CENTRE
DE RECHERCHES
MATHÉMATIQUES

Programme thématique

« Points rationnels, courbes rationnelles et courbes entières sur les variétés algébriques »

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Moduli spaces for some Fano manifolds

1. Moduli spaces of cubics using period maps.

Smooth cubics in $\mathbb{P}^{n+1}$ have a GIT affine moduli space $\mathcal{C}_n$ (singular cubics are parametrized by a $SL(n+1)$-invariant divisor) with comp. $\overline{\mathcal{C}_n}$

These moduli spaces may be studied via the period map. (in low dimensions).

Best example is $n=4$

Hodge structure is of K3 type

$$H^4(X,\mathbb{C})_{prim} = \underset{(1)}{H^{1,3}} \oplus \underset{(20)}{H^{3,2}} \oplus \underset{(1)}{H^{3,1}}$$

This Hodge structure is determined by the line $H^{1,3}$.

Period domain

$$Q = \{ [\omega] \in \mathbb{P}^{21} \mid \omega \cdot \omega = 0, \ \omega \cdot \bar{\omega} > 0 \}$$

(signature (20, 2))

bounded symmetric domain of type IV dim (20)

Period map

$$p : \underset{(20)}{\mathcal{C}_4} \longrightarrow \underset{(20)}{\mathcal{D}} = \Gamma \backslash Q$$

Γ ↳ subgroup of finite index of $O(\Lambda)$

Λ lattice $H^4(X,\mathbb{Z})_{prim}$

$2U \oplus 2E_8 \oplus A_2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

- T_p injective (easy)
- p injective (Voisin 1986, hard; original proof had a gap filled using subsequent work of Laza)
- compactify p to study its image (Laza, 2007)

- Image is complement of divisors $\mathcal{D}_2$ and $\mathcal{D}_6$ ("special" cubic fourfolds (Hassett) of discriminant 2 and 6; i.e. $H^4(X,\mathbb{Z}) \cap H^{2,2}(X)$ contains rank-2 lattices of disc. 2 and 6)

"C" due to Hassett

Heegner divisors

This identifies $\mathcal{C}_4$ with explicit complement of 2 hypersurfaces in $\overline{\mathcal{D}}$ (Baily-Borel compactification, obtained by adding 6 in components of dim ≤ 3)

- To extend p, one needs to blow-up in $\overline{\mathcal{C}_4}$ a particular point (the "determinantal cubic" $\begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} = 0$)

big auto. group $SL(3)$

unique semi-stable orbit corresponding to cubics with 2 dim'l sing locus) then a curve

- Nodal cubics go to $\mathcal{D}_6$
- Blown-up det'l cubic go onto $\mathcal{D}_2$ via K3 surfaces of degree 2

- Cubics of other dimensions

<u>$n=3$</u> Hodge structure of $H^3(X)$ has weight one

$\rightsquigarrow$ 5 dim'l pp intermediate Jacobian

This does give an injective map

$$\mathcal{C}_3 \hookrightarrow \mathcal{A}_5$$

but image has large codimension.

Associate with X a cubic fourfold $X' \xrightarrow[\mathbb{Z}/3]{} \mathbb{P}^4 \supset X$ with order 3 automorphism

$$\begin{array}{ccc} \mathcal{C}_3 & \overset{p}{\hookrightarrow} & \Gamma \backslash \mathbb{B}^{10} \\ \uparrow & & \cap \\ \widetilde{\mathcal{C}}_3 & \simeq & \overline{\Gamma \backslash \mathbb{B}^{10}}^{BB} \end{array}$$

blow-up of one point (on previous curve blown-up in $\overline{\mathcal{C}_4}$)

Add to $n=4$

F(X) is an irreducible hol. sympl. (IHS) variety with "same period"

$$\begin{array}{ccc} \mathcal{C}_4 & \longrightarrow & \mathcal{M} \\ & \searrow \quad \swarrow & \\ & \mathcal{D} & \end{array}$$

$\mathcal{M}$ ← moduli space of these IHS (with a degree 6 polarization)

Verbitsky

$\cup$ contains $S^{[2]}$ (form a divisor)

whose image in $\mathcal{D}$ is $\mathcal{D}_2$ S K3 of degree 2

These IHS "correspond" to det'l cubic, they are not $S^{[2]}$ varieties of lines on a cubic fourfold

[F(X) can be $S^{[2]}$ for infinitely many polarized S ; see BD or Hassett]

$n=2$ no interesting Hodge structure
⇝ similar construction of a 3-fold.

Associated

2. Quadratic line complexes

Varieties $G(2,n) \cap Q$ dim $2n-5$
Fano of index $n-2$ $(n>3)$

Classically $n=4$

Here $n=5$ → index 3

- Hodge structure has level 1
⇝ intermediate Jacobian, dim (10)
too large!

We use the same trick and consider
$Z \xrightarrow{2:1} G(2,5)$ branched along X
(Fano of dim 6, index 4, $(\rho=1)$
→ all are like that)

Hodge structure of K3 type on $H^6(Z)$

$H^6(Z)_{van} = H^{2,4} \oplus H^{3,3} \oplus H^{4,2}$
(1) (20) (1)

(but different lattice $2U \oplus 2E_8(-1) \oplus 2A_1(-1)$

Moduli space: affine GIT quotient

$p: \mathcal{Z} \longrightarrow \mathcal{D} := \Gamma \backslash \Omega$
(25) (20)

T_p always surjective ($\Rightarrow p$ dominant)

We would like to use this map to describe $\mathcal{Z}$

- As for cubics, one can associate an IHS (Iliev-Manivel). This is a bit more complicated

$I_G(2)$ quadrics containing G, all of rank 6

$I_X(2) = I_G(2) \oplus \mathbb{C}Q$

(if X smooth, we may take Q smooth)

Discriminant $= 4|I_G(2)| + Y_X$

$Y_X \subset \mathbb{P}^5$ is a (possibly non-integral) sextic ⇝ EPW sextic.

Brief definition

$A_{10} \subset \Lambda^3 V_6$ Lagrangian

$Y_A = \{ v \in \mathbb{P}(V_6) \mid (\Lambda^2 V_6 \wedge v) \cap A \neq 0 \}$

is either all or a (singular sextic)

$\mathrm{Sing}(Y_A) = Y_A[2] \cup \bigcup_{[V_3] \in \Theta_A} \mathbb{P}(V_3)$

$\Theta_A = \{ V_3 \mid [\Lambda^3 V_3] \in A \}$

$\tilde{X}_A \xrightarrow{2:1} Y_A$ ↳ $\dim(\cdots) \geq 2$

↳ IHS for A general

There is a GIT moduli space for EPW sextics

$\overline{EPW} = LG(\Lambda^3 V_6) /\!\!/ SL(V_6)$ (O'Grady)

EPW (corr. to smooth double EPW) is the complement of 2 (ample) divisors

$\Sigma = \{ A \mid \Theta_A \neq \emptyset \}$

$\Delta = \{ A \mid Y_A[3] \neq \emptyset \}$

Theorem

The association $X \rightsquigarrow Y_X$ induces a morphism

$\Phi : \mathcal{Z} \longrightarrow \overline{EPW} - \Sigma$ (hence possibly finite singular locus)

affine

Fibers are $\mathbb{P}(V_6^\vee) - Y_A^\vee / \mathrm{Aut}$

↳ dual EPW $= \{ H \mid \Lambda^3 H \cap A \neq 0 \}$

On the other hand (by Verbitsky again),

$$EPW \xhookrightarrow{\text{period}} \mathcal{D} \quad \text{open embedding}$$

In fact same remains true for

$$\overline{EPW} - \Sigma \xhookrightarrow{p} \mathcal{D}$$

[Q) can we relate that to special 6-folds?]

Δ is sent onto divisor $\mathcal{D}'_{10}$

($X_A \dashrightarrow S^{[2]}_{\text{degree } 10}$ $\rightsquigarrow$ period of S)

<u>Corollary</u> $\mathcal{Z}$ is an affine bundle in $\mathbb{P}^5$ - EPW over the image of p

<u>Rmk</u> $p \circ \Phi$ should be the period map of $\mathcal{Z}$

<u>Q)</u> What is the image of p (or $p \circ \Phi$) ?

<u>Conj</u> (Debarre, O'Grady)

The image of p is the complement of

Known: it is contained

$\mathcal{D}_2$

$\mathcal{D}_4$

$\mathcal{D}_8$ is the image of nodal X or, equivalently, of Σ

($\mathcal{D}_2$ is the image of the orbit $A(U_3)$

$\mathcal{D}_4$ ——————— $A(U_4)$

These EPW sextics are 3x smooth quadric.

We will explain the corresponding X

(both are semi-stable non stable points, when X_t approaches X, the period should approach period of S quartic in $\mathbb{P}^3$)

D_2 These EPW sextics are $2\times$ det'l cubic in $\mathbb{P}(V_6)$ (one orbit!)

ss not stable

period of S K3 of degree 2

I don't know the corresponding X (should be singular along $v_3(\mathbb{P}^2) \subset G(2, V_5)$

Tangential line complex

$q \subset \mathbb{P}(V_5)$ smooth

$X \subset G(3, V_5)$ lines tangent to q

singular along $OG(3, V_5)$ = lines in q

$= v_2(\mathbb{P}^3)$

Y_X = $3\times$ smooth quadric

3. Other dimensions

We consider linear sections of quadratic line complexes

	dim	index
$G(3\ V_5) \cap Q \cap \mathbb{P}^8$	4	2
$\cap\ \mathbb{P}^7$	3	1
$\cap\ \mathbb{P}^6$	2	general K3 of degree 10

Easy to show that

$X_4 \rightsquigarrow X_5 \xrightarrow{2:1} G \cap \mathbb{P}^8$ branched along X_4

limit of quadratic line complexes