Corrigendum to “FANO VARIETIES”

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March 19, 2012

As Benjamin Nill pointed out, there is a slight imprecision in the notation and the proof of Theorem 8 of [D]. One has to treat the case $a = 0$ separately. The conclusion of the Theorem holds as stated. Here is a corrected proof.[1]

Theorem 1 *The Picard number of a smooth toric Fano variety of dimension n is at most*

$$2 + 2\sqrt{(n^2 - 1)(2n - 1)}$$

Proof. Let u be a vertex of P. The half line $\mathbf{R}^- u$ meets the boundary of P at a point that belongs to some face of dimension $k \le n - 1$ (choose k minimal) hence is in the relative interior of the convex hull of some vertices $u_1, \dots, u_d$, with $d \le k + 1$ by Helly's theorem. One can therefore write

$$0 = \lambda u + \lambda_1 u_1 + \cdots + \lambda_d u_d$$

with $\lambda_1 \ge \cdots \ge \lambda_d > 0$ and $\lambda > 0$. Let $a \in \{0, \dots, d\}$, let $\mathcal{V}_a(Q)$ be the set of vertices of Q that are on the facets $F_u, F_{u_1}, \dots, F_{u_a}$ or on a facet adjacent to them, and let v be a vertex of Q in the complement $\mathcal{V}_a(Q)^c$ of $\mathcal{V}_a(Q)$ in the set $\mathcal{V}(Q)$ of all vertices of Q.

Note first that for each $i \in \{2, \dots, d\}$, the intersection $F_{u_1} \cap F_{u_i}$ is a face of Q of dimension at least $n - k - 1$, hence contains at least $n - k$ vertices. Hence for each $b > 0$,

$$F_{u_1} \cup \cdots \cup F_{u_b} \text{ contains at most } n + (b-1)k \text{ vertices.} \qquad (1)$$

[1] Note that Cinzia Casagrande proved later in [C] a strong version of the Batyrev conjecture: the Picard number of a normal projective $\mathbf{Q}$-factorial Gorenstein toric Fano variety X of dimension n is at most $2n$, with equality if and only if n is even and X is isomorphic to the $n/2$-th self-product of the blow-up of $\mathbf{P}^2$ at three noncollinear points.

Now by Remark 5(2), we have

$$\langle u, v\rangle \geq 1 \qquad\qquad \langle u_i, v\rangle \geq 1$$

for $1 \leq i \leq a$, hence

$$\begin{aligned} -\lambda_{a+1}\langle u_{a+1}, v\rangle - \cdots - \lambda_d\langle u_d, v\rangle &= \lambda\langle u, v\rangle + \sum_{i=1}^{a} \lambda_i\langle u, v_i\rangle \\ &\geq \lambda + \sum_{i=1}^{a} \lambda_i > \sum_{i=1}^{a} \lambda_i \end{aligned}$$

This implies that the integer $\langle u_i, v\rangle$ must be equal to -1 for at least $a+1$ indices i in $\{a+1, \ldots, d\}$, i.e., that v must be on at least $a+1$ faces among $F_{u_{a+1}}, \ldots, F_{u_d}$.

For $a = 0$, we obtain that $\mathcal{V}_0(Q)^c$ is contained in $F_{u_1} \cup \cdots \cup F_{u_d}$, hence

$$\mathrm{Card}(\mathcal{V}_0(Q)^c) \leq n + (d-1)k$$

by (1). For $a > 0$, consider the set

$$I_a = \{(v, i) \in \mathcal{V}_a(Q)^c \times \{a+1, \ldots, d\} \mid v \in F_{u_i}\}$$

The fiber of i for the second projection $I_a \to \{a+1, \ldots, d\}$ consists of vertices that are on F_{u_i} but that are not on the intersection $\bigcap_{j=1}^{d} F_{u_j}$, which is an $(n-k-1)$-dimensional face of Q, hence has $n-k$ vertices. We obtain

$$\mathrm{Card}(I_a) \leq k(d-a)$$

Since we proved that each fiber of the first projection has at least $a+1$ elements, we get

$$\mathrm{Card}(\mathcal{V}_a(Q)^c) \leq \frac{\mathrm{Card}(I_a)}{a+1} \leq \frac{k(d-a)}{a+1}$$

Using (1) again, we obtain

$$\begin{array}{rlcl} \mathrm{Card}(\mathcal{V}_a(Q)) & \leq & n & \text{vertices on } F_u \\ & + & n + (a-1)k & \text{vertices on } F_{u_1} \cup \cdots \cup F_{u_a} \\ & + & (a+1)n & \text{vertices adjacent to } F_u,\, F_{u_1}, \ldots, F_{u_a} \end{array}$$

and

$$\begin{array}{rcll} \mathrm{Card}(\mathcal{V}_0(Q)) & \leq & n & \text{vertices on } F_u \\ & + & n & \text{vertices adjacent to } F_u \end{array}$$

All in all, we obtain

$$\begin{aligned} \mathrm{Card}(\mathcal{V}(Q)) &\leq 2n + (a-1)k + (a+1)n + \frac{k(d-a)}{a+1} \\ &= 3n - 2k + a(k+n) + \frac{k(d+1)}{a+1} \end{aligned}$$

for *all* $a \in \{0, \dots, d\}$. Taking

$$a = \left[\sqrt{\frac{k(d+1)}{k+n}} \right]$$

we get

$$\mathrm{Card}(\mathcal{V}(Q)) \leq 3n - 2k + 2\sqrt{k(d+1)(k+n)}$$

Therefore,

$$\begin{aligned} \mathrm{Card}(\mathcal{V}(Q)) &\leq \max_{1 \leq d \leq k+1 \leq n} \left(3n - 2k + 2\sqrt{k(d+1)(k+n)} \right) \\ &= n + 2 + 2\sqrt{(n^2-1)(2n-1)} \end{aligned}$$

□

References

[C] Casagrande, C., The number of vertices of a Fano polytope, *Ann. Inst. Fourier (Grenoble)* **56** (2006), 121–130.

[D] Debarre, O., Fano Varieties, in *Higher Dimensional Varieties and Rational Points, Budapest, 2001,* K. Böröczky Jr., J. Kollár and T. Szamuely editors, Bolyai Society Mathematical Studies **12**, Springer–Verlag, Berlin, 2003, 93–132.