On the geometry of hypersurfaces of low degrees in the projective space

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Introduction

The sets of zeroes in the projective space of homogeneous polynomials of degree d with coefficients in a field are called projective hypersurfaces of degree d and the study of their geometry is a very classical subject.

For example, Cayley wrote in a 1869 memoir that a smooth complex cubic ($d = 3$) surface contains exactly 27 projective lines. We start this series of lectures by studying more generally, in Chapter 1, the family of projective linear spaces contained in a smooth hypersurface of any degree and any dimension. We take this opportunity to introduce Grassmannians and a little bit of Schubert calculus.

In Chapter 2, we concentrate on the family of projective lines contained in a smooth hypersurface, with particular attention to the case of cubic hypersurfaces. We discuss several base fields: $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$, and finite fields.

In Chapter 3, we discuss cubic surfaces. We show that the projective plane blown up at six points in general position is isomorphic to a smooth cubic surface, and that, over an algebraically closed field, every smooth cubic surface is obtained in this way. It is in particular rational, i.e., birationally isomorphic to a projective space. Over a non-algebraically closed field, this is no longer the case. We introduce the Picard group of a scheme and explain a criterion of Segre for non-rationality of a smooth cubic surface.

In Chapter 4, we discuss unirationality, a weakening of rationality, and prove that all smooth cubic hypersurfaces are unirational as soon as they contain a line (a condition which is always satisfied if the field is algebraically closed and the dimension is at least 2).

In Chapter 5, we explain why cubic hypersurfaces of dimension 3 are not rational. For that, we introduce several fundamental objects in algebraic geometry: the intermediate Jacobian, the Albanese variety, principally polarized abelian varieties and theta divisors, Abel–Jacobi maps, conic bundles, and Prym varieties, in order to go through the Clemens–Griffiths proof of this fact over the complex numbers. The pace is considerably faster as more and more sophisticated material is presented.

We end these notes in Chapter 6 with the study of geometrical properties of the variety of lines contained in a smooth complex cubic hypersurface of dimension 4. This is a smooth projective variety, also of dimension 4, with trivial canonical bundle, and we use the Beauville–Bogomolov classification theorem to prove that it is a simply connected holomorphic symplectic variety. We define Pfaffian cubics and prove the Beauville–Donagi result that this variety is a deformation of the Hilbert square of a K3 surface. This is a beautiful argument in classical algebraic geometry.

Instead of proving every result that we state, we have tried instead to give a taste of the many tools that are used in modern classical algebraic geometry. The bibliography provides a few references where the reader can find more detailed expositions. We offer a few exercises, especially in the first chapters, and even a couple of open questions which can be easily stated.

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Chapter 1

Projective spaces and Grassmannians

Let $\mathbf{k}$ be a field and let V be a $\mathbf{k}$-vector space of dimension N.

1.1 Projective spaces

The projective space

$$\mathbf{P}(V) := \{\text{1-dimensional vector subspaces in } V\}$$

is a $\mathbf{k}$-variety[1] of dimension $N-1$. It is endowed with a very ample invertible sheaf $\mathscr{O}_{\mathbf{P}(V)}(1)$; seen as a line bundle, its fiber at a point $[V_1]$ is the dual vector space $V_1^\vee$. Its space of global sections is isomorphic to $V^\vee$, by the map

$$\begin{array}{rcl} V^\vee & \longrightarrow & H^0(\mathbf{P}(V), \mathscr{O}_{\mathbf{P}(V)}(1)) \\ v^\vee & \longmapsto & ([V_1] \mapsto v^\vee|_{V_1}). \end{array}$$

More generally, for any $m \in \mathbf{N}$, the space of global sections of $\mathscr{O}_{\mathbf{P}(V)}(m) := \mathscr{O}_{\mathbf{P}(V)}(1)^{\otimes m}$ is isomorphic to the symmetric product $\operatorname{Sym}^m V^\vee$.

1.2 The Euler sequence

The variety $\mathbf{P}(V)$ is smooth and its tangent bundle $T_{\mathbf{P}(V)}$ fits into an exact sequence

$$0 \to \mathscr{O}_{\mathbf{P}(V)} \to \mathscr{O}_{\mathbf{P}(V)}(1) \otimes_{\mathbf{k}} V \to T_{\mathbf{P}(V)} \to 0. \tag{1.1}$$

At a point $[V_1]$, this exact sequence is the following exact sequence of $\mathbf{k}$-vector spaces:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{k} & \longrightarrow & V_1^\vee \otimes_{\mathbf{k}} V & \longrightarrow & T_{\mathbf{P}(V),[V_1]} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \to & \operatorname{Hom}_{\mathbf{k}}(V_1, V_1) & \to & \operatorname{Hom}_{\mathbf{k}}(V_1, V) & \to & \operatorname{Hom}_{\mathbf{k}}(V_1, V/V_1) & \to & 0 \end{array}$$

[1] A $\mathbf{k}$-*variety* is an integral and separated scheme of finite type over $\mathbf{k}$.

1.3 Grassmannians

For any integer r such that $0 \le r \le N = \dim_{\mathbf{k}}(V)$, the Grassmannian

$$G := G(r, V) := \{r\text{-dimensional vector subspaces } V_r \text{ in } V\}$$

is a smooth projective $\mathbf{k}$-variety of dimension $r(N-r)$ (when $r = 1$, this is just $\mathbf{P}(V)$; when $r = N-1$, this is the *dual projective space* $\mathbf{P}(V^\vee)$).

There is on G a *tautological rank-r subbundle* $\mathscr{S}$ whose fiber at a point $[V_r]$ of G is V_r (when $r = 1$, so that $G = \mathbf{P}(V)$, this is $\mathscr{O}_{\mathbf{P}(V)}(-1) := \mathscr{O}_{\mathbf{P}(V)}(1)^\vee$). It fits into an exact sequence

$$0 \to \mathscr{S} \to \mathscr{O}_G \otimes_{\mathbf{k}} V \to \mathscr{Q} \to 0,$$

where $\mathscr{Q}$ is the *tautological rank-$(N-r)$ quotient bundle.*

As in the case of the projective space ($r = 1$), one shows that for any $m \in \mathbf{N}$, the space of global sections of $\mathscr{S}ym^m \mathscr{S}^\vee$ is isomorphic to $\operatorname{Sym}^m V^\vee$.

Let $[V_r]$ be a point of G and choose a decomposition $V = V_r \oplus V_{N-r}$. The subset of G consisting of subspaces complementary to V_{N-r} is an open subset of G whose elements can be written as $\{x + u(x) \mid x \in V_r\}$ for some uniquely defined $u \in \operatorname{Hom}_{\mathbf{k}}(V_r, V_{N-r})$. In fact, we have a canonical identification

$$T_{G,[V_r]} \simeq \operatorname{Hom}_{\mathbf{k}}(V_r, V/V_r), \tag{1.2}$$

or

$$T_G \simeq \mathscr{H}om_{\mathscr{O}_S}(\mathscr{S}, \mathscr{Q}) \simeq \mathscr{S}^\vee \otimes_{\mathscr{O}_S} \mathscr{Q}.$$

The analog of the Euler sequence (1.1) for the Grassmannian is therefore

$$0 \to \mathscr{S}^\vee \otimes_{\mathscr{O}_S} \mathscr{S} \to \mathscr{S}^\vee \otimes_{\mathbf{k}} V \to T_G \to 0.$$

The invertible sheaf

$$\mathscr{O}_G(1) := \textstyle\bigwedge^r \mathscr{S}^\vee \tag{1.3}$$

is again very ample, with space of global sections isomorphic to $\bigwedge^r V^\vee$. It induces the *Plücker embedding*

$$\begin{array}{rcl} G(r, V) & \longrightarrow & \mathbf{P}(\bigwedge^r V) \\ {[V_r]} & \longmapsto & [\bigwedge^r V_r]. \end{array}$$

Example 1.1 When $N = 4$, the image of the Plücker embedding $G(2, V) \hookrightarrow \mathbf{P}(\bigwedge^2 V) \simeq \mathbf{P}^5_{\mathbf{k}}$ is the smooth quadric with equation $\eta \wedge \eta = 0$ (it consists of the decomposable tensors in $\bigwedge^2 V$).

1.4 Linear spaces contained in a subscheme of $\mathbf{P}(V)$

We can also interpret the isomorphism (1.2) as follows. Let us write the Euler exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{O}_{\mathbf{P}(V_r)} & \to & \mathscr{O}_{\mathbf{P}(V_r)}(1)\otimes V_r & \longrightarrow & T_{\mathbf{P}(V_r)} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{O}_{\mathbf{P}(V_r)} & \to & \mathscr{O}_{\mathbf{P}(V_r)}(1)\otimes V & \to & T_{\mathbf{P}(V)}|_{V_r} & \to & 0, \end{array}$$

from which we obtain a formula for the normal bundle of $\mathbf{P}(V_r)$ in $\mathbf{P}(V)$ (the cokernel of the rightmost vertical map)

$$N_{\mathbf{P}(V_r)/\mathbf{P}(V)} \simeq \mathscr{O}_{\mathbf{P}(V_r)}(1)\otimes (V/V_r). \tag{1.4}$$

We can therefore rewrite (1.2) as

$$T_{G,[V_r]} \simeq H^0(\mathbf{P}(V_r), N_{\mathbf{P}(V_r)/\mathbf{P}(V)}).$$

This is a particular case of a more general result.

Theorem 1.2 *Let $X \subset \mathbf{P}(V)$ be a subscheme containing $\mathbf{P}(V_r)$. Define*

$$F_r(X) := \{[V_r] \in G \mid V_r \subset X\} \subset G(r, V).$$

If X is smooth along $\mathbf{P}(V_r)$, one has

$$T_{F_r(X),[V_r]} \simeq H^0(\mathbf{P}(V_r), N_{\mathbf{P}(V_r)/X}).$$

What is the scheme structure on $F_r(X)$? Assume first that $X \subset \mathbf{P}(V)$ is a hypersurface $Z(f)$ defined by one equation $f = 0$, where $f \in \operatorname{Sym}^d V^\vee$ (a homogeneous polynomial in degree d). Then $[V_r] \in F(X)$ if and only if $f|_{V_r}$ is identically 0. Note that f defines (by restriction) a section s_f of $\mathscr{S}ym^d\mathscr{S}^\vee$. Then we define

$$F_r(X) := Z(s_f) \subset G(r, V) \tag{1.5}$$

(the scheme of zeroes of the section s_f) as a scheme.

In general, for a subscheme $X \subset \mathbf{P}(V)$ defined by equations $f_1 = \cdots = f_m = 0$, we set

$$F_r(X) := F_r(Z(f_1)) \cap \cdots \cap F_r(Z(f_m)) \subset G(r, V)$$

as a (projective) scheme.

Going back to the case where X is a hypersurface of degree d, we see that the *expected codimension* of $F_r(X)$ in G is the rank of $\mathscr{S}ym^d\mathscr{S}^\vee$, which is

$$\binom{d+r-1}{r-1}.$$

When $r = 2$ (so that $F(X) := F_2(X)$ is the scheme of projective lines $L \subset X$), the expected dimension of $F(X)$ is

$$2(N-2) - d - 1 = 2N - 5 - d. \tag{1.6}$$

Assume that X is smooth along L and consider the normal exact sequence

$$0 \to N_{L/X} \to N_{L/\mathbf{P}(V)} \to N_{X/\mathbf{P}(V)}|_L \to 0. \tag{1.7}$$

By (1.4), the normal bundle $N_{L/\mathbf{P}(V)}$ is isomorphic to $\mathscr{O}_L(1)^{\oplus(N-2)}$, hence has degree (or first Chern class) $N-2$, whereas $N_{X/\mathbf{P}(V)}$ is isomorphic to $\mathscr{O}_X(d)$. It follows that $N_{L/X}$ has rank $N-3$ and degree $N-2-d$.

Theorem 1.2 has a more precise form, which we will only state for projective lines ($r = 2$).

Theorem 1.3 *Let $X \subset \mathbf{P}(V)$ be a subscheme containing a projective line L. If X is smooth along L, the scheme $F(X)$ can be defined, in a neighborhood of its point $[L]$, by $h^1(L, N_{L/X})$ equations in a smooth scheme of dimension $h^0(L, N_{L/X})$. In particular, every component of $F(X)$ has dimension at least*

$$\chi(L, N_{L/X}) := h^0(L, N_{L/X}) - h^1(L, N_{L/X}) = \deg(N_{L/X}) + \dim(X) - 1.$$

The last equality follows from the Riemann–Roch theorem applied to the vector bundle $N_{L/X}$ on the genus-0 curve L.

The number $\deg(N_{L/X}) + \dim(X) - 1$ is called the *expected dimension* of $F(X)$ (when $X \subset \mathbf{P}(V)$ is a hypersurface, this number is the same as in (1.6), obtained without the assumption that X be smooth). When $H^1(L, N_{L/X}) = 0$, the scheme $F(X)$ is smooth of the expected dimension at $[L]$.

1.5 Schubert calculus

Note that none of our results so far say anything about the *existence* of a line in a hypersurface. We will use cohomological calculations to that effect. The argument is based on the following result.

Theorem 1.4 *Let X be a smooth irreducible projective scheme and let $\mathscr{E}$ be a locally free sheaf on X of rank r. Assume that the zero set $Z(s)$ of some global section s of $\mathscr{E}$ is empty or has codimension exactly r in X. Then the class $[Z(s)] \in CH^r(X)$ is equal to $c_r(\mathscr{E})$. In particular, if $c_r(\mathscr{E})$ is non-zero, $Z(s)$ is non-empty.*

The group $CH^r(X)$ in the theorem is the Chow group of codimension-r cycles on X modulo rational equivalence. I do not want to explain here the theory of Chow groups. For our purposes, it can be replaced with the corresponding group in any good cohomology theory that you like, such as the group $H^{2r}(X, \mathbf{Z})$ in singular cohomology when $\mathbf{k} = \mathbf{C}$.

If X is a hypersurface of degree d of $\mathbf{P}(V)$, recall from (1.5) that the subscheme $F(X) \subset G := G(2, V)$ of lines contained in X is defined as the zero locus of a section of $\mathscr{S}ym^d\mathscr{S}^\vee$, a locally free sheaf on G of rank $d+1$.

To compute $c_{d+1}(\mathscr{S}ym^d\mathscr{S}^\vee)$, we need to know the ring $CH(G)$. To describe it, we define the Schubert cycles.

Let a and b be integers such that $N-2 \geq a \geq b \geq 0$. Choose vector subspaces

$$V_{N-1-a} \subset V_{N-b} \subset V$$

such that $\dim(V_{N-1-a}) = N-1-a$ and $\dim(V_{N-b}) = N-b$. We define a subvariety of G, called a *Schubert variety,* by

$$\Sigma_{a,b} := \{[V_2] \in G \mid V_2 \cap V_{N-1-a} \neq 0,\ V_2 \subset V_{N-b}\}.$$

It is irreducible of codimension $a+b$ in G and its class $\sigma_{a,b} := [\Sigma_{a,b}] \in CH^{a+b}(G)$ only depends on a and b. It is usual to write σ_a for $\sigma_{a,0}$ and to set $\sigma_{a,b} = 0$ whenever (a, b) does not satisfy $N-2 \geq a \geq b \geq 0$.

Theorem 1.5 *The group $CH(G(2, V))$ is a free abelian group with basis $(\sigma_{a,b})_{N-2\geq a\geq b\geq 0}$.*

For example, the group $CH^1(G)$ of isomorphism classes of invertible sheaves on G has rank 1, generated by σ_1. This class is the first Chern class of the invertible sheaf $\mathscr{O}_G(1)$ defined in (1.3). We also have

$$c(\mathscr{Q}) = 1 + \sigma_1 + \cdots + \sigma_{N-2},$$

hence

$$c(\mathscr{S}) = (1 + \sigma_1 + \cdots + \sigma_{N-2})^{-1} = 1 - \sigma_1 + \sigma_1^2 - \sigma_2.$$

(The rank of $\mathscr{S}$ is 2 so there are no higher Chern classes.) To compute this class in the basis $(\sigma_{a,b})$, we need to know the multiplicative structure of $CH(G)$: whenever $N-2 \geq a \geq b \geq 0$ and $N-2 \geq c \geq d \geq 0$, there are formulas

$$\sigma_{a,b} \cdot \sigma_{c,d} = \sum_{\substack{x+y=a+b+c+d \\ N-2\geq x\geq y\geq 0}} n_{a,b,c,d,x,y}\sigma_{x,y},$$

where the $n_{a,b,c,d,x,y}$ are integers. This is the content of Schubert calculus, which we will only illustrate in some particular cases (the combinatorics are quite involved in general).

Poincaré duality. If $a+b+c+d = 2N-4$, one has

$$\sigma_{a,b} \cdot \sigma_{c,d} = \begin{cases} 1 & \text{if } a+d = b+c = N-2, \\ 0 & \text{otherwise.} \end{cases}$$

(The class $\sigma_{N-2,N-2}$ is the class of a point and generates $CH^{2N-4}(G)$; we usually drop it.) In other words, the Poincaré dual of $\sigma_{a,b}$ is $\sigma_{N-2-b,N-2-a}$.

Pieri's formula. This is the relation

$$\sigma_{a,b} \cdot \sigma_m = \sum_{\substack{x+y=a+b+m \\ x \geq a \geq y \geq b}} \sigma_{x,y}.$$

For example, we have

$$\sigma_{a,b} \cdot \sigma_1 = \sigma_{a+1,b} + \sigma_{a,b+1} \tag{1.8}$$

(where the last term is 0 when $a = b$), which implies

$$c(\mathscr{S}^\vee) = 1 + \sigma_1 + \sigma_{1,1}.$$

The following formula can be deduced from Pieri's formula (using $\sigma_{1,1} = \sigma_1^2 - \sigma_2$):

$$\sigma_{a,b} \cdot \sigma_{1,1} = \sigma_{a+1,b+1}. \tag{1.9}$$

Example 1.6 How many lines meet 4 general lines L_1, L_2, L_3, and L_4 in $\mathbf{P}^3_{\mathbf{C}}$? One can answer this question geometrically as follows: through any point of L_3, there is a unique line meeting L_1 and L_2 and one checks by explicit calculations that the union of these lines is a smooth quadric surface, which therefore meets L_4 in 2 points ("counted with multiplicities"). Through each of these 2 points, there is a unique line meeting all 4 lines.

But we can also use Schubert calculus: the set of lines meeting L_i has class σ_1, hence the answer is (use (1.8))

$$\sigma_1^4 = \sigma_1^2(\sigma_2 + \sigma_{1,1}) = \sigma_1(\sigma_{2,1} + \sigma_{2,1}) = 2\sigma_{2,2}.$$

(To be honest, this calculation only shows that either there are 2 such lines "counted with multiplicities," or there are infinitely many of them.)

Chapter 2

Projective lines contained in a hypersurface

2.1 The scheme of lines contained in a hypersurface

We use Schubert calculus to show the existence of lines in hypersurfaces of small enough degrees.

Theorem 2.1 *When* $\mathbf{k}$ *is algebraically closed and* $d \leq 2N-5$*, any hypersurface of degree* d *in* $\mathbf{P}^{N-1}_{\mathbf{k}}$ *contains a projective line.*

Proof. According to Theorem 1.4, it is enough to prove that the top Chern class $c_{d+1}(\mathscr{S}ym^d\mathscr{S}^\vee)$ does not vanish.

The method for computing the Chern classes of the symmetric powers of $\mathscr{S}^\vee$ is the following: pretend that $\mathscr{S}^\vee$ is the direct sum of two invertible sheaves $\mathscr{L}_1$ and $\mathscr{L}_2$, with first Chern classes ℓ_1 and ℓ_2 (the *Chern roots* of $\mathscr{S}^\vee$), so that

$$c(\mathscr{S}^\vee) = (1+\ell_1)(1+\ell_2).$$

Then

$$\mathscr{S}ym^d\mathscr{S}^\vee \simeq \bigoplus_{i=0}^{d}(\mathscr{L}_1^{\otimes i} \otimes \mathscr{L}_2^{\otimes(d-i)})$$

and

$$c(\mathscr{S}ym^d\mathscr{S}^\vee) = \prod_{i=0}^{d}(1+i\ell_1+(d-i)\ell_2).$$

This symmetric polynomial in ℓ_1 and ℓ_2 can be expressed as a polynomial in

$$\begin{aligned} \ell_1+\ell_2 &= c_1(\mathscr{S}^\vee) = \sigma_1 \\ \ell_1\ell_2 &= c_2(\mathscr{S}^\vee) = \sigma_{1,1}. \end{aligned}$$

One obtains in particular

$$\begin{aligned}
c_{d+1}(\mathrm{Sym}^d \mathscr{S}^\vee) &= \prod_{i=0}^{d} \big(i\ell_1 + (d-i)\ell_2\big) \\
&= \prod_{0\le i<d/2} \big(i(d-i)\big(\ell_1+\ell_2\big)^2 + (d-2i)^2\ell_1\ell_2\big) \times \left[\frac{d}{2}(\ell_1+\ell_2)\right] \\
&= \prod_{0\le i<d/2} \big(i(d-i)\sigma_1^2 + (d-2i)^2\sigma_{1,1}\big) \times \left[\frac{d}{2}\sigma_1\right] \\
&= \prod_{0\le i<d/2} \big(i(d-i)\sigma_2 + \big((d-2i)^2 + i(d-i)\big)\sigma_{1,1}\big) \times \left[\frac{d}{2}\sigma_1\right],
\end{aligned}$$

where the expressions between brackets are only there when d is even. Formulas (1.8) and (1.9) imply that this is a sum of Schubert classes with non-negative coefficients which, since $(d-2i)^2 + i(d-i) \ge 1$ for all i, is "greater than or equal to"

$$\prod_{0\le i<d/2} \sigma_{1,1} \times [\sigma_1] = \sigma_{\lceil d/2\rceil,\lceil d/2\rceil} \times [\sigma_1] \ge \sigma_{\lceil (d+1)/2\rceil,\lfloor (d+1)/2\rfloor},$$

which is non-zero for $(d+1)/2 \le N-2$. If $d \le 2N-5$, we have therefore proved that $c_{d+1}(\mathrm{Sym}^d\mathscr{S}^\vee)$ is non-zero, hence $F(X)$ is non-empty by Theorem 1.4. When $\mathbf{k}$ is algebraically closed, $F(X)$ has $\mathbf{k}$-point, which means that X contains a line defined over $\mathbf{k}$. $\square$

Exercise 2.2 Prove the relations

1) $c_4(\mathscr{S}ym^3\mathscr{S}^\vee) = 9(2\sigma_{3,1} + 3\sigma_{2,2})$, $c_5(\mathscr{S}ym^4\mathscr{S}^\vee) = 32(3\sigma_{4,1} + 10\sigma_{3,2})$.
2) $c_1(\mathscr{S}ym^d\mathscr{S}^\vee) = \frac{d(d+1)}{2}\sigma_1$.

Assume $2N-5-d \ge 0$. Under the hypotheses of the theorem, it follows from (1.6) that $F(X)$ has everywhere dimension $\ge 2N-5-d$.

One can show that when $X \subset \mathbf{P}^{N-1}_{\mathbf{k}}$ is a *general* hypersurface of degree d, the scheme $F(X)$ is a smooth variety of dimension $2N-5-d$ (hence empty whenever this number is < 0). But when $d \ge 4$, the scheme $F(X)$ may be singular for some smooth X, or even reducible or non-reduced (this does not happen when $d = 2$ or 3; see §2.2). However, we have the following conjecture.

Conjecture 2.3 (Debarre, de Jong) *Assume $N > d \ge 3$ and that* $\mathrm{char}(\mathbf{k})$ *is either 0 or $\ge d$. For any smooth hypersurface $X \subset \mathbf{P}^{N-1}_{\mathbf{k}}$ of degree d, the scheme $F(X)$ has the expected dimension* $2N-5-d$.

We will see in Section 2.2 that the conjecture holds for $d = 3$. When $\mathrm{char}(\mathbf{k}) = 0$, the conjecture is known for $d \le 6$ or for $d \ll N$ (Collino ($d = 4$), Debarre ($d \le 5$), Beheshti ($d \le 6$), Harris et al. ($d \ll N$)). Example 2.5 shows why the hypothesis $\mathrm{char}(\mathbf{k}) \ge d$ is necessary.

Example 2.4 (Real lines) When d is even, it is clear that the Fermat hypersurface

$$x_1^d + \cdots + x_N^d = 0$$

contains no real points, hence no real lines, whereas the diagonal hypersurface

$$x_1^d + \cdots + x_{N-1}^d - x_N^d = 0$$

contains infinitely many real points, but no real lines.

Example 2.5 (Positive characteristic) Over an algebraically closed field $\mathbf{k}$ of characteristic $p > 0$, let us consider the smooth Fermat hypersurface $X \subset \mathbf{P}_{\mathbf{k}}^{N-1}$ with equation

$$x_1^{p^r+1} + \cdots + x_N^{p^r+1} = 0.$$

The line joining two points x and y of X is contained in X if and only if

$$\begin{aligned} 0 &= \sum_{j=1}^N (x_j + ty_j)^{p^r+1} \\ &= \sum_{j=1}^N (x_j^{p^r} + t^{p^r} y_j^{p^r})(x_j + ty_j) \\ &= \sum_{j=1}^N (x_j^{p^r+1} + t x_j^{p^r} y_j + t^{p^r} x_j y_j^{p^r} + t^{p^r+1} y_j^{p^r+1}) \\ &= t \sum_{j=1}^N x_j^{p^r} y_j + t^{p^r} \sum_{j=1}^N x_j y_j^{p^r} \end{aligned} \tag{2.1}$$

for all t. This is equivalent to the two equations

$$\sum_{j=1}^N x_j^{p^r} y_j = \sum_{j=1}^N x_j y_j^{p^r} = 0,$$

hence $F(X)$ has dimension at least $2\dim(X) - 2 - 2 = 2N - 8$ at every point $[L]$.

It is known that any locally free sheaf on $\mathbf{P}^1$ split as a direct sum of invertible sheaves, so we can write

$$N_{L/X} \simeq \bigoplus_{i=1}^{N-3} \mathscr{O}_L(a_i), \tag{2.2}$$

where $a_1 \geq \cdots \geq a_{N-3}$ and $a_1 + \cdots + a_{N-3} = N - p^r - 3$. By (1.7), $N_{L/X}$ is a subsheaf of $N_{L/\mathbf{P}_{\mathbf{k}}^{N-1}} \simeq \mathscr{O}_L(1)^{\oplus(N-2)}$, hence $a_1 \leq 1$. We have

$$\begin{aligned} 2N - 8 &\leq \dim(F(X)) \\ &\leq h^0(L, N_{L/X}) && \text{by Theorem 1.2} \\ &= 2\,\mathrm{Card}\{i \mid a_i = 1\} + \mathrm{Card}\{i \mid a_i = 0\} \\ &\leq \mathrm{Card}\{i \mid a_i = 1\} + N - 3. \end{aligned}$$

The only possibility is

$$N_{L/X} \simeq \mathscr{O}_L(1)^{\oplus(N-4)} \oplus \mathscr{O}_L(1-p^r), \tag{2.3}$$

which implies $h^0(L, N_{L/X}) = 2N - 8$. Since this is $\leq \dim(F(X))$, Theorem 1.2 implies that $F(X)$ is smooth of (non-expected if $p^r \neq 2$) dimension $2N - 8$.

Remark 2.6 (Free rational curves) Assume that the hypersurface X is smooth. We say that a line $L \subset X$ is *free* if all the a_i that appear in the decomposition (2.2) are non-negative (with the notation above, this means $a_{N-3} \geq 0$). Over an algebraically closed field of *characteristic 0,* one can show that when X is *covered by lines* (meaning that through any point of X, there is a line contained in X; this happens as soon as $d \leq N-2$ (why?)), a general line contained in X is free.

The last example shows that in characteristic $p > 0$, the Fermat hypersurface of degree $p^r + 1 > 3$ contains no free lines (although these lines cover the hypersurface).

More generally, we say that a non-constant morphism $f\colon \mathbf{P}^1_{\mathbf{k}} \to X$ is *free* if all the a_i that appear in the decomposition of the locally free sheaf f^*T_X are non-negative. *It is conjectured that any hypersurface of degree $d < N$ in $\mathbf{P}^{N-1}_{\mathbf{k}}$ contains a free rational curve.* This holds in characteristic zero, because X is covered by lines when $d \leq N-2$ (as we saw above), or by conics when $d \leq N-1$; so the problem is in positive characteristic.

For the Fermat hypersurface of degree $p^r + 1$ in $\mathbf{P}^{N-1}_{\mathbf{F}_p}$ (which often seems to exhibit the strangest behavior), one can prove that it contains no free rational curves of degree $\leq p^r$; however, when $p^r + 1 \leq N/2$, it contains a free rational curve of degree $2p^r + 1$ defined over $\mathbf{F}_p$ (Conduché).

Exercise 2.7 (Pfaffian hypersurfaces) Let $\mathbf{k}$ be an algebraically closed field of characteristic $\neq 2$ and let $W := \mathbf{k}^{2d}$. In $\mathbf{P}(\bigwedge^2 W^\vee)$, the *Pfaffian hypersurface* X_d of degenerate skew-symmetric bilinear forms (defined by the vanishing of the Pfaffian polynomial) has degree d.

1) Let m be a positive integer. Given a 2-dimensional vector space of skew-symmetric forms on $\mathbf{k}^m$, prove that there exists a subspace of dimension $\lfloor (m+1)/2 \rfloor$ which is isotropic for all forms in that space (*Hint:* proceed by induction on m).
2) Given a 2-dimensional vector space of *degenerate* skew-symmetric forms on $\mathbf{k}^{2d}$, prove that there exists a subspace of dimension $d+1$ which is isotropic for all forms in that space (*Hint:* proceed by induction on d and use 1)).
3) Show that the scheme $F(X_d)$ of projective lines contained in X_d is irreducible of the expected dimension (see (1.6))[1] (*Hint:* prove that the locus $\{([L], [V_{d+1}]) \in G(2, \bigwedge^2 W^\vee) \times G(d+1, W) \mid V_{d+1}$ is isotropic for all forms in $L\}$ is irreducible of dimension $4d^2 - 3d - 5$ and apply 2)).

Exercise 2.8 (Finite fields) Let $X \subset \mathbf{P}^{N-1}_{\mathbf{k}}$ be a hypersurface of degree $d \leq N-1$ defined over a finite field $\mathbf{k}$ with q elements. Show that the number of $\mathbf{k}$-points of X is at least $q^{N-1-d} + \cdots + q + 1$ (*Hint:* use the Chevalley–Warning theorem[2]).

---

[1]Note that X_d is singular for $d \geq 3$: its singular locus is the set of skew-symmetric forms of rank $\leq 2d-4$.

[2]This theorem says that if $f_1, \ldots, f_r$ are homogeneous polynomials in N variables with coefficients in a finite field $\mathbf{k}$, of respective degrees $d_1, \ldots, d_r$, and if $d_1 + \cdots + d_r < N$, the number of solutions in $\mathbf{k}^N$ of the system of equations $f_1(x) = \cdots = f_r(x) = 0$ is divisible by the characteristic of $\mathbf{k}$. The proof is clever but elementary. A refinement by Ax–Katz says that this number is divisible by $\mathrm{Card}(\mathbf{k})^{\lceil (N - \sum d_i)/\max(d_i) \rceil}$.

2.2 Projective lines contained in a cubic hypersurface

Assume that $\mathbf{k}$ is algebraically closed and let $X \subset \mathbf{P}(V)$ be a smooth cubic hypersurface. When $N = \dim(V) \geq 4$, it follows from Theorem 2.1 that X contains a line L. From (1.7), we have an exact sequence

$$0 \to N_{L/X} \to \mathscr{O}_L(1)^{\oplus(N-2)} \to \mathscr{O}_L(3) \to 0. \tag{2.4}$$

We write as in (2.2)

$$N_{L/X} \simeq \bigoplus_{i=1}^{N-3} \mathscr{O}_L(a_i), \tag{2.5}$$

where $a_1 \geq \cdots \geq a_{N-3}$ and $a_1 + \cdots + a_{N-3} = N-5$. By (2.4), we have $a_1 \leq 1$, hence

$$a_{N-3} = (N-5) - a_1 - \cdots - a_{N-4} \geq -1.$$

This implies $H^1(L, N_{L/X}) = 0$, hence $F(X)$ is smooth of the expected dimension $2N-8$ (Theorem 1.3).[3]

We have proved the following.

Theorem 2.9 *Let $X \subset \mathbf{P}(V)$ be a smooth cubic hypersurface. If $N \geq 4$, the scheme $F(X)$ of lines contained in X is a smooth projective variety of dimension $2N-8$.*

Remark 2.10 When $N \geq 5$, the scheme $F(X)$ is connected. Indeed, $F(X)$ is the zero locus of a section s_f of the locally free sheaf $\mathscr{E}^\vee := \mathscr{S}ym^3 \mathscr{S}^\vee$ on $G := G(2,N)$ and it has the expected codimension $\operatorname{rank}(\mathscr{E}) = 4$ (see (1.5)). In this situation, we have a *Koszul resolution*

$$0 \longrightarrow \bigwedge^4 \mathscr{E} \longrightarrow \bigwedge^3 \mathscr{E} \longrightarrow \bigwedge^2 \mathscr{E} \longrightarrow \mathscr{E} \xrightarrow{s_f^\vee} \mathscr{O}_G \longrightarrow \mathscr{O}_{F(X)} \longrightarrow 0. \tag{2.6}$$

of its structure sheaf (this complex is exact because locally, $\mathscr{E}$ is free and the components of s_f in a basis form a regular sequence). Using this sequence and, *in characteristic zero,* the Borel–Weil theorem, which computes the cohomology of homogeneous vector bundles such as $\bigwedge^r \mathscr{E}$ on G, one can compute some of the cohomology of $\mathscr{O}_{F(X)}$ and obtain for example $h^0(F(X), \mathscr{O}_{F(X)}) = 1$ for $N \geq 5$ ([DM, th. 3.4]), hence the connectedness of $F(X)$. This is obtained in all characteristics in [AK, Theorem (5.1)] by direct computations.

[3] A little more work shows there are two possible types for normal bundles:

$$\begin{array}{rcll} N_{L/X} & \simeq & \mathscr{O}_L(1)^{\oplus(N-5)} \oplus \mathscr{O}_L^{\oplus 2} & \text{for lines “of the first type;”} \\ N_{L/X} & \simeq & \mathscr{O}_L(1)^{\oplus(N-4)} \oplus \mathscr{O}_L(-1) & \text{for lines “of the second type.”} \end{array}$$

When $N \geq 4$, there are always lines of the second type. When $N \geq 5$, a general line in X is of the first type if $\operatorname{char}(\mathbf{k}) \neq 2$ (this is not true for the Fermat cubic in characteristic 2 by (2.3)).

Under the hypotheses of the theorem, by Exercice 2.2.1), the subscheme $F(X) \subset G(2,V)$ has class $9(2\sigma_{3,1} + 3\sigma_{2,2})$. When $N = 4$, the class $\sigma_{3,1}$ vanishes in $G(2,4)$ and $\sigma_{2,2}$ is the class of a point.

This proves the very famous classical result:

Every smooth cubic surface over an algebraically closed field contains 27 lines.

Remark 2.11 It can be shown that, over an algebraically closed field, a normal cubic surface that contains infinitely many lines is a cone. If a normal cubic surface X is not a cone, the scheme of lines $F(X)$ still has class $27\sigma_{2,2}$, but might not be reduced, so that X contains at most 27 lines. In fact, X is smooth if and only if it contains exactly 27 lines. It follows that when X is normal and singular (but not a cone), $F(X)$ is not reduced.

Example 2.12 (Real lines) The 27 complex lines contained in a smooth real cubic surface X are either real or complex conjugate. Since 27 is odd, X always contains a real line. One can prove that the set of real lines contained in X has either 3, 7, 15, or 27 elements (see Example 3.1).[4] In many mathematics departments around the world, there are plaster models of (real!) cubic surfaces with 27 (real) lines on them; it is usually the Clebsch cubic (1871), with equations in $\mathbf{P}^4$:

$$x_0 + \cdots + x_4 = x_0^3 + \cdots + x_4^3 = 0.$$

Among these 27 lines, 15 are defined over $\mathbf{Q}$, and the other 12 over the field $\mathbf{Q}(\sqrt{5})$.[5]

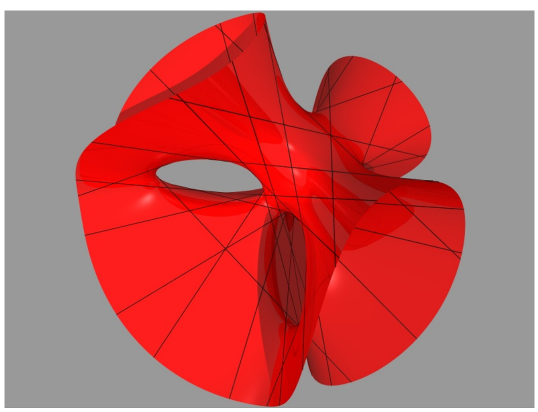

Figure 2.1: The Clebsch cubic with its 27 real lines

[4]Actually, lines on real cubic surfaces should be counted with signs, in which case one gets that the total number is always 3.

[5]The permutation group $\mathfrak{S}_5$ acts on X. The lines defined over $\mathbf{Q}$ are $\langle (1,-1,0,0,0),(0,0,1,-1,0)\rangle$ and its images by $\mathfrak{S}_5$, and the 12 other lines are the real line $\langle (1,\zeta,\zeta^2,\zeta^3,\zeta^4),(1,\bar\zeta,\bar\zeta^2,\bar\zeta^3,\bar\zeta^4)\rangle$, where $\zeta := \exp(2i\pi/5)$, and its images by $\mathfrak{S}_5$.

Example 2.13 (Rational lines) It is only recently that a *rational* cubic surface with all its 27 lines rational was found (by Tetsuji Shioda in 1995). Its equation is

$$x_2^2x_4 + 2x_2x_3^2 = x_1^3 - x_1(59475x_4^2 + 78x_3^2) + 2848750x_4^3 + 18226x_3^2x_4.$$

All 27 lines have explicit rational equations.

Example 2.14 (Finite fields) By the Chevalley–Warning theorem (see footnote 2), any cubic hypersurface of dimension ≥ 2 defined over a finite field $\mathbf{k}$ has a $\mathbf{k}$-point. What about lines defined over $\mathbf{k}$?

Consider a diagonal cubic surface $X \subset \mathbf{P}^3_{\mathbf{k}}$ with equation

$$a_1x_1^3 + a_2x_2^3 + a_3x_3^3 + a_4x_4^3 = 0,$$

where $a_1, \dots, a_4 \in \mathbf{k}$ are all non-zero. It is smooth whenever $\mathbf{k}$ is not of characteristic 3, which we assume. Let b_{ij} be such that $b_{ij}^3 = a_i/a_j$. Then, if $\{1,2,3,4\} = \{i,j,k,l\}$, the projective line joining $e_i - b_{ij}e_j$ and $e_k - b_{kl}e_l$ is contained in X. Since we have 3 choices for $\{i,j\}$ and 3 choices for each b_{ij}, the 27 lines of the cubic $X \times_{\mathbf{k}} \overline{\mathbf{k}}$ are all obtained in this fashion hence are defined over $\mathbf{k}[\sqrt[3]{a_i/a_j}, 1 \leq i < j \leq 4]$.

In particular, the 27 lines of the Fermat cubic

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$$

in characteristic 2 are defined over $\mathbf{F}_4$ (but only 3 over $\mathbf{F}_2$), whereas, if $a \in \mathbf{F}_4 \smallsetminus \{0,1\}$, the cubic surface defined by

$$x_1^3 + x_2^3 + x_3^3 + ax_4^3 = 0 \tag{2.7}$$

contains no lines defined over $\mathbf{F}_4$.

The general phenomenon is that when $\mathbf{k}$ is not algebraically closed, the Galois group $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ acts on the set of 27 lines contained in $X_{\overline{\mathbf{k}}} := X \times_{\mathbf{k}} \overline{\mathbf{k}}$ and (when $\mathbf{k}$ is perfect) the fixed points are the lines defined over $\mathbf{k}$ contained in X. For the surface defined by the equation (2.7), all the lines are defined over $\mathbf{F}_{64}$ and the orbits all consist of 3 points.

Exercise 2.15 Let X be a cubic hypersurface of dimension ≥ 6 defined over a finite field $\mathbf{k}$. Show that any $\mathbf{k}$-point of X is on a line contained in X and defined over $\mathbf{k}$ (*Hint:* use the Chevalley–Warning theorem).

This leaves the case of cubic hypersurfaces of dimensions 3, 4 or 5 open: are there (smooth) cubic hypersurfaces of dimensions 3, 4 or 5, defined over a finite field $\mathbf{k}$, which contain no $\mathbf{k}$-lines? Some answers are given in [DLR].

Exercise 2.16 Show that the cubic surface $X \subset \mathbf{P}^3_{\mathbf{F}_2}$ defined by the equation

$$x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_2^2x_3 + x_3^2x_1 + x_1x_2x_3 + x_1x_4^2 + x_1^2x_4 = 0$$

is smooth and has a unique $\mathbf{F}_2$-point; in particular, it does not contain any line defined over $\mathbf{F}_2$ (*Hint:* for smoothness, you may make explicit computations, or else prove that X contains 27 lines defined over $\overline{\mathbf{F}}_2$ and use Remark 2.11).

Chapter 3

Cubic surfaces

3.1 The plane blown up at six points

Let X be the blow up of the projective plane $\mathbf{P}^2_{\mathbf{k}}$ at 6 distinct $\mathbf{k}$-points *in general position* (no 3 on a line, no 6 on a conic). One checks that the linear system of cubic plane curves passing through these 6 points has projective dimension 3 (we say that these 6 points impose independent conditions on cubics) and that the resulting rational map $\mathbf{P}^2_{\mathbf{k}} \dashrightarrow \mathbf{P}^3_{\mathbf{k}}$ induces an embedding

$$X \hookrightarrow \mathbf{P}^3_{\mathbf{k}}$$

whose image is a smooth cubic surface defined over $\mathbf{k}$.

The 27 lines on X are then

- the images of the 6 exceptional divisors;
- the images of the strict transforms on X of the 15 lines passing through 2 of the 6 points;
- the images of the strict transforms on X of the 5 conics passing through 5 of the 6 points.

They are all defined over $\mathbf{k}$.

Example 3.1 Note that for X to be defined over $\mathbf{k}$, we only need the whole set of 6 points to be defined over $\mathbf{k}$. So if we take set a set, defined over $\mathbf{R}$, of 6 points in $\mathbf{P}^2_{\mathbf{C}}$,

- either the 6 points are real and the cubic surface contains 27 real lines;
- or only 4 points are real and the other 2 are complex conjugate, and the cubic surface contains 15 real lines (why?);

- or only 2 points are real and the other 4 form 2 pairs of complex conjugate points, and the cubic surface contains 7 real lines (why?);
- or the 6 points are not real but form 3 pairs of complex conjugate points, and the cubic surface contains 3 real lines (why?).

The following converse was proved by Arthur Clebsch in 1871.

Theorem 3.2 (Clebsch) *Let $\mathbf{k}$ be an algebraically closed field. Any smooth cubic surface $X \subset \mathbf{P}^3_{\mathbf{k}}$ is isomorphic to the plane $\mathbf{P}^2_{\mathbf{k}}$ blown up at 6 distinct points in general position.*

Sketch of proof. Since $\mathbf{k}$ is algebraically closed, X contains 27 lines and one can show (by direct computation) that it contains two disjoint lines L_1 and L_2. We define rational maps

$$\begin{array}{rcl} \Phi\colon L_1 \times L_2 & \dashrightarrow & X \\ (x_1, x_2) & \longmapsto & \begin{array}{c}\text{3rd point of intersection of}\\ \text{the line } \langle x_1, x_2\rangle \text{ with } X\end{array} \end{array}$$

and

$$\begin{array}{rcl} \Psi\colon X \smallsetminus L_1 \smallsetminus L_2 & \dashrightarrow & L_1 \times L_2 \\ x & \longmapsto & (\langle x, L_2\rangle \cap L_1, \langle x, L_1\rangle \cap L_2). \end{array}$$

It is clear that Φ and Ψ are mutually inverse. Moreover, Ψ can be extended to a morphism

$$\Psi\colon X \longrightarrow L_1 \times L_2.$$

(When $x \in L_i$, juste replace the plane $\langle x, L_i\rangle$ with the plane tangent to X at x.)

By the general theory of birational morphisms between smooth projective surfaces, we know that Ψ is a composition of blow ups. On the other hand, it blows down exactly the 5 lines contained in X that meet both L_1 and L_2 (the existence of these lines is again obtained by direct computation); Ψ is therefore the blow up of 5 distinct points on $L_1 \times L_2 \simeq \mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}}$. On the other hand, the blow up of a point on $\mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}}$ is isomorphic to $\mathbf{P}^2_{\mathbf{k}}$ blown up at two distinct points.[1]

It follows that X is isomorphic to $\mathbf{P}^2$ blown up at 6 distinct points $x_1, \dots, x_6$. By adjunction, the hyperplane linear system coming from the embedding $X \subset \mathbf{P}^3_{\mathbf{k}}$ is $|-K_X|$. It is therefore the linear system of cubics passing through $x_1, \dots, x_6$. This linear system must be very ample; this implies that no lines pass through 3 of these points and no conic through all 6 points (they would otherwise be contracted). □

[1] This can be obtained by direct calculation, or by considering the projection of a smooth quadric $Q \subset \mathbf{P}^3_{\mathbf{k}}$ (isomorphic to $\mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}}$) from a point $x \in Q$: it induces a birational morphism from the blow up of Q at x to $\mathbf{P}^2_{\mathbf{k}}$ to $\mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}}$ which contracts the two generators of Q passing through x.

The proof above implies that a cubic surface is isomorphic to a blow up of the plane $\mathbf{P}^2_{\mathbf{k}}$ at 6 points as soon as it contains two skew lines defined over $\mathbf{k}$ (the points need not be defined over $\mathbf{k}$: only the whole set is; when X contains 27 lines defined over $\mathbf{k}$, the 6 points are all defined over $\mathbf{k}$).

Example 3.3 For $p \in \{2, 3\}$, no smooth cubic surfaces in $\mathbf{P}^3_{\mathbf{F}_p}$ contain 27 lines defined over $\mathbf{F}_p$ (the plane $\mathbf{P}^2_{\mathbf{F}_p}$ is too small to contains six $\mathbf{F}_p$-points in general position!).

Up to the action of $\mathrm{PGL}_3(\mathbf{F}_4)$, there is only one set of six $\mathbf{F}_4$-points in $\mathbf{P}^2_{\mathbf{F}_4}$ which are in general position: if $a \in \mathbf{F}_4 \smallsetminus \{0, 1\}$, they are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$, $(1, a, a^2)$, and $(1, a^2, a)$.

Since the Fermat cubic $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$ contains 27 lines defined over $\mathbf{F}_4$ (see Example 2.14), it is isomorphic to the plane $\mathbf{P}^2_{\mathbf{F}_4}$ blown up in these 6 points.

More results on smooth cubic surfaces defined over an algebraically closed field can be found in [H, Section V.4].

3.2 Rationality

One important consequence of Theorem 3.2 is that smooth cubic surfaces defined over an algebraically closed field are *rational.*

Definition 3.4 Let X be a variety of dimension n defined over a field $\mathbf{k}$. We say that X is $\mathbf{k}$-rational if there is a birational isomorphism $\mathbf{P}^n_{\mathbf{k}} \dashrightarrow X$ defined over $\mathbf{k}$.

When $\mathbf{k}$ is algebraically closed, we say only "rational."

In terms of field extensions, $\mathbf{k}$-rationality means that the field $\mathbf{k}(X)$ of rational functions on X is a purely transcendental extension of $\mathbf{k}$.

Remark 3.5 For a smooth cubic surface defined over a field $\mathbf{k}$ to be rational, it is enough that it contain two skew lines defined over $\mathbf{k}$ (see the proof of Theorem 3.2).

Remark 3.6 The set of real points of some real smooth cubic surfaces is not connected: such a surface cannot be $\mathbf{R}$-rational. It is known that real smooth cubic surfaces containing 7, 15, or 27 lines are $\mathbf{R}$-rational; some real smooth cubic surfaces containing only 3 lines are $\mathbf{R}$-rational, while some others are not.

Exercise 3.7 Prove that the cubic surface $X \subset \mathbf{P}^4_{\mathbf{R}}$ defined by the equations

$$x_0 + \cdots + x_4 = \tfrac{1}{8}x_0^3 + x_1^3 + \cdots + x_4^3 = 0$$

is smooth and that the set of its real points is not connected. Find all the real lines contained in X.

Exercise 3.8 Show that any smooth cubic hypersurface $X \subset \mathbf{P}^{2m+1}_{\mathbf{k}}$ which contains two disjoint m-planes defined over $\mathbf{k}$ is $\mathbf{k}$-rational. Find an example of such a cubic for each m, defined over $\mathbf{Q}$.

No examples of smooth rational complex cubic hypersurfaces of odd dimensions are known.

3.3 Picard groups

To state a result of Segre about non-rationality of cubic surfaces, we need to define the Picard group of a scheme.

Definition 3.9 Let X be a scheme. Its Picard group $\mathrm{Pic}(X)$ is the group of isomorphism classes of invertible sheaves on X, under the operation given by tensor product.

When X is integral, $\mathrm{Pic}(X)$ is also the group of Cartier divisors on X modulo linear equivalence.[2] A computation in Cech cohomology shows that $\mathrm{Pic}(X)$ is also isomorphic to the cohomology group $H^1(X, \mathscr{O}_X^*)$.

Example 3.10 The Picard group of $\mathbf{P}^n_{\mathbf{k}}$ is isomorphic to $\mathbf{Z}$ since the invertible sheaves on $\mathbf{P}^n_{\mathbf{k}}$ are the $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(d)$, for $d \in \mathbf{Z}$.

If $\widetilde{X} \to X$ is a smooth blow up, with exceptional divisor E, we have

$$\mathrm{Pic}(\widetilde{X}) \simeq \mathrm{Pic}(X) \oplus \mathbf{Z}[\mathscr{O}_{\widetilde{X}}(E)].$$

In particular, by Theorem 3.2, the Picard group of a cubic surface defined over an algebraically closed field is isomorphic to $\mathbf{Z}^7$.[3]

The Picard group is not invariant under extensions of the base field $\mathbf{k}$. Let X be a smooth cubic surface defined over a field $\mathbf{k}$. Then $\mathrm{Pic}(X)$ is a subgroup of $\mathrm{Pic}(X_{\overline{\mathbf{k}}}) \simeq \mathbf{Z}^7$. Its rank is the *Picard number* $\rho(X)$; since $\mathrm{Pic}(X)$ always contains the (non-zero) class of $\mathscr{O}_X(1)$ (the "hyperplane class"), the Picard number is in $\{1, \dots, 7\}$.

Let p be the characteristic exponent of $\mathbf{k}$ (this is characteristic of $\mathbf{k}$ if it is positive, 1 otherwise). If $\mathbf{k}^{p^{-\infty}} := \bigcup_e \mathbf{k}^{p^{-e}} \subset \overline{\mathbf{k}}$ is the perfect closure of $\mathbf{k}$ and $G := \mathrm{Aut}(\overline{\mathbf{k}}/\mathbf{k}) = \mathrm{Aut}(\overline{\mathbf{k}}/\mathbf{k}^{p^{-\infty}})$ the Galois group, we have:

- any divisor defined over $\mathbf{k}^{p^{-\infty}}$ is defined over some purely inseparable extension of $\mathbf{k}$, so that some p^e-th multiple is defined over $\mathbf{k}$. This implies $\rho(X) = \rho(X_{\mathbf{k}^{p^{-\infty}}})$;

[2] Two Cartier divisors are linear equivalent if their difference is the divisor of a rational function.

[3] In general, the Picard group is an extension of a finitely generated abelian group by a "continuous component" (which is zero in the cubic surface case). We will come back to that in Section 5.1.

- divisors of X defined over $\mathbf{k}^{p^{-\infty}}$ can be identified with the G-invariant divisors of $X_{\overline{\mathbf{k}}}$, which in turn can be identified with the G-orbits of divisors defined over $\overline{\mathbf{k}}$. In particular, the Picard group of $X_{\mathbf{k}^{p^{-\infty}}}$ is the subgroup of the Picard group of $X_{\overline{\mathbf{k}}}$ invariant under the natural action of G. If $\{L_{i_1}, \dots, L_{i_t}\}$ is an orbit of G on the 27 lines, $L_{i_1} + \cdots + L_{i_t}$ is an effective curve defined over $\mathbf{k}^{p^{-\infty}}$ and these orbit sums span $\operatorname{Pic}(X_{\mathbf{k}^{p^{-\infty}}})$.

Theorem 3.11 (B. Segre) *Let X be a smooth cubic surface defined over a field $\mathbf{k}$, with algebraic closure $\overline{\mathbf{k}}$. Consider the action of the Galois group $G := \operatorname{Aut}(\overline{\mathbf{k}}/\mathbf{k})$ on the 27 lines of $X_{\overline{\mathbf{k}}}$.*

The following conditions are equivalent:

(i) *the Picard number $\rho(X)$ is one;*

(ii) *the sum of the lines in each G-orbit is linearly equivalent to a multiple of the hyperplane class on X;*

(iii) *no G-orbit consists of pairwise disjoint lines on X.*

If these conditions hold, X is not $\mathbf{k}$-rational.

For complete proofs, we refer to [KSR]. The implication (i)⇒(ii) is easy. The converse follows from the discussion above since the orbit sums span a free abelian group of rank $\rho(X)$. To see that (i) implies (iii), suppose that $\{L_{i_1}, \dots, L_{i_t}\}$ is a G-orbit consisting of pairwise disjoint lines. If $\rho(X) = 1$, any other orbit sum $L_{j_1} + \cdots + L_{j_s}$ is linearly equivalent to $a(L_{i_1} + \cdots + L_{i_t})$, for some $a \in \mathbf{Q}^+$. But then,

$$-t = (L_{i_1} + \cdots + L_{i_t})^2 = \frac{1}{a}(L_{i_1} + \cdots + L_{i_t})(L_{j_1} + \cdots + L_{j_s}),$$

which is non-negative since the lines in the first orbit are different from the line in the second orbit. This is a contradiction.

The proof that (iii) implies (i) is not too difficult ([KSR, p. 13]), but the fact that the conditions (i)–(iii) imply the irrationality of X is harder ([KSR, p. 17–21]).

Exercise 3.12 1) Let $\mathbf{k}$ be a field of characteristic $\neq 3$ and let $a_1, \dots, a_4 \in \mathbf{k}^*$. Following Segre ([S]), show (using Theorem 3.11) that the smooth cubic surface over $\mathbf{k}$ defined by the equation

$$a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 + a_4 x_4^3 = 0 \tag{3.1}$$

has Picard number one if and only if, for all permutations $\sigma \in \mathfrak{S}_4$,

$$\frac{a_{\sigma(1)} a_{\sigma(2)}}{a_{\sigma(3)} a_{\sigma(4)}}$$

is not a cube in $\mathbf{k}$.

2) Let $a \in \mathbf{F}_4 \smallsetminus \{0,1\}$. Prove that the cubic surface defined by the equation

$$x_1^3 + x_2^3 + x_3^3 + a x_4^3 = 0$$

in $\mathbf{P}^3_{\mathbf{F}_4}$ is smooth, has $\mathbf{F}_4$-points, but is not $\mathbf{F}_4$-rational.

In general, it is known that a cubic surface as (3.1) is rational if and only if

1. it has a $\mathbf{k}$-point (which always holds if $\mathbf{k}$ is finite by the Chevalley–Warning Theorem) and
2. one of the $\frac{a_{\sigma(1)} a_{\sigma(2)}}{a_{\sigma(3)} a_{\sigma(4)}}$ is a cube in $\mathbf{k}$.

Chapter 4

Unirationality

4.1 Unirationality

Let X be a $\mathbf{k}$-variety. We defined the $\mathbf{k}$-rationality of X in Definition 3.4. The following weaker property is also central.

Definition 4.1 Let X be a variety defined over a field $\mathbf{k}$. We say that X is $\mathbf{k}$-*unirational* if there are an integer n and a dominant morphism $\mathbf{P}^n_{\mathbf{k}} \dashrightarrow X$ defined over $\mathbf{k}$.

We say that X is *separably* $\mathbf{k}$-*unirational* if there are an integer n and a dominant separable morphism $\mathbf{P}^n_{\mathbf{k}} \dashrightarrow X$ defined over $\mathbf{k}$.

The integer n in this definition can be taken to be the dimension of X. In characteristic 0, $\mathbf{k}$-unirationality and separable $\mathbf{k}$-unirationality are of course equivalent.

In terms of field extensions, (separable) $\mathbf{k}$-unirationality means that $\mathbf{k}(X)$ has an algebraic (separable) extension which is a purely transcendental extension of $\mathbf{k}$.

Exercise 4.2 Show that the smooth real (non-rational) cubic surface defined by the equations

$$x_0 + \cdots + x_4 = \tfrac{1}{8}x_0^3 + \cdots + x_4^3 = 0$$

in $\mathbf{P}^4_{\mathbf{R}}$ is $\mathbf{R}$-unirational (recall from Exercise 3.7 that it is not $\mathbf{R}$-rational).

Theorem 4.3 *Let $X \subset \mathbf{P}^{N-1}_{\mathbf{k}}$ be a smooth cubic hypersurface containing a $\mathbf{k}$-line L. There exists a double cover $\pi_L \colon \mathbf{P}^{N-2}_{\mathbf{k}} \dashrightarrow X$ defined over $\mathbf{k}$; in particular, X is $\mathbf{k}$-unirational.*

When $\mathbf{k}$ is algebraically closed, the existence of the line L is automatic as soon as $N \geq 4$.

Proof. Let us consider the restriction $T_X|_L$ of the locally free tangent sheaf T_X to L and the total space $\mathbf{P}$ of the projectification of the associated vector bundle. Since $T_X|_L$ is trivial of rank $N-2$ on any affine open subset of L, the $\mathbf{k}$-variety $\mathbf{P}$ is rational, of dimension $N-2$. We define a rational map

$$\pi_L\colon \mathbf{P} \dashrightarrow X$$

as follows. A point of $\mathbf{P}$ is a pair (x, L_x), where x is a (geometric) point of L and L_x is a projective line tangent to X at x. If L_x is not contained in X, since it has contact of order ≥ 2 with X at x, it meets X in a third point, which we call $\pi_L(x, L_x)$. Note that there always exists a line L_x not contained in X, since otherwise, the projective tangent space $\mathbf{T}_{X,x}$ would be contained in X, contradicting the irreducibility of X. We have therefore defined a rational map π_L, defined over $\mathbf{k}$.

Let y be a point in X such that the plane $P_y := \langle L, y\rangle$ is not contained in X (this is the case for y general, since otherwise, X would be a cone with vertex L). The scheme-theoretic intersection $P_y \cap X$ is then the union of L and a residual conic C_y passing through y. Any point $x \in L \cap C_y$ is then singular on $L \cup C_y$, hence $P_y \subset \mathbf{T}_{L\cup C_y,x} \subset \mathbf{T}_{X,x}$. It follows that $y = \pi_L(x, \langle x, y\rangle)$, hence π_L is dominant, of degree 2. □

Remark 4.4 One can show that if $\mathbf{k}$ is algebraically closed and $N \geq 4$, there always exists a line $L \subset X$ such the map π_L defined in the proof above is separable, except if the characteristic is 2 and X is projectively equivalent to the Fermat cubic $x_1^3 + \cdots + x_N^3 = 0$. This implies that X is $\mathbf{k}$-unirational (note that in characteristic 2, Fermat cubics of even dimensions are in fact rational by Exercice 3.8).

We now show that $\mathbf{k}$-unirationality implies the existence of a $\mathbf{k}$-point.

Proposition 4.5 *If a variety is* $\mathbf{k}$*-unirational, it has a* $\mathbf{k}$*-point.*

Proof. This is obvious when $\mathbf{k}$ is infinite: any rational map $\pi\colon \mathbf{P}^n_{\mathbf{k}} \dashrightarrow X$ is defined on some dense Zariski open subset of $\mathbf{P}^n_{\mathbf{k}}$ and, because $\mathbf{P}^n_{\mathbf{k}}$ has plenty of $\mathbf{k}$-points in every dense open set, the image of any one of them will be a $\mathbf{k}$-point of X. This is not so obvious when $\mathbf{k}$ is finite, because $\mathbf{P}^n_{\mathbf{k}}$ has dense open subsets with no $\mathbf{k}$-points.

To prove the proposition, we proceed by induction on n (we do not assume that π is dominant). When $n = 1$, this holds because π is actually a morphism.

Assume $n \geq 2$ and let $\widetilde{\mathbf{P}^n_{\mathbf{k}}} \to \mathbf{P}^n_{\mathbf{k}}$ be the blow up of a $\mathbf{k}$-point, with exceptional divisor E isomorphic to $\mathbf{P}^{n-1}_{\mathbf{k}}$. The induced rational map $\widetilde{\pi}\colon \widetilde{\mathbf{P}^n_{\mathbf{k}}} \to \mathbf{P}^n_{\mathbf{k}} \dashrightarrow X$ is defined over an open subset U of $\widetilde{\mathbf{P}^n_{\mathbf{k}}}$ whose complement has codimension ≥ 2. In particular, U meets E, hence $\widetilde{\pi}$ restricts to E to induce a rational map $\mathbf{P}^{n-1}_{\mathbf{k}} \dashrightarrow X$. We now apply the induction hypothesis to conclude. □

Remark 4.6 Conversely, Kollár proved that over an arbitrary field $\mathbf{k}$, a cubic hypersurface with a $\mathbf{k}$-point is always $\mathbf{k}$-unirational ([K]). Cubic hypersurfaces of dimension ≥ 2 over a finite field $\mathbf{k}$ always have $\mathbf{k}$-points (Chevalley–Warning Theorem), so they are $\mathbf{k}$-unirational.

Exercise 4.7 (Positive characteristics) Over an algebraically closed field $\mathbf{k}$ of characteristic $p > 0$, let us consider as in Example 2.5 the smooth Fermat hypersurface $X \subset \mathbf{P}^{N-1}_{\mathbf{k}}$ with equation

$$x_1^{p^r+1} + \cdots + x_N^{p^r+1} = 0.$$

We assume $N \geq 4$ and that $\mathbf{k}$ contains all (p^r+1)-th roots of -1; let u be such a root. Then X contains the line L joining the points $(1, u, 0, 0, \ldots, 0)$ and $(0, 0, 1, u, 0, \ldots, 0)$. The pencil $-tux_1 + tx_2 - ux_3 + x_4 = 0$ of hyperplanes containing L induces a dominant rational map $\pi \colon X \dashrightarrow A^1_{\mathbf{k}}$ which makes $\mathbf{k}(X)$ into an extension of $\mathbf{k}(t)$.

Show that the generic fiber of π is isomorphic over $\mathbf{k}(t^{1/p^r})$ to

- if $N = 4$, the $\mathbf{k}$-rational plane curve with equation $y_3^{p^r-1} y_4 + y_2^{p^r} = 0$,
- if $N \geq 5$, the singular $\mathbf{k}$-rational hypersurface with equation $y_3^{p^r} y_4 + y_3 y_2^{p^r} + y_5^{p^r+1} + \cdots + y_N^{p^r+1} = 0$ in $\mathbf{P}^{N-2}$.

Deduce that X has a purely inseparable cover of degree p^r which is $\mathbf{k}$-rational.

In contrast with the last exercise, we will show that hypersurfaces of high degree cannot be *separably* unirational.

Theorem 4.8 *If X is a smooth projective variety of dimension $d \geq 1$ that is separably unirational, $H^0(X, (\Omega^r_X)^{\otimes m}) = 0$ for all $r, m \geq 1$.*

Here, Ω_X is the sheaf of Kähler differentials on X ([H, Section II.8]; since X is smooth, this is also the dual of the tangent sheaf T_X), locally free of rank d because X is smooth, and $\Omega^r_X := \bigwedge^r \Omega_X$.

Let K_X be the *canonical divisor* of X, so that $\mathscr{O}(K_X) \simeq \Omega^d_X$. For all positive integers m, the numbers $h^0(X, \mathscr{O}(mK_X))$ are called the *plurigenera* of X and are very important in the classification theory of algebraic varieties. The theorem says that they vanish for a separably unirational smooth projective variety X, and so do the Hodge numbers $h^0(X, \Omega^r_X)$.

PROOF. Let $\pi \colon \mathbf{P}^n \dashrightarrow X$ be a dominant and separable map. It is defined on an open subset $U \subset \mathbf{P}^n$ whose complement has codimension ≥ 2. We have an exact sequence ([H, Proposition II.8.11])

$$\pi^*\Omega_X \to \Omega_U \to \Omega_{U/X} \to 0,$$

where $\pi^*\Omega_X$ and Ω_U are locally free of respective ranks d and n. At the generic point of U, this sequence is the sequence

$$\Omega_{\mathbf{k}(X)/\mathbf{k}} \otimes_{\mathbf{k}} \mathbf{k}(U) \to \Omega_{\mathbf{k}(U)/\mathbf{k}} \to \Omega_{\mathbf{k}(U)/\mathbf{k}(X)} \to 0$$

of $\mathbf{k}(U)$-vector spaces of respective dimensions d, n, and $n - d$ (because the extension $\mathbf{k}(U)/\mathbf{k}(X)$ is separable; [H, Theorem II.8.6A]).

All this implies that the map $\pi^*\Omega_X \to \Omega_U$ is injective (its kernel is torsion-free and 0 at the generic point). In particular, there is an injection

$$H^0(X, (\Omega^r_X)^{\otimes m}) \hookrightarrow H^0(U, (\Omega^r_U)^{\otimes m}),$$

where the latter space is isomorphic to $H^0(\mathbf{P}^n, (\Omega^r_{\mathbf{P}^n})^{\otimes m})$, because the complement of U has codimension ≥ 2.

We now prove that this last space is 0: the dual of the Euler sequence (1.1) gives an inclusion $\Omega_{\mathbf{P}^n} \hookrightarrow \mathscr{O}_{\mathbf{P}^n}(-1)^{\oplus(n+1)}$, hence an inclusion of $\Omega^r_{\mathbf{P}^n} = \bigwedge^r \Omega_{\mathbf{P}^n}$ into a direct sum of copies of $\mathscr{O}_{\mathbf{P}^n}(-r)$, and an inclusion of $(\Omega^r_{\mathbf{P}^n})^{\otimes m}$ into a direct sum of copies of $\mathscr{O}_{\mathbf{P}^n}(-rm)$, a sheaf which has no non-zero sections since $rm > 0$.

This ends the proof of the theorem. $\square$

Corollary 4.9 *A smooth projective hypersurface of degree* $\geq \dim(X) + 2$ *is not separably unirational.*

Proof. This is because the canonical sheaf $\mathscr{O}_X(K_X)$ is $\mathscr{O}_X(\deg(X) - \dim(X) - 2)$, hence it has non-zero sections under our hypothesis. $\square$

Chapter 5

Cubic threefolds

Let $\mathbf{k}$ be an algebraically closed field and let $X \subset \mathbf{P}^4_{\mathbf{k}}$ be a smooth cubic threefold. As we saw in Theorem 4.3, X is unirational: there exists a double cover $\mathbf{P}^3_{\mathbf{k}} \dashrightarrow X$. The question of the rationality of X was a longstanding question until it was solved negatively in 1972 by Clemens–Griffiths (over $\mathbf{C}$).[1] We will explain the tools used in their proof, or rather, in the simpler proofs that appeared later (such as [B2]).

5.1 Jacobians

5.1.1 The Picard group

Let X be a *complex* projective variety. As explained in Section 3.3, the group $\mathrm{Pic}(X)$ can be identified with $H^1(X, \mathscr{O}_X^*)$. Consider the *exponential exact sequence*

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathscr{O}_{X_{\mathrm{an}}} \xrightarrow{\exp} \mathscr{O}^*_{X_{\mathrm{an}}} \longrightarrow 0$$

of sheaves of analytic functions on the underlying complex variety X_{an}. The associated cohomology sequence reads

$$\begin{aligned} 0 \to H^1(X_{\mathrm{an}}, \mathbf{Z}) \to H^1(X_{\mathrm{an}}, \mathscr{O}_{X_{\mathrm{an}}}) \to H^1(X_{\mathrm{an}}, \mathscr{O}^*_{X_{\mathrm{an}}}) & \\ \to H^2(X_{\mathrm{an}}, \mathbf{Z}) \to H^2(X_{\mathrm{an}}, \mathscr{O}_{X_{\mathrm{an}}}). & \qquad (5.1) \end{aligned}$$

By Serre's GAGA theorems, we have $H^q(X_{\mathrm{an}}, \mathscr{O}_{X_{\mathrm{an}}}) \simeq H^q(X, \mathscr{O}_X)$ for all q, and $H^1(X_{\mathrm{an}}, \mathscr{O}^*_{X_{\mathrm{an}}}) \simeq \mathrm{Pic}(X)$.

The image of $\mathrm{Pic}(X)$ in the finitely generated abelian group $H^2(X, \mathbf{Z})$ is again a finitely generated abelian group called the *Néron–Severi group* of X and denoted by $\mathrm{NS}(X)$. We have an exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0,$$

[1] This non-rationality result still holds over any field of characteristic other than 2 ([M]).

where

$$\mathrm{Pic}^0(X) := H^1(X, \mathscr{O}_X)/H^1(X, \mathbf{Z}). \tag{5.2}$$

Any morphism $f\colon X \to Y$ induces compatible morphisms

$$f^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(Y) \quad , \quad f^*\colon \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y) \quad , \quad f^*\colon \mathrm{NS}(X) \to \mathrm{NS}(Y).$$

When X is moreover smooth, by Hodge theory, $\mathrm{Pic}^0(X)$ is an *abelian variety,* a smooth projective variety with a group structure.

In the case of a cubic surface, studied in Section 5.1, $H^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X) = 0$, so that $\mathrm{Pic}^0(X) = 0$ and $\mathrm{Pic}(X) \simeq \mathrm{NS}(X) \simeq H^2(X, \mathbf{Z}) \simeq \mathbf{Z}^7$.

When C is a smooth projective curve, the abelian variety $\mathrm{Pic}^0(C)$ is *principally polarized* of dimension $g := h^1(C, \mathscr{O}_C)$, the *genus* of C, and $\mathrm{NS}(C) \simeq \mathbf{Z}$. The abelian variety $\mathrm{Pic}^0(C)$ is also refered to as the *Jacobian* of C and denoted by $J(C)$.

5.1.2 Intermediate Jacobians

If we want to mimic the construction (5.2) of the abelian variety $\mathrm{Pic}^0(X)$ with $H^3(X, \mathbf{Z})$ instead of $H^1(X, \mathbf{Z})$, the problem is that the Hodge decomposition

$$H^3(X, \mathbf{C}) \simeq H^{0,3}(X) \oplus H^{1,2}(X) \oplus H^{2,1}(X) \oplus H^{3,0}(X)$$

is now more complicated. The quotient

$$J(X) := (H^{0,3}(X) \oplus H^{1,2}(X))/H^3(X, \mathbf{Z})$$

is still a complex torus,[2] but not in general an abelian variety, unless, for example, $H^{0,3}(X)$ or $H^{1,2}(X)$ vanishes. It is called the *intermediate Jacobian* of X.

5.1.3 The Albanese variety

At the other end, if X has dimension n, we may also consider the complex torus

$$\mathrm{Alb}(X) := H^{n-1,n}(X)/H^{2n-1}(X, \mathbf{Z}), \tag{5.3}$$

called the *Albanese variety* of X (this is always an abelian variety). By Poincaré and Serre dualities, this is also

$$\mathrm{Alb}(X) \simeq H^0(X, \Omega_X)^\vee / H_1(X, \mathbf{Z}),$$

[2] What we denote by $H^{2n-1}(X, \mathbf{Z})$ is actually (here and in (5.3)) the image of that group by the composition

$$H^{2n-1}(X, \mathbf{Z}) \to H^{2n-1}(X, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = H^{2n-1}(X, \mathbf{C}) \twoheadrightarrow H^{0,2n-1}(X) \oplus \cdots \oplus H^{n-1,n}(X),$$

which kills exactly the torsion.

where $H_1(X, \mathbf{Z})$ maps to $H^0(X, \Omega_X)^\vee$ by the morphism

$$[\gamma] \longmapsto \Big[\omega \mapsto \int_\gamma \omega\Big].$$

Contrary to the intermediate Jacobian, it is always algebraic (Theorem 5.5).

Any morphism $f\colon X \to Y$ induces compatible morphisms $f_*\colon H_1(X, \mathbf{Z}) \to H_1(Y, \mathbf{Z})$ and $f^{*T}\colon H^0(X, \Omega_X)^\vee \to H^0(Y, \Omega_Y)^\vee$, hence a group morphism $\mathrm{Alb}(f)\colon \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$.

5.1.4 Principally polarized abelian varieties

A *principal polarization* on an abelian variety A is an effective divisor Θ (aptly named *theta divisor*) with the following two properties:

- Θ is ample;
- $h^0(A, \mathscr{O}_A(\Theta)) = 1$.

A *principally polarized abelian variety* is a pair (A, Θ), where A is an abelian variety and Θ a theta divisor on A (defined up to translation by an element of A).

When A is expressed as a complex torus V/Γ, where V is a complex vector space and Γ a lattice in V, a principal polarization on A is exactly the same as a definite positive Hermitian form on V whose imaginary part is an integral unimodular non-degenerate skew-symmetric form on Γ.[3]

When C is a smooth projective curve, such a form is provided by the intersection form on $H^1(C, \mathbf{Z})$ (unimodularity follows from Poincaré duality), hence $J(C)$ is a principally polarized abelian variety.

When X has dimension 3 and $H^{0,3}(X) = 0$, such a form is provided again by the intersection form on $H^3(X, \mathbf{Z})$, hence $J(X)$ is a principally polarized abelian variety. This is the case when X is a smooth cubic threefold, because

$$H^{0,3}(X) \simeq H^3(X, \mathscr{O}_X) \simeq H^0(X, \mathscr{O}_X(-2))^\vee = 0$$

[3]This can be understood through the exact sequence (5.1) for A:

$$\begin{array}{ccccccccccc}
0 \to & H^1(A, \mathbf{Z}) & \to & H^1(A, \mathscr{O}_A) & \to & H^1(X_A, \mathscr{O}_A^*) & \to & H^2(A, \mathbf{Z}) & \xrightarrow{a} & H^2(A, \mathscr{O}_A) \\
& \| & & \| & & \| & & \| & & \| \\
& \Gamma^\vee & \to & \overline{V}^\vee & \to & \mathrm{Pic}(A) & \to & \bigwedge^2 \Gamma^\vee & \to & \bigwedge^2 \overline{V}^\vee.
\end{array}$$

The kernel of the map a is identified with the group of Hermitian forms H on V whose imaginary part is an integral skew-symmetric form on Γ (i.e., an element of $\bigwedge^2 \Gamma^\vee$). Such a form therefore defines a (non-unique) invertible sheaf on A. It is ample if and only if H is positive definite, and its space of global sections has dimension 1 if and only if the integral skew-symmetric form on Γ is unimodular.

by Serre duality (this is also a consequence of Theorems 4.3 and 4.8). As shown in the exercise below, $J(X)$ has dimension $h^{1,2}(X) = h^2(X, \Omega_X) = 5$.

In both cases, the corresponding theta divisors have geometric constructions which we will explain in Section 5.3.

Exercise 5.1 Let V be a 5-dimensional vector space and let $X \subset \mathbf{P}(V)$ be a smooth cubic hypersurface. Using the (dual of the) Euler sequence (1.1), the conormal exact sequence

$$0 \to \mathscr{O}_X(-3) \to \Omega_{\mathbf{P}(V)}|_X \to \Omega_X \to 0,$$

various vanishing theorems, and Serre duality, show that there is an isomorphism

$$H^2(X, \Omega_X) \simeq H^0(X, \mathscr{O}_X(1))^\vee \simeq V. \tag{5.4}$$

In particular, the intermediate Jacobian $J(X)$ has dimension 5.

5.2 The Clemens–Griffiths method

Let X be a smooth complex projective threefold. Assume that X is rational, so that there is a birational isomorphism $\mathbf{P}^3 \overset{\sim}{\dashrightarrow} X$. By Theorem 4.8, we have $H^{0,3}(X) = 0$, so we may consider the principally polarized abelian variety $J(X)$. A typical birational map is the blow up of a smooth subvariety of codimension ≥ 2. We will examine how the intermediate Jacobian changes under this operation.

Proposition 5.2 *Let X be a smooth complex projective threefold with $H^{0,3}(X) = 0$, let $Z \subset X$ be a smooth projective subvariety of codimension ≥ 2, and let $\widetilde{X} \to X$ be the blow up of Z. Then $H^{0,3}(\widetilde{X}) = 0$ and*

$$J(\widetilde{X}) \simeq \begin{cases} J(X) & \text{if } Z \text{ is a point,} \\ J(X) \times J(Z) & \text{if } Z \text{ is a curve,} \end{cases}$$

as principally polarized abelian varieties.

Proof. There is a general formula for the cohomology of the blow up $\widetilde{X} \to X$ of a smooth subvariety $Z \subset X$ of codimension r ([V, Theorem 7.31]): there is an isomorphism of Hodge structures

$$H^k(X, \mathbf{Z}) \oplus \Big(\bigoplus_{i=1}^{r+1} H^{k-2i}(Z, \mathbf{Z})(-i) \Big) \longrightarrow H^k(\widetilde{X}, \mathbf{Z}).$$

In particular, $H^3(\widetilde{X}, \mathbf{Z}) \simeq H^3(X, \mathbf{Z}) \oplus H^1(Z, \mathbf{Z})(-1)$ as Hodge structures and $H^{0,3}(\widetilde{X}, \mathbf{Z}) \simeq H^{0,3}(X, \mathbf{Z}) = 0$.

Furthermore, $H^{1,2}(\widetilde{X}) \simeq H^{1,2}(X) \oplus H^{0,1}(Z)$ and $H^{0,1}(Z) = 0$ when Z is a point. $\square$

Now, any principally polarized abelian variety A decomposes as a product of principally polarized indecomposable factors, and these factors only depend on A. Jacobians of smooth curves are indecomposable, so we may define the *Griffiths component* A_G of A as the product of the indecomposable factors of A which are not Jacobians of curves.[4]

The theorem then says that the Griffiths component of the intermediate Jacobian of a threefold does not change under blow ups.

Let again X be a smooth complex projective threefold with $H^{0,3}(X) = 0$ and let $X \dashrightarrow Y$ be a birational map. By Hironaka's theorem on resolution of singularities, there is a birational morphism $\widetilde{X} \to X$ which is a composition of blow ups with smooth centers (either points or smooth curves) such that the composition

$$f\colon \widetilde{X} \longrightarrow X \dashrightarrow Y$$

is a (birational) morphism. The induced map $f^*\colon H^3(Y, \mathbf{Z}) \to H^3(\widetilde{X}, \mathbf{Z})$ is an injective morphism of Hodge structures. In particular, $H^{0,3}(Y) = 0$ and $J(Y)$ injects into $J(\widetilde{X})$. One checks that more precisely, $J(\widetilde{X})$ decomposes as the product of $J(Y)$ and another principally polarized abelian variety. In particular, the Griffiths component of Y injects into (more precisely, is a factor of) the Griffiths component of $\widetilde{X}$. Using Theorem 5.2, we obtain the following.

Theorem 5.3 *Let X and Y be a smooth complex projective threefolds which are birationally isomorphic. If $H^{0,3}(X) = 0$, we have $H^{0,3}(Y) = 0$ and $J(X)_G \simeq J(Y)_G$.*

In particular, the Griffiths component of the intermediate Jacobian of a rational smooth complex projective threefold is 0.

The last statement follows from $J(\mathbf{P}^3) = 0$.

To prove the unirationality of a smooth cubic threefold, it is therefore "enough" to prove that the Griffiths component of its intermediate Jacobian is non-zero; in other words, that it is not a product of Jacobians of curves. Following Clemens and Griffiths, we will proceed as follows.

As explained earlier, a principally polarized abelian variety has a theta divisor Θ, uniquely determined up to translation. For a general principally polarized abelian variety A, the divisor Θ is smooth, but for any product of Jacobians of curves, the singular locus of Θ has codimension ≤ 4 in A.[5]

So, one way to prove that the (5-dimensional) intermediate Jacobian of a smooth cubic threefold is not a product of Jacobians of curves is to prove that the singular locus of its theta divisor has dimension ≤ 0. We will explain that this singular locus consists of exactly one point.

[4] It should be mentioned here that all principally polarized abelian varieties of dimension ≤ 3 are product of Jacobians of curves, hence their Griffiths component vanishes. In dimensions ≥ 4, a general principally polarized abelian variety is not a Jacobian of curve. This follows easily from a "count of parameters:" curves of genus g depend on $\max(g, 3g-3)$ parameters and principally polarized abelian varieties of dimension g on $g(g+1)/2$ parameters, and $3g - 3 < g(g+1)/2$ for $g \geq 4$.

[5] This follows from the Riemann singularity theorem, which relates the singularities of the theta divisor of the Jacobian of a smooth projective curve C to specific linear systems on the curve C.

5.3 Abel–Jacobi maps

5.3.1 For curves

Let C be a smooth complex projective curve of genus g. We explained earlier that its Jacobian

$$J(C) := H^{0,1}(C)/H^1(C, \mathbf{Z})$$

is a principally polarized abelian variety of dimension g.

The Abel–Jacobi map is a regular map

$$C \to J(C)$$

which we now explain. One way to see it is to remember that $J(C)$ is isomorphic to $\mathrm{Pic}^0(C)$, the group of divisors of degree 0 modulo linear equivalence. If we choose $p_0 \in C$, the Abel–Jacobi map (which depends on the choice of p_0) is then just the map

$$\begin{array}{rcl} u\colon C & \longrightarrow & \mathrm{Pic}^0(C) \\ p & \longmapsto & [p] - [p_0]. \end{array} \tag{5.5}$$

This map is injective when $g \geq 1$ (when $g = 1$, it identifies the elliptic curve C with the 1-dimensional abelian variety $\mathrm{Pic}^0(C)$). It induces for all integers $m \geq 1$ maps

$$\begin{array}{rcl} u_m\colon C^m & \longrightarrow & \mathrm{Pic}^0(C) \\ (p_1, \dots, p_m) & \longmapsto & u(p_1) + \cdots + u(p_m). \end{array}$$

Theorem 5.4 (Riemann) *Let C be a smooth projective curve of genus g. The image $u_{g-1}(C^{g-1})$ is a theta divisor on $J(C)$.*

5.3.2 The Albanese map

Another way to see the map u is to interpret $\mathrm{Pic}^0(C)$ as the Albanese variety of C. Recall from Section 5.1.3 that for X smooth projective, one has

$$\mathrm{Alb}(X) \simeq H^0(X, \Omega_X)^\vee / H_1(X, \mathbf{Z}),$$

where the map $H_1(X, \mathbf{Z}) \to H^1(X, \Omega_X)^\vee$ sends a loop γ to the linear form $\omega \mapsto \int_\gamma \omega$.

Fixing $x_0 \in X$, we consider the well-defined holomorphic map (the Albanese map)

$$\begin{array}{rcl} a\colon X & \longrightarrow & \mathrm{Alb}(X) \simeq H^0(X, \Omega_X)^\vee / H_1(X, \mathbf{Z}) \\ x & \longmapsto & [\omega \mapsto \displaystyle\int_{x_0}^{x} \omega]. \end{array}$$

When X is a curve, this is the same map as u in (5.5).

Theorem 5.5 *Let X be a smooth projective complex variety. The image $a(X)$ generates the group* $\mathrm{Alb}(X)$. *More precisely, for $m \gg 0$, the morphism*

$$\begin{aligned} a_m \colon X^m &\longrightarrow \mathrm{Alb}(X) \\ (x_1, \dots, x_m) &\longmapsto a(x_1) + \dots + a(x_m) \end{aligned}$$

is surjective. In particular, $\mathrm{Alb}(X)$ *is an abelian variety and a is a regular map.*

Proof. The tangent map to a at a point $x \in X$ is the linear map

$$\begin{aligned} T_{X,x} &\longrightarrow T_{\mathrm{Alb}(X),u(x)} \simeq H^0(X, \Omega_X)^\vee \\ t &\longmapsto [\omega \mapsto \omega(t)]. \end{aligned}$$

To prove that a_m is surjective, it is enough to prove that its tangent map

$$\begin{aligned} T_{X,x_1} \oplus \dots \oplus T_{X,x_m} &\longrightarrow H^0(X, \Omega_X)^\vee \\ (t_1, \dots, t_m) &\longmapsto [\omega \mapsto \omega(t_1) + \dots + \omega(t_m)] \end{aligned}$$

is surjective at some point, or that its transpose, the evaluation map

$$H^0(X, \Omega_X) \longrightarrow \Omega_{X,x_1} \oplus \dots \oplus \Omega_{X,x_m}$$

is injective. This follows from the facts that $H^0(X, \Omega_X)$ is finite-dimensional and that, given any non-zero subspace $V \subset H^0(X, \Omega_X)$, the kernel of the evaluation map $V \to \Omega_{X,x}$ has dimension $< \dim(V)$ for $x \in X$ general.

The fact that $\mathrm{Alb}(X)$ is a projective algebraic variety then follows from general principles: a Kähler variety (such as any complex torus) which is the image by a holomorphic map of a projective algebraic variety, is projective algebraic. The fact that a is regular then follows from GAGA principles. □

The Albanese map a of X has a universal property:

any regular map $f \colon X \to B$ to a complex torus B factors through $a \colon X \to \mathrm{Alb}(X)$.

Indeed, by Section 5.1.3, and since $\mathrm{Alb}(B) = B$, there is an induced morphism

$$\mathrm{Alb}(f) \colon \mathrm{Alb}(X) \to \mathrm{Alb}(B) = B$$

and one checks that f is the composition of $\mathrm{Alb}(f) \circ a$ and the translation by $f(x_0)$.

5.3.3 For cubic threefolds

We want to construct an Abel–Jacobi map for a smooth complex cubic threefold X with (principally polarized) intermediate Jacobian

$$J(X) := H^{1,2}(X)/H^3(X, \mathbf{Z}) \simeq H^{2,1}(X)^\vee / H_3(X, \mathbf{Z}).$$

Lines on X are parametrized by a smooth connected surface $F(X)$ (Theorem 2.9 and Remark 2.10). Fix a line $L_0 \subset X$. I claim that we can define a map

$$\begin{aligned} v\colon F(X) &\longrightarrow J(X) \\ [L] &\longmapsto \Big[[\omega] \mapsto \int_{L_0}^{L} \omega \Big]. \end{aligned} \tag{5.6}$$

We need to explain several things.

- The lines L_0 and L have same class in $H_2(X, \mathbf{Z})$ (one can pass continuously from one to the other because $F(X)$ is connected) hence there exists a differentiable real 3-chain Z in X such that $\partial Z = L - L_0$. The integral $\int_{L_0}^{L}$ means $\int_Z$. It is independent of the choice of Z.
- We view $[\omega] \in H^{2,1}(X)$ as an element of $H^3(X, \mathbf{R})$, represented by a closed differential 3-form ω on X, which we can integrate on Z.

The map v is then well-defined and regular (Griffiths, 1968). The following theorem, which we will not prove, gives a beautiful geometric description of one theta divisor in $J(X)$.

Theorem 5.6 (Tyurin, Beauville) *Let $X \subset \mathbf{P}^4_{\mathbf{C}}$ be a smooth complex cubic threefold, with surface of lines $F(X)$ and intermediate Jacobian $J(X)$. The Abel–Jacobi morphism v is an embedding and the image of*

$$\begin{aligned} F(X) \times F(X) &\longrightarrow J(X) \\ (x, y) &\longmapsto v(x) - v(y) \end{aligned} \tag{5.7}$$

is a theta divisor on $J(X)$.

Unfortunately, it does not seem easy to compute the singularities of the theta divisor using this description (recall that our aim is to prove that a theta divisor has only finitely many singular points).

5.4 Conic bundles and Prym varieties

Let as before $X \subset \mathbf{P}^4_{\mathbf{C}}$ be a smooth complex cubic threefold and let $L \subset X$ be a line. Consider the projection $X \smallsetminus L \to \mathbf{P}^2$ from L. If $X_L \to X$ is the blow up of L, it induces a surjective morphism

$$p_L \colon X_L \to \mathbf{P}^2$$

whose fibers are the conics obtained by intersecting planes containing L with X. One checks that for L general, the fibers are either smooth conics (this is the general case) or unions of two distinct lines. The latter happens for fibers of points in the *discriminant curve* $C \subset \mathbf{P}^2$. We say that X_L is a *conic bundle* over $\mathbf{P}^2$.

Lemma 5.7 *The curve C is a smooth quintic curve.*

PROOF. The smoothness of C is a local calculation and follows from our assumption (valid for general L) that all fibers of p_L are reduced. We will skip that and only compute the degree of C.

Choose coordinates $x_1, \dots, x_5$ such that L is given by $x_1 = x_2 = x_3 = 0$. The equation of X can then be written

$$f(x) = x_1 Q_1(x) + x_2 Q_2(x) + x_3 Q_3(x),$$

where Q_1, Q_2, and Q_3 are quadrics. The fiber of (a_1, a_2, a_3) can be found by intersecting X with the plane spanned by $(a_1, a_2, a_3, 0, 0)$ and L. A typical element of that plane is $(\alpha a_1, \alpha a_2, \alpha a_3, \beta, \gamma)$ so we compute

$$\begin{aligned} f(\alpha a_1, \alpha a_2, \alpha a_3, \beta, \gamma) = {} & \alpha a_1 Q_1(\alpha a_1, \alpha a_2, \alpha a_3, \beta, \gamma) \\ & + \alpha a_2 Q_2(\alpha a_1, \alpha a_2, \alpha a_3, \beta, \gamma) + \alpha a_3 Q_3(\alpha a_1, \alpha a_2, \alpha a_3, \beta, \gamma). \end{aligned}$$

This is the product of α (whose vanishing defines L) and a degree-2 homogeneous polynomial in (α, β, γ). The locus $C \subset \mathbf{P}^2$ (with coordinates (a_1, a_2, a_3)) where the corresponding conic is singular is defined by the vanishing of a symmetric 3×3 determinant whose entries are homogeneous polynomials in (a_1, a_2, a_3) of degrees

$$\begin{vmatrix} 3 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix}.$$

It is therefore a quintic curve. $\square$

The preimage of C in X_L is the union of the (strict transforms of the) lines in X which are incident to L. The family of these lines is a curve $\widetilde{C} \subset F(X)$ endowed with an involution σ and a double étale cover $\pi \colon \widetilde{C} \to C$. In particular, $\widetilde{C}$ is smooth and can be shown to be connected. Its genus is (by the Riemann–Hurwitz formula) $g(\widetilde{C}) = 2g(C) - 1 = 11$.

Consider the restriction

$$\widetilde{C} \to J(X)$$

of the Abel–Jacobi map v in (5.6). Since the Jacobian $J(\widetilde{C})$ is also the Albanese variety of $\widetilde{C}$, this map factors, by the universal property of the Albanese map (Section 5.3.2), as

$$\widetilde{C} \xrightarrow{u} J(\widetilde{C}) \longrightarrow J(X).$$

On the other hand, we consider the pullback $\pi^* \colon J(C) \to J(\widetilde{C})$ and the quotient

$$P := J(\widetilde{C}) / \pi^* J(C).$$

It is called the *Prym variety* associated with the double étale covering π and is a principally polarized abelian variety of dimension $g(\widetilde{C}) - g(C) = 11 - 6 = 5$. The theta divisor in P

can be described geometrically and Mumford related its singular points to specific linear systems on $\widetilde{C}$, very much as in the Riemann singularity theorem for Jacobian of curves (see footnote 5). The conclusion of all this is the following ([B2, Proposition 2]).

Theorem 5.8 *The theta divisor of the intermediate Jacobian of a smooth complex cubic threefold has a unique singular point.*

This singular point is, by the way, the image 0 of the diagonal of $F(X) \times F(X)$ by the morphism (5.7).

Combining this with our previous "results," we finally obtain the celebrated following result.

Corollary 5.9 (Clemens–Griffiths) *Every smooth complex cubic threefold is irrational.*

Chapter 6

Cubic fourfolds

In this chapter, we will consider smooth cubic fourfolds $X \subset \mathbf{P}^5_{\mathbf{k}}$.

6.1 The fourfold $F(X)$

The scheme $F(X) \subset G(2,6)$ of lines contained in X is a smooth connected projective fourfold (Theorem 2.9 and Remark 2.10) obtained as the zero locus of a section s of the dual of the rank-4 sheaf $\mathscr{E} := \mathscr{S}ym^3\mathscr{S}$ (see (1.5)). The normal bundle of $F(X)$ in $G := G(2,6)$ is $\mathscr{E}^\vee|_{F(X)}$, hence the normal exact sequence

$$0 \to T_{F(X)} \to T_G|_{F(X)} \to N_{F(X)/G} \to 0$$

gives (using 2.2.2))

$$c_1(F(X)) := -c_1(T_{F(X)}) = -c_1(T_G)|_{F(X)} + c_1(\mathscr{E}^\vee)|_{F(X)} = -6\sigma_1|_{F(X)} + 6\sigma_1|_{F(X)} = 0. \quad (6.1)$$

Moreover, we have an exact sequence

$$0 \longrightarrow \textstyle\bigwedge^4 \mathscr{E} \longrightarrow \cdots \longrightarrow \textstyle\bigwedge^2 \mathscr{E} \longrightarrow \mathscr{E} \xrightarrow{s^\vee} \mathscr{O}_G \longrightarrow \mathscr{O}_{F(X)} \longrightarrow 0$$

of sheaves on G (the Koszul resolution; see (2.6)) which gives

$$\chi(F(X), \mathscr{O}_{F(X)}) = \sum_{i=0}^{4} (-1)^i \chi(G, \textstyle\bigwedge^i \mathscr{E}).$$

There are computer programs (such as Macaulay2) which compute this sort of things (unfortunately, they can only do it on "small" Grassmannians) and we find

$$\chi(F(X), \mathscr{O}_{F(X)}) = 3.$$

6.2 Varieties with vanishing first Chern class

We assume here $\mathbf{k} = \mathbf{C}$. Smooth projective complex varieties with vanishing first Chern class were classified by Beauville and Bogomolov.

Theorem 6.1 (Beauville–Bogomolov Decomposition Theorem) *Let F be a smooth projective complex variety with $c_1(F)_{\mathbf{R}} = 0$. There exists a finite étale cover of F which is isomorphic to the product of*

- *non-zero abelian varieties;*
- *simply connected* Calabi–Yau varieties;
- *simply connected* holomorphic symplectic varieties.

Here, a *Calabi–Yau variety* is a (complex) variety Y of dimension $n \geq 3$ such that $H^i(Y, \mathscr{O}_Y) = 0$ for all $0 < i < n$. In particular, $\chi(Y, \mathscr{O}_Y) = 1 + (-1)^n$.

A *holomorphic symplectic* variety Y is a (complex) variety carrying a holomorphic 2-form η which is everywhere non-degenerate and such that $H^0(Y, \Omega_Y^2) = \mathbf{C}\eta$. The dimension of Y is even, Y is simply connected, and

$$H^0(Y, \Omega_Y^r) = \begin{cases} \mathbf{C}\eta^{\wedge(r/2)} & \text{if } r \text{ is even and } 0 \leq r \leq \dim(Y), \\ 0 & \text{otherwise,} \end{cases}$$

so that $\chi(Y, \mathscr{O}_Y) = 1 + \frac{1}{2}\dim(Y)$. When Y is a surface, it is called a K3 surface.

If the holomorphic Euler characteristic $\chi(F, \mathscr{O}_F)$ is non-zero, the same holds for the étale cover in the theorem. Since the holomorphic Euler characteristic of a non-zero abelian variety vanishes, there can be no such factors in the decomposition of the theorem. It follows that the universal cover of F is a product of simply connected Calabi–Yau varieties and simply connected holomorphic symplectic varieties.

For our fourfold $F(X)$, there are 3 possibilities for its universal cover $\pi\colon \widetilde{F}(X) \to F(X)$:

- either $\widetilde{F}(X)$ is a Calabi–Yau fourfold, in which case
$$2 = \chi(\widetilde{F}(X), \mathscr{O}_{\widetilde{F}(X)}) = \deg(\pi)\chi(F(X), \mathscr{O}_{F(X)}) = 3\deg(\pi),$$
which is impossible;
- or $\widetilde{F}(X)$ is a product of two K3 surfaces, in which case $\chi(\widetilde{F}(X), \mathscr{O}_{\widetilde{F}(X)}) = 4 = 3\deg(\pi)$, which is also impossible;
- or $\widetilde{F}(X)$ is a holomorphic symplectic fourfold, in which case
$$3 = \chi(\widetilde{F}(X), \mathscr{O}_{\widetilde{F}(X)}) = \deg(\pi)\chi(F(X), \mathscr{O}_{F(X)}) = 3\deg(\pi).$$

It follows that we are in the third case and that

$F(X)$ is a holomorphic symplectic fourfold.

6.3 The Hilbert square of a smooth variety

Let Y be a smooth projective variety. A subscheme of length 2 of Y is either reduced, in which case it is just a subset of two distinct points in Y, or non-reduced, in which case it consists of a point $y \in Y$ and a tangent direction to Y at y.

Consider now the blow up $\varepsilon\colon \widetilde{Y \times Y} \to Y \times Y$ of the diagonal $\Delta := \{(y_1, y_2) \in Y \mid y_1 = y_2\}$. Outside of the exceptional divisor E, a point of $\widetilde{Y \times Y}$ is just a pair (y_1, y_2) of two distinct points of Y, whereas a point of E is a tangent direction at some point of Y (this is because the normal bundle $N_{\Delta/Y\times Y}$ is the tangent bundle T_Y). The involution ι of $Y \times Y$ which exchanges the two factors lifts to an involution $\widetilde{\iota}$ of $\widetilde{Y \times Y}$ whose fixed locus is E.

If follows that the subschemes of length 2 of Y are in one-to-one correspondence with the quotient $\widetilde{Y \times Y}/\widetilde{\iota}$. This is a smooth projective variety of dimension $2\dim(Y)$ which we call the *Hilbert square* of Y and denote by $Y^{[2]}$. We still denote by $E \subset Y^{[2]}$ the smooth hypersurface which parametrizes non-reduced subschemes.

Theorem 6.2 (Fujiki, Beauville) *Let S be a complex K3 surface. The Hilbert square $S^{[2]}$ is a holomorphic symplectic fourfold and*

$$H^2(S^{[2]}, \mathbf{Z}) \simeq H^2(S, \mathbf{Z}) \oplus \mathbf{Z}\tfrac{1}{2}[E]. \tag{6.2}$$

Sketch of proof. The surface S carries a nowhere vanishing 2-form η_S. It induces the 2-form $p_1^*\eta_S + p_2^*\eta_S$ on $S \times S$, which we pull back by ε on $\widetilde{S \times S}$. The resulting form is invariant by the involution $\widetilde{\iota}$ described above hence comes from a (non-zero) 2-form η on $S^{[2]}$. Since the double cover $\pi\colon \widetilde{S \times S} \to S^{[2]}$ is simply ramified along E, we have

$$\begin{aligned}\pi^* \operatorname{div}(\eta \wedge \eta) &= \operatorname{div}(\pi^*\eta \wedge \pi^*\eta) - E = \operatorname{div}(\varepsilon^*({\textstyle\bigwedge}^2(p_1^*\eta_S + p_2^*\eta_S))) - E\\ &= \varepsilon^*(\operatorname{div}({\textstyle\bigwedge}^2(p_1^*\eta_S + p_2^*\eta_S))) = \varepsilon^*(\operatorname{div}(p_1^*\eta_S \wedge p_2^*\eta_S)) = 0.\end{aligned}$$

This proves that the 2-form η is everywhere non-degenerate. One checks that it spans $H^0(S^{[2]}, \Omega^2_{S^{[2]}})$, so that $S^{[2]}$ is a holomorphic symplectic variety.

Finally, (6.2) follows from the explicit construction that we gave of $S^{[2]}$. $\square$

6.4 Pfaffian cubics

For these interesting cubics, already studied by Fano in 1942, we relate the fourfold $F(X)$ with a certain K3 surface.

The construction is the following. Let W_6 be a 6-dimensional complex vector space. We defined and studied in Exercise 2.7 the cubic hypersurface $X_3 \subset \mathbf{P}(\bigwedge^2 W_6^\vee)$ of degenerate skew-symmetric bilinear forms on W_6. One can show that its singular set corresponds to skew forms of corank 4, which, since W_6 has dimension 6, is just the Grassmannian $G(2, W_6^\vee)$. It has codimension 6 in $\mathbf{P}(\bigwedge^2 W_6^\vee)$. It follows from the Bertini theorem that for a general 6-dimensional vector subspace $V_6 \subset \bigwedge^2 W_6^\vee$,

$$X := \mathbf{P}(V_6) \cap X_3 \subset \mathbf{P}(\bigwedge^2 W_6^\vee)$$

is a smooth cubic fourfold. We study here these *Pfaffian cubic fourfolds.*

Consider in the dual space the intersection

$$S := G(2, W_6) \cap \mathbf{P}(V_6^\perp) \subset \mathbf{P}(\bigwedge^2 W_6).$$

Since V_6 is general and $\operatorname{codim}(V_6^\perp) = \dim(V_6) = 6$, we obtain a surface and since $K_{G(2,W_6)} \equiv -6H$, its canonical sheaf is trivial (by adjunction). It is in fact a K3 surface (see Section 6.2).[1]

Proposition 6.3 (Beauville–Donagi) *Let $X \subset \mathbf{P}(V_6)$ be a smooth complex Pfaffian cubic fourfold. Then,*

- *X is rational;*
- *when X is general,[2] the fourfold $F(X)$ is isomorphic to the Hilbert square $S^{[2]}$.[3]*

PROOF. To prove that X is rational, we consider

$$Z := \{([w], [\phi]) \in \mathbf{P}(W_6) \times X \mid w \in \operatorname{Ker}(\phi)\},$$

where ϕ is seen as a (rank-4) skew-symmetric form on W_6. The second projection $p_2 \colon Z \to X$ is a $\mathbf{P}^1$-bundle, hence Z is smooth irreducible of dimension 5. On the other hand, the fiber of $[w] \in \mathbf{P}(W_6)$ under the first projection $p_1 \colon Z \to \mathbf{P}(W_6)$ is $\{\phi \in \mathbf{P}(V_6) \mid \phi(w, W_6) \equiv 0\}$ (such a form is necessarily degenerate, hence in X_3, hence in X), which is a non-empty projective linear space. It follows that p_1 is a birational isomorphism.

Since the fibers of p_2 are mapped by p_1 to lines in $\mathbf{P}(W_6)$, the inverse image by p_1 of a general hyperplane in $\mathbf{P}(W_6)$ is birationally isomorphic, on the one hand, by p_1, to $\mathbf{P}^4_{\mathbf{C}}$, and on the other hand, by p_2, to X. The latter is therefore rational.

We now construct a morphism $F(X) \to S^{[2]}$.

A line contained in X corresponds to a pencil of skew-symmetric forms on W_6, all degenerate of rank 4. There exists a $W_4 \subset W_6$ which is isotropic for all these forms (Exercise 2.7.2)) and one can show that it is unique.[4] The pencil is then contained in $(\bigwedge^2 W_4)^\perp \cap V_6$.

[1] As usual in algebraic geometry, "general" means that the property holds for V_6 in a Zariski dense open subset of $G(6, \bigwedge^2 W_6^\vee)$.

[2] More precisely, we assume that X contains no projective planes and S contains no projective lines.

[3] See Section 6.3 for the construction of $S^{[2]}$.

[4] One way to check that is to use a result of Jordan–Kronecker which gives normal forms for any pair of skew-symmetric forms on a finite-dimensional vector space over an algebraically closed field of characteristic $\neq 2$. In our case, one sees that there is a basis of W_6 in which any pair of generators of the pencil is given by the matrices $\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0 \end{pmatrix}$ and either $\begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ -1&0&0&0&0&0\\ 0&-1&0&0&0&0 \end{pmatrix}$, or $\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0 \end{pmatrix}$. In both cases, W_4 is spanned by the kernels of all the forms in the pencil.

Conversely, this intersection defines a linear space contained in X, since any form in $(\bigwedge^2 W_4)^\perp$ must be degenerate.

A "count of parameters" shows that *for a general choice of* V_6*, the fourfold* X *contains no projective planes.* If we make this assumption, we obtain

$$\dim\big((\bigwedge^2 W_4)^\perp \cap V_6\big) = 2.$$

By duality, this means $\dim\big(\bigwedge^2 W_4 + V_6^\perp\big) = 13$, hence

$$\dim\big((\bigwedge^2 W_4) \cap V_6^\perp\big) = 2.$$

In other words, $\mathbf{P}(\bigwedge^2 W_4) \cap \mathbf{P}(V_6^\perp)$ is a projective line. Its intersection with the quadric $G(2, W_4) \subset G(2, W_6)$ (see Exercice 1.1) is then contained in S. Again, a "count of parameters" shows that *for a general choice of* V_6*, the surface* S *contains no projective lines.* If we make this further assumption, the intersection is a subscheme of S of length 2, hence a point of $S^{[2]}$.

Let us show that ϕ is birational by constructing an inverse. Consider two distinct points in S. We can see them as distinct vector subspaces P_1 and P_2 of dimension 2 of W_6. They also define a line in $\mathbf{P}(V_6^\perp)$. Since S contains no lines, this line cannot be contained in $G(2, P_1 + P_2)$. This implies in particular that $P_1 + P_2$ has dimension 4. Any skew-symmetric form in V_6 vanishes on P_1 and P_2, hence those forms that vanish on $P_1 + P_2$ form a vector space of dimension ≥ 2 which corresponds, because of the assumption on X, to a projective line contained in X, hence to a point of $F(X)$. This defines an inverse to ϕ on the complement of the divisor E in $S^{[2]}$.

To finish the proof, one can either see that this construction of the inverse extends to the whole of $S^{[2]}$, so that ϕ is an isomorphism, or argue that the pullback by ϕ of a nowhere vanishing 4-form on $S^{[2]}$ (which exists because $S^{[2]}$ is a symplectic variety) is a non-identically zero 4-form on $F(X)$. This form cannot vanish anywhere, because $F(X)$ is also a symplectic variety. This implies that the tangent map to ϕ is everywhere an isomorphism, hence the birational morphism ϕ is an isomorphism. □

Corollary 6.4 *Let* X *be a smooth complex cubic fourfold. The Hodge numbers* $h^{p,q}(F(X))$ *of the variety* $F(X)$ *of lines contained in* X *are as follows*

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 21 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 21 & & 231 & & 21 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 1 & & 21 & & 1 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

PROOF. Hodge numbers are invariant by smooth deformations. It follows that the Hodge numbers of $F(X)$ are the same as the Hodge numbers of the Hilbert square of a K3 surface and those can be computed using the explicit construction we gave in Section 6.3. $\square$

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