ON PRIME FANO VARIETIES OF DEGREE 10 AND COINDEX 3

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Abstract. We discuss the period map of certain (complex) Fano varieties. The first part of this talk deals with cubic hypersurfaces. We recall their Hodge structures and certain classical results of Clemens-Griffiths, Carlson-Griffiths, Voisin, and Donagi, and more recent results of Allcock-Carlson-Toledo and Laza, about their period maps. We also mention the Beauville-Donagi construction of an associated irreducible symplectic fourfold.

In the second part, we discuss Fano varieties of degree 10, dimension n, and coindex 3 (index $n-2$). According to Gushel and Mukai, most of them are obtained as linear sections of the intersection with a quadric of the Grassmannian $G(2,5)$ in its Plücker embedding. In particular, n can only be 3, 4, or 5, and they depend on 22, 24, and 25 parameters respectively. Their period maps have a more complex behavior, because the Torelli property does not hold. We also mention a construction of Iliev-Manivel of an associated irreducible symplectic fourfold called a double EPW-sextic and studied by O'Grady. This is joint work in progress with A. Iliev and L. Manivel.

1. Periods of cubic hypersurfaces

1.1. **Hodge structure.** Let $X \subset \mathbf{P}^{n+1}$ be a smooth complex cubic hypersurface and let $h \in H^2(X, \mathbf{Z})$ be the class of a hyperplane section. We define the primitive (middle) cohomology groups as

$$H_0^{p,n-p}(X) := \{\eta \in H^{p,n-p}(X) \mid \eta \cdot h = 0\}$$

(the other cohomology is not interesting). This defines a Hodge structure on $H_0^n(X) := \{\eta \in H^n(X) \mid \eta \cdot h = 0\}$ which is polarized by the (unimodular) intersection form.

Deligne (and Hirzebruch) proved the following generating formula for the dimensions of these spaces

$$\sum_{p\geq 0,\ q\geq 0} h_0^{p,q} y^p z^q = \frac{2+y+z}{1-3yz-y^2z-yz^2}.$$

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So for small n, we get the following numbers

n	$h_0^{0,n}$	$h_0^{1,n-1}$	$h_0^{2,n-2}$	$h_0^{3,n-3}$	$h_0^{4,n-4}$	$h_0^{5,n-5}$	$h_0^{6,n-6}$	$h_0^{7,n-7}$
2	0	6	0					
3	0	5	5	0				
4	0	1	20	1	0			
5	0	0	21	21	0	0		
6	0	0	8	70	8	0	0	
7	0	0	1	84	84	1	0	0

Note that $h_0^{p,n-p}$ is non-zero if and only if $\left|p-\frac{n}{2}\right| \le \frac{1}{3}\left(\frac{n}{2}+1\right)$.

1.2. **Period domains and period maps.** There is a *period domain* $\mathscr{D}_n$ which classifies polarized Hodge structures of the type corresponding to smooth cubic hypersurfaces of dimension n. It is a complex homogeneous space.

- For $n=2$, the period domain is just a point.
- For $n=3$ (resp. $n=5$), the period domain is the Siegel upper half space $\mathscr{H}_5$ (resp. $\mathscr{H}_{21}$).
- For $n=4$, the period domain is Hermitian symmetric:
$$\mathscr{D}_4 = \{\omega \in \mathbf{P}^{21} \mid Q(\omega)=0, Q(\omega+\bar{\omega})<0\},$$
where Q is a non-degenerate quadratic form of real signature $(20,2)$.
- For $n \ge 6$, the period domain is not Hermitian symmetric anymore.

Let $\mathscr{C}_n$ be the (quasi-projective) moduli space for smooth cubic hypersurfaces of dimension n. It is quasi-projective of dimension $\binom{n+4}{3} - (n+2)^2$. There is a holomorphic *period map*

$$p_n : \mathscr{C}_n \to \mathscr{D}_n/\Gamma_n,$$

where Γ_n is a group that acts properly discontinuously on $\mathscr{D}_n$, so that the quotient $\mathscr{D}_n/\Gamma_n$ is a complex space. Here are the dimensions

n	2	3	4	5
$\dim(\mathscr{C}_n)$	4	10	20	84
$\dim(\mathscr{D}_n)$	0	15	20	231

The injectivity (or lack of) of the period map is usually called the Torelli problem for the corresponding class of varieties.

Theorem 1.1. *The period map is injective for $n = 3$ (Clemens-Griffiths) or $n = 4$ (Voisin). For $n \geq 5$, it is generically injective for $n \equiv 0 \pmod 3$ (Carlson-Griffiths) or $n \equiv 2 \pmod 3$ (Donagi).*

Remark 1.2 ($n = 2$)**.** The period map for cubic surfaces is constant. However, with such a (smooth) surface $X \subset \mathbf{P}^3$ with equation $f(x_0, \dots, x_3) = 0$, one can associate the (smooth) cubic threefold $Y \subset \mathbf{P}^4$ with equation $f(x_0, \dots, x_3) + x_4^3 = 0$. This defines a map $\mathscr{C}_2 \to \mathscr{C}_3$, hence we get a period map $\mathscr{C}_2 \to \mathscr{D}_3/\Gamma_3$. Moreover, because of the presence of an automorphism of order 3 of Y which acts on its Hodge structure, one can, following Allcock-Carlson-Toledo, restrict the period domain and construct a period map

$$\mathscr{C}_2 \to \mathbf{B}_4/\Gamma$$

which induces *an isomorphism onto a Zariski open subset of the quasi-projective variety* $\mathbf{B}_4/\Gamma$. Moreover, the period map extends and induces an isomorphism between the GIT compactification of $\mathscr{C}_2$ and the Baily-Borel compactification of $\mathbf{B}_4/\Gamma$ (Allcock-Carlson-Toledo).

Remark 1.3 ($n = 3$)**.** The injective period map p_3 is certainly not surjective, but the same construction as above yields a different period map

$$\mathscr{C}_3 \to \mathbf{B}_{10}/\Gamma$$

which Allcock-Carlson-Toledo show also induces an isomorphism onto a Zariski open subset of the quasi-projective variety $\mathbf{B}_{10}/\Gamma$. Again, the period map extends and induces a morphism from a suitable compactification of $\mathscr{C}_3$ (which is the blow-up at one point of the GIT compactification) and the Baily-Borel compactification of $\mathbf{B}_{10}/\Gamma$. This morphism is an isomorphism except over a single point, whose preimage is a rational curve (Allcock-Carlson-Toledo).

Remark 1.4 ($n = 4$)**.** The complement of the image of the period map $p_4 : \mathscr{C}_4 \to \mathscr{D}_4/\Gamma_4$ in the Baily-Borel compactification of $\mathscr{D}_4/\Gamma_4$ is known by recent work of Laza: it is the union of two irreducible hypersurfaces. Laza also extends the period map to the GIT compactification of $\mathscr{C}_4$ and proves that it induces an isomorphism between this compactification and an explicit birational modification of the Baily-Borel compactification of $\mathscr{D}_4/\Gamma_4$.

1.3. **The associated irreducible symplectic fourfold.** Let $X \subset \mathbf{P}^5$ be a smooth cubic fourfold and let $F(X) \subset G(1, \mathbf{P}^5)$ be the (smooth projective) variety that parametrizes lines contained in X. It is an irreducible symplectic fourfold. As X describes $\mathscr{C}_4$, the family of $F(X)$ is a locally complete family of deformations.

Beauville-Donagi proved that the primitive cohomology $H^4_0(X,\mathbf{Z})$ is isomorphic to the primitive cohomology $H^4_0(F(X),\mathbf{Z})$. More precisely, consider the incidence variety

$$I := \{(x,\ell)\in X\times F(X)\mid x\in\ell\}$$

with its projections $p : I\to X$ and $q : I\to F(X)$. Then the *Abel-Jacobi map*

$$a := q_*p^* : H^4_0(X,\mathbf{Z})\to H^2_0(F(X),\mathbf{Z})$$

is an isomorphism of polarized Hodge structures, where $H^2_0(F(X),\mathbf{Z})$ *is endowed with the Beauville-Bogomolov quadratic form.* The injectivity of the period map for cubic fourfolds therefore implies the injectivity of the period map for deformations of $F(X)$

The Torelli problem for these varieties was recently solved in general by Verbitsky. We will come back to that.

2. Prime Fano varieties of degree 10 and coindex 3

2.1. **Construction.** A *Fano variety* is a smooth projective variety X whose canonical bundle is anti-ample. It is *prime* if the Picard group of X is isomorphic to $\mathbf{Z}$; if H is a generator and $n := \dim(X)$, the *degree* of X is the positive number H^n and its *coindex* is the number r such that $-K_X \underset{\text{lin}}{\equiv} (n+1-r)H$. So for example, cubic hypersurfaces of dimension ≥ 2 are Fano varieties of degree 3 and coindex 2.

According to Gushel and Mukai, any prime Fano n-fold X of coindex 3 and degree 10 is obtained as follows. Consider the Grassmannian $G := G(2,V_5)\subset\mathbf{P}(\wedge^2 V_5)=\mathbf{P}^9$ in its Plücker embedding. Then $n\in\{3,4,5\}$ and X is

- either a smooth linear section by a $\mathbf{P}^{n+4}$ of the intersection of $G\subset\mathbf{P}^9$ with a quadric;
- or a double cover of a smooth linear section by a $\mathbf{P}^{n+3}$ of $G\subset\mathbf{P}^9$, branched along its intersection with a quadric.

Note that the second case (the " Gushel" case) is a degeneration of the first case: consider, in $\mathbf{P}^{10}$, the 7-dimensional cone CG over $G\subset\mathbf{P}^9$, with vertex v, and its intersection with a general quadric and a $\mathbf{P}^{n+4}$. If the $\mathbf{P}^{n+4}$ does not contain v, everything takes place in that linear space and we are in the first case; if the $\mathbf{P}^{n+4}$ does contain v, we are in the second case.

We let $\mathscr{X}_n$ be the corresponding moduli space. Points corresponding to Gushel n-folds form a subfamily which is birational to $\mathscr{X}_{n-1}$. When $n=2$, the same construction gives K3 surfaces of degree 10, and when $n=1$, we get canonical curves of genus 6.

2.2. **Hodge structure, period domains, and period maps.** Let us look at the middle primitive Hodge numbers:

n	$h_0^{0,n}$	$h_0^{1,n-1}$	$h_0^{2,n-2}$	$h_0^{3,n-3}$	$h_0^{4,n-4}$	$h_0^{5,n-5}$
1	6	6				
2	1	19	1			
3	0	10	10	0		
4	0	1	20	1	0	
5	0	0	10	10	0	0

In all these cases, the period domain $\mathscr{D}_n$ is a Hermitian symmetric domain, so we get algebraic period maps

$$p_n : \mathscr{X}_n \to \mathscr{D}_n/\Gamma_n.$$

Here are the dimensions:

n	1	2	3	4	5
$\dim(\mathscr{X}_n)$	15	19	22	24	25
$\dim(\mathscr{D}_n)$	21	19	55	20	55

The map p_1 is of course injective (this is the Torelli theorem for curves). Similarly, p_2 induces an isomorphism between $\mathscr{X}_2$ and an explicit Zariski open subset of $\mathscr{D}_2/\Gamma_2$ (by the Torelli theorem for K3 surfaces). For $n = 3$ and $n = 5$, the period maps are however *not injective.*

Theorem 2.1. *The fibers of the period map p_3 are disjoint unions of pairs of smooth connected projective surfaces. Any two threefolds in the same pair are birationally isomorphic one to another.*

The period map p_4 is dominant.

The fibers of p_5 are 5-dimensional and the closure of the images of p_3 and p_5 are the same.

The dimensions of the fibers are obtained by computing the differential of the period map.

We conjecture that

- the general fibers of p_3 are just a pair of connected surfaces;
- the fibers of p_5 are rational.

2.3. **The associated irreducible symplectic fourfold.** Let X be a fourfold of the type studied above. Just as in the case of cubic fourfolds, there is an associated irreducible symplectic manifold (Iliev-Manivel).

Assume for simplicity that we are in the non-Gushel case: X is the (smooth) intersection, in $\mathbf{P}^9$, of $G := G(2, V_5)$, a hyperplane $\mathbf{P}^8$, and a quadric Q.

2.3.1. *The associated sextic.* The vector space $I_G(2)$ of quadrics containing G is isomorphic to V_5 (Plücker quadrics). Therefore, the vector space $I_X(2) \simeq I_G(2) \oplus \mathbf{C}Q$ has dimension 6.

Let $V_4 \subset V_5$ be a hyperplane. All Plücker quadrics restrict in $\mathbf{P}(\wedge^2 V_4)$ to the sole (smooth) Plücker quadric defining $G(2, V_4) \subset \mathbf{P}(\wedge^2 V_4)$, hence elements of $I_X(2)$ restrict to a pencil. A general element of that pencil is smooth, and a finite number of elements are singular. These elements define a finite number of hyperplanes in $I_X(2)$, hence points in $\mathbf{P}(I_X(2)^\vee)$. When V_4 varies, we get a subvariety

$$Z_X \subset \mathbf{P}(I_X(2)^\vee) = \mathbf{P}^5.$$

Theorem 2.2 (Iliev-Manivel)**.** *For X general in $\mathscr{X}_4$, the subvariety $Z_X \subset \mathbf{P}^5$ is an EPW sextic hypersurface.*

2.3.2. *Double EPW sextics.* What is an EPW sextic? Let U_6 be a 6-dimensional vector space. If we choose an isomorphism $\wedge^6 U_6 \simeq \mathbf{C}$, the vector space $\wedge^3 U_6$ inherits a non-degenerate skew-symmetric form given by wedge product. Let $A \subset \wedge^3 U_6$ be a general Lagrangian subspace. Then

$$Z_A := \{U_5 \subset U_6 \mid \wedge^3 U_5 \cap A \neq 0\} \subset \mathbf{P}(U_6^\vee)$$

is an *EPW sextic.* Its singular locus is the smooth surface

$$S_A := \{U_5 \subset U_6 \mid \dim(\wedge^3 U_5 \cap A) \geq 2\}.$$

There is a canonically defined double cover (O'Grady)

$$\pi : Y_A \to Z_A,$$

such that

- the branch locus of π is the surface S_A;
- the fourfold Y_A is a (smooth) irreducible symplectic fourfold.

2.3.3. *A conjecture.* Iliev and Manivel proved that the variety $C(X)$ of (possibly degenerate) conics contained in a general X in $\mathscr{X}_4$ is a smooth projective fivefold.

A general conic $c \subset X$ is contained in a unique $G(2, V_4)$. As explained above, elements of $I_X(2)$ restrict to a pencil of quadrics which all contain c, but not all the 2-plane $\langle c \rangle$ that it spans (because $\langle c \rangle \not\subset X$). A single element of that pencil contains $\langle c \rangle$. It is therefore singular, and it defines a hyperplane in $I_X(2)$, hence a morphism $C(X) \to Z_X$ whose Stein factorisation is

$$C(X) \xrightarrow{\beta} Y_X \xrightarrow{\pi} Z_X,$$

where π is the double cover mentioned above and β is generically a $\mathbf{P}^1$-bundle. As in the case of cubic fourfolds, we may define an Abel-Jacobi map

$$a : H^4(X, \mathbf{Z}) \to H^2(C(X), \mathbf{Z})$$

which is a morphism of Hodge structures.

Conjecture 2.3. On the primitive cohomology, the Abel-Jacobi map factors as

$$a : H^4_0(X, \mathbf{Z}) \xrightarrow{u} H^2_0(Y_X, \mathbf{Z}) \overset{\beta^*}{\hookrightarrow} H^2(C(X), \mathbf{Z}),$$

where u is an isomorphism of Hodge structures.

We have checked that the lattices $H^4_0(X, \mathbf{Z})$ and $H^2_0(Y_X, \mathbf{Z})$ are isomorphic.

2.4. **Periods of EPW sextics.** Let me rephrase the conjecture. O'Grady proved that the moduli space of EPW sextics is quasi-projective and has dimension 20. By the last theorem, there is a rational map

$$\begin{array}{rcl} \mathrm{epw}_4 : \mathscr{X}_4 & \dashrightarrow & \mathscr{E}\mathscr{P}\mathscr{W} \\ {[X]} & \longmapsto & Z_X . \end{array}$$

Iliev and Manivel proved that it is dominant. Its general fibers therefore have dimension 4.

On the other hand, there is also a global period map

$$p : \mathscr{E}\mathscr{P}\mathscr{W} \to \mathscr{D}_4/\Gamma_4$$

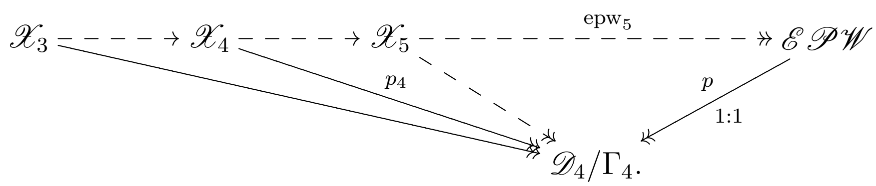
with values in the same period domain as for our fourfolds, which takes an EPW sextic to the period of its canonical double cover. By work of Verbitsky on the Torelli problem for irreducible symplectic manifolds, *p is dominant and birational.*

The conjecture above says that the following diagram is commutative

$$\begin{array}{ccc} \mathscr{X}_4 & \overset{\mathrm{epw}}{\dashrightarrow\!\!\!\!\rightarrow} & \mathscr{E}\mathscr{P}\mathscr{W} \\ & {\scriptstyle p_4}\searrow \quad \swarrow{\scriptstyle p}\ {\scriptstyle 1:1} & \\ & \mathscr{D}_4/\Gamma_4 . & \end{array}$$

The general (4-dimensional) fibers of p_4 and epw are then the same (birationally).

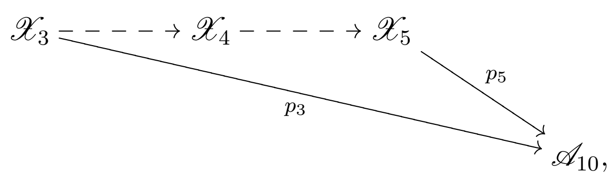
2.5. **The Gushel construction.** The Gushel construction induces morphisms $\mathscr{X}_3 \dashrightarrow \mathscr{X}_4$ and $\mathscr{X}_4 \dashrightarrow \mathscr{X}_5$ (one should view this as the analog of the morphisms $\mathscr{C}_n \to \mathscr{C}_{n+1}$ explained earlier) and Iliev-Manivel have constructed compatible dominant maps $\mathrm{epw}_n : \mathscr{X}_n \dashrightarrow \mathscr{E}\mathscr{P}\mathscr{W}$ so that we get a commutative diagram (assuming the conjecture above)



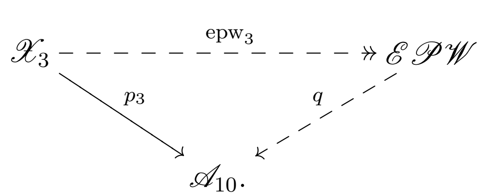
We can prove that the general fibers of epw_3 are smooth connected surfaces.[1] We conjecture that the fibers of epw_4 are EPW sextics, and that the fibers of epw_5 are $\mathbf{P}^5$.

Also, it would be interesting to describe (as did Allcock-Carlson-Toledo and Laza for cubics) the precise images of these various maps in $\mathscr{D}_4/\Gamma_4$. One problem is that constructing compactifications of $\mathscr{X}_n$ seems much more difficult.

On the other hand, we also have period maps p_3 and p_5, and we can prove that the following diagram commutes:

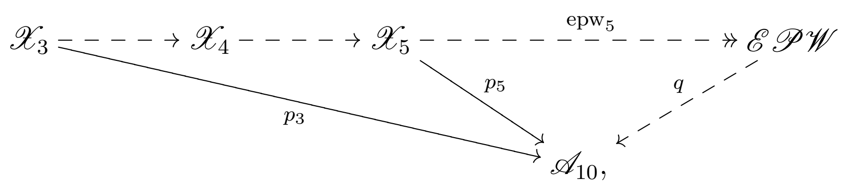


In particular, since $p_3(\mathscr{X}_3)$ and $p_5(\mathscr{X}_5)$ both have dimension 20 (this is obtained by computing the ranks of the tangent maps), their closures are equal. Moreover, we can prove that the general fibers of epw_3 are contracted by p_3, hence we obtain a morphism $q : \mathscr{E}\mathscr{P}\mathscr{W} \dashrightarrow \mathscr{A}_{10}$ and another commutative diagram:



[1]This, together with the smoothness of p_4, should imply that the general fibers of epw_4 are also connected fourfolds.

We conjecture $q \circ \mathrm{epw}_5 = p_5$, so that the two diagrams would fit into a single commutative diagram:



which is commutative except that we do not know how to prove $q \circ \mathrm{epw}_5 = p_5$ (however, $q \circ \mathrm{epw}_3 = p_3$). We expect q to have degree 2 onto its image (it factors through the map that takes an EPW sextic to its projective dual).

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