Problem set 3

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Due Tuesday May 2, 2017

Problem 1. Let $\mathbf{k}$ be a field. We consider two copies $U_1 := \mathrm{Spec}(\mathbf{k}[T_1])$ and $U_2 := \mathrm{Spec}(\mathbf{k}[T_2])$ of the affine line $\mathbf{A}^1_{\mathbf{k}}$.

a) Compute the Picard groups of $\mathbf{A}^1_{\mathbf{k}}$ and $\mathbf{A}^1_{\mathbf{k}} \smallsetminus \{0\}$ (*Hint:* you may use without proof the fact that if A is a unique factorization domain, the Picard group of $\mathrm{Spec}(A)$ is trivial).

b) Let X be the scheme obtained by glueing U_1 and U_2 along the open subsets $U_1 \smallsetminus \{0\} = \mathrm{Spec}(\mathbf{k}[T_1, T_1^{-1}])$ and $U_2 \smallsetminus \{0\} = \mathrm{Spec}(\mathbf{k}[T_2, T_2^{-1}])$ by the isomorphism $\mathbf{k}[T_1, T_1^{-1}] \xrightarrow{\sim} \mathbf{k}[T_2, T_2^{-1}]$ of $\mathbf{k}$-algebras sending T_1 to T_2^{-1}. Which scheme is X?

c) Compute the Picard group of X (*Hint:* explain that you may use Leray's theorem to compute $H^1(X, \mathscr{O}_X^*)$).

d) Find the global sections of each invertible sheaf on X.

e) Let Y be the scheme obtained by glueing U_1 and U_2 as in b), but using now the isomorphism $\mathbf{k}[T_1, T_1^{-1}] \xrightarrow{\sim} \mathbf{k}[T_2, T_2^{-1}]$ that sends T_1 to T_2. Compute the Picard group of Y (*Hint:* proceed as in c)).

f) Find the global sections of each invertible sheaf on Y.

g) Prove that there are no ample invertible sheaves on Y.

Problem 2. Prove that the scheme $Y_n := \mathbf{A}^n_{\mathbf{k}} \smallsetminus \{0\}$ is not an affine scheme for any $n \geq 2$ (*Hint:* use Leray's theorem to compute $H^1(Y_2, \mathscr{O}_{Y_2})$).

Problem 3. Let X be a projective scheme over a field and let $\mathscr{L}$ and $\mathscr{M}$ be invertible sheaves on X.

a) If $\mathscr{L}$ is generated by global sections and $\mathscr{M}$ is very ample, the invertible sheaf $\mathscr{L} \otimes \mathscr{M}$ is very ample (*Hint:* use a Segre embedding).

b) If $\mathscr{M}$ is ample, the invertible sheaf $\mathscr{L} \otimes \mathscr{M}^{\otimes r}$ is very ample for *all* sufficiently large integers r (*Hint:* we proved in class that $\mathscr{L} \otimes \mathscr{M}^{\otimes r}$ is ample for some integer $r > 0$).