PERIODS OF CUBIC FOURFOLDS AND OF DEBARRE–VOISIN VARIETIES

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Abstract. Beauville and Donagi showed that the primitive Hodge structure of a smooth complex cubic hypersurface of dimension four is isomorphic to that of its variety of lines, a smooth hyperkähler variety of dimension also four. This gives a way to study the period map for cubic fourfolds, a rational map from the projective GIT moduli space of semistable cubic fourfolds to the period domain. As shown by Hassett, this map is not defined at the very special (semistable) point corresponding to the (singular) chordal cubic, but it is well defined (generically) on the blow up of this point and the exceptional divisor maps onto a well understood divisor (called a Hassett–Looijenga–Shah, or HLS divisor) in the period domain.

In this talk, I would like to describe an analogous situation (although it has no direct relationship with cubic hypersurfaces): given a general 3-form on a complex vector space of dimension ten, one can construct a smooth hyperkähler variety of dimension four (called a Debarre–Voisin variety). We exhibit several HLS divisors and identify the corresponding very special 3-forms.

This is work in progress in collaboration with Frédéric Han, Kieran O'Grady, and Claire Voisin.

1. The variety of lines on a cubic fourfold

Throughout the talk, V_r denotes a complex vector space of dimension r.

1.1. **The Beauville–Donagi theorem.** Let $f \in \mathrm{Sym}^3 V_6^\vee$ be nonzero. It defines a cubic hypersurface

$$Y_f \subseteq \mathbf{P}(V_6).$$

We denote by

$$F(Y_f) \subseteq \mathsf{Gr}(2, V_6)$$

the scheme of lines contained in Y_f (defined as the zero-locus of the section of the rank-4 vector bundle $\mathrm{Sym}^3 \mathscr{S}_2^\vee$ corresponding to f).

Theorem 1.1 (Beauville–Donagi, 1985)**.** *Assume that $Y := Y_f$ is smooth.*

(1) *The scheme $F(Y)$ is a smooth hyperkähler fourfold.*

(2) *The incidence correspondence*

$$Y \xleftarrow{\;q\;} \{(x, [\ell]) \in Y \times F(Y) \mid x \in \ell\} \xrightarrow{\;p\;} F(Y)$$

induces an isomorphism

$$p_* q^* \colon H^4(Y, \mathbf{Z})_0 \xrightarrow{\ \sim\ } H^2(F(Y), \mathbf{Z})_0(-1)$$

of polarized Hodge structures.

The subscript 0 means that we consider primitive Hodge structures (the respective orthogonal complements of $[\mathscr{O}_Y(1)]$ and $[\mathscr{O}_{F(Y)}(1)]^3$).

1.2. **Moduli spaces and period maps.** One can define a GIT moduli space

$$\mathscr{M}_{\mathrm{cubic}} := \mathbf{P}(H^0(\mathbf{P}(V_6), \mathscr{O}_{\mathbf{P}(V_6)}(3)))/\!\!/ \mathrm{SL}(V_6) = \mathbf{P}(\mathrm{Sym}^3 V_6^\vee)/\!\!/ \mathrm{SL}(V_6)$$

for cubic fourfolds, a 20-dimensional irreducible projective scheme. All smooth cubics are stable and define a dense affine open subset

$$\mathscr{M}_{\mathrm{cubic}}^{\mathrm{smooth}} \subseteq \mathscr{M}_{\mathrm{cubic}}.$$

There is also (Viehweg) a 20-dimensional irreducible quasi-projective moduli space

$$\mathscr{M}_6^{(2)}$$

for polarized (smooth) hyperkähler fourfolds which are deformations of $(F(Y), \mathscr{O}_{F(Y)}(1))$ (the polarization has Beauville–Bogomolov degree 6 and divisibility 2).

There are associated periods maps and Theorem 1.1 can be rephrased as saying that the diagram

(1)
$$\begin{array}{ccc} \mathscr{M}_{\mathrm{cubic}}^{\mathrm{smooth}} & \xhookrightarrow{F} & \mathscr{M}_6^{(2)} \\ & {\scriptstyle \mathfrak{p}_{\mathrm{cubic}}} \searrow \quad \swarrow {\scriptstyle \mathfrak{p}} & \\ & \mathscr{F} & \end{array}$$

is commutative, where $\mathscr{F}$ is the period domain, quotient of a bounded symmetric domain of type IV by an arithmetic group. Moreover,

- F is an open embedding (one can recover Y from $F(Y)$);
- $\mathfrak{p}_{\mathrm{cubic}}$ is an open embedding (Voisin's thesis and erratum);
- $\mathfrak{p}$ is an open embedding (Verbitsky and Markman).

In order to identity precisely $\mathscr{M}_{\mathrm{cubic}}^{\mathrm{smooth}}$ and $\mathscr{M}_6^{(2)}$ as open subsets of $\mathscr{F}$, one can ask about the images of $\mathfrak{p}_{\mathrm{cubic}}$ and $\mathfrak{p}$:

- the image of $\mathfrak{p}_{\mathrm{cubic}}$ is the complement of two irreducible Heegner divisors $\mathscr{D}_2$ and $\mathscr{D}_6$ (Laza, Looijenga);
- the image of $\mathfrak{p}$ is the complement of $\mathscr{D}_6$.

Laza's results are much more precise (see the end of § 1.4) and explain how to resolve the singularities of the rational map

$$\mathfrak{p}_{\mathrm{cubic}} \colon \mathscr{M}_{\mathrm{cubic}} \dashrightarrow \overline{\mathscr{F}}^{\mathrm{BB}}$$

between compactifications. We now explain which cubics the divisors $\mathscr{D}_2$ and $\mathscr{D}_6$ "correspond to."

1.3. **The divisor $\mathscr{D}_6$ and nodal cubics.** A cubic hypersurface $Y \subseteq \mathbf{P}(V_6)$ has a node at $p := (1, 0, \dots, 0)$ if and only if its equation can be written as

$$F_2(x_1, \dots, x_5)x_0 + F_3(x_1, \dots, x_5) = 0.$$

Lines through p contained in Y are then parametrized by the K3 surface

$$S := (F_2 = F_3 = 0) \subseteq \mathbf{P}^4$$

so we have an embedding $S \subseteq F(Y)$.

The relation between the variety of lines $F(Y)$ and S is the following: there is a birational morphism

$$\varphi \colon S^{[2]} \longrightarrow F(Y)$$

that takes a point $(s,s')\in S^{[2]}$ to the residual line of the intersection $\langle \ell_s,\ell_{s'}\rangle\cap Y$. It contracts a uniruled divisor in $S^{[2]}$ to the surface $S\subseteq F(Y)$, which is the singular locus of $F(Y)$. Moreover, $\varphi^*\mathscr{O}_{F(Y)}(1)=2L-3\delta$.

Theorem 1.2 (Laza)**.** *The period map extend to an open embedding*

$$\mathfrak{p}_{\mathrm{cubic}}\colon \mathscr{M}_{\mathrm{cubic}}^{\text{isolated sing}} \longrightarrow \mathscr{F}$$

and the divisor of singular cubics dominates the divisor $\mathscr{D}_6$.

The extended period map sends a nodal $[Y]$ as above to the period of the pair $(S^{[2]},2L-3\delta)$ (where the line bundle $2L-3\delta$ is nef (base-point free) and big on $S^{[2]}$ but not ample).

I am not mentioning an extension of diagram (1) because I am not sure about the existence of a moduli space for polarized singular hyperkähler varieties (like $(F(Y),\mathscr{O}_{F(Y)}(1))$) or for smooth hyperkähler varieties with a nef and big line bundle (like $(S^{[2]},2L-3\delta)$).

1.4. **The divisor $\mathscr{D}_2$ and the chordal cubic.** The situation with the divisor $\mathscr{D}_2$ is a bit more complicated: to extend the period map $\mathfrak{p}_{\mathrm{cubic}}$ in order to get periods in $\mathscr{D}_2$, one needs to blow up a very special point of $\mathscr{M}_{\mathrm{cubic}}$.

Write $V_6=\mathrm{Sym}^2V_3$ and consider the Veronese surface, defined as the image T of

$$\begin{array}{rcl}\psi\colon \mathbf{P}(V_3) & \hookrightarrow & \mathbf{P}(\mathrm{Sym}^2V_3)=\mathbf{P}(V_6)\\ y & \longmapsto & y^2.\end{array}$$

It is the set of quadratic forms of rank 1. Its secant variety $Y_0\subseteq\mathbf{P}(V_6)$ (the set of degenerate quadratic forms) is a cubic hypersurface, defined by the equation

$$f_0:=\begin{vmatrix}x_0 & x_1 & x_2\\ x_1 & x_3 & x_4\\ x_2 & x_4 & x_5\end{vmatrix}.$$

Its singular locus is T. The scheme $F(Y_0)$ of lines contained in Y_0 has two irreducible components

$$F(Y_0)=F_1\cup F_2,$$

where F_1 is the (closure of the) family of lines secant to T and F_2 is isomorphic to $\mathbf{P}(V_3)\times\mathbf{P}(V_3^\vee)$. Teh multiplicities of $F(Y_0)$ at a general point of F_1 (resp. F_2) is 4 (resp. 1) (Van den Dries).

Let $Y_f\subseteq\mathbf{P}(V_6)$ be a general cubic hypersurface. The inverse image $\psi^{-1}(Y_f)$ is a smooth sextic curve in $\mathbf{P}(V_3)$ hence define a double cover $\pi\colon S\to\mathbf{P}(V_3)$, where S is a K3 surface with a polarization of degree 2. When $t\to 0$, the family of polarized hyperkähler fourfolds $F(Y_{f_0+tf})$ converges to $(S^{[2]},2L-\delta)$, with $L:=\pi^*\mathscr{O}_{\mathbf{P}(V_3)}(1)$. The linear system $|2L-\delta|$ is ample but has base-points: it defines a rational map that factors as

$$\varphi_{2L-\delta}\colon S^{[2]} \underset{\pi^{[2]}}{\overset{4:1}{\dashrightarrow}} T^{[2]} \xrightarrow{\ \sim\ } F_1\subseteq \mathsf{Gr}(2,V_6)\subseteq\mathbf{P}^{14}.$$

Its indeterminacy locus is the image of the embedding $\mathbf{P}(V_3)\hookrightarrow S^{[2]}$ coming from the double cover $\pi\colon S\to\mathbf{P}(V_3)$.

Theorem 1.3 (Laza)**.** *The period map $\mathfrak{p}_{\mathrm{cubic}}$ factors through the blow up $\widetilde{\mathscr{M}}_{\mathrm{cubic}}\to\mathscr{M}_{\mathrm{cubic}}$ of the point $[f_0]$ as a rational map*

$$\widetilde{\mathfrak{p}}_{\mathrm{cubic}}\colon \widetilde{\mathscr{M}}_{\mathrm{cubic}} \dashrightarrow \mathscr{F}$$

and the exceptional divisor dominates the divisor $\mathscr{D}_2$.

Remark 1.4. Let (X, L) correspond to a point of $\mathscr{M}_6^{(2)}$. Then,

$$\begin{aligned} L \text{ very ample} &\iff L \text{ base-point free} \\ &\iff X = F(Y), \text{ with } Y \text{ smooth} \\ &\iff \mathfrak{p}(X, L) \notin \mathscr{D}_2. \end{aligned}$$

Let us describe briefly Laza's results. There is a 1-dimensional semistable stratum $\chi \subseteq \mathscr{M}_{\text{cubic}}$ (corresponding to cubic fourfolds whose singular locus contains a rational normal quartic curve) containing the point $[f_0]$. Let $\widehat{\mathscr{M}}_{\text{cubic}} \to \widetilde{\mathscr{M}}_{\text{cubic}}$ be the blow up of the strict transform of χ. There is a commutative diagram

$$\begin{array}{ccc} \widehat{\mathscr{M}}_{\text{cubic}} & \longrightarrow & \widehat{\mathscr{F}^{\text{BB}}} \\ \big\downarrow & & \big\downarrow \text{small modification} \\ \mathscr{M}_{\text{cubic}} & \overset{\mathfrak{p}_{\text{cubic}}}{\dashrightarrow} & \mathscr{F}^{\text{BB}}, \end{array}$$

where $\mathscr{F}^{\text{BB}}$ is the Baily–Borel compactification (the left vertical map is known as the Kirwan blow up).

2. Debarre–Voisin varieties

2.1. **Definition.** Let $\sigma \in \bigwedge^3 V_{10}^\vee$ be a nonzero alternating 3-form (which we call a trivector). It defines a Plücker hyperplane section

$$Y_\sigma := \{[W_3] \in \mathsf{Gr}(3, V_{10}) \mid \sigma|_{W_3} = 0\}.$$

The subscheme

$$K_\sigma := \{[W_6] \in \mathsf{Gr}(6, V_{10}) \mid \sigma|_{W_6} = 0\}$$

of 6-dimensional vector subspaces of V_{10} that are totally isotropic for σ (defined as the zero-locus of the section of the rank-20 vector bundle $\bigwedge^3 \mathscr{S}_6^\vee$ corresponding to σ) is called the Debarre–Voisin variety associated with σ.

Theorem 2.1 (Debarre–Voisin, 2010)**.** *Assume that σ is general in $\bigwedge^3 V_{10}^\vee$.*

(1) *The scheme Y_σ is smooth and K_σ is a smooth hyperkähler fourfold.*

(2) *The incidence correspondence*

$$\begin{array}{ccccc} & & \{([W_3], [W_6]) \in Y_\sigma \times K_\sigma \mid W_3 \subseteq W_6\} & & \\ & \overset{q}{\swarrow} & & \overset{p}{\searrow} & \\ Y_\sigma & & & & K_\sigma \end{array}$$

induces an isomorphism

$$p_* q^* \colon H^{20}(Y_\sigma, \mathbf{Q})_{\text{van}} \xrightarrow{\sim} H^2(K_\sigma, \mathbf{Q})_{\text{van}}(-9) \tag{2}$$

of rational Hodge structures.

The subscript "van" means that we consider vanishing Hodge structures (the respective orthogonal complements of $H^{20}(\mathsf{Gr}(3, V_{10}), \mathbf{Q})$ and $H^2(\mathsf{Gr}(6, V_{10}), \mathbf{Q})$).

2.2. **Moduli spaces and period maps.** One can define a GIT moduli space

$$\mathscr{M}_{\mathrm{DV}} := \mathbf{P}(\bigwedge^3 V_{10}^\vee) /\!\!/ \mathrm{SL}(V_{10})$$

for the varieties Y_σ, a projective 20-dimensional irreducible scheme. There is also again a 20-dimensional irreducible quasi-projective moduli space

$$\mathscr{M}_{22}^{(2)}$$

for (smooth) polarized hyperkähler fourfolds which are deformations of $(K_\sigma, \mathscr{O}_{K_\sigma}(1))$ (the polarization has Beauville–Bogomolov degree 22 and divisibility 2) and a generically finite rational map

$$\begin{array}{rcl} K\colon \mathscr{M}_{\mathrm{DV}} & \dashrightarrow & \mathscr{M}_{22}^{(2)} \\ [\sigma] & \longmapsto & K_\sigma \end{array}$$

(whose degree is unfortunately unknown) whose domain of definition is the open subset $\mathscr{M}_{\mathrm{DV}}^{\mathrm{smooth}} \subseteq \mathscr{M}_{\mathrm{DV}}$ corresponding to points $[\sigma]$ such that K_σ is a smooth fourfold (these points are all semistable).

The period map

$$\mathfrak{p}\colon \mathscr{M}_{22}^{(2)} \longrightarrow \mathscr{F}$$

is again an open embedding (Verbitsky and Markman) and its image is the complement of the (irreducible) Heegner divisor $\mathscr{D}_{22}$ (Debarre–Macrì).

We want to geometrically analyze the composition

$$\mathfrak{p} \circ K\colon \mathscr{M}_{\mathrm{DV}} \dashrightarrow \mathscr{M}_{22}^{(2)} \longrightarrow \mathscr{F}$$

(we do not know whether this is the period map for the varieties F_σ, because the isomorphism (2) is only known over $\mathbf{Q}$ and says nothing about polarizations).

More precisely, we will describe loci in $\mathscr{M}_{\mathrm{DV}}$ where $\mathfrak{p} \circ K$ is not defined and whose strict transforms in $\mathscr{F}$ are Heegner divisors (in the same way that the chordal cubic point was sent to the Heegner divisor $\mathscr{D}_2$ that parametrizes periods of pairs $(S^{[2]}, 2L - \delta)$) called HLS divisors (for Hassett–Looijenga–Shah).

Our starting point will be the (infinite) list of Heegner divisors $\mathscr{D}_{2e} \subseteq \mathscr{F}$ that parametrize points that are isomorphic to Hilbert squares of K3 surfaces (with a suitable polarization). A general point of such a divisor corresponds to a pair $(S^{[2]}, 2bL - a\delta)$, where (S, L) is a polarized K3 surface of degree $L^2 = 2e$ and Picard group $\mathbf{Z}[L]$, and

$$8eb^2 - 2a^2 = 22,$$

or

$$a^2 - 4eb^2 = -11. \tag{3}$$

Moreover, the class $2bL - a\delta$ must be ample on $S^{[2]}$. There are infinitely many such divisors (take for example $e = m^2 + m + 3$, with $m \geq 0$, so that $(a, b) = (2m + 1, 1)$). We indicate in the following table the positive integers $e \leq 26$ for which the equation (3) has solutions and, for these values of e, the classes of square 22 and divisibility 2 which are ample, or only movable (ample on a birational model).

e	1	3	5	9	11	15
(a,b)	$(5,3)$	$(1,1)$	$(3,1)$	$(5,1)$	$(33,5)$	$(7,1)$
movable classes of div. 2 and square 22	$6L-5\delta$	$2L-\delta$	$2L-3\delta$ $6L-13\delta$	$2L-5\delta$	$10L-33\delta$	$2L-7\delta$
ample classes of div. 2 and square 22	–	$2L-\delta$	$2L-3\delta$	$2L-5\delta$	–	$2L-7\delta$

TABLE 1. Movable and nef classes of square 22 and divisibility 2

When $e=1$, the class $6L-5\delta$ is ample on the unique nontrivial hyperkähler birational model of $S^{[2]}$.

When $e=5$, the fourfold $S^{[2]}$ has a (unique) nontrivial birational involution ι, and $\iota^*(2L-3\delta)=6L-13\delta$.

When $e=11$, the class $10L-33\delta$ is nef and big (but not ample) on the unique nontrivial hyperkähler birational model of $S^{[2]}$.

2.3. **The divisor $\mathscr{D}_{22}$ (the nodal case).** This is the analog of the divisor $\mathscr{D}_6$ in the cubic fourfold case and is not an HLS divisor.

The trivectors σ such that F_σ is singular form a divisor $\mathscr{M}_{\mathrm{DV}}^{\mathrm{sing}}$ in $\mathscr{M}_{\mathrm{DV}}$. Debarre–Voisin proved that for a general such σ,

- the scheme K_σ is a normal irreducible fourfold;
- its singular locus contains a general K3 surface S with a polarization L of degree 22;
- there is a birational isomorphism
$$\varphi\colon S^{[2]} \dashrightarrow K_\sigma$$
such that $\varphi^*\mathscr{O}_{K_\sigma(1)}=10L-33\delta$.

The map $\mathscr{M}_{\mathrm{DV}} \dashrightarrow \mathscr{F}$ is defined at a general point $[\sigma]$ of the divisor $\mathscr{M}_{\mathrm{DV}}^{\mathrm{sing}}$ and sends it to the pair $(S^{[2]\prime}, 10L-33\delta)$, where $S^{[2]\prime}$ is the unique nontrivial hyperkähler birational model of $S^{[2]}$ (on which $10L-33\delta$ is nef and big but not ample; I do not know whether it is base-point-free, that is, if the composition $S^{[2]\prime} \dashrightarrow S^{[2]} \overset{\varphi}{\dashrightarrow} K_\sigma$ is a morphism). The image of $\mathscr{M}_{\mathrm{DV}}^{\mathrm{sing}}$ is the divisor $\mathscr{D}_{22}$.

2.4. **The divisor $\mathscr{D}_6$.** We will construct a trivector σ_0 with stabilizer $\mathrm{Sp}(4)$ such that the variety K_{σ_0} has excessive dimension 6 but is still smooth.

For the chordal cubic, we worked in the vector space $V_6=\mathrm{Sym}^2V_3$ and considered a point with infinite stabilizer $\mathrm{SL}(V_3)$. Following this clue, we will consider trivectors with large stabilizers.

Let V_4 be a 4-dimensional vector space equipped with a symplectic form ω and let $V_5\subseteq \bigwedge^2V_4$ be the hyperplane defined by ω, endowed with the nondegenerate quadratic form q defined by $q(x,y)=(\omega\wedge\omega)(x\wedge y)$. The space
$$V_{10}:=\bigwedge{}^2V_5\simeq \mathrm{Sym}^2V_4$$

can be seen as the space of endomorphisms of V_5 which are skew-symmetric with respect to q. We define a trivector σ_0 on V_{10} by

$$\sigma_0(a,b,c) = \mathrm{Tr}(a \circ b \circ c).$$

It is invariant for the canonical action of the group $\mathrm{Sp}(V_4) = O(V_5)$ on $\bigwedge^3 V_{10}^\vee$. This is a particular case of a general situation studied by Hivert.

Theorem 2.2 (Hivert)**.** *Let $Q \subseteq \mathbf{P}(V_5)$ be the quadric defined by q. The Debarre–Voisin variety*

$$K_{\sigma_0} = \{[W] \in \mathsf{Gr}(6, V_{10}) \mid \sigma_0|_W = 0\}$$

is smooth of dimension 6 and isomorphic to $Q^{[2]}$.

Proof. Let $x, y \in Q$ be general points. Since $x \in x^{\perp_q}$ and $y \in y^{\perp_q}$, the subspace

$$x \wedge x^{\perp_q} + y \wedge y^{\perp_q} \subseteq \textstyle\bigwedge^2 V_5 = V_{10}$$

has dimension 6 and one checks that it is totally isotropic for σ_0, hence defines a point of K_{σ_0}.

This gives an $\mathrm{Sp}(4)$-equivariant rational map $Q^{[2]} \dashrightarrow K_{\sigma_0}$ and Hivert proves that it is an isomorphism. □

We consider now a general 1-parameter family $(\sigma_t)_{t\in\Delta}$ of deformations of σ_0 and we want to find the limit of the DV varieties $K_{\sigma_t} \subseteq \mathsf{Gr}(6, V_{10})$ as $t \to 0$. In other words, we consider the family

$$\mathscr{K} := \{([W_6], t) \in \mathsf{Gr}(6, V_{10}) \times \Delta \mid \sigma_t|_{W_6} = 0\} \longrightarrow \Delta$$

of Debarre–Voisin varieties; it has irreducible general 4-dimensional fibers over Δ, hence a unique irreducible 5-dimensional component $\mathscr{K}^0$ that dominates Δ, and we want to find the central fiber $\mathscr{K}^0_0$.

This is an excess computation that we describe in a general context.

Let M be a smooth variety of dimension n, let $\mathscr{E}$ be a globally generated vector bundle of rank r on M, and let σ_0 be a section of $\mathscr{E}$, with smooth zero-locus $Z \subseteq M$ of codimension $s \le r$. The differential of σ_0 then defines a morphism

$$d\sigma_0 \colon T_M|_Z \longrightarrow \mathscr{E}|_Z$$

whose kernel is T_Z. We define the *excess bundle* $\mathscr{F}$ as its cokernel. It is isomorphic to $\mathscr{E}|_Z/N_{Z/M}$, hence has rank $r-s$ on Z and is globally generated.

Let $(\sigma_t)_{t\in\Delta}$ be a general 1-parameter family of sections of $\mathscr{E}$. For $t \in \Delta$ general, the zero-set of σ_t is smooth irreducible of codimension r in M. Let $\overline{\sigma'}$ the image of $\frac{d\sigma_t}{dt}\big|_{t=0} \in H^0(M, \mathscr{E})$ in $H^0(Z, \mathscr{F})$; it is a general section, hence its zero-locus is smooth of codimension $r-s$ in Z.

Consider the closed subset

$$\mathscr{Z} = \{(x,t) \in M \times \Delta \mid \sigma_t(x) = 0\}.$$

The general fibers of the projection $\pi \colon \mathscr{Z} \to \Delta$ are smooth of dimension $n-r$ and the central fiber is Z. It follows that the union $\mathscr{Z}^0$ of all dominating components of $\mathscr{Z}$ has dimension $n+1-r$. The central fiber of the restriction $\pi^0 \colon \mathscr{Z}^0 \to \Delta$ is contained in Z.

Proposition 2.3. *The map $\pi^0 \colon \mathscr{Z}^0 \to \Delta$ is smooth and its central fiber is the zero-locus of $\overline{\sigma'}$ in Z.*

In our situation, the derivative $\frac{d\sigma_t}{dt}\big|_{t=0}$ provides a nonzero element of the normal space to the $\mathrm{GL}(V_{10})$-orbit of σ_0, which can be checked to be $H^0(Q, \mathscr{O}_Q(3))$. It defines a K3 surface $S \subseteq Q \subseteq \mathbf{P}(V_5)$ with a polarization L of degree 6.

After identifying the rank-2 excess bundle on $K_{\sigma_0} = Q^{[2]} \subseteq \mathsf{Gr}(6, V_{10})$, one obtains—using the above proposition—that the central fiber of the family $\mathscr{K}^0 \to \Delta$ is isomorphic to $S^{[2]} \subseteq Q^{[2]}$ and that the Plücker polarization restricts to $2L - \delta$.

It is not difficult to see that all this implies that after blowing up the (semistable) point $[\sigma_0] \in \mathscr{M}_{\mathrm{DV}}$, the exceptional divisor is mapped onto the Heegner divisor $\mathscr{D}_6$ in the period domain $\mathscr{F}$, making it an HLS divisor.

Along this divisor of $\mathscr{M}_{22}^{(2)}$, the polarization is very ample: it defines the embedding

$$\varphi_{2L-\delta}\colon S^{[2]} \subseteq Q^{[2]} = K_{\sigma_0} \subseteq \mathsf{Gr}(6, V_{10}) \subseteq \mathbf{P}(\textstyle\bigwedge^6 V_{10}).$$

2.5. **The divisors $\mathscr{D}_2$, $\mathscr{D}_{10}$, $\mathscr{D}_{18}$, and $\mathscr{D}_{30}$.** We have constructions of various degrees of complexity that prove that $\mathscr{D}_2$, $\mathscr{D}_{10}$, $\mathscr{D}_{18}$, and $\mathscr{D}_{30}$ are HLS divisors. They are all based on Mukai's description of general polarized K3 surfaces of these degrees.

For $\mathscr{D}_{18}$, the trivector σ_0 has stabilizer $G_2 \times \mathrm{SL}(3)$ and K_{σ_0} is smooth of dimension 10. Under a general 1-parameter deformation, the Debarre–Voisin fourfolds K_{σ_t} specialize to a smooth fourfold K_0 isomorphic to $S^{[2]}$, where S is a general degree-18 $K3$ surface. The polarization is very ample in general.

For $\mathscr{D}_2$, we take $V_{10} = \mathrm{Sym}^3 V_3$ and, for σ_0, a generator of the (1-dimensional) space of $\mathrm{SL}(V_3)$-invariants of $\bigwedge^3 V_{10}^\vee$. Its stabilizer is $\mathrm{SL}(V_3)$ and the DV variety K_{σ_0} is the union of two irreducible 4-dimensional varieties.

For $\mathscr{D}_{10}$, we take $V_{10} = \bigwedge^2 V_5$ and σ_0 with stabilizer $\mathrm{SL}(2)$. The corresponding DV variety K_{σ_0} has a smooth 6-dimensional component. The polarization has base-points.

The situation for $\mathscr{D}_{30}$ is not yet clear.

Question 2.4 (Hassett)**.** On the (open) image of $\mathscr{M}_{\mathrm{DV}}^{\mathrm{smooth}}$ in $\mathscr{M}_{22}^{(2)}$, the polarization is very ample. Are the sets

$$\{(K, M) \in \mathscr{M}_{22}^{(2)} \mid M \text{ is not very ample }\}$$

and

$$\{(K, M) \in \mathscr{M}_{22}^{(2)} \mid |M| \text{ has base-points }\}$$

unions of (Heegner) divisors? This question can of course be asked for any moduli space of polarized hyperkähler manifolds.

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