) = \mathbf{C}\omega \subset H^2(X,\mathbf{C})$ as a point $[\omega]$ in the complex 21-dimensional projective space $\mathbf{P}(H^2(X,\mathbf{C}))$ and call it the *period point of $X$*. Note that it determines the whole Hodge decomposition, because $H^{1,1}(X)$ is the orthogonal, for the *intersection form $Q$* on $H^2(X,\mathbf{C})$ (given by cup-product or, at the level of differential forms, by integration of the wedge product), of $\mathbf{C}\overline{\omega}\oplus\mathbf{C}\omega$.

There are constraints on the point $[\omega]$. First, $[\omega]$ lies in the orthogonal $H^2(X,\mathbf{C})_{\rm prim}$ (called the primitive cohomology) of the class of a plane section $C$ of $X$ because

$$[\omega]\cdot[C] = \int_C \omega = 0.$$

Moreover,

$$[\omega]\cdot[\omega] = \int_X \omega\wedge\omega = 0 \quad , \quad [\omega]\cdot[\overline{\omega}] = \int_X \omega\wedge\overline{\omega} > 0.$$

The intersection form $Q$ is a non-degenerate quadratic form of signature $(3,19)$. The lattice $(H^2(X,\mathbf{Z}),Q)$ is the unique even modular lattice with this signature, denoted by $II_{3,19}$. The lattice $(H^2(X,\mathbf{Z})_{\rm prim},Q)$ has signature $(2,19)$ and is isomorphic to[3]

$$\Lambda_{\rm quartic} := E_8(-1)\oplus E_8(-1)\oplus U\oplus U\oplus(-4).$$

*Once an isometry between* $H^2(X,\mathbf{Z})_{\rm prim}$ *and* $\Lambda_{\rm quartic}$ *is chosen,* we may view the period $[\omega]$ as a point in

$$\mathscr{Q}_{\rm quartic} := \{[\alpha]\in \mathbf{P}(\Lambda_{\rm quartic}\otimes\mathbf{C}) \mid Q([\alpha])=0\ ,\ Q([\alpha+\overline{\alpha}])>0\}.$$

This space has two connected components, which are both 19-dimensional and isomorphic to $\mathrm{SO}(2,19)^0/\,\mathrm{SO}(2)\times\mathrm{SO}(2,19)$, a homogeneous symmetric domain of type IV in Cartan's classification.

There is again some ambiguity in the definition of this point coming from the choice of the isometry and the conclusion is that, given a family $(X_s)_{s\in S}$ of smooth quartic surfaces parametrized by a complex variety $S$, this construction defines a single-valued holomorphic *period map*

$$S\longrightarrow \mathscr{D}_{\rm quartic} := \widetilde{O}(\Lambda_{\rm quartic})\backslash\mathscr{Q}_{\rm quartic},$$

where $\widetilde{O}(\Lambda_{\rm quartic})\subset O(\Lambda_{\rm quartic})$ is a subgroup of finite index.[4] The action of $\widetilde{O}(\Lambda_{\rm quartic})$ on $\mathscr{Q}_{\rm quartic}$ is nice (it is properly discontinuous; moreover, a subgroup of $\widetilde{O}(\Lambda_{\rm quartic})$ of finite index acts freely) and one can put a structure of normal quasi-projective algebraic variety on the quotient $\mathscr{D}_{\rm quartic}$. For this structure, *the period map is algebraic.*

One can take as parameter space for all smooth quartic surfaces a Zariski open subset $V_{35}^0$ of the 35-dimensional complex vector space $V_{35}$ parametrizing homogeneous polynomials of degree 4 in 4 variables. Even better, one may want to use the 19-dimensional quotient $V_{35}^0/\,\mathrm{GL}_4$: using Geometric Invariant Theory (GIT), there is a way to put a structure of affine algebraic variety on $V_{35}^0/\,\mathrm{GL}_4$, so all in all, we obtain an algebraic period map

$$p_{\rm quartic}: V_{35}^0/\,\mathrm{GL}_4\longrightarrow \mathscr{D}_{\rm quartic}$$

between 19-dimensional quasi-projective varieties.

[3]The lattice $U$ is the hyperbolic plane, with matrix $\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

[4]This is the subgroup of automorphisms of $\Lambda_{\rm quartic}$ which are restrictions of automorphisms of the larger lattice $II_{3,19}$ acting as the identity on $\Lambda_{\rm quartic}^{\perp}$.

**Theorem 1** (Piatetski-Shapiro, Shafarevich, Shah, Kulikov). *The map $p_{\text{quartic}}$ is an open immersion whose image can be explicitly described: it is the complement of a prime Heegner divisor.*[5]

One should view the quotient $V_{35}^0/\operatorname{GL}_4$ as *the moduli space* for smooth quartic surfaces (its points are in one-to-one correspondence with isomorphism classes of such surfaces) and the theorem as an explicit description and uniformization of this moduli space. The description of the image of $p_{\text{quartic}}$ can be used very efficiently to produce such surfaces containing certain configurations of curves or with certain automorphisms.

## 3. Cubic fourfolds

A cubic fourfold $Y \subset \mathbf{P}^5$ is defined as the set of zeroes of a homogeneous polynomial of degree 3 in 6 variables. We ask that it be smooth (which is equivalent to saying that the partial derivatives of this polynomial have no common zeroes). This is the case for example for the Fermat cubic, defined by the equation $x_0^3 + \cdots + x_5^3 = 0$. There are no non-zero holomorphic differential forms on $Y$, but, following Griffiths, we will still construct a period map using the Hodge decomposition

$$\begin{array}{ccccccc} H^4(Y,\mathbf{C}) & = & H^{1,3}(Y) & \oplus & H^{2,2}(Y) & \oplus & H^{3,1}(Y) \\ \text{dimensions:} & & 1 & & 21 & & 1 \end{array}$$

or rather, again, its primitive part

$$\begin{array}{ccccccc} H^4(Y,\mathbf{C})_{\text{prim}} & = & H^{1,3}(Y) & \oplus & H^{2,2}(Y)_{\text{prim}} & \oplus & H^{3,1}(Y) \\ \text{dimensions:} & & 1 & & 20 & & 1 \end{array}$$

(again, "primitive" means "orthogonal to the class of a linear section"). Indeed, we may view the line $H^{3,1}(Y) \subset H^4(Y,\mathbf{C})_{\text{prim}}$ as a point in the 21-dimensional projective space $\mathbf{P}(H^4(Y,\mathbf{C})_{\text{prim}})$, called the *period point* of $Y$.

The lattice $H^4(Y,\mathbf{Z})_{\text{prim}}$ endowed with the intersection form has signature $(20,2)$ and is isomorphic to[6]

$$\Lambda_{\text{cubic}} := E_8 \oplus E_8 \oplus U \oplus U \oplus A_2.$$

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[5]In modular form theory, a Heegner divisor is defined as the image $\mathscr{D}_m$ in a quotient $\Gamma\backslash\mathscr{D}$ as above of the $\Gamma$-invariant divisor

$$\sum_{x\in\mathscr{D},\, Q(x)=m} x^\perp,$$

where $m \in \mathbf{Z}$. In our case, the image is the complement of $\mathscr{D}_{-2}$.

[6]The lattice $A_2$ is the rank-2 lattice with matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Again, once an isometry between $H^4(Y,\mathbf{Z})_{\mathrm{prim}}$ and $\Lambda_{\mathrm{cubic}}$ is chosen, the period point lies in

$$\mathscr{Q}_{\mathrm{cubic}} := \{[\alpha] \in \mathbf{P}(\Lambda_{\mathrm{cubic}} \otimes \mathbf{C}) \mid Q([\alpha]) = 0\ ,\ Q([\alpha + \overline{\alpha}]) < 0\}$$

This variety again has two connected components, which are homogeneous symmetric domains of type IV. If we consider the affine 20-dimensional *moduli space* $V_{56}^0/\,\mathrm{GL}_6$ which parametrizes all smooth cubic fourfolds, this leads to an algebraic *period map*

$$p_{\mathrm{cubic}} : V_{56}^0/\,\mathrm{GL}_6 \longrightarrow \mathscr{D}_{\mathrm{cubic}} := \widetilde{O}(\Lambda_{\mathrm{cubic}})\backslash\mathscr{Q}_{\mathrm{cubic}}$$

between 20-dimensional quasi-projective varieties, where $\widetilde{O}(\Lambda_{\mathrm{cubic}}) \subset O(\Lambda_{\mathrm{cubic}})$ is a subgroup of finite index.

**Theorem 2** (Voisin, Looijenga, Laza)**.** *The map $p_{\mathrm{cubic}}$ is an open immersion whose image can be explicitly described: it is the complement of two prime Heegner divisors.*

The injectivity of $p_{\mathrm{cubic}}$ (Voisin) can be translated into the following period-map-free statement: *if $Y$ and $Y'$ are smooth cubic fourfolds, any isomorphism $H^4(Y,\mathbf{Z}) \to H^4(Y',\mathbf{Z})$ of polarized Hodge structures that maps the class of a linear section to the class of a linear section is induced by an isomorphism $Y' \xrightarrow{\sim} Y$.*

The image of $p_{\mathrm{cubic}}$ is determined by studying the extension of this map to the GIT compactification of $V_{56}^0/\,\mathrm{GL}_6$. It is not well-defined on this compactification: the point corresponding to the (singular) determinantal cubic $\begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} = 0$ needs to be blown up; it is then sent onto one of the Heegner divisors (the other Heegner divisor corresponds to nodal cubics).

## 4. Irreducible holomorphic symplectic varieties (IHS)

For us, an irreducible holomorphic symplectic variety is a smooth projective simply-connected variety $X$ such that the vector space of holomorphic differential 2-forms on $X$ is generated by a symplectic (i.e., non-degenerate at each point of $X$) 2-form $\omega$. The dimension of $X$ is necessarily even and when $X$ is a surface, it is simply a K3 surface. Examples were constructed by Beauville and O'Grady in every even dimension (but there are not many).

Contrary to what we did earlier, where we always looked at the middle cohomology, we will work on $H^2(X,\mathbf{C})$. Beauville constructed a nondegenerate quadratic form $Q_{BB}$ on this vector space. We let

$H^2(X, \mathbf{C})_{\mathrm{prim}}$ be the orthogonal of the hyperplane class $h_X$. One then has

$$\begin{array}{ccccccc} H^2(X, \mathbf{C})_{\mathrm{prim}} & = & H^{0,2}(X) & \oplus & H^{1,1}(X)_{\mathrm{prim}} & \oplus & H^{2,0}(X). \\ \text{dimensions:} & & 1 & & b_2 - 3 & & 1 \end{array}$$

The quadratic form $Q_{BB}$ has signature $(2, b_2 - 3)$ on $H^2(X, \mathbf{C})_{\mathrm{prim}}$. Let

$$\mathscr{Q} := \{[\alpha] \in \mathbf{P}(H^2(X, \mathbf{C})_{\mathrm{prim}}) \mid Q_{BB}([\alpha]) = 0 \ , \ Q_{BB}([\alpha + \overline{\alpha}]) > 0\}.$$

Verbitski recently proved that the period map

$$\mathscr{M} \longrightarrow \mathscr{D}$$

from the connected component $\mathscr{M}$ of the moduli space corresponding to smooth polarized deformations of $X$ to the quotient $\mathscr{D}$ of $\mathscr{Q}$ by a suitable arithmetic group $\Gamma$, is in some cases an isomorphism.[7]

**Example 3** (Beauville-Donagi)**.** Let $Y \subset \mathbf{P}^5$ be a smooth cubic fourfold. Let $L(Y) \subset G(1, \mathbf{P}^5)$ be the (smooth projective) variety that parametrizes lines contained in $Y$. It is an irreducible symplectic fourfold endowed with the polarization $h_{L(Y)}$ coming from the Plücker embedding of $G(1, \mathbf{P}^5)$ in $\mathbf{P}^{14}$.

Consider the incidence variety

$$I := \{(y, \ell) \in Y \times L(Y) \mid y \in \ell\}$$

with its projections $p_1 : I \to Y$ and $p_2 : I \to L(Y)$. Beauville & Donagi proved that the *Abel-Jacobi map*

$$p_{2*}p_1^* : (H^4(Y, \mathbf{Z})_{\mathrm{prim}}, Q) \to (H^2(L(Y), \mathbf{Z})_{\mathrm{prim}}, Q_{BB})$$

is an anti-isometry of polarized Hodge structures. In other words, the period maps are compatible: the following diagram is commutative

$$\begin{array}{ccc} \left\{\begin{matrix}\text{moduli space of} \\ \text{smooth cubic fourfolds}\end{matrix}\right\} & \xrightarrow{\quad L \quad} & \mathscr{M} \\ & \searrow^{p_{\mathrm{cubic}}} \qquad {}^{p_{\mathrm{IHS}}}\swarrow \wr & \\ & \mathscr{D}_{\mathrm{cubic}}, & \end{array}$$

where $\mathscr{M}$ is the moduli space of smooth IHS fourfolds which are deformations of $(L(Y), h_{L(Y)})$.

Since $L$ is injective, the fact that $p_{\mathrm{IHS}}$ is an open immersion on the family of all $L(Y)$ (Verbitsky) can be used (Charles) to reprove the Torelli theorem for cubics (Voisin) mentioned earlier (the fact that $p_{\mathrm{cubic}}$ is an open immersion).

[7]To be precise, one must allow in $\mathscr{M}$ all smooth deformations of the pair $(X, h_X)$, including those where $h_X$ may not be ample anymore.

## 5. Prime Fano fourfolds of degree 10 and index 2

Let $X$ be a smooth projective fourfold with $\mathrm{Pic}(X) \simeq \mathbf{Z}[H]$, where $H$ is ample, $H^4 = 10$, and $K_X \underset{\text{lin}}{\equiv} -2H$.

In most cases, $X$ is obtained as follows. Let $V_5$ be a complex 5-dimensional vector space. Then $X$ is isomorphic to the intersection of $G(2, V_5) \subset \mathbf{P}^9$ with a hyperplane and a quadric $q$ (Mukai).

Instead of the primitive cohomology, we consider here the vanishing cohomology $H^4(X, \mathbf{Z})_{\mathrm{van}}$, defined as the orthogonal in $H^4(X, \mathbf{Z})$ (for the intersection form $Q$) of $H^4(G(2, V_5), \mathbf{Z})$. The Hodge structure is again "of K3 type":

$$\begin{array}{lccccccc} H^4(X, \mathbf{C})_{\mathrm{van}} & = & H^{1,3}(X) & \oplus & H^{2,2}(X)_{\mathrm{van}} & \oplus & H^{3,1}(X) \\ \text{dimensions:} & & 1 & & 20 & & 1 \end{array}$$

The lattice $(H^4(X, \mathbf{Z})_{\mathrm{van}}, Q)$ is isomorphic to

$$\Lambda := E_8 \oplus E_8 \oplus U \oplus U \oplus (2) \oplus (2)$$

and we construct as before a 20-dimensional period domain $\mathscr{D} = \widetilde{O}(\Lambda) \backslash \mathscr{Q}$ and a period map

$$p : \mathscr{N} \to \mathscr{D},$$

where $\mathscr{N}$ is the 24-dimensional irreducible coarse moduli space for the smooth deformations of the pair $(X, h_X)$.

**Theorem 4.** *The map $p$ is dominant with 4-dimensional fibers.*

The situation here is therefore different: the period point will not be enough to characterize $X$.

As in the case of cubic fourfolds, one can associate with any smooth $X$ as above an IHS fourfold as follows. The vector space $I_X(2)$ of quadrics in $\mathbf{P}^8$ containing $X$ splits as

$$I_X(2) = I_{G(2,V_5)} \oplus \mathbf{C}q,$$

where $I_{G(2,V_5)}$ is the space of (restrictions to $\mathbf{P}^8$ of) Plücker quadrics $P_v : \omega \mapsto \omega \wedge \omega \wedge v$, for $v \in V_5$. One checks that $X$ is always contained in a smooth quadric, hence the scheme of singular quadrics containing $X$ is a degree-9 hypersurface in $|I_X(2)| = \mathbf{P}^5$. Since restrictions of Plücker quadrics have rank 6, it splits as

$$3|I_{G(2,V_5)}(2)| + Y_X,$$

where $Y_X$ is a sextic which can be shown to be integral but singular along a surface. This type of sextics was studied by Eisenbud, Popescu, and Walter; they are called EPW sextics.

O'Grady showed that there is a canonical double covering

$$\widetilde{Y}_X \twoheadrightarrow Y_X \subset \mathbf{P}^5,$$

where $\widetilde{Y}_X$ is a (smooth) IHS fourfold called a *double EPW sextic.*

As in the case of cubics, one can show that we have an isomorphism of polarized Hodge structures

$$H^4(X, \mathbf{Z})_{\mathrm{van}} \xrightarrow{\sim} H^2(\widetilde{Y}_X, \mathbf{Z})_{\mathrm{prim}},$$

hence a commutative diagram

$$\begin{array}{ccc} \mathscr{N} & \xrightarrow{\ \varphi\ } & \mathscr{M}_{\mathrm{EPW}} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p_{\mathrm{IHS}}}\,\wr & \\ & \mathscr{D}, & \end{array}$$

where $\mathscr{M}_{\mathrm{EPW}}$ is the (quasi-projective) moduli space of double EPW sextics.

**Theorem 5.** *The map $\varphi$ is "almost" isomorphic to the universal dual EPW sextic.*

To be more precise, the fiber of any point $[Y]$ outside of a divisor in $\mathscr{M}_{\mathrm{EPW}}$ is isomorphic to the projective dual $Y^\vee \subset (\mathbf{P}^5)^\vee$ (which is itself an EPW sextic!). Over this divisor, the fiber is obtained from $Y^\vee$ by removing the (finite) singular set of the surface $\mathrm{Sing}(Y^\vee)$.

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