



**Programme thématique**

**« Points rationnels, courbes rationnelles et courbes entières sur les variétés algébriques »**

**24 juin 2013**

Olivier Debarre  
(École normale supérieure)

**Moduli spaces for some Fano manifolds**

# 1. Moduli spaces of cubics using period maps.

Smooth cubics in  $\mathbb{P}^{n+1}$  have a GIT affine moduli space  $\mathcal{C}_n$  (singular cubics are parametrized by a  $SL(n+1)$ -invariant discriminant) with comp.  $\overline{\mathcal{C}}_n$

These moduli spaces may be studied via the period map. (in low dimensions).

Best example is  $n=4$

Hodge structure is of K3 type

$$H^4(X, \mathbb{C})_{\text{prim}} = H^{1,3} \oplus H^{3,2} \oplus H^{2,2}$$

(1)                      (20)                      (1)

This Hodge structure is determined by the line  $H^{1,3}$ .

Period domain

$$Q = \{ [\omega] \in \mathbb{P}^{21} \mid \omega \cdot \omega = 0, \omega \cdot \bar{\omega} > 0 \}$$

signature (20, 2)

bounded symmetric domain of type IV disc (20)

Period map

$$p: \mathcal{C}_4 \rightarrow \mathcal{D} = \Gamma \backslash Q$$

(20)                      (20)                      ↪ subgroup of finite index of  $O(1, 2)$

$\Lambda$  lattice  $H^4(X, \mathbb{Z})_{\text{prim}}$

$$2U \oplus 2E_8 \oplus A_2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

- $T_p$  injective (easy)
- $p$  injective (Voisin, hard; original proof had a gap filled using subsequent work of Laza, 1986)
- compactify  $p$  to study its image (Laza, 2007)

"C" due to Hassett

Image is complement of divisors  $D_2$  and  $D_6$  ("special" cubic fourfolds (Hassett) of discriminant 2 and 6, i.e.  $H^4(X, \mathbb{Z}) \cap H^3(X)$  contains rank-2 lattices of disc. 2 and 6)

This identifies  $\mathbb{C}_4$  with explicit complement of 2 hypersurfaces in  $\bar{D}$  (Bruhat-Borel compactification, obtained by adding 6 in components of dim  $\leq 3$ )

To extend  $p$ , one needs to blow-up  $\bar{\mathbb{C}}_4$  a particular point (the "determinantal cubic"  $\begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} = 0$ )

big auto. group  $SL(3)$

unique semi-stable orbit corresponding to cubics with 2 dual sing locus) then a curve  
• Nodal cubics go to  $D_6$   
• Blown-up det'l cubic go onto  $D_2$  via  $K3$  surfaces of degree 2

Cubics of other dimensions

$n=3$  Hodge structure, has weight one  $H^3(X)$

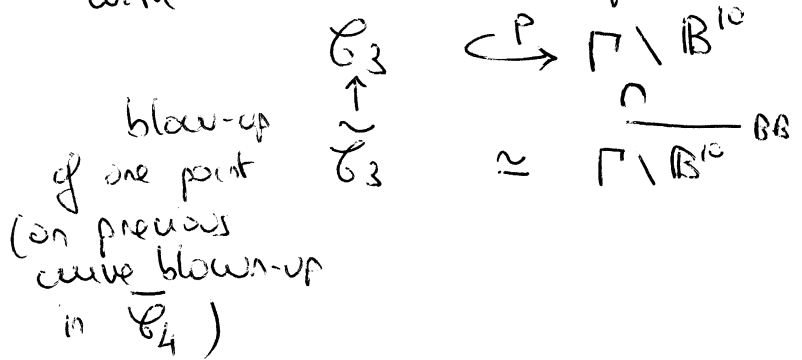
$\rightsquigarrow$  5 dual pp intermediate Jacobian

This does give an injective map

$\mathbb{C}_3 \hookrightarrow A_5$  but image has large codimension.

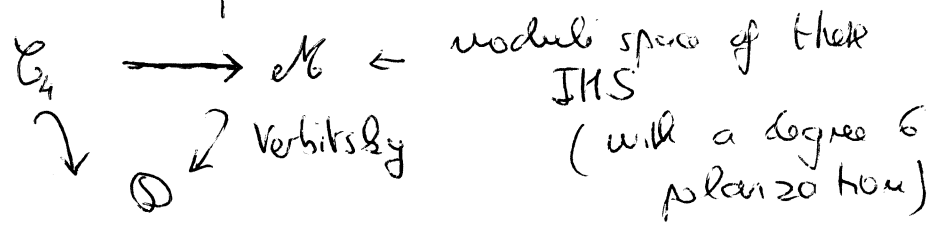
Associate with  $X$  a cubic fourfold with order 3 automorphism.

$$X' \xrightarrow{\mathbb{Z}/3} \mathbb{P}^4 \supset X$$



Add to  $n=4$

$F(X)$  is an indecomposable hol. sympl. (IHS) variety with "same period"



whose image in  $\mathbb{D}$  is  $S^{[2]}$  (from a divisor) of degree 2

These IHS "correspond" to det<sup>2</sup> cubic; they are not varieties of lines on a cubic fourfold

[  $F(X)$  can be  $S^{[2]}$  for infinitely many polarized  $S$ ; see BD or Hassett ]

n=2 no interesting Hodge structure  
→ similar construction of a 3-fbl.

Associated

## 2. Quadratic line complexes

Varieties  $G(2, n) \cap Q$  dim  $2n-5$   
Fano of index  $n-2$   
( $n > 3$ )  
Classically  $n=4$   
Here  $n=5 \rightarrow$  index 3

• Hodge structure has level 1  
→ intermediate Jacobian, dim 10  
too large!

We use the same trick and consider

$Z \xrightarrow{2:1} G(2, 5)$  branched along  $X$   
(Fano of dim 6, index 4,  $(p=1)$   
→ all are like that)

Hodge structure of K3 type  
on  $H^6(Z)$

$$H^6(Z)_{\text{van}} = H^{2,4} \oplus H^{3,3} \oplus H^{4,2}$$

(1)                      (20)                      (1)

(but different lattice  $2U \oplus 2E_8(-1) \oplus 2A_1(-1)$ )

Moduli space: affine GIT quotient

$$p: Z \rightarrow D := \mathbb{P}^1 \setminus Q$$

(25)                      (20)

$T_p$  always surjective ( $\Rightarrow p$  dominant)

We would like to use this map to describe  $Z$

• As for cubics, one can associate an IHS (Iliev-Macriavel). This is a bit more complicated

$I_G(2)$  quadrics containing  $G$ , all of rank 6

$I_X(2) = I_G(2) \oplus \mathbb{C}Q$   
 (if  $X$  smooth, we may take  $Q$  smooth)

Discriminant =  $4|I_G(2)| + \gamma_X$

$\gamma_X \subset \mathbb{P}^5$  is a (possibly non-integral) sextic  $\rightsquigarrow$  EPW sextic.

Brief definition

$A_{10} \subset \Lambda^3 V_6$  Lagrangian

$$\gamma_A = \{ v \in \mathbb{P}(V_6) \mid (\Lambda^2 V_6 \wedge v) \cap A \neq \emptyset \}$$

is either all or a (singular) sextic

$$\text{Sing}(\gamma_A) = \gamma_A[2] \cup \bigcup_{[V_3] \in \mathbb{P}^2} \mathbb{P}(V_3)$$

$$G = \{ V_3 \mid [\Lambda^3 V_3] \in A \}$$

$$\overline{X}_A \xrightarrow{2:1} \gamma_A \quad \hookrightarrow \dim(\dots) \geq 2$$

$\hookrightarrow$  IHS for  $A$  general.

There is a GIT moduli space for EPW sextics.

$$\overline{\text{EPW}} = \text{LG}(\Lambda^3 V_6) // \text{SL}(V_6) \quad (\text{O'Grady})$$

EPW (com. to smooth double EPW) is the

complement of 2 (ample) divisors

$$\Sigma = \{ A \mid \mathbb{K}_A \neq \emptyset \}$$

$$\Delta = \{ A \mid \gamma_A[3] \neq \emptyset \}$$

Theorem

The association  $X \rightsquigarrow \gamma_X$  induces a morphism

$$\Phi: \mathbb{Z} \rightarrow \overline{\text{EPW}} - \Sigma$$

affine

(hence possibly finite singular locus)

Fibers are

$$\mathbb{P}(V_6^v) - \gamma_A^v / \text{Aut}$$

$$\hookrightarrow \text{dual EPW} = \{ H \mid \Lambda^3 H \cap A \neq \emptyset \}$$

On the other hand (by Verbitsky again),

$$EPW \xrightarrow{\text{period}} \mathbb{D} \quad \text{open embedding}$$

In fact same remains true for

$$\overline{EPW} - \Sigma \xrightarrow{P} \mathbb{D}$$

$\Delta$  is sent onto divisor  $\mathbb{D}'_{10}$   
( $X_A \dashrightarrow S^{[2]}$  degree 10)

[Q] can we relate that to special 6-folds?

$\rightsquigarrow$  period of  $S$ )

Corollary  $Z$  is an affine bundle in  $\mathbb{P}^5$ -EPW over the image of  $P$

Rms  $p_0: \mathbb{P}^5$  should be the period map of  $Z$

Q1 What is the image of  $P$  (or  $p_0: \mathbb{P}^5$ )?

Conj (Debarre, O'Grady) The image of  $P$  is the complement of

Known: it is contained

$$\mathbb{D}_2$$

$$\mathbb{D}_4$$

$$\mathbb{D}_8$$

is the image of nodal  $X$  or, equivalently, of  $\Sigma$

$\mathbb{D}_2$  is the image of the orbit  $A(U_3)$

$\mathbb{D}_4$   $A(U_4)$

These EPW sextics are 3x smooth quadric. We will explain the corresponding  $X$  (both are semi-stable non stable points) when  $X_t$  approaches  $X$ , the period should approach period of  $S$  quartic in  $\mathbb{P}^3$ )

$\textcircled{2}$  Those EPW sextics are  $2 \times$  det'l cubic  
 in  $\mathbb{P}(V_6)$  (one orbit!)  
 ss not stable  
 period of  $S$  K3 of degree 2  
 I don't know the corresponding  $X$  (should be  
 singular along  $V_3(\mathbb{P}^2) \subset G(2, V_5)$ )

Tangential line complex  
 $q \subset \mathbb{P}(V_5)$  smooth  
 $X \subset G(2, V_5)$  lines tangent to  $q$   
 singular along  $OG(2, V_5) =$  lines in  $q$   
 $= V_2(\mathbb{P}^3)$   
 $Y_X = 3 \times$  smooth quadric

3. Other dimensions

We consider linear sections of quadratic line complexes

|                                      | dim | index                      |
|--------------------------------------|-----|----------------------------|
| $G(2, V_5) \cap Q \cap \mathbb{P}^8$ | 4   | 2                          |
| $\cap \mathbb{P}^7$                  | 3   | 1                          |
| $\cap \mathbb{P}^6$                  | 2   | general K3<br>of degree 10 |

Easy to show that

$X_4 \rightsquigarrow X_5 \xrightarrow{2:1} G \cap \mathbb{P}^8$  branched along  
 locus of quadratic line  $X_4$  complexes