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DEGENERATIONS OF DEBARRE–VOISIN VARIETIES

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Abstract. Given a general 3-form on a complex vector space of dimension ten, one can construct a smooth hyperkähler variety of dimension four (called a Debarre–Voisin variety). For some 3-forms, the associated Debarre–Voisin variety has excessive dimension and the construction breaks down. We show that after blowing up (the orbits of) these points, the Debarre–Voisin construction can be extended and produces the Hilbert squares of various polarized K3 surfaces of low degrees. This work, in collaboration with Frédéric Han, Kieran O'Grady, and Claire Voisin, ultimately rests on Mukai's description of these surfaces.

1. Hyperkähler manifolds

Definition 1.1. A hyperkähler manifold is a compact complex simply connected kähler manifold whose space of holomorphic 2-forms is generated by a symplectic form.

The dimension of a hyperkähler manifold is even and we will exclusively be concerned with hyperkähler fourfolds. Examples were first constructed in that dimension by Fujiki in 1982 and in all even dimensions by Beauville in 1983. If S is a K3 surface, on may consider the diagram

$$\begin{tikzcd} \mathrm{Bl}_\Delta(S\times S) \arrow[r] \arrow[d, "/\mathbf{Z}/2\mathbf{Z}"] & S\times S \arrow[d, "/\mathbf{Z}/2\mathbf{Z}"] \\ S^{[2]} \arrow[r] & S^{(2)}. \end{tikzcd}$$

The fourfold $S^{[2]}$, called the Hilbert square of S, is a hyperkähler fourfold. One has

$$H^2(S^{[2]},\mathbf{Z})\simeq H^2(S,\mathbf{Z})\oplus\mathbf{Z}\delta,$$

where 2δ is the class of the image in $S^{[2]}$ of the exceptional divisor of $\mathrm{Bl}_\Delta(S\times S)$.

A hyperkähler fourfold is said to be of $\mathrm{K3}^{[2]}$-type if it is a deformation of the Hilbert square of a K3 surface. These fourfolds form a 21-dimensional family; most of them are not projective.

We will be concerned with a particular 20-dimensional family of such fourfolds, which are projective with a polarization of fixed degree and for which a geometrical description is available (there are very few known such examples).

2. Debarre–Voisin varieties

2.1. **Definition.** Let $\sigma\in\bigwedge^3 V_{10}^\vee$ be a nonzero alternating 3-form (which we call a trivector). The subscheme

$$K_\sigma := \{[W_6]\in \mathsf{Gr}(6,V_{10}) \mid \sigma|_{W_6}=0\}$$

of 6-dimensional vector subspaces of V_{10} that are totally isotropic for σ (defined as the zero-locus of the section of the rank-20 vector bundle $\bigwedge^3\mathscr{S}_6^\vee$ corresponding to σ) is called the Debarre–Voisin variety associated with σ.

Theorem 2.1 (Debarre–Voisin, 2010)**.** *When σ is general in $\bigwedge^3 V_{10}^\vee$, the scheme K_σ is a smooth hyperkähler fourfold of* $\mathrm{K3}^{[2]}$*-type.*

2.2. **Moduli spaces.** There is a 20-dimensional irreducible quasi-projective moduli space

$$\mathscr{M}_{\mathrm{HK}}$$

for (smooth) polarized hyperkähler fourfolds which are deformations of $(K_\sigma, \mathscr{O}_{K_\sigma}(1))$. There is also a 20-dimensional projective irreducible GIT quotient

$$\mathscr{M}_{\mathrm{DV}} := \mathbf{P}(\textstyle\bigwedge^3 V_{10}^\vee) /\!/ \mathrm{SL}(V_{10}).$$

Finally, there is dominant (hence generically finite) rational map

$$\begin{array}{rccc} K\colon & \mathscr{M}_{\mathrm{DV}} & \dashrightarrow & \mathscr{M}_{\mathrm{HK}} \\ & [\sigma] & \longmapsto & K_\sigma \end{array}$$

(which we suspect is birational) whose domain of definition is the open subset of $\mathscr{M}_{\mathrm{DV}}$ corresponding to points $[\sigma]$ such that K_σ is a smooth fourfold (these points are all semistable).

3. HLS divisors

Our purpose is to describe some points $[\sigma] \in \mathscr{M}_{\mathrm{DV}}$ with large stabilizers, at which K is not defined, and whose total images in $\mathscr{M}_{\mathrm{HK}}$ are divisors (which we will call HLS divisors, for Hassett–Looijenga–Shah).

The very general points of these divisors will correspond to pairs $(S^{[2]}, 2bL - a\delta)$, where (S, L) is a polarized K3 surface of degree $L^2 = 2e$ and Picard group $\mathbf{Z}[L]$, satisfying the degree condition

$$a^2 - 4eb^2 = -11. \tag{1}$$

Moreover, the class $2bL - a\delta$ must be ample on $S^{[2]}$. For a given $e > 0$, there is either 0, 1, or 2 such pairs (a, b); when there is (at least) one, the fourfolds $(S^{[2]}, 2bL - a\delta)$ describe a divisor in $\mathscr{M}_{\mathrm{HK}}$ (with 1 or 2 components) which we denote by $\mathscr{D}_{2e}$.

We indicate in the following table the positive integers $e \leq 26$ for which the equation (1) has solutions and, for these values of e, the corresponding classes that are ample, or only movable (nef and big on a hyperkähler birational model of $S^{[2]}$).

e	1	3	5	9	11	15
(a,b)	$(5,3)$	$(1,1)$	$(3,1)$	$(5,1)$	$(33,5)$	$(7,1)$
movable classes	$6L-5\delta$	$2L-\delta$	$2L-3\delta$ $6L-13\delta$	$2L-5\delta$	$10L-33\delta$	$2L-7\delta$
ample classes	–	$2L-\delta$	$2L-3\delta$	$2L-5\delta$	–	$2L-7\delta$
nontrivial birational hyperkähler models	$S^{[2]} \dashrightarrow S^{[2]\prime}$	–	$S^{[2]} \overset{\iota}{\dashrightarrow} S^{[2]}$	–	$S^{[2]} \dashrightarrow S^{[2]\prime}$	–

There are infinitely many divisors $\mathscr{D}_{2e}$: take for example $e = m^2 + m + 3$, with $m \geq 0$, and $(a, b) = (2m + 1, 1)$; the class $2L - (2m + 1)\delta$ is always ample on $S^{[2]}$.

When $e = 1$, the class $6L - 5\delta$ is ample on the unique nontrivial hyperkähler birational model $S^{[2]\prime}$ of $S^{[2]}$.

When $e = 5$, the fourfold $S^{[2]}$ has a (unique) nontrivial birational involution ι, and $\iota^*(2L - 3\delta) = 6L - 13\delta$.

When $e = 11$, the class $10L - 33\delta$ is nef and big (but not ample) on the unique nontrivial hyperkähler birational model $S^{[2]\prime}$ of $S^{[2]}$.

For each value $e \in \{1, 3, 5, 9, 11, 15\}$, we will use the projective models of general K3 surfaces S of degree $2e$ that are either classical or constructed by Mukai. The trick is to construct, in each case, a globally generated rank-4 vector bundle on $S^{[2]}$ which could be the restriction of the rank-4 tautological quotient bundle via an embedding $S^{[2]} \subseteq K_{\sigma_0} \subseteq \mathsf{Gr}(6, V_{10})$ (one necessary condition is that it has the same Chern numbers).

3.1. **The nodal case.** The trivectors σ such that the Plücker hyperplane section

$$X_\sigma := \{[W_3] \in \mathsf{Gr}(3, V_{10}) \mid \sigma|_{W_3} = 0\}.$$

(which, by Bertini's theorem, is, for σ general, smooth irreducible) is singular form an irreducible divisor $\mathscr{M}_{\mathrm{DV}}^{\mathrm{sing}}$ in $\mathscr{M}_{\mathrm{DV}}$. Debarre–Voisin proved that for σ general in $\mathscr{M}_{\mathrm{DV}}^{\mathrm{sing}}$,

- the scheme K_σ is a normal integral fourfold;
- its singular locus contains a general K3 surface S with a polarization L of degree 22;
- there is a birational isomorphism

$$\varphi\colon S^{[2]} \dashrightarrow K_\sigma$$

  such that $\varphi^*\mathscr{O}_{K_\sigma(1)} = 10L - 33\delta$.

The map $\mathscr{M}_{\mathrm{DV}} \dashrightarrow \mathscr{M}_{\mathrm{HK}}$ is defined at a general point $[\sigma]$ of the divisor $\mathscr{M}_{\mathrm{DV}}^{\mathrm{sing}}$ and sends it to the pair $(S^{[2]\prime}, 10L - 33\delta)$, where $S^{[2]\prime}$ is the unique nontrivial hyperkähler birational model of $S^{[2]}$ (on which $10L - 33\delta$ is nef and big but not ample; so this is not quite an element of $\mathscr{M}_{\mathrm{HK}}$ but of some compactification that allows quasipolarizations). The image of $\mathscr{M}_{\mathrm{DV}}^{\mathrm{sing}}$ is therefore the divisor $\mathscr{D}_{22}$. It is not an HLS divisor.

3.2. **The divisor $\mathscr{D}_6$.** We construct polystable (semistable with closed orbit in the semistable locus) trivectors σ_0 with stabilizer $\mathrm{Sp}(4)$ such that the scheme K_{σ_0} has excessive dimension 6 but is still smooth.

Let V_4 be a 4-dimensional vector space equipped with a symplectic form $\omega \in \bigwedge^2 V_4^\vee$ and let $V_5 \subseteq \bigwedge^2 V_4$ be the hyperplane defined by ω, endowed with the nondegenerate quadratic form q defined by $q(x, y) = (\omega \wedge \omega)(x \wedge y)$. The space

$$V_{10} := \bigwedge^2 V_5 \simeq \mathrm{Sym}^2 V_4$$

can be seen as the space of endomorphisms of V_5 which are skew-symmetric with respect to q. The space of $\mathrm{Sp}(4)$-invariant trivectors on V_{10} is generated by the trivector σ_0 defined by

$$\sigma_0(a, b, c) = \mathrm{Tr}(a \circ b \circ c).$$

This is a particular case of a general situation studied by Hivert.

Theorem 3.1 (Hivert)**.** *Let $Q \subseteq \mathbf{P}(V_5)$ be the quadric defined by q. The Debarre–Voisin variety K_{σ_0} is smooth of dimension 6 and isomorphic to the Hilbert square $Q^{[2]}$.*

Proof. Let $x, y \in Q$ be general points. Since $x \in x^{\perp_q}$ and $y \in y^{\perp_q}$, the subspace

$$x \wedge x^{\perp_q} + y \wedge y^{\perp_q} \subseteq \bigwedge^2 V_5 = V_{10}$$

has dimension 6 and one checks that it is totally isotropic for σ_0, hence defines a point of K_{σ_0}.

This gives an Sp(4)-equivariant rational map $Q^{[2]} \dashrightarrow K_{\sigma_0}$ and Hivert proves that it is an isomorphism. $\square$

We consider now a general 1-parameter family $(\sigma_t)_{t\in\Delta}$ of deformations of σ_0 and we want to find the limit of the Debarre–Voisin varieties $K_{\sigma_t} \subseteq \mathsf{Gr}(6, V_{10})$ as $t \to 0$. In other words, we consider the family

$$\mathscr{K} := \{([W_6], t) \in \mathsf{Gr}(6, V_{10}) \times \Delta \mid \sigma_t|_{W_6} = 0\} \longrightarrow \Delta$$

of Debarre–Voisin varieties; it has irreducible general 4-dimensional fibers over Δ, hence a unique irreducible 5-dimensional component $\mathscr{K}^0$ that dominates Δ, and we want to find the central 4-dimensional fiber $\mathscr{K}^0_0 \subseteq K_{\sigma_0}$.

This is an excess computation: the differential of σ_0 (viewed as a section of the vector bundle $\mathscr{E} := \bigwedge^3 \mathscr{S}_6^\vee$ on $G := \mathsf{Gr}(6, V_{10})$) defines an exact sequence

$$0 \to T_{K_{\sigma_0}} \to T_G|_{K_{\sigma_0}} \xrightarrow{d\sigma_0} \mathscr{E}|_{K_{\sigma_0}} \to \mathscr{F} \to 0$$

of vector bundles on K_{σ_0}, where $\mathscr{F} := \mathscr{E}|_{K_{\sigma_0}}/N_{K_{\sigma_0}/G}$ is the *excess bundle*. It has rank 2 and is globally generated.

The image of $\frac{d\sigma_t}{dt}\big|_{t=0} \in H^0(G, \mathscr{E})$ in $H^0(K_{\sigma_0}, \mathscr{F})$ is a general section, its zero-locus is the central 4-dimensional fiber $\mathscr{K}^0_0 \subseteq K_{\sigma_0}$ and it is smooth of codimension 2.

The derivative $\frac{d\sigma_t}{dt}\big|_{t=0}$ provides a nonzero element of the normal space to the $\mathrm{GL}(V_{10})$-orbit of σ_0, which can be checked to be $H^0(Q, \mathscr{O}_Q(3))$. It defines a K3 surface $S \subseteq Q \subseteq \mathbf{P}(V_5)$ with a polarization L of degree 6.

After identifying the rank-2 excess bundle on $K_{\sigma_0} = Q^{[2]} \subseteq \mathsf{Gr}(6, V_{10})$, one checks that the central fiber $\mathscr{K}^0_0 \subseteq K_{\sigma_0}$ is isomorphic to $S^{[2]} \subseteq Q^{[2]}$ and that the Plücker polarization restricts to $2L - \delta$. General polarized K3 surfaces of degree 6 are obtained in this fashion. This makes $\mathscr{D}_6$ into an HLS divisor.

At a general point of the divisor $\mathscr{D}_6$, the polarization $2L - \delta$ on $S^{[2]}$ (induced by the polarization $\mathscr{O}_{K_{\sigma_0}}(1)$) is very ample: it defines the embedding

$$\varphi_{2L-\delta} \colon S^{[2]} \subseteq Q^{[2]} \xrightarrow{\sim} K_{\sigma_0} \subseteq \mathsf{Gr}(6, V_{10}) \subseteq \mathbf{P}(\textstyle\bigwedge^6 V_{10}).$$

3.3. **The divisor $\mathscr{D}_{18}$.** We construct polystable trivectors σ_0 with stabilizer $G_2 \times \mathrm{SL}(3)$ such that the scheme K_{σ_0} has excessive dimension 10 but is still smooth.

The group G_2 is the subgroup of $\mathrm{GL}(V_7)$ leaving a general 3-form α invariant. There is a G_2-invariant Fano 5-fold $X \subseteq \mathsf{Gr}(2, V_7) \subseteq \mathbf{P}(\bigwedge^2 V_7)$ which has index 3 and Mukai proved that general degree-18 $K3$ surfaces are obtained by intersecting X with a general 3-codimensional linear subspace $\mathbf{P}(W_3^\perp) \subseteq \mathbf{P}(\bigwedge^2 V_7^\vee)$.

The vector space $V_{10} := V_7 \oplus W_3$ is acted on diagonally by the group $G_2 \times \mathrm{SL}(W_3)$ and we consider $G_2 \times \mathrm{SL}(W_3)$-invariant trivectors $\sigma_0 = \alpha + \beta$, where β spans $\bigwedge^3 W_3^\vee$. The corresponding points $[\sigma_0]$ of $\mathbf{P}(\bigwedge^3 V_{10}^\vee)$ are all in the same $\mathrm{SL}(V_{10})$-orbit and the corresponding Debarre–Voisin variety K_{σ_0} splits as a product of a smooth variety of dimension 8 and of $\mathbf{P}(W_3^\vee)$.

An excess bundle analysis shows the following.

Theorem 3.2. *Under a general* 1*-parameter deformation* $(\sigma_t)_{t\in\Delta}$*, the Debarre–Voisin fourfolds* K_{σ_t} *specialize to a smooth subscheme of* K_{σ_0} *isomorphic to* $S^{[2]}$*, where* $S \subseteq X$ *is a general degree-*18 $K3$ *surface.*

This makes $\mathscr{D}_{18}$ into an HLS divisor. At a general point of $\mathscr{D}_{18}$, the polarization $2L - 5\delta$ on $S^{[2]}$ (induced by the polarization $\mathscr{O}_{K_{\sigma_0}}(1)$) is very ample: it defines the embedding

$$S^{[2]} \subseteq K_{\sigma_0} \subseteq \mathsf{Gr}(6, V_{10}) \subseteq \mathbf{P}(\textstyle\bigwedge^6 V_{10}).$$

3.4. **The divisor $\mathscr{D}_{10}$.** The variety $X \subseteq \mathsf{Gr}(2, V_5^\vee) \subseteq \mathbf{P}(\bigwedge^2 V_5^\vee)$ cut out by a general 3-codimensional linear subspace $\mathbf{P}(W_3^\perp) \subseteq \mathbf{P}(\bigwedge^2 V_5^\vee)$ is a degree-5 Fano threefold. Mukai proved that general degree-10 $K3$ surfaces are quadratic sections of X.

The automorphism group of X is PGL(2). If U_2 is the standard self-dual irreducible representation of SL(2) and $V_5 := \mathrm{Sym}^4 U_2$, there is a direct sum decomposition

$$V_{10} := \textstyle\bigwedge^2 V_5 = V_7 \oplus W_3, \qquad \text{with } V_7 = \mathrm{Sym}^6 U_2 \quad \text{and} \quad W_3 = \mathrm{Sym}^2 U_2,$$

into irreducible representations of SL(2), so that X is the unique SL(2)-invariant section of $\mathsf{Gr}(2, V_5^\vee)$ by a linear subspace of codimension 3.

We define an SL(2)-invariant trivector σ_0 (we do not prove that it is polystable, but the neutral component of its stabilizer is SL(2)). The Debarre–Voisin K_{σ_0} has one component which is generically smooth and birationally isomorphic to $X^{[2]}$.

Theorem 3.3. *Under a general* 1*-parameter deformation* $(\sigma_t)_{t\in\Delta}$*, the Debarre–Voisin fourfolds* K_{σ_t} *specialize, after finite base change, to a smooth subscheme of* K_{σ_0} *which is isomorphic to* $S^{[2]}$*, where* $S \subseteq X$ *is a general degree-*10 *K3 surface.*

The proof uses a recent result of Kollár–Laza–Saccà–Voisin, which roughly says that if the general fibers of a projective flat family are hyperkähler manifolds and the central fiber has one reduced component which is birationally isomorphic to a hyperkähler manifold K, then, after finite base change, one may arrange that the central fiber be isomorphic to K.

The limit on $S^{[2]}$ of the line bundles on $\mathscr{O}_{K_{\sigma_t}}(1)$ is the ample line bundle $2L - 3\delta$. We show that it is not globally generated.

3.5. **The divisor $\mathscr{D}_2$.** We construct polystable trivectors σ_0 with stabilizer SL(3) such that the scheme K_{σ_0} has dimension 4 but is reducible and nonreduced.

A general degree-2 K3 surface (S, L) is a double cover of $\mathbf{P}(W_3)$ branched along a smooth sextic curve. We take $V_{10} := \mathrm{Sym}^3 W_3$ and we let $\sigma_0 \in \bigwedge^3 V_{10}^\vee$ be a generator of the space of $\mathrm{SL}(W_3)$-invariant trivectors on V_{10}.

We show that the Debarre–Voisin variety K_{σ_0} has two 4-dimensional rational components and is nonreduced along one component K_1. Using the Kollár–Laza–Saccà–Voisin result, one can show that there are degenerations of smooth Debarre–Voisin fourfolds to the unique nontrivial hyperkähler birational model $\mathscr{M}_S(0, L, 1)$ of $S^{[2]}$. There is a 4:1 map $\mathscr{M}_S(0, L, 1) \to K_1 \overset{\sim}{\dashrightarrow} \mathbf{P}^4$ induced by the double cover $S \to \mathbf{P}^2$.

This makes $\mathscr{D}_2$ into an HLS divisor. At a general point of $\mathscr{D}_2$, the polarization $6L - 5\delta$ on $\mathscr{M}_S(0, L, 1)$ is base-point-free but not very ample: it defines the morphism

$$\mathscr{M}_S(0, L, 1) \xrightarrow{4:1} K_1 \subseteq K_{\sigma_0} \subseteq \mathsf{Gr}(6, V_{10}) \subseteq \mathbf{P}(\textstyle\bigwedge^6 V_{10}).$$

3.6. **The divisor $\mathscr{D}_{30}$.** Noncanonical geometric descriptions of general polarized K3 surfaces S of degree 30 were given by Mukai. Unfortunately, we were not able to find a trivector on some 10-dimensional vector space that would relate $S^{[2]}$ to Debarre–Voisin varieties. We were however able to construct on $S^{[2]}$ a rank 4-vector bundle with the same numerical invariants as the restriction of the tautological quotient bundle of $\mathsf{Gr}(6, V_{10})$ to a Debarre–Voisin variety.

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