CURVES OF LOW DEGREES IN FANO VARIETIES

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Abstract. We work over the complex numbers. Fano manifolds are smooth projective varieties whose canonical bundle is anti-ample. In dimensions at most 3, they are all classified. However, for most of them, their very rich geometry remains mysterious. Classically, classification was obtained using curves of low degrees. Although more efficient methods are now available, it is important, for other purposes, to understand the varieties that parametrize these curves.

I did not attempt to mention all the (numerous) results in this field, nor did I try to give a complete bibliography nor to give complete proofs of every statement that I make. Instead, my aim was more to give an idea of some of the questions that one can ask about Fano varieties and of the various techniques that are used in this field.

Contents

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1. Cubic hypersurfaces

In this first section, we study the family of lines $F(X)$ contained in a smooth n-dimensional cubic hypersurface $X \subset \mathbf{P}^{n+1}$ and prove that it is always smooth of the expected dimension $2n-4$.

In the case $n=4$, we explain, after Beauville and Donagi, that $F(X)$ is an irreducible symplectic manifold and that the Hodge structure on $H^4(X)$ is isomorphic to the Hodge structure on $H^2(F(X))$, with naturally defined polarizations: the usual intersection product on $H^4(X)$, and the Beauville-Bogomolov form on $H^2(F(X))$.

1.1. **Lines on a cubic surface.** Let $X \subset \mathbf{P}^3 = \mathbf{P}(V)$ be a smooth cubic surface with equation $f(x_0,x_1,x_2,x_3)=0$. We consider the polynomial f as an element of $\operatorname{Sym}^3 V^\vee$.

On the Grassmannian $G := G(2,V)$ parametrizing lines in $\mathbf{P}(V)$, we have an exact sequence of locally free sheaves

$$0 \to \mathscr{S} \to V \otimes \mathscr{O}_G \to \mathscr{Q} \to 0,$$

where $\mathscr{S}$ is the tautological rank-2 subbundle and $\mathscr{Q}$ is the tautological rank-3 quotient bundle. It induces a surjection

$$\operatorname{Sym}^3 V^\vee \otimes \mathscr{O}_G \twoheadrightarrow \operatorname{Sym}^3 \mathscr{S}^\vee$$

hence f defines a section s_f of $\operatorname{Sym}^3 \mathscr{S}^\vee$. We have

$$\begin{aligned} s_f([V_2]) = 0 &\iff f|_{V_2} \equiv 0 \\ &\iff \text{the line } \mathbf{P}(V_2) \text{ is contained in } X. \end{aligned}$$

Define

$$F(X) := \text{zero locus of } s_f \subset G.$$

Since $\dim(G)=4$ and $\operatorname{rank}(\operatorname{Sym}^3 \mathscr{S}^\vee)=4$, we expect $F(X)$ to have dimension 0. Moreover, Schubert calculus gives $c_4(\operatorname{Sym}^3 \mathscr{S}^\vee)=27$, hence $F(X)$ is non-empty.

Local structure of $F(X)$. It is a general fact that at a point corresponding to a line $\ell \subset X$, *the Zariski tangent space to the scheme $F(X)$*

is isomorphic to $H^0(\ell, N_{\ell/X})$. Moreover, *locally at the point* $[\ell]$, *the scheme* $F(X)$ *is defined by* $h^1(\ell, N_{\ell/X})$ *equations in a smooth scheme of dimension* $h^0(\ell, N_{\ell/X})$. In particular, by Riemann-Roch, its dimension at $[\ell]$ is at least $\chi(\ell, N_{\ell/X}) = \deg(N_{\ell/X}) + \mathrm{rank}(N_{\ell/X})$.

Let us look at possible normal bundles $N_{\ell/X}$. We have an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & N_{\ell/X} & \to & N_{\ell/\mathbf{P}^3} & \to & N_{X/\mathbf{P}^3}|_\ell & \to & 0 \\ & & & & \| & & \| & & \\ & & & & \mathscr{O}_\ell(1)^{\oplus 2} & & \mathscr{O}_\ell(3) & & \end{array}$$

hence $N_{\ell/X}$ has degree -1: it is $\mathscr{O}_\ell(-1)$. In particular, $F(X)$ is smooth ($H^1(\ell, N_{\ell/X}) = 0$), of dimension 0 ($\chi(\ell, N_{\ell/X}) = 0$), and consists of 27 smooth points.

1.2. **Lines on a cubic hypersurface.** Let $X \subset \mathbf{P}^{n+1} = \mathbf{P}(V_{n+2})$ be a smooth n-dimensional cubic hypersurface. We follow the same reasoning and expect $F(X) \subset G := G(2, V_{n+2})$ to have codimension 4 (hence dimension $2n-4$) and cohomology class $c_4(\mathrm{Sym}^3 \mathscr{S}^\vee) = 9(2\sigma_{3,1}+3\sigma_{2,2})$.

To check for smoothness, we write the normal exact sequence

$$\begin{array}{ccccccccc} 0 & \to & N_{\ell/X} & \to & N_{\ell/\mathbf{P}^{n+1}} & \to & N_{X/\mathbf{P}^{n+1}}|_\ell & \to & 0. \\ & & & & \| & & \| & & \\ & & & & \mathscr{O}_\ell(1)^{\oplus n} & & \mathscr{O}_\ell(3) & & \end{array}$$

The normal bundle $N_{\ell/X}$ therefore has degree $n-3$ and rank $n-1$. Write it as

$$\mathscr{O}_\ell(a_1) \oplus \cdots \oplus \mathscr{O}_\ell(a_{n-1}),$$

with $a_1 \le \cdots \le a_{n-1}$.

We have $a_{n-1} \le 1$ and $a_1 + \cdots + a_{n-1} = n-3$, hence

$$a_1 = (n-3) - a_2 - \cdots - a_{n-1} \ge -1.$$

This implies as above $H^1(\ell, N_{\ell/X}) = 0$, hence $F(X)$ is smooth of dimension $\chi(\ell, N_{\ell/X}) = (n-3) + (n-1) = 2n-4$. One can show that it is connected when $n \ge 3$.

More precisely, there are two possible types for normal bundles:

$$\begin{array}{rcl} N_{\ell/X} \simeq \mathscr{O}_\ell \oplus \mathscr{O}_\ell \oplus \mathscr{O}_\ell(1)^{\oplus(n-3)} & : & \text{lines of the first type;} \\ N_{\ell/X} \simeq \mathscr{O}_\ell(-1) \oplus \mathscr{O}_\ell(1)^{\oplus(n-2)} & : & \text{lines of the second type.} \end{array}$$

For $n \ge 3$, lines cover X (this is easy: lines through a point $x \in X$ are parametrized by $F_x =$ zero locus (f_1, f_2, f_3) inside the $\mathbf{P}^n$ parametrizing lines through x; this is non-empty for $n \ge 3$).

A line ℓ through a general point $x \in X$ must be *free:* all factors in $N_{\ell/X}$ must have non-negative degree; hence ℓ if of the first type.

Line of the second type are characterized by the fact that

$$H^0(\ell, N_{\ell/\mathbf{P}^{n+1}}(-1)) \to H^0(\ell, N_{X/\mathbf{P}^{n+1}}|_\ell(-1))$$

is *not* surjective. We can globalize the above exact sequence: if $\ell = \mathbf{P}(V_2)$, the vector space $H^0(\ell, N_{\ell/\mathbf{P}^{n+1}}(-1))$ is the fiber at $[V_2]$ of the vector bundle V_{n+2}/V_2, whereas $H^0(\ell, N_{X/\mathbf{P}^{n+1}}|_\ell(-1))$ is the fiber at $[V_2]$ of the vector bundle $\operatorname{Sym}^2 \mathscr{S}^\vee$. So the locus $F_{\rm st}(X) \subset F(X)$ of lines of the second type is the degeneracy locus where the rank of the morphism

$$\begin{array}{rcl} \mathscr{Q} & \longrightarrow & \operatorname{Sym}^2 \mathscr{S}^\vee \\ \overline{v} & \longmapsto & f(v, \cdot, \cdot)|_{V_2} \end{array}$$

is ≤ 2. Its expected codimension is therefore $n-2$. For X general, one can prove that $F_{\rm st}(X)$ is smooth of that dimension.

1.3. **Lines on a cubic fourfold.** Let $X \subset \mathbf{P}^5$ be a smooth cubic hypersurface. The variety $F(X)$ is then also a smooth projective fourfold. By adjunction, we have

$$c_1(F(X)) = c_1(G(2,5)) + c_1(\operatorname{Sym}^3 \mathscr{S}^\vee) = 0.$$

Beauville and Donagi proved in fact that $F(X)$ is an *irreducible symplectic variety:*

- $F(X)$ is simply connected;
- it carries an everywhere non-degenerate 2-form that spans $H^0(F(X), \Omega^2_{F(X)})$ (this implies $c_1 = 0$, of course).

Their proof is indirect; in particular, they do not construct the symplectic form explicitly on $F(X)$ (this was done later by Iliev and Manivel).

Beauville and Donagi show that $F(X)$ is isomorphic, for some particular X called *Pfaffian cubics* (whose equation is given by the Pfaffian of a 6×6, skew-symmetric matrix who coefficients are (general) linear forms; they are rational) to the Hilbert scheme $S^{[2]}$ parametrizing length-2 subschemes of a K3 surface S (of genus 8).

The variety $S^{[2]}$ is known to be irreducible symplectic by another result of Beauville, and secondly, any smooth deformation of an irreducible symplectic variety is still an irreducible symplectic variety.

Another interesting feature of this set-up is that *the Hodge structures $H^4(X, \mathbf{Z})$ and $H^2(F(X), \mathbf{Z})$ are isomorphic.* The (upper) Hodge diamond for X is as follows:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 21 & & 1 & & 0 \end{array}$$

We say that the Hodge structure on $H^4(X)$ is of K3 type.

The isomorphism comes from the *Abel-Jacobi mapping,* defined as follows.

Let $I = \{(x, \ell) \in X \times F(X) \mid x \in L\}$ be the incidence variety. It is a smooth projective fivefold which comes with two projections $p : I \to X$ and $q : I \to F(X)$ (a $\mathbf{P}^1$-bundle). Then we have the following.

Theorem 1.1. *The Abel-Jacobi map*

$$a := q_* p^* : H^4(X, \mathbf{Z}) \longrightarrow H^2(F(X), \mathbf{Z})$$

is an isomorphism.

Proof. First of all, if H is a hyperplane section of X, note that $a([H]^2)$ is the locus of lines meeting a general codimension-2 linear space: this is $s_F := s_1|_{F(X)}$.

Let now $u \in H^4(X, \mathbf{Z})$. Since $q : I \to F(X)$ is a $\mathbf{P}^1$-bundle and p^*H is a birational section of q (a general line contained in X meets H is one point), we can write

$$p^*u = q^*v_1(u) \cdot p^*[H] + q^*v_2(u) \in H^4(I, \mathbf{Z}), \tag{1}$$

with $v_i \in H^{2i}(F(X), \mathbf{Z})$. Applying q_*, we obtain

$$a(u) = v_1(u) q_* p^*[H] = v_1(u).$$

In particular,

$$p^*[H]^2 = q^*s_F \cdot p^*[H] + q^*v_2([H]^2). \tag{2}$$

Assume $u \cdot [H] = 0$ (this equation defines the *primitive cohomology* $H^4(X, \mathbf{Z})_0 \subset H^4(X, \mathbf{Z})$). Multiplying (1) by $p^*[H]$ and using (2), we get

$$\begin{aligned} 0 &= q^*v_1(u) \cdot p^*[H]^2 + q^*v_2(u) \cdot p^*[H] \\ &= q^*v_1(u) \cdot (q^*s_F \cdot p^*[H] + q^*v_2([H]^2)) + q^*v_2(u) \cdot p^*[H] \\ &= q^*(a(u) \cdot s_F + v_2(u)) \cdot p^*[H] + q^*(a(u) \cdot v_2([H]^2)). \end{aligned}$$

Because of the structure of the cohomology of a $\mathbf{P}^1$-bundle, this implies

$$a(u) \cdot s_F + v_2(u) = 0 \quad \text{and} \quad a(u) \cdot v_2([H]^2) = 0. \tag{3}$$

In particular, $v_2([H]^2) = -a([H]^2) \cdot s_F = -s_F^2$ and *a maps the primitive cohomology* $H^4(X, \mathbf{Z})_0$ *to the primitive cohomology* $H^2(F, \mathbf{Z})_0$, defined as the orthogonal of s_F^2. Squaring

$$p^*u = q^*a(u) \cdot p^*[H] - q^*(a(u) \cdot s_F),$$

and using (2) and (3), we obtain

$$\begin{aligned} p^*u^2 &= q^*a(u)^2 \cdot p^*[H]^2 - 2q^*(a(u)^2 \cdot s_F) \cdot p^*[H] + q^*(a(u)^2 \cdot s_F^2) \\ &= q^*a(u)^2 \cdot (q^*s_F \cdot p^*[H] + q^*v_2([H]^2)) \\ &\qquad\qquad -2q^*(a(u)^2 \cdot s_F) \cdot p^*[H] + q^*(a(u)^2 \cdot s_F^2) \\ &= -q^*(a(u)^2 \cdot s_F) \cdot p^*[H] + q^*(a(u)^2 \cdot s_F^2) \in H^8(I, \mathbf{Z}). \end{aligned}$$

Multiply by q^*s_F:

$$p^*u^2 \cdot q^*s_F = -q^*(a(u)^2 \cdot s_F^2) \cdot p^*[H] \in H^{10}(I, \mathbf{Z}) \simeq \mathbf{Z}.$$

Since p^*H is a birational section of q, the right side is $-a(u)^2 \cdot s_F^2$. What is the degree of $q^*\Sigma_1 \to X$? This is the number of lines through a general $x \in X$ meeting a general codimension-2 linear space P; in other words, the number of lines through x contained in $X \cap \langle x, P\rangle$. We saw above that this number is 6. Finally, we have proven

$$u^2 = -\frac{1}{6}a(u)^2 \cdot s_F^2 \in \mathbf{Z}.$$

The non-degeneracy of the product $(\alpha, \beta) \mapsto \frac{1}{6}\alpha \cdot \beta \cdot s_F^2$ on $H^2(F, \mathbf{Z})_0$ (a fact that comes from Hodge-Lefschetz theory) implies that a is injective.

One can go further and check that this product is the (integral) *Beauville-Bogomolov form* q_F on $H^2(F, \mathbf{Z})_0$. The computation above then shows that

$$a : (H^4(X, \mathbf{Z})_0, \cdot) \longrightarrow (H^2(F, \mathbf{Z})_0, q_F)$$

is an isometry. Using the forms of these lattices, we conclude that a is an isomorphism. □

Just a few words on the Beauville-Bogomolov form. Let F be an irreducible symplectic variety of dimension $2n$. There is a unique integral and nondivisible quadratic form q_F on $H^2(F, \mathbf{Z})$ and a positive rational constant c_F such that

$$\forall \alpha \in H^2(F, \mathbf{R}) \qquad \alpha^{2n} = c_F q_F(\alpha)^n.$$

The signature of q_F is $(3, b_2(F) - 3)$. When F is a deformation of the Hilbert scheme $S^{[n]}$ parametrizing length-n subschemes of a K3 surface S, we have $c_F = (2n)!/(n!2^n)$ (in particular $c_{S^{[2]}} = 3$). Beauville's original definition of this form was slightly different (the normalization was also different); on the primitive cohomology $H^2(F, \mathbf{Z})_0$, the form q_F is proportional, *for any Kähler form ω on F,* to the more standard form

$$q_\omega(\alpha) = \alpha^2 \cdot \omega^{2n-4},$$

a fact that we used above.

2. Fano manifolds

This short section contains the definition of Fano manifolds, together with some elementary properties.

2.1. **Definition.** A Fano manifold is a smooth projective variety X whose anticanonical divisor $-K_X$ is ample. A smooth cubic hypersurface $X \subset \mathbf{P}^{n+1}$ is a Fano manifold for $n \geq 2$ because, by adjunction, $K_X = -(n-1)H$.

Note that

$$\forall i > 0 \qquad H^i(X, \mathscr{O}_X) = H^i(X, K_X + (-K_X)) = 0$$

by Kodaira's vanishing.

2.2. **The Picard number.** In particular, $\mathrm{Pic}(X) \simeq H^2(X, \mathbf{Z})$, and this is a finitely generated abelian group. We now prove that it is *torsion-free,* or that numerical equivalence is the same as linear equivalence.

If $D \underset{\text{num}}{\equiv} 0$, we have $H^i(X, D) = H^i(X, K_X + (-K_X + D)) = 0$ for all $i > 0$ by Kodaira vanishing, and $\chi(X, D) = \chi(X, \mathscr{O}_X) = 1$, hence $h^0(X, D) = 1$. Any effective divisor $D' \underset{\text{lin}}{\equiv} D$ has zero intersection number with any curve, hence is 0 (because X is projective). We denote by ρ_X the rank of $\mathrm{Pic}(X)$ (this is also $b_2(X)$) and call it the *Picard number* of X.

2.3. **The index.** The *index* of X is the largest positive integer r such that the class $-K_X$ is divisible by r in $\mathrm{Pic}(X)$. We usually write $-K_X = rH$; the ample divisor H is called a fundamental divisor. We say that X is a *prime* Fano variety if its index is 1.

If $n := \dim(X)$, we have $r \leq n+1$. This follows from Mori's Cone Theorem, but also from the following elementary argument: for $-r < m < 0$, we have

$$\forall i \in \mathbf{Z} \qquad H^i(X, mH) = H^i(X, K_X + (m+r)H) = 0$$

(for $i = 0$ because $m + r < 0$, for $i > 0$ by Kodaira's vanishing). Hence the non-indentically-zero degree-n polynomial map $t \mapsto \chi(X, tH)$ vanishes at $r - 1$ points. This implies $r - 1 \leq n$.

3. Del Pezzo varieties

The "only" Fano manifold of dimension 1 is $\mathbf{P}^1$. In dimension 2, Fano manifolds are called del Pezzo surfaces. They are all rational and are classified: $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{P}^2$ blown-up in at most 8 points in general position.

There is also a complete classification in dimension 3, but it is long. It is possible to classify Fano n-folds X with "high" index r.

If $r = n+1$, one shows that X is isomorphic to $\mathbf{P}^n$.

If $r = n$, one can show that X is isomorphic to a smooth quadric in $\mathbf{P}^{n+1}$.

If $r = n-1$, these varieties are called *del Pezzo varieties.* They were classified by Iskovskikh and Fujita. Among them, we find smooth cubic hypersurfaces in $\mathbf{P}^{n+1}$, smooth intersections of two quadrics in $\mathbf{P}^{n+2}$, and double covers of $\mathbf{P}^n$ branched along a smooth quartic hypersurface. The complete list is in [IP], Theorem 3.3.1.

Write $-K_X = (n-1)H$ and set $d := H^n > 0$. Riemman-Roch then gives $h^0(X,H) = \chi(X,H) = d+n-1$. We will not explain the complete classification, but concentrate on the case $\rho_X = 1$ and $d \geq 5$ (this excludes the infinite series of examples given above). The final results are Corollary 3.2 and Theorem 3.3 below.

When $d \geq 3$, Fujita first proves that $\varphi_H : X \to \mathbf{P}^{d+n-2}$ is an embedding. Taking $n-2$ hyperplane sections, we end up with a smooth del Pezzo surface S, for which we know that the degree satisfied $d \leq 9$. Cases $d = 8$ and $d = 9$ are then excluded.

Since $d \leq 7$, it follows from classification of del Pezzo surfaces that S, hence also X, contains a line ℓ. We have $-K_S \cdot \ell = 1$, hence, by adjunction, $\ell^2 = -1$, hence $N_{\ell/S} \simeq \mathscr{O}_\ell(-1)$.

Assume now $n = 3$ and consider the normal exact sequence

$$\begin{array}{ccccccccc} 0 & \to & N_{\ell/S} & \to & N_{\ell/X} & \to & N_{S/X}|_\ell & \to & 0. \\ & & \| & & & & \| & & \\ & & \mathscr{O}_\ell(-1) & & & & \mathscr{O}_\ell(1) & & \end{array}$$

It implies that there are only two possibilities:

$$\begin{array}{rcl} N_{\ell/X} \simeq \mathscr{O}_\ell \oplus \mathscr{O}_\ell & : & \text{lines of the first type;} \\ N_{\ell/X} \simeq \mathscr{O}_\ell(-1) \oplus \mathscr{O}_\ell(1) & : & \text{lines of the second type.} \end{array}$$

The scheme $F(X) \subset G(1, \mathbf{P}^{d+1})$ of lines contained in $X \subset \mathbf{P}^{d+1}$ is thus smooth of dimension 2.

Let $\varepsilon : \widetilde{X} \to X$ be the blow-up of a line ℓ of the first type. Then $\widetilde{X}$ is smooth and

$$-K_{\widetilde{X}} = 2\varepsilon^* H - E = \varepsilon^* H + (\varepsilon^* H - E)$$

is ample. In other words, $\widetilde{X}$ is a Fano threefold with $\rho_{\widetilde{X}} \geq 2$. By Mori's theory, there is $K_{\widetilde{X}}$-negative contraction $\varepsilon^+ : \widetilde{X} \to X^+$ other than ε. We will describe this contraction.

Assume now $d \geq 5$. Riemann-Roch and Kawamata-Viehweg vanishing give $h^0(\widetilde{X}, \varepsilon^*H - 2E) = d - 4$.

Assume further $\rho_X = 1$, so that $\widetilde{X}$ has Picard number 2. Computing intersection numbers of irreducible divisors in the linear systems $|\varepsilon^*H - kE|$, for $k \geq 2$, one easily concludes $d = 5$.

Take now $D \underset{\text{lin}}{\equiv} \varepsilon^*H - 2E$ and write $D = D' + mE$, with D' irreducible. Then

$$0 \leq D' \cdot (\varepsilon^*H - E)^2 = d - 5 - 2m,$$

hence $m = 0$ (i.e., D is irreducible) and the morphism $\varphi_{\varepsilon^*H-E} : \widetilde{X} \twoheadrightarrow X^+ \subset \mathbf{P}^4$ (which is nothing else but the projection from ℓ) contracts D. It must therefore be the other contraction ε^+. Note that $h^0(\widetilde{X}, (\varepsilon^*H - E) - D) = h^0(\widetilde{X}, E) = 1$, hence $\varepsilon^+(D)$ is not a point, hence a curve. The classification of extremal K-negative contractions gives that X^+ is smooth and $\varepsilon^+ : \widetilde{X} \to X^+$ is the blow-up of a smooth curve $C := \varepsilon^+(D) \subset X^+$. One then has

$$\varepsilon^{+*}(-K_{X^+}) = -K_{\widetilde{X}} + D = 3(\varepsilon^*H - E).$$

It follows that $X^+ \subset \mathbf{P}^4$ is a Fano threefold of index 3, hence a smooth quadric. One then checks that C a twisted cubic contained in a smooth hyperplane section of X^+. We have therefore proved the following.

Theorem 3.1. *Let X be a smooth Fano threefold with Picard number 1, $-K_X = 2H$, and $H^3 \geq 5$. Then X contains a line of the first type and the blow-up of this line is the blow-up of a smooth quadric $X^+ \subset \mathbf{P}^4$ along a smooth twisted cubic contained in a smooth hyperplane section of X^+.*

Since the automorphism group of X^+ acts transitively on the set of such curves $C \subset X^+$, one sees that all these X are actually isomorphic. On the other hand, it is easy to construct another one of these, as a smooth linear section of $G(2,5) \subset \mathbf{P}^9$ by a $\mathbf{P}^6$.

Corollary 3.2. *Let X be a smooth Fano threefold with Picard number 1, $-K_X = 2H$, and $d := H^3 \geq 5$. Then $d = 5$ and X is isomorphic to a smooth linear section of $G(2,5) \subset \mathbf{P}^9$ by a $\mathbf{P}^6$.*

Similar techniques allowed Fujita to prove the following.

Theorem 3.3. *Let X be a smooth Fano manifold of dimension $n \geq 3$ with Picard number 1, $-K_X = (n-1)H$, and $H^n = 5$. Then $n \in \{3,4,5,6\}$ and X is isomorphic to a smooth linear section of $G(2,5) \subset \mathbf{P}^9$ by a $\mathbf{P}^{n+3}$.*

Again, given n, these smooth linear sections are all isomorphic.

4. Fano varieties of coindex 3

We will be concerned with Fano manifolds of dimension n, Picard number one, and index $n-2$ (the "coindex" is $n+1-\text{index}$). The original classification, due to Iskovskikh and his students, was very geometric and relied heavily on techniques similar to the ones used by Fujita, using the existence of lines and conics (by no means obvious).

Later, Gushel (in dimension 3) and Mukai discovered a very elegant and clever method using vector bundles. It had also the advantage of working over non-algebraically closed fields (of characteristic zero). We will briefly explain this method and give more details in the degree-10 case.

The idea is that since many Fano manifolds are eventually contained in a Grassmannian G, one should try to construct directly the embedding $X \hookrightarrow G$ by producing a rank-2 vector bundle $\mathscr{E}$ on X generated by global sections. This then gives a morphism

$$f : X \to G(2, H^0(X, \mathscr{E})^\vee)$$

defined by sending $x \in X$ to the codimension-2 linear subspace of $H^0(X, \mathscr{E})$ of sections that vanish at x. It satisfies $f^*\mathscr{S}^\vee \simeq \mathscr{E}$.

So assume $-K_X = (n-2)H$. When H is very ample (which often occurs; see (4) below), taking $n-2$ hyperplane sections now gives a smooth K3 surface, and taking one more section gives a canonical curve of genus $g := \frac{1}{2}H^n + 1$. This is why this number is called the *genus* of X.

Again, Riemman-Roch gives $h^0(X, H) = \chi(X, H) = g + n - 1$. One can then show (Shokurov, Mella) that

$$\varphi_H : X \to \mathbf{P}^{g+n-2} \tag{4}$$

is an embedding for $g \geq 4$.

Here are some examples: smooth quartic hypersurfaces in $\mathbf{P}^{n+1}$ (genus 3), smooth intersections of a quadric and a cubic in $\mathbf{P}^{n+2}$ (genus 4), and smooth linear sections by a $\mathbf{P}^{n+4}$ of the intersection of $G(2,5) \subset \mathbf{P}^9$ with a quadric (genus 6). The complete list is in [IP], Theorem 5.2.3.

Theorem 4.1 (Mukai). *Let X be a smooth Fano manifold of dimension $n \geq 3$ with Picard number 1, $-K_X = (n-2)H$, and $g := \frac{1}{2}H^n + 1 \geq 6$. Then $n \in \{3, \dots, 10\}$ and $g \leq 12$, $g \neq 11$.*

We will only sketch the proof given in [M1] (not all details are given there...). First of all, by taking hyperplane sections, we may assume $n = 3$, so that $X \subset \mathbf{P}^{g+1}$ is a smooth threefold of degree $2g-2$ and $-K_X \underset{\text{lin}}{\equiv} H$.

Mukai says that a deformation argument shows that if X is general, a general hyperplane section of X is a general polarized K3 surface (of degree $2g-2$). Now take a general pencil $\Lambda \subset |H|$ of such hyperplane sections. We obtain a rational map

$$\Lambda \dashrightarrow \mathscr{F}_g := \left\{ \begin{matrix} \text{19-dimensional moduli space of} \\ \text{polarized K3 surfaces of degree } 2g-2 \end{matrix} \right\}$$

which is not constant since Λ contains singular members. The base locus C_Λ of Λ is a genus-g canonical curve. This implies that the rational map

$$\begin{matrix} \mathscr{P}_g := \bigcup_{(S,H_S)\in\mathscr{F}_g} |H_S| & \stackrel{\varphi_g}{\dashrightarrow} & \mathscr{M}_g \\ \text{dimension } 19+g & & \text{dimension } 3g-3 \end{matrix}$$

is not finite over the point $[C_\Lambda]$, which is general in the image of φ_g. In other words, φ_g *is not generically finite onto its image.* For $g \le 10$, this is not surprising since $19+g > 3g-3$.

Mukai then proves that φ_g is generically finite if and only if $g = 11$ or $g \ge 13$ (actually, he only gives a proof for $g = 11$, and says that the case $g \ge 13$ is similar). This implies the part of the theorem concerning the possible values of g.

We will now show how Mukai proceeds in the case $g = 6$.

Theorem 4.2 (Mukai). *Let X be a smooth Fano manifold of dimension $n \ge 3$ with X has Picard number 1, $-K_X = (n-2)H$, and $g := \frac{1}{2}H^n + 1 = 6$. Then $n \in \{3,4,5\}$ and X is*

- *either a smooth linear section by a $\mathbf{P}^{n+4}$ of the intersection of $G(2,5) \subset \mathbf{P}^9$ with a quadric;*
- *or a double cover of a smooth linear section by a $\mathbf{P}^{n+3}$ of $G(2,5) \subset \mathbf{P}^9$, branched along its intersection with a quadric.*

Note that the second case is a degeneration of the first case: consider, in $\mathbf{P}^{10}$, the 7-dimensional cone $CG(2,5)$ over $G(2,5) \subset \mathbf{P}^9$, with vertex v, and its intersection with a general quadric and a $\mathbf{P}^{n+4}$. If the $\mathbf{P}^{n+4}$ does not contain v, everything takes place in that linear space and we are in the first case; if the $\mathbf{P}^{n+4}$ does contain v, we are in the second case.

Proof. The first step is to construct the morphism $X \to G(2,5)$ i.e., a globally generated rank-2 vector bundle $\mathscr{E}$ on X. Its restriction $\mathscr{E}_S$ to a K3 surface section S must satisfy $c_1(\mathscr{E}_S) = H_S$ and $c_2(\mathscr{E}_S) = 4$.

So we start out by constructing such a vector bundle on the genus-6 K3 surface $S \subset \mathbf{P}^6$ by Serre's construction. A general hyperplane section of S is a canonical curve C of genus 6, hence has a g^1_4. Let $Z \subset C$ be an element of this pencil, so that

$$\begin{array}{rcl} h^0(S, \mathscr{I}_Z(H_S)) & = & 1 + h^0(C, H_S - Z) \\ & = & 1 + h^1(C, g^1_4) \\ & = & 1 + h^0(C, g^1_4) - (4 + 1 - 6) = 4. \end{array}$$

Now by Serre's construction, there is a vector bundle $\mathscr{E}_S$ on S that fits into an exact sequence

$$0 \to \mathscr{O}_S \to \mathscr{E}_S \to \mathscr{I}_Z(H_S) \to 0$$

We then have $c_1(\mathscr{E}_S) = [H_S]$ and $c_2(\mathscr{E}_S) = [Z]$ and

- $h^0(S, \mathscr{E}_S) = 1 + h^0(S, \mathscr{I}_Z(H_S)) = 5$;
- $H^i(S, \mathscr{E}_S) = 0$ for all $i > 0$;
- $\mathscr{E}_S$ is stable and rigid.

Assume $n = 3$. A Lefschetz-type result of Fujita ([F], Proposition (2.1)) says that $\mathscr{E}_S$ first extends to a locally free sheaf $\widehat{\mathscr{E}_S}$ on the formal completion $\widehat{X}$ of X along S, provided $H^2(S, \mathscr{E}nd\,\mathscr{E}_S(tH_S)) = 0$ for all $t < 0$, which holds in our case, then to a locally free sheaf $\mathscr{E}$ on X provided $H^1(S, \mathscr{E}_S(tH_S)) = 0$ for all $t \in \mathbf{Z}$, which also holds in our case.

Then one checks that $V_5 := H^0(X, \mathscr{E})^\vee$ has dimension 5 and that $\mathscr{E}$ is generated by global sections hence defines a morphism $\psi : X \to G(2, V_5)$.

We have a linear map

$$\eta : \wedge^2 H^0(X, \mathscr{E}) \to H^0(X, \wedge^2 \mathscr{E}) \simeq H^0(X, H)$$

and a commutative diagram (recall from(4) that φ_H is an embedding)

$$\begin{array}{ccc} X & \xrightarrow{\quad\psi\quad} & G(2, V_5) \\ {\scriptstyle \varphi_H}\big\downarrow & & \big\downarrow \\ \mathbf{P}(H^0(X, H)^\vee) & \overset{\eta^\vee}{\dashrightarrow} & \mathbf{P}(\wedge^2 V_5). \end{array}$$

If η is surjective, the map $\eta^\vee$ in the diagram is a morphism, which is moreover injective. The image of $\eta^\vee$ is a $\mathbf{P}^7 \subset \mathbf{P}(\wedge^2 V_5)$. The intersection $W_4 := G(2, V_5) \cap \mathbf{P}^7$ has dimension 4 (because all hyperplane sections of $G(2, V_5)$ are irreducible) and degree 5, and $\varphi_H(X) \subset W_4$ is a smooth divisor. It follows that W_4 must be smooth (otherwise, one easily sees that its singular locus would have positive dimension hence would meet $\varphi_H(X)$, which is impossible). Thus, $\varphi_H(X)$ is a Cartier divisor, of degree 10, hence is the intersection of W_4 with a quadric.

If the corank of η *is 1,* consider the cone $CG \subset \mathbf{P}(\mathbf{C} \oplus \wedge^2 V_5)$, with vertex $v = \mathbf{P}(\mathbf{C})$, over $G(2, V_5)$. In the above diagram, we may view $H^0(X, H)^\vee$ as embedded in $\mathbf{C} \oplus \wedge^2 V_5$ by mapping $(\mathrm{Im}(\eta))^\perp$ to $\mathbf{C}$, and $\eta^\vee$ as the projection from v, with image a $\mathbf{P}^7 \subset \mathbf{P}(\wedge^2 V_5)$. Again, the intersection $W_4 := G(2, V_5) \cap \mathbf{P}^7 = \mathrm{Im}(\psi)$ is smooth and $\varphi_H(X)$ is the intersection of the (smooth locus of the) degree-5 cone CW_4 with a quadric.

If the corank of η *is* ≥ 2, one shows that $\varphi_H(X)$ must be singular, which is impossible. So this proves the theorem when $n = 3$.

For $n \geq 4$, one extends $\mathscr{E}$ to successive hyperplane sections of X all the way to X and proceed similarly (see [IP], Proposition 5.2.7; Mukai does not provide any details). □

5. Prime Fano threefolds of genus 6

Assume that the smooth Fano threefold $X \subset \mathbf{P}^7$ falls into the first case of Mukai's theorem above: X is the (smooth) intersection of $G := G(2, V_5) \subset \mathbf{P}^9$ with a quadric Q and a $\mathbf{P}^7$. Note that since G is an intersection of quadrics, so is X. Also, by Lefschetz theorem, $\mathrm{Pic}(X) = \mathbf{Z}[H]$; in particular, the degree of any surface contained in X is divisible by 10 and X contains no planes.

In this section, we study the variety $F(X)$ of lines contained in X, the variety $C(X)$ of conics contained in X, and the period map, which associates with X its intermediate Jacobian $J(X)$, a 10-dimensional principally polarized abelian variety, and we briefly discuss rationality problems.

5.1. **Lines.** As in the case of cubic hypersurfaces (see §1.2), one can construct the scheme $F(X)$ of lines contained in X as the zero locus of some vector bundle of rank 44 on the Grassmannian $G(2, \wedge^2 V_5)$, whose top Chern class is non-zero. In particular, $F(X)$ is non-empty (and everywhere of dimension ≥ 1): X contains a line ℓ.

Now, *a general hyperplane section of* X *containing* ℓ *is a smooth K3 surface* S. To prove that, one needs to apply Bertini on X and on the blow-up $\varepsilon : \widetilde{X} \to X$ of X along ℓ: a general member $\widetilde{S}$ of the base-point-free linear system $|\varepsilon^* H - E|$ is a smooth surface and, since $|\varepsilon^* H - E||_E$ is also base-point-free, $\widetilde{S}|_E$ is also smooth. Non-trivial fibers of $\widetilde{S} \to S$ can only be fibers of $E \to \ell$, but this contradicts the smoothness of $\widetilde{S}|_E$ (which cannot be a union of fibers). It follows that ε induces an isomorphism from $\widetilde{S}$ to S; the latter is thus smooth.

We have $K_S \cdot \ell = 0$, hence, by adjunction, $\ell^2 = -2$, hence $N_{\ell/S} \simeq \mathscr{O}_\ell(-2)$. Consider the normal exact sequence

$$\begin{array}{ccccccccc} 0 & \to & N_{\ell/S} & \to & N_{\ell/X} & \to & N_{S/X}|_\ell & \to & 0. \\ & & \| & & & & \| & & \\ & & \mathscr{O}_\ell(-2) & & & & \mathscr{O}_\ell(1) & & \end{array}$$

It implies that there are only two possibilities:

$$\begin{array}{rcl} N_{\ell/X} \simeq \mathscr{O}_\ell(-1) \oplus \mathscr{O}_\ell & : & \text{lines of the first type;} \\ N_{\ell/X} \simeq \mathscr{O}_\ell(-2) \oplus \mathscr{O}_\ell(1) & : & \text{lines of the second type.} \end{array}$$

Proposition 5.1. *For any X as above, the scheme $F(X)$ of lines contained in X has pure dimension 1. Its singular locus corresponds to lines of the second type. Through any point of X, there pass only finitely many lines.*

When X is general, $F(X)$ is a smooth connected curve of genus 71.

Proof. From the shape of their normal bundles, we see that lines of the first type correspond to smooth one-dimensional points of $F(X)$. If $F(X)$ has a component of dimension ≥ 2, it is everyhere smooth of dimension 2 and corresponds entirely to lines of the second type. Since these lines are not free, they cannot cover X, hence cover a surface which contains ∞^2 lines, hence must be a plane. But this is impossible since X contains no planes. We have proved that $F(X)$ has pure dimension 1. Furthermore, lines of the second type correspond to (planar) singular points since the Zariski tangent space to $F(X)$ has dimension 2 at the corresponding points. This does not exclude the possibility (although it does not actually happen) that there be infinitely many such lines; they would then form a non-reduced 1-dimensional component of $F(X)$.

The general theory of Hilbert schemes says that the tangent space to the subscheme of $F(X)$ consisting of lines passing through a fixed point $x \in X$ is isomorphic to $H^0(\ell, N_{\ell/X}(-1))$. This is at most 1-dimensional. One checks that if that family is actually 1-dimensional, the surface these lines sweep out in X has degree < 10, which is impossible. Therefore, only finitely many lines contained in X pass through x. In fact, there are at most 4 such lines ([IP], Proposition 4.2.2).

Generic smoothness is enough to show that $F(X)$ is smooth for X general. Connectedness and the computation of the genus is more difficult. □

One can use lines contained in X to construct interesting birational maps, a little in the spirit of the proof of Theorem 3.1.

Let $\ell \subset X$ be a line of the first type and consider the projection $\pi_\ell : X \dashrightarrow \mathbf{P}^5$. If $\varepsilon : \widetilde{X} \to X$ is the blow-up of ℓ, with exceptional divisor E, the composition

$$\varphi : \widetilde{X} \xrightarrow{\varepsilon} X \overset{\pi_\ell}{\dashrightarrow} \mathbf{P}^5$$

is a morphism, associated with the base-point-free linear system $|\varepsilon^* H - E|$. One has

$$-K_{\widetilde{X}} = \varepsilon^*(-K_X) - E = \varphi^*\mathscr{O}(1).$$

Moreover, $(\varepsilon^* H - E)^3 = 10$ hence φ is generically finite.

What are the fibers of φ? If $x \in \widetilde{X} \smallsetminus E$, we have

$$\varphi^{-1}(\varphi(x)) \smallsetminus E = (\langle \ell, x\rangle \cap X) \smallsetminus \ell.$$

Since X is an intersection of quadrics containing ℓ and contains no planes, the right side is an intersection of lines through x, hence is

- either $\{x\}$;
- or a line through x, contained in X and meeting ℓ.

Since X is not covered by lines, we are in the first case for x general, hence φ is birational. We have therefore

$$\operatorname{Exc}(\varphi) \smallsetminus E = \Big(\bigcup \text{lines in } X \text{ meeting } \ell\Big) \smallsetminus \ell.$$

One can prove that there is only a finite, non-zero number of such lines ([IP], Proposition 4.3.1); this means (after checking that nothing goes wrong on $E \simeq \mathbf{P}(N_{\ell/X}) \simeq \ell \times \mathbf{P}^1$) that φ *is a small contraction.* Note however that since $-K_{\widetilde{X}} = \varphi^*\mathscr{O}(1)$, it is *not* $-K_{\widetilde{X}}$-negative, but $-K_{\widetilde{X}}$-trivial.

If ℓ' is a line contracted by φ, we have $\varphi^* H \cdot \ell' = 0$ and $E \cdot \ell' = 1$. Since $\varphi(E) \subsetneq \varphi(\widetilde{X})$, we have $|m\varphi^* H - E| \neq \varnothing$ for $m \gg 0$, and for D in that linear system, $D \cdot \ell' < 0$. Since the Picard number of $\widetilde{X}$ is 2, the relative Picard number of φ is 1; this means that D is φ-antiample. In that situation, we can perform a D-flop (which is nothing else than a flip for the klt pair $(\widetilde{X}, tD)$ for $t > 0$ small). This is a commutative diagram

$$\begin{array}{ccccc}
\widetilde{X} & & \overset{\chi}{\dashrightarrow} & & \widetilde{X}^+ \\
\Big\downarrow \varepsilon & \overset{\varphi}{\searrow} & & \overset{\varphi^+}{\swarrow} & \\
 & & \bar{X} & & \\
 & \overset{\pi_\ell}{\nearrow} & & & \\
X & & & &
\end{array}$$

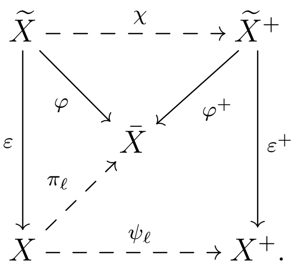
where

- $\bar{X}$ is the normalization of $\varphi(\widetilde{X})$;
- χ is an isomorphism in codimension 1;
- the projective threefold $\widetilde{X}^+$ is smooth;
- if $\bar{H}$ is a hyperplane section of $\bar{X}$, we have $-K_{\widetilde{X}^+} \underset{\text{lin}}{\equiv} \varphi^{+*}\bar{H}$ and $\chi_*(D)$ is φ^+-ample, or equivalently, $\chi_*(E)$ is φ^+-antiample.

We have $\rho(\widetilde{X}^+) = \rho(\widetilde{X}) = 2$. Since the extremal ray generated by the class of curves contracted by φ^+ has $K_{\widetilde{X}^+}$-degree 0 and $K_{\widetilde{X}^+}$ is not nef, the other extremal ray is $K_{\widetilde{X}^+}$-negative and defines a contraction $\varepsilon^+ : \widetilde{X}^+ \to X^+$. One can prove ([IP], Proposition 4.3.3.(iii)):

- X^+ is again a smooth Fano threefold of degree 10 in $\mathbf{P}^7$;
- ε^+ is the blow-up of a line $\ell^+ \subset X^+$, with exceptional divisor $E^+ \underset{\text{lin}}{\equiv} -K_{\widetilde{X}^+} - \chi_*(E)$.

So we may complete the diagram above as follows:



If H^+ is a hyperplane section of X^+, we have

$$\chi^*\varepsilon^{+*}H^+ \underset{\text{lin}}{\equiv} \chi^*(-K_{\widetilde{X}^+} + E^+) \underset{\text{lin}}{\equiv} \chi^*(-2K_{\widetilde{X}^+} - \chi_*(E)) \underset{\text{lin}}{\equiv} -2\varepsilon^*K_X - 3E,$$

hence ψ_ℓ is associated with a linear subsystem of $|\mathscr{I}_\ell^3(2)|$.

Note that the process can be reversed: the elementary transformation of X^+ along the line ℓ^+ is $\psi_\ell^{-1} : X^+ \dashrightarrow X$.

This very important construction was one of the cornerstones of Iskovskikh's method of classification (see [IP], Theorem 4.3.3). It can be performed on all Fano threefolds and can be used to

- prove the existence of lines or conics;
- prove the condition on the genus: $g \leq 12$, $g \neq 11$ in Mukai's theorem;
- construct interesting birational isomorphisms between various threefolds.

One can also use this transform to construct Fano threefolds, starting from the end result (often simpler) and reversing the construction. Genus-6 prime Fano threefolds are very peculiar to this respect: they are the only prime Fano threefolds for which the transform is still in the same family.

Note that it is not obvious whether the various "line-transforms" are isomorphic to X (they are not, in general!) or mutually isomorphic.

5.2. **Conics.** In some sense, there are not enough lines on our threefold X. It will be more fruitful to look at conics contained in X. I will skip the details (which can be found in [L] or [DIM]), just mentioning the results (analogous to what we saw for lines).

Let $c \subset X$ be a smooth conic. The possible normal bundles are

$$\begin{aligned} N_{c/X} &\simeq \mathscr{O}_c \oplus \mathscr{O}_c, \\ N_{c/X} &\simeq \mathscr{O}_c(-1) \oplus \mathscr{O}_c(1), \\ N_{c/X} &\simeq \mathscr{O}_c(-2) \oplus \mathscr{O}_c(2). \end{aligned}$$

Let $C(X)$ be the projective variety that parametrizes (possibly degenerate) conics in X. It has dimension $\geq \chi(c, N_{c/X}) = 2$ at $[c]$ (and in fact everywhere). It is smooth of dimension 2 at $[c]$ for the first two types of normal bundles and by the same argument as for lines, we obtain that $C(X)$ *has pure dimension 2;* one can show that it is also irreducible. When X is general, the third type does not occur and $C(X)$ is smooth. In fact, we have more.

Theorem 5.2 (Logachev)**.** *For X general, $C(X)$ is a smooth connected surface of general type, not minimal.*

I will go into more details to explain why this surface is not minimal. First of all, there are only two kinds of planes contained in $G(2, 5)$:

- α-planes: $\{[V_2] \mid V_1 \subset V_2 \subset V_4\}$;
- β-planes: $\{[V_2] \mid V_2 \subset V_3\}$.

Then, there are only three kinds of (possibly non-reduced or reducible) conics c in $G(2, 5)$:

- τ-*conics*: the 2-plane $\langle c \rangle$ is not contained in $G(2, V_5)$, there is a unique hyperplane $V_4 \subset V_5$ such that $c \subset G(2, V_4)$, the conic c is reduced and, if it is smooth, the union of the corresponding lines in $\mathbf{P}(V_5)$ is a smooth quadric surface in $\mathbf{P}(V_4)$;
- σ-*conics*: the 2-plane $\langle c \rangle$ is an α-plane, there is a unique hyperplane $V_4 \subset V_5$ such that $c \subset G(2, V_4)$, and the union of the corresponding lines in $\mathbf{P}(V_5)$ is a quadric cone in $\mathbf{P}(V_4)$;
- ρ-*conics*: the 2-plane $\langle c \rangle$ is a β-plane and the union of the corresponding lines in $\mathbf{P}(V_5)$ is this 2-plane.

One checks that X contains a unique ρ-conic, which we denote by c_X. It is smooth when X is general. Let us consider

$$\widetilde{C}(X) := \{(c, V_4) \in C(X) \times \mathbf{P}(V_5^\vee) \mid c \subset G(2, V_4) \cap X\}.$$

The first projection $\eta : \widetilde{C}(X) \to C(X)$ is the blow-up of the point $[c_X]$ of $C(X)$.

There is an involution ι on the surface $\widetilde{C}(X)$, constructed as follows. The intersection

$$G(2, V_4) \cap X = G(2, V_4) \cap (\text{codim. 2 linear space}) \cap Q$$

in $\mathbf{P}(\wedge^2 V_4) = \mathbf{P}^5$ is a degree-4, genus-1, 1-cycle in $\mathbf{P}^3$. If $c \subset G(2, V_4) \cap X$, it is the union of c and another conic $c' \subset G(2, V_4) \cap X$ meeting c in two points. The involution $\iota : c \mapsto c'$ on $\widetilde{C}(X)$ is base-point-free when X is general (more precisely, its fixed points correspond to conics of the third type).

So we have two disjoint exceptional (-1)-curves on the surface $\widetilde{C}(X)$: $\eta^{-1}([c_X])$ and its image by ι. The image of the latter on the surface $C(X)$ is an exceptional (-1)-curve which we can contract to obtain a *minimal surface* $C_m(X)$.

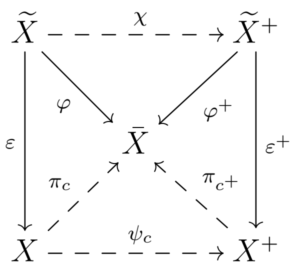
As in the case of lines, we can perform an elementary transformation along a smooth conic $c \subset X$. Keeping analogous notation, we just remark that if $x \in \widetilde{X} - E$, we have

$$\varphi^{-1}(\varphi(x)) = \langle c, x \rangle \cap X,$$

outside of E. Since X is an intersection of quadrics and contains no planes nor quadric surfaces, this is

- either $\{x\}$;
- or a line through x, contained in X and meeting c;
- or a conic through x, contained in X and meeting c in two points.

Again, only finitely many lines contained in X meet c (at least when c or X is general), and in the third case, the only possible conic is $\iota(c)$. In particular, φ is a small contraction and we can perform a flop and an extremal contraction to obtain a commutative diagram



Again, X^+ is a smooth Fano threefold of the same family which contains a conic $c^+ = \varepsilon^+(E^+)$. The rational map ψ_c is associated with a linear subsystem of $|\mathscr{I}_c^4(3)|$.

5.3. **Intermediate Jacobians and the period map.** Given a (smooth) Fano threefold, or more generally any smooth threefold X with $H^{0,3}(X) = 0$, one can define a complex torus by setting

$$J(X) := H^{1,2}(X)/H^3(X, \mathbf{Z})$$

(with a slight abuse of notation). The unimodular intersection form on $H^3(X, \mathbf{Z})$ defines a *principal polarization* that makes $J(X)$ into a principally polarized abelian variety.

For example, *when* $X \subset \mathbf{P}^4$ *is a smooth cubic threefold,* we have $h^{1,2}(X) = 5$, so that $J(X)$ is a 5-dimensional principally polarized abelian variety. The geometry of $J(X)$ and its theta divisors are very much related to the geometry of X because of the following classical construction: as we saw earlier, the variety $F(X)$ of lines contained in X is a smooth projective surface. One defines classically the Abel-Jacobi map $a : F(X) \to J(X)$. Clemens and Griffiths proved that it induces an isomorphism between the Albanese variety of $F(X)$ and $J(X)$, and that the fourfold $\Theta := a(F(X)) - a(F(X)) \subset J(X)$ is a theta divisor (see §5.4 for a nice application).

Let $\mathscr{C}_3$ be the (quasi-projective, irreducible) 10-dimensional moduli space of smooth cubic threefolds. Griffiths proved that the intermediate Jacobian construction induces an algebraic *period map*

$$\begin{aligned} \wp_{\mathscr{C}} : \mathscr{C}_3 &\longrightarrow \mathscr{A}_5 \\ [X] &\longmapsto J(X), \end{aligned}$$

where $\mathscr{A}_5$ is the moduli space of 5-dimensional principally polarized abelian varieties. A local calculation shows that its differential is injective, and Clemens and Griffiths proved that $\wp_{\mathscr{C}}$ itself is injective: one can recover a cubic threefold from its intermediate Jacobian and its theta divisor (X is isomorphic to the projectified tangent cone to Θ at 0).

When $X = G(2,5) \cap \mathbf{P}^7 \cap Q$ *is a prime Fano threefold of genus 6,* we have $h^{1,2}(X) = 10$, so that $J(X)$ is a 10-dimensional principally polarized abelian variety.

Let $\mathscr{X}_3$ be the moduli stack of smooth prime Fano threefolds of genus 6 (we will not bother with the precise nature of this object). It is irreducible of dimension 22 and again, there is a *period map*

$$\begin{aligned} \wp : \mathscr{X}_3 &\longrightarrow \mathscr{A}_{10} \\ [X] &\longmapsto J(X), \end{aligned}$$

where $\mathscr{A}_{10}$ is the moduli stack of 10-dimensional principally polarized abelian varieties.

One computes $H^i(X, T_X) = 0$ for $i \neq 1$. In particular, the group of automorphisms of X is finite (in fact trivial for X general) and the local deformation space of X is smooth, with tangent space $H^1(X, T_X)$. The differential of the period map at $[X]$ is then the map

$$d\wp : H^1(X, T_X) \to \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X))$$

defined by the natural pairing $H^1(X, T_X) \otimes H^1(X, \Omega_X^2) \to H^2(X, \Omega_X^1)$. One checks that its kernel has dimension 2, hence $\wp$ has smooth 2-dimensional fibers and its image has dimension 20.

We want to describe the (2-dimensional) fibers of $\wp$. Recall that for any line $\ell \subset X$, we defined a line transformation $\psi_\ell : X \to X^+$, where $[X^+] \in \mathscr{X}_4$. This line transformation is a composition of two blow-ups of lines, which do not change the intermediate Jacobian, as we will see soon (because is line is a rational curve, with trivial Jacobian), and a flop, which does not change the intermediate Jacobian either (in dimension 3!). In other words, $[X^+]$ and $[X]$ are in the same fiber of $\wp$. Note that this still holds for conic transforms (since a conic is also rational).

Theorem 5.3. *For $[X] \in \mathscr{X}_3$ general, the component of the fiber of $\wp$ through $[X]$ is isomorphic to the surface $C_m(X)/\iota$.*

Proof. The point is to show that the various conic transforms are not isomorphic, except for c and $\iota(c)$. We will only show that for $[c]$ general in $C(X)$, the conic transform X^+ can only be isomorphic to finitely many other conic transforms (to determine precisely the fiber as in the theorem, we need a difficult theorem of Logachev (the "reconstruction theorem"; see [DIM], Theorem 9.1) which implies that the line transforms corresponding to conics $c, c' \subset X$ are isomorphic if and only if there exists an automorphism σ of $C_m(X)$ such that $\sigma([c]) = [c']$ and that σ is either trivial or the involution ι).

This is done by describing the surface $C(X^+)$ of conics contained in X^+. For that, we construct a birational map

$$\varphi_c : C(X) \dashrightarrow C(X^+)$$

as follows. Let $c' \subset X$ general. Then c and c' span a 5-plane $\langle c, c' \rangle$, whose intersection with X is a canonically embedded, genus-6 curve. This curve splits as

$$c + c' + \Gamma_{c,c'},$$

where $\Gamma_{c,c'}$ is a smooth irreducible sextic meeting c (and c') in 4 points. Since the conic transformation ψ_c is associated with a linear subsystem of $|\mathscr{I}_c^4(3)|$, the degree of $\psi_c(\Gamma_{c,c'})$ is $3 \times 6 - 4 \times 4 = 2$: this is a conic on X^+ which we call $\varphi_c([c'])$.

One then checks that

- the inverse of φ_c is φ_{c^+}, hence φ_c is birational;
- φ_c therefore induces an isomorphism $C_m(X) \xrightarrow{\sim} C_m(X^+)$ between the minimal models;
- this isomorphism sends $[c_X]$ to $\iota^+([c^+])$ and $[c]$ to $\iota^+([c_{X^+}])$.

In other words, $C(X^+)$ is isomorphic to the blow-up of $C_m(X)$ at the point $[c]$.

Since the group of automorphisms of the surface of general type $C_m(X)$ is finite (and in fact generated by the involution ι for X general), the conic transforms of X, as $[c]$ describes $C(X)$, do form a 2-dimensional family of varieties in a fiber of the period map $\wp$. With a bit more care, can define a conic transform for *any* conic $c \subset X$, hence we get a proper component of the fiber.

This component passes through $[X]$ (it corresponds to $c = c_X$). Line transforms actually land in another component of the fiber of $\wp([X])$ (this is not obvious; we show it by a degeneration argument to the nodal case). □

5.4. **Rationality problems.** Let X and Y be smooth threefolds with $H^{0,3}(X) = H^0(X, K_X) = 0$. Assume that there exist a birational isomorphism $f : X \dashrightarrow Y$. Then $0 = H^0(Y, K_Y) = H^{0,3}(Y)$ and by Hironaka's theorem, there is a composition $\widetilde{X} \to X$ of blow-ups with smooth centers (points or curves) whose composition with f is a *morphism* $\widetilde{X} \to Y$. Then, one shows that $J(Y)$ is a direct factor in the principally polarized abelian variety $J(\widetilde{X})$: there exists a principally polarized abelian variety A such that $J(\widetilde{X}) \simeq J(Y) \times A$.

On the other hand, if $X' \to X$ is a blow-up with a smooth center, one still has $H^{0,3}(X') = 0$. Moreover, either $J(X') \simeq J(X)$ if the center is a point, or $J(X') \simeq J(X) \times J(C)$ if the center is a smooth curve $C \subset X$.

Any principally polarized abelian variety A has a unique (up to permutations) decomposition as a product of indecomposable principally polarized abelian varieties; in that product, throw away the Jacobians of curves and call the result principally polarized abelian variety A_G. We obtain that if X and Y are birationally isomorphic, $J(X)_G$ and $J(Y)_G$ are isomorphic. In particular, *if a smooth Fano threefold X is rational, $J(X)$ is a product of Jacobians of curves.* Note that this property is closed, whereas it is unkown whether rationality is either closed or open.

Being a product of Jacobians of curves imposes a very large singular locus for theta divisors. This is how Clemens and Griffiths, using the description of a theta divisor of the intermediate Jacobian of any smooth cubic threefold X given above to prove that it has a single singular point, deduced that X is irrational.

For prime Fano threefolds of genus 6, the situation is more complicated: there is only a conjectural description of the singular locus of a theta divisor. However, one can prove that a general nodal degeneration has a (generalized) Jacobian which is not in the closure of the set of product of Jacobians. This implies that a general prime Fano threefold of genus 6 is irrational. However, no explicit (smooth) example of an irrational (or rational) such threefold is known.

Note that Enriques proved one hundred years ago that these threefolds are *unirational* (they are dominated by a rational threefold).

6. Prime Fano fourfolds of genus 6

Assume that the smooth Fano fourfold $X \subset \mathbf{P}^8$ falls into the first case of Mukai's theorem: X is the (smooth) intersection of $G := G(2,5) \subset \mathbf{P}^9$ with a quadric Q and a hyperplane $\mathbf{P}^8$. Note that since G is an intersection of quadrics, so is X. Also, by Lefschetz theorem, $\mathrm{Pic}(X) = \mathbf{Z}[H]$. When X is general, a dimension count shows that X contains no planes.

In this section, we briefly discuss the variety $F(X)$ of lines contained in X and the variety $C(X)$ of conics contained in X. We show how one can associate with X a doube EPW sextic, an irreducible symplectic fourfold, and study the related period maps.

6.1. **Lines.** A dimension count shows that X is covered by lines. A general hyperplane section containing a fixed line $\ell \subset X$ is a prime Fano threefold of genus 6 as studied in the last section. One checks that the possible normal bundles are

$$\mathcal{O}_\ell \oplus \mathcal{O}_\ell \oplus \mathcal{O}_\ell \ , \quad \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell \oplus \mathcal{O}_\ell(1) \ , \quad \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(1).$$

The variety $F(X)$ of lines contained in X has everywhere dimension ≥ 3, and only the last type could correspond to singular points. Since a line passing through a general point of X must be free, hence of the first type, we deduce as usual that $F(X)$ has pure dimension 3. In fact, one shows that for X general, $F(X)$ is smooth irreducible of that dimension.

6.2. **Conics in X and double EPW-sextics.** Conics are another matter. It is true, but quite difficult to show ([IM], Theorem 3.2), that when X is general, the projective scheme $C(X)$ of (possibly degenerate) conics contained in X is smooth irreducible of dimension 5.

There is a very interesting relationship between this variety and *double EPW sextics,* symplectic fourfolds discovered by O'Grady, which we will now describe.

The vector space $I_G(2)$ of quadrics containing $G := G(2, V_5) \subset \mathbf{P}(\wedge^2 V_5)$ has dimension 5 and consists of (rank-6) *Plücker quadrics.* More precisely, we have an isomorphism

$$\begin{aligned} V_5 &\xrightarrow{\sim} I_G(2) \\ \alpha &\longmapsto (\omega \mapsto \omega \wedge \omega \wedge \alpha). \end{aligned}$$

Then, $I_X(2) \simeq I_G(2) \oplus \mathbf{C}Q$. Let $V_4 \subset V_5$ be a hyperplane. All Plücker quadrics restrict in $\mathbf{P}(\wedge^2 V_4)$ to the sole (smooth) Plücker quadric defining $G(2, V_4) \subset \mathbf{P}(\wedge^2 V_4)$, hence elements of $I_X(2)$ restrict to a pencil. A general element of that pencil is smooth, and a finite number of elements are singular. These elements define a finite number of hyperplanes in $I_X(2)$, hence points in $\mathbf{P}(I_X(2)^\vee)$. When V_4 varies, we get a subvariety

$$Z_X \subset \mathbf{P}(I_X(2)^\vee) = \mathbf{P}^5.$$

Theorem 6.1. *For X general, the fourfold $Z_X \subset \mathbf{P}^5$ is an EPW sextic hypersurface.*

What is an EPW sextic? Let U_6 be a 6-dimensional vector space. If we choose an isomorphism $\wedge^6 U_6 \simeq \mathbf{C}$, the vector space $\wedge^3 U_6$ inherits a non-degenerate skew-symmetric form given by wedge product. Let $A \subset \wedge^3 U_6$ be a general Lagrangian subspace. Then

$$Z_A := \{U_5 \subset U_6 \mid \wedge^3 U_5 \cap A \neq 0\} \subset \mathbf{P}(U_6^\vee)$$

is a *EPW sextic.* Its singular locus is the smooth surface

$$S_A := \{U_5 \subset U_6 \mid \dim(\wedge^3 U_5 \cap A) \geq 2\}.$$

Let $\mathscr{X}_4$ be the 24-dimensional moduli stack of smooth prime Fano fourfolds of genus 6 (again, we will not bother with the precise nature of this object) and let $\mathscr{EPW}$ be the 20-dimensional moduli space of EPW sextics (O'Grady studied it in great detail and proved that it is quasi-projective). By the theorem, there is a rational map

$$\begin{aligned} \mathrm{epw} : \mathscr{X}_4 &\dashrightarrow \mathscr{EPW} \\ [X] &\longmapsto Z_X. \end{aligned}$$

Iliev and Manivel proved in [IM] that it is dominant. Its general fibers therefore have dimension 4.

Let us go back to conics: we define a morphism

$$C(X) \longrightarrow Z_X$$

as follows. Let $(c, V_4) \in \widetilde{C}(X)$ and let us look at the 4-plane $\mathbf{P}(\wedge^2 V_4) \cap \mathbf{P}^8$. It contains two quadrics:

- its intersection with $G(2, V_4)$,
- its intersection with Q,

whose intersection is contained is $X \cap \mathbf{P}(\wedge^2 V_4)$. They both contain c, but not both $\langle c \rangle$, because X contains no 2-planes. Thus, there exists a unique quadric in this pencil that contains $\langle c \rangle$, and this quadric must be singular (because a smooth quadric in $\mathbf{P}^4$ contains no 2-planes). This defines a morphism $\widetilde{C}(X) \to Z_X$ which induces

$$\alpha : C(X) \to Z_X.$$

Let us look at the general (1-dimensional) fibers of this map. For $Q \supset X$ general in Z_X, the hyperplane V_4 such that $Q|_{\mathbf{P}(\wedge^2 V_4)}$ is singular is unique. The quadric $Q|_{\mathbf{P}(\wedge^2 V_4)}$ has rank 4, hence is a cone over a smooth quadric in a 3-plane; it contains two disjoint families of 2-planes (each parametrized by a $\mathbf{P}^1$). Given one of these 2-planes P, its intersection with another quadric in the pencil is a conic contained in X whose image by α is still $[Q]$. The general fiber of α is therefore the disjoint union of two rational curves. Its Stein factorization is therefore

$$\alpha : C(X) \xrightarrow{\beta} Y_X \xrightarrow{\pi} Z_X,$$

where π is a double cover. It turns out ([IM]) that

- the branch locus of π is the surface $S_X^\vee = \mathrm{Sing}(Z_X)$;
- the fourfold Y_X is smooth;
- the fourfold Y_X is an irreducible symplectic variety.

The construction of the "canonical" double cover π of an EPW sextic, and the proof of the fact that it is irreducible symplectic, were first done by O'Grady.

6.3. **Periods of prime Fano fourfolds of genus 6.** The Hodge diamond for a smooth Fano fourfold X in $\mathscr{X}_4$ was computed in [IM], Lemma 4.1; its upper-half is as follows:

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 0 && 0 &&& \\ && 0 && 1 && 0 && \\ & 0 && 0 && 0 && 0 & \\ 0 && 1 && 22 && 1 && 0 \end{array} \tag{5}$$

In particular, the Hodge structure on $H^4(X)$ is of K3 type. We define a local period map as follows. One computes $H^i(X, T_X) = 0$ for $i \neq 1$. In particular, the group of automorphisms of X is finite and the local deformation space T of X is smooth, with tangent space $H^1(X, T_X) \simeq \mathbf{C}^{24}$.

Take for T a small polydisk, with universal family $f : \mathscr{X} \to T$. This family is then $\mathscr{C}^\infty$-trivial. This means that we can identify all the lattices $H^4(X_t, \mathbf{Z})$ (with their intersection form) with a fixed lattice Λ_X of rank 24, with a unimodular quadratic form q of (real) signature $(22, 2)$. Inside the fixed vector space $\Lambda_X \otimes_{\mathbf{Z}} \mathbf{C}$, the line $H^{3,1}(X_t)$ moves. This gives a *local period map*

$$\wp : T \to \mathbf{P}(\Lambda_X \otimes_{\mathbf{Z}} \mathbf{C}),$$

which Griffiths proved is *holomorphic.* Because of Riemann's bilinear relations, the period actually lands into the smaller 22-dimensional subset

$$\{\omega \in \mathbf{P}(\Lambda_X \otimes_{\mathbf{Z}} \mathbf{C}) \mid q(\omega) = 0, q(\omega + \bar{\omega}) > 0\}$$

called the *local period domain.* Since the X_t actually all sit into the fixed Grassmannian $G := G(2, V_5)$, the period is always orthogonal to the subspace $H^4(G, \mathbf{C})$ of $H^4(X, \mathbf{C})$, so it makes more sense to work in the primitive cohomology lattice

$$\Lambda := \Lambda_X \cap H^4(G, \mathbf{Z})^\perp.$$

One checks that this lattice is even, with discriminant group $(\mathbf{Z}/2\mathbf{Z})^2$, and signature $(20, 2)$. There is only one such lattice:

$$\Lambda \simeq 2E_8 \oplus 2U \oplus 2\langle 2 \rangle.$$

We also restrict the local period map to

$$\wp : T \to \Omega := \{\omega \in \mathbf{P}(\Lambda \otimes_{\mathbf{Z}} \mathbf{C}) \mid q(\omega) = 0, q(\omega + \bar{\omega}) > 0\},$$

where Ω has dimension 20. It is a bounded symmetric domain of type IV (actually, Ω has two connected components).

Theorem 6.2. *The local period map $\wp : T \to \Omega$ is a submersion.*

Proof. This follows from the computation of the kernel of its differential, which is the map

$$\begin{aligned} H^1(X, T_X) &\to \operatorname{Hom}(H^{3,1}(X), H^{3,1}(X)^\perp / H^{3,1}(X)) \\ &\simeq \operatorname{Hom}(H^1(X, \Omega_X^3), H^2(X, \Omega_X^2)) \end{aligned}$$

defined by the natural pairing $H^1(X, T_X) \otimes H^1(X, \Omega_X^3) \to H^2(X, \Omega_X^2)$ (recall that $H^1(X, \Omega_X^3)$ is one-dimensional). One checks that it has dimension 4. □

There is also a global period map

$$\mathscr{X}_4 \to \mathscr{D} := O(\Lambda) \backslash \Omega,$$

which is therefore dominant (the group $O(\Lambda)$ is some subgroup of finite index of the group of automorphisms of the lattice Λ).

6.4. **Periods of double EPW sextics.** Periods of double EPW sextics were studied by O'Grady. First of all, he proved that double EPW sextics are deformations of the Hilbert scheme $S^{[2]}$, where S is a K3 surface of genus 2 (this is how he proved that double EPW sextics are irreducible symplectic). In particular, the lattice $H^2(Y_A, \mathbf{Z})$ (equipped with the Beauville-Bogomolov form), is isomorphic to $2E_8 \oplus 3U \oplus \langle 2 \rangle$, and the primitive cohomology lattice $H^2(Y_A, \mathbf{Z})_0$ (i.e., the orthogonal of the rank-1 lattice $\pi^* H^2(Z_A, \mathbf{Z})$) is isomorphic to $2E_8 \oplus 2U \oplus 2\langle 2 \rangle$. Notice that this is precisely the lattice Λ introduced above.

The Hodge structure on $H^2(Y_A, \mathbf{Z})_0$ is of K3 type, so we can then define a global period map

$$\wp' : \mathscr{E}\mathscr{P}\mathscr{W} \to \mathscr{D}$$

exactly as before. By work of Verbitsky on the Torelli problem for irreducible symplectic manifolds, *$\wp'$ is dominant and birational.*

On the other hand, the fact that the lattice is the same as the lattice Λ we introduced for prime Fano fourfold of genus 6 makes our situation look very similar to the one we encountered in §1.3! Let us make that more precise.

Let X be a prime Fano fourfold of genus 6. Introduce the incidence variety

$$I := \{(x, [c]) \in X \times C(X) \mid x \in c\}$$

and the two projections $p : I \to X$ and $q : I \to C(X)$. We define the *Abel-Jacobi map*

$$a := q_* p^* : H^4(X, \mathbf{Z}) \to H^2(C(X), \mathbf{Z}).$$

Recall that there is a map $\beta : C(X) \to Y_X$.

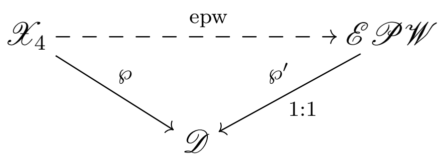
Conjecture 6.3. On the primitive cohomology, the Abel-Jacobi map factors as

$$a : H^4(X, \mathbf{Z})_0 \xrightarrow{\sim} H^2(Y_X, \mathbf{Z})_0 \overset{\beta^*}{\hookrightarrow} H^2(C(X), \mathbf{Z}).$$

Moreover,

$$\forall u, v \in H^4(X, \mathbf{Z})_0 \qquad q_{Y_X}(a(u), a(v)) = -u \cdot v.$$

If we believe this conjecture, we get a commutative diagram



and (at least birationally), the general (4-dimensional) fibers of $\wp$ and epw are the same.

6.5. **Periods of prime Fano fourfolds of genus 6 of Gushel type.** In this section, we will try to make a connection with the results of §5.3 on the period map

$$\wp_3 : \mathscr{X}_3 \to \mathscr{A}_{10}$$

for prime Fano threefolds of genus 6. Recall that from Theorem 4.2 that there are two kinds of prime Fano manifolds of genus 6: they are either complete intersections in $G := G(2, V_5)$, or double covers of Fano manifolds of genus 6 and index 2. We call the second type the "Gushel type."

So a fourfold of Gushel type is a double cover $X \to W$, where $W = G \cap \mathbf{P}^8$, branched along the intersection B of W with a quadric. Note that B is a prime Fano threefold of genus 6.

In this way, we get an injective morphism $\mathscr{X}_3 \to \mathscr{X}_4$ and two period maps:

$$\wp_3 : \mathscr{X}_3 \to \mathscr{A}_{10} \qquad \text{and} \qquad \wp_3' : \mathscr{X}_3 \to \mathscr{X}_4 \xrightarrow{\wp_4} \mathscr{D}.$$

One can compute the differential of $\wp_3'$ and we find that $\wp_3'$ is still dominant, hence its general fibers have dimension 2 (the same as the fibers of $\wp_3$!). However, the link between those two maps still remains quite mysterious.

On the other hand, the composition $\mathrm{epw}_3 : \mathscr{X}_3 \to \mathscr{X}_4 \xrightarrow{\mathrm{epw}} \mathscr{E}\mathscr{P}\mathscr{W}$ can be very simply described in terms of quadrics as in the 4-dimensional case, and one finds ([IM]) that it is still dominant.

Let us stop for a moment and go back briefly to an EPW sextic $Z_A \subset \mathbf{P}(U_6^\vee)$. O'Grady showed that *its projective dual* $Z_A^\vee \subset \mathbf{P}(U_6)$ *is still an EPW sextic* (it is the EPW sextic $Z_{A^\perp}$, where $A^\perp \subset \wedge^3 U_6^\vee$ is the orthogonal of $A \subset \wedge^3 U_6$). This operation defines an involution on $\mathscr{E}\mathscr{P}\mathscr{W}$ and a non-trivial involution r on $\mathscr{D}$.

We have the following result that we will not prove (see the proof of [IM], Proposition 4.16).

Proposition 6.4. *Let B be a prime Fano threefold of genus 6 with associated EPW sextic $Z_B \subset \mathbf{P}(I_B(2))$. There is a natural embedding $C_m(B)/\iota \hookrightarrow \mathbf{P}(I_B(2)^\vee)$ whose image is* $\mathrm{Sing}(Z_B^\vee)$.

We can now describe the general fibers of epw_3.

Theorem 6.5. *Let B be a general prime Fano threefold of genus 6. The fiber of* epw_3 *through $[B] \in \mathscr{X}_3$ is isomorphic to $C_m(B)/\iota$. It corresponds to all conic transforms of B.*

Proof. By the proposition above, we have an embedding $C_m(B)/\iota \hookrightarrow \mathbf{P}(I_B(2)^\vee)$ whose image is $\mathrm{Sing}(Z_B^\vee)$.

Let H be the pull-back of the hyperplane class on $C_m(B)/\iota$. Logachev proved ([L], paragraph right after (3.17))

$$K_{C_m(B)/\iota} \underset{\mathrm{lin}}{\equiv} 3H + \sigma,$$

where σ is the 2-torsion divisor class that defines the double étale covering $\pi : C_m(B) \to C_m(B)/\iota$. From Z, we can therefore recover the minimal surface $C_m(B)$.

Now it follows from Logachev's reconstruction theorem ([DIM], §7 and Theorem 9.1) that the threefolds in $\mathscr{X}_3$ with the same minimal surface of conics as B are all the "conic-transforms" of B, and their isomorphism classes are parametrized by $C_m(B)/\iota$ (see the proof of Theorem 5.3). Thus, we have isomorphisms $\mathrm{epw}_3^{-1}([Z_B]) \simeq C_m(B)/\iota \simeq \mathrm{Sing}(Z_B^\vee)$, which prove the proposition. □

In particular, since all conic transforms have isomorphic intermediate Jacobians, the period map $\wp_3 : \mathscr{X}_3 \to \mathscr{A}_{10}$ factors as follows:

$$\wp_3 : \mathscr{X}_3 \xrightarrow{\mathrm{epw}_3} \mathscr{E}\mathscr{P}\mathscr{W} \xrightarrow{\gamma} \mathscr{A}_{10}.$$

Using a result of Logachev, γ can be described as follows: given an EPW sextic Z, the principally polarized abelian variety $\gamma([Z])$ is the Albanese variety of the double étale cover of the singular locus of its dual (constructed as above).

One checks that the EPW sextic associated with a line transform is the dual EPW sextic. Since again, all line transforms have isomorphic intermediate Jacobians, we get a further factorization

$$\wp_3 : \mathscr{X}_3 \xrightarrow{\mathrm{epw}_3} \mathscr{E}\mathscr{P}\mathscr{W} \longrightarrow \mathscr{E}\mathscr{P}\mathscr{W}/\mathrm{duality} \xrightarrow{\overline{\gamma}} \mathscr{A}_{10}.$$

The map $\overline{\gamma}$ is generically finite (onto its image) and we expect it to be birational.

Since the period map $p : \mathscr{E}\mathscr{P}\mathscr{W} \to \mathscr{D}$ is birational, we may also write it as

$$\wp_3 : \mathscr{X}_3 \xrightarrow{\wp_3'} \mathscr{D} \longrightarrow \mathscr{D}/r \dashrightarrow \mathscr{A}_{10}.$$

So the situation is subtle: given a threefold B and a line transform B', their Hodge structures are the same, but not the Hodge structures of the Gushel fourfolds that they define.

We finish with a conjecture. Recall from Theorem 6.5 that the fiber of $\mathrm{epw}_3 : \mathscr{X}_3 \hookrightarrow \mathscr{X}_4 \xrightarrow{\mathrm{epw}_4} \mathscr{E}\mathscr{P}\mathscr{W}$ at a general EPW sextic Z is isomorphic to $\mathrm{Sing}(Z^\vee)$. *Iliev and Manivel conjecture that the (4-dimensional) fiber of* epw_4 *should be isomorphic to the sextic fourfold* $Z^\vee$.

One way to study this fiber is to define conic and line transforms for fourfolds. This is work under progress (one of the problems is that a flop of a smooth fourfold is usually not smooth anymore; another one is that it is not clear whether this operation preserves the Hodge structures...). The story also continues with prime Fano fivefolds of genus 6...

References

[DIM] Debarre, O., Iliev, A., Manivel, L., On the period map for prime Fano threefolds of degree 10, *J. Algebraic Geom.* **21** (2012), 21–59.

[F] Fujita, T., Vector bundles on ample divisors, *J. Math. Soc. Japan* **33** (1981), 405–414.

[IP] Iskovskikh, V. A., Prokhorov, Yu., Fano varieties, *Algebraic geometry, V*, 1–247, Encyclopaedia Math. Sci. **47**, Springer-Verlag, Berlin, 1999.

[IM] Iliev, A., Manivel, L., Fano manifolds of degree 10 and EPW sextics, *Ann. Sci. École Norm. Sup.* **44** (2011), 393–426.

[L] Logachev, D., Fano threefolds of genus 6, `eprint arXiv:math/0407147v1`.

[M1] Mukai, S., Fano 3-folds, London Math. Soc. Lect. Note Ser. **179**, Cambridge Univ. Press (1992), 255–263.

[M2] Mukai, S., Curves, $K3$ surfaces and Fano 3-folds of genus ≤ 10, *Algebraic geometry and commutative algebra,* Vol. I, 357–377, Kinokuniya, Tokyo, 1988.

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