TWO OR THREE THINGS I KNOW ABOUT ABELIAN VARIETIES

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ABSTRACT. We discuss, mostly without proofs, classical facts about abelian varieties (proper algebraic groups defined over a field). The general theory is explained over arbitrary fields. We define and describe the properties of various examples of principally polarized abelian varieties: Jacobians of curves, Prym varieties, intermediate Jacobians.

CONTENTS

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1. Why abelian varieties?

1.1. **Chevalley's structure theorem.** If one wants to study arbitrary algebraic groups (of finite type over a field $\mathbf{k}$), the first step is the following result of Chevalley. It indicates that, over a perfect field $\mathbf{k}$, the theory splits into two very different cases:

- affine algebraic groups, which are subgroups of some general linear group $\mathrm{GL}_n(\mathbf{k})$;
- proper algebraic groups, which are called *abelian varieties* and are the object of study of these notes.

Theorem 1 (Chevalley)**.** *Let G be an algebraic group defined over a perfect field $\mathbf{k}$. There exist a canonical exact sequence of morphisms of algebraic groups*

$$1 \longrightarrow G_{\mathrm{aff}} \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where G_{aff} is an affine algebraic group and A is a proper algebraic group, both defined over $\mathbf{k}$.

This sequence is not split in general.

1.2. **Basic properties of abelian varieties.** We prove two fundamental properties of abelian varieties (one of which justifies their name!).

Theorem 2. *Any abelian variety is a commutative group and a projective variety.*

Proof. To prove commutativity, we rely on the following rigidity lemma.

Lemma 3. *Let X be an irreducible proper variety defined over an algebraically closed field, let Y be an irreducible variety, let y_0 be a point of Y and let $u\colon X \times Y \to Z$ be a regular map such that $u(X \times \{y_0\})$ is a point. Then $u(X \times \{y\})$ is a point for all $y \in Y$.*

Proof. Let

$$\Gamma := \{(x, y, z) \in X \times Y \times Z \mid z = u(x, y)\}$$

be the graph of u and let $q\colon X \times Y \times Z \to Y \times Z$ be the projection. Since X is proper,

$$q(\Gamma) = \{(y, z) \in Y \times Z \mid \exists x \in X \quad z = u(x, y)\}$$

is a closed subvariety, which is irreducible by hypothesis. The projection $q(\Gamma) \to Y$ is surjective and the fibre of y_0 is a point. This implies that the variety $q(\Gamma)$ as same dimension as Y. Let x_0 be a point of X; the graph $\{(y, u(x_0, y)) \mid y \in Y\}$ of the regular map $y \mapsto u(x_0, y)$ is closed in $q(\Gamma)$ and has same dimension: they are therefore equal and, for all x in X and all y in Y, we have $u(x, y) = u(x_0, y)$. □

The conclusion of the lemma may not hold if X is not proper, as shown by the regular map $u\colon \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ defined by $u(x, y) = xy$: we have $u(\mathbf{A}^1 \times \{0\}) = \{0\}$ but $u(\mathbf{A}^1 \times \{t\}) = \mathbf{A}^1$ for $t \neq 0$. In the proof above, $q(\Gamma) = \{(y, xy)\}$ is not a variety (only a constructible set).

Going back to the proof of the theorem, given an abelian variety A, we work over the algebraic closure of $\mathbf{k}$ and apply the lemma to the map $u\colon A \times A \to A$ defined by $u(x, x') = x^{-1}x'x$. It contracts $A \times \{e\}$ to the point e. The lemma implies that for all x, x' in A, we have $u(x, x') = u(e, x') = x'$, so that A is an abelian group.

Another useful consequence of the lemma is the following.

Proposition 4. *Let A be an abelian variety and let G be an algebraic group. A regular map $u\colon A \to G$ that satisfies $u(e_A) = e_G$ is a group morphism.*

Proof. Define a regular map $u'\colon A \times A \to G$ by

$$u'(x, x') = u(x^{-1})u(x')u(x^{-1}x')^{-1}.$$

We have $u'(A \times \{e_A\}) = \{e_G\}$ and the rigidity lemma implies that for all x, x' in A, we have $u'(x, x') = u'(e_A, x') = e_G$, which implies the proposition. $\square$

To prove that A is projective, we need to show that it contains an ample divisor. For that, we may assume that the base field $\mathbf{k}$ is algebraically closed. Following Weil, we first construct hypersurfaces $D_1, \dots, D_r$ in A such that $\bigcap_{i=1}^r D_i = \{0\}$ as follows.

Let x_1 be a (closed) point in A distinct from the origin 0_A. The exists an affine open neighborhood V of 0_A in A. If $z \in V \cap (V + x_1)$, both 0_A and x_1 are in the affine open subset $U := V + x_1 - z$. Embed U in some affine space $\mathbf{A}^N_{\mathbf{k}}$ and choose a hyperplane $H \subset \mathbf{A}^N_{\mathbf{k}}$ such that $0_A \in H$ but $x_1 \notin H$. Set D_1 be the closure in X of $U \cap H$; this is a hypersurface in A which contains 0_A but not x_1.

If there is a point x_2 in $D_1 \smallsetminus \{0_A\}$, construct, using the same process, a hypersurface $D_2 \subset X$ which contains 0_A but not x_2. Repeat the process and construct hypersurfaces $D_1, D_2, \dots$. We obtain a strictly decreasing chain $D_1 \supsetneq D_1 \cap D_2 \supsetneq \cdots$ of closed subsets of A which, since A is noetherian, must stop at some point where $D_1 \cap \cdots \cap D_r = \{0\}$.

We will now prove that the divisor $D := D_1 \cap \cdots \cap D_r$ is ample on A.

Let a and b be distinct points of A. Since $b - a \neq 0_A$, there exists $i \in \{1, \dots, r\}$, say $i = 1$, such that $b - a \notin D_i$. Set $a_1 = a$; then $D_1 + a_1$ contains a but not b. Take b_1 outside of $(b - D_1) \cup (D_1 - a_1 - b)$, so that

$$b \notin (D_1 + a_1) \cup (D_1 + b_1) \cup (D_1 - a_1 - b_1).$$

Similarly, for each $j > 1$, choose a_j outside of $b - D_j$ and b_j outside of $(b - D_j) \cup (D_j - a_j - b) \cup (b - a_1 - D_1)$, so that

$$b \notin (D_j + a_j) \cup (D_j + b_j) \cup (D_j - a_j - b_j).$$

The effective divisor

$$\sum_{i=1}^{r} (D_j + a_j) + (D_j + b_j) + (D_j - a_j - b_j)$$

contains a but not b. By the theorem of the square (see Theorem 5 below), it is linearly equivalent to $3D$. This implies that the map

$$\psi_{3D}\colon A \longrightarrow \mathbf{P}(H^0(A, \mathscr{O}_A(3D))^\vee)$$

associated with the global sections of $\mathscr{O}_A(3D)$ is an injective morphism. It is in particular finite and $3D$ (hence also D) is ample. This finishes the proof of the theorem. $\square$

The main ingredient of the second part of the proof above is the following crucial (and difficult) theorem (Weil).

Theorem 5 (Theorem of the square)**.** *Let D be a divisor on an abelian variety A defined over a field $\mathbf{k}$. For any $x, y \in A(\mathbf{k})$, we have*

$$(D+x)+(D+y) \underset{\text{lin}}{\equiv} D+(D+x+y).$$

Given an integer $n \in \mathbf{Z}$, we will denote the morphism multiplication by n on A by

$$\mathbf{n}_A \colon A \longrightarrow A$$

and by $A[n]$ its kernel (so that $A[n](\mathbf{k})$ is the set of n-torsion $\mathbf{k}$-points).

1.3. **Torsion points.** Let A be an abelian variety of dimension g defined over a field $\mathbf{k}$ and let n be a non-zero integer. When $\mathbf{k} = \mathbf{C}$, we will see in Section 2.1 that the morphism $\mathbf{n}_A$ defined above has degree n^{2g} and that the subgroup $A[n](\mathbf{C})$ of n-torsion points in the group $A(\mathbf{C})$ is isomorphic to $(\mathbf{Z}/n\mathbf{Z})^{2g}$. In positive characteristics, the situation is different.

Theorem 6. *Let A be an abelian variety of dimension g defined over a field $\mathbf{k}$ of characteristic $p > 0$ and let n be a non-zero integer. The map $\mathbf{n}_A$ is surjective of degree n^{2g}. Moreover,*

- *if n is prime to p, the map $\mathbf{n}_A$ is étale and, when $\mathbf{k}$ is separably closed, the group $A[n](\mathbf{k})$ is isomorphic to $(\mathbf{Z}/n\mathbf{Z})^{2g}$;*
- *the map p_A is not étale and, when $\mathbf{k}$ is separably closed, there is an integer $r \in \{0, \dots, g\}$ (called the p-rank of A) such that, for any positive integer m, the group $A[p^m](\mathbf{k})$ is isomorphic to $(\mathbf{Z}/p^m\mathbf{Z})^r$.*

The p-rank r can take any value between 0 and g but the "typical" value is $r = g$ (the smaller the p-rank, the more special A is).

2. Where does one find abelian varieties?

2.1. **Complex tori.** Let V be a complex vector space of dimension g and let $\Gamma \subset V$ be a lattice (of rank $2g$). Then $X := V/\Gamma$ is a compact complex manifold (diffeomorphic to $(\mathbf{S}^1)^{2g}$) which is a group. It is called a complex torus. Note that for any positive integer n, the morphism $\mathbf{n}_X$ is surjective of degree n^{2g} and that its kernel, the group of n-torsion points of X, is isomorphic to $\frac{1}{n}\Gamma/\Gamma \simeq (\mathbf{Z}/n\mathbf{Z})^{2g}$.

One can show that any complex abelian variety is a complex torus. But conversely, when is a complex torus algebraic? This is a non-trivial question.

When $g = 1$, the classical analytic theory of elliptic curves tells us that the Weierstraß function $\wp$ induces an analytic map

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{P}^2_{\mathbf{C}} \\ z & \longmapsto & (\wp(z), \wp'(z), 1) \end{array}$$

which is an isomorphism between X and a (smooth) plane cubic curve with affine equation

$$y^2 = 4x^3 - g_2 x - g_3,$$

where

$$g_2 := 60 \sum_{\gamma \in \Gamma \smallsetminus \{0\}} \frac{1}{\gamma^4} \quad \text{and} \quad g_3 := 140 \sum_{\gamma \in \Gamma \smallsetminus \{0\}} \frac{1}{\gamma^6}.$$

So X is always algebraic, hence an abelian variety. Note that over any field, any smooth plane cubic curve with affine equation $y^2 = x^3 + ax + b$ defines an abelian variety of dimension 1, where the group law has neutral element the point at infinity $e := (0, 1, 0)$ and $P+Q+R = e$ if and only if P, Q, and R are on a line.

Let now X be a complex torus (of dimension g) and let us look at holomorphic maps $X \to \mathbf{P}^n_{\mathbf{C}}$ which are embeddings. Consider the Fubini–Study (Kähler) metric on $\mathbf{P}^n_{\mathbf{C}}$ and its associated $(1,1)$-form ω. The class $[\omega]$ generates $H^2(\mathbf{P}^n_{\mathbf{C}}, \mathbf{Z})$. The pullback $u^*\omega$ to X is cohomologous to a constant form ω_X, real of type $(1,1)$, positive.

Since X is diffeomorphic to $(\mathbf{S}^1)^{2g}$, its integral cohomology ring is $\bigwedge^\bullet H^1(X, \mathbf{Z}) = \bigwedge^\bullet \Gamma^\vee$, where $\Gamma^\vee := \mathrm{Hom}_{\mathbf{Z}}(\Gamma, \mathbf{Z})$. In particular,

$$H^1(X, \mathbf{C}) = \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C}) = V^\vee \oplus \bar{V}^\vee,$$

where $V^\vee := \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C})$ is the space of $\mathbf{C}$-linear forms on V and $\bar{V}^\vee$ the space of $\mathbf{C}$-antilinear forms. This decomposition is the Hodge decomposition

$$H^1_{\mathrm{DR}}(X) = H^{1,0}(X) \oplus H^{0,1}(X).$$

More generally, we have isomorphisms

$$H^{p,q}(X) \simeq \textstyle\bigwedge^{p,q} V := \bigwedge^p V^\vee \otimes \bigwedge^q \bar{V}^\vee$$

and the $(1,1)$-form ω_X can be viewed as an element of $H^{1,1}(X) \simeq V^\vee \otimes \bar{V}^\vee$. The fact that it is real means that it induces a Hermitian form H on V; since it is positive, H is positive definite. The fact that it is an integral class (as the pullback of the integral class ω) means that the skew symmetric form $\mathrm{Im}(H)$ takes integral values on the lattice Γ.

This shows that if X can be embedded as a subvariety of a projective space, there must exist a positive definite Hermitian form H on V with the property that its imaginary part takes integral values on the lattice Γ; the converse holds (this follows either from the classical theory of theta functions or from a more recent theorem of Kodaira). When $g > 1$, this integrality condition is very restrictive and for most lattices $\Gamma \subset V$, it cannot be realized; more precisely, one has then $X_{\mathrm{ab}} = \{0\}$ in the following theorem.

Theorem 7. *Let X be a complex torus. There exists a complex abelian variety X_{ab} and a quotient morphism $\rho\colon X \to X_{\mathrm{ab}}$ such that any regular map from X to a projective space factors through ρ.*

2.2. **Jacobian of smooth projective curves.** Again, we begin with the case $\mathbf{k} = \mathbf{C}$. Let C be a smooth (complex) projective curve of genus $g := h^1(C, \mathscr{O}_C)$. The Hodge decomposition reads

$$H^1(C, \mathbf{C}) = H^{0,1}(C) \oplus H^{1,0}(C)$$

where both factors are (complex conjugate) complex vector spaces of dimension g. We can form the g-dimensional complex torus

$$J(C) := H^{0,1}(C)/H^1(C, \mathbf{Z}).$$

The intersection form on $H^1(C, \mathbf{Z})$ given by cup-product defines a Hermitian form on $H^{0,1}(C)$ which has the properties described in Section 2.1. The torus $J(C)$ is therefore an abelian variety.

If C is a smooth projective curve defined over a field $\mathbf{k}$, one can construct (by very different methods) a Jacobian, which is an abelian variety defined over $\mathbf{k}$ of dimension $g := \dim_{\mathbf{k}}(H^1(C, \mathscr{O}_C))$. This was first achieved by Weil in 1948.

2.3. **Picard varieties.** Let X be a compact Kähler manifold. The set

$$\mathrm{Pic}_{\mathrm{an}}(X) := \{\text{holomorphic line bundles on } X\}/\text{isomorphism}$$

endowed with the composition law given by tensor product, is an abelian group called the (analytic) *Picard group* of X. This group is isomorphic to $H^1(X, \mathscr{O}^*_{X,\mathrm{an}})$, where $\mathscr{O}^*_{X,\mathrm{an}}$ is the sheaf of nowhere vanishing holomorphic functions on X.

The long exact sequence in cohomology associated with the exponential exact sequence

$$0 \longrightarrow \underline{\mathbf{Z}} \longrightarrow \mathscr{O}_{X,\mathrm{an}} \xrightarrow{\exp(2i\pi\cdot)} \mathscr{O}^*_{X,\mathrm{an}} \longrightarrow 1$$

includes

$$0 \longrightarrow H^1(X, \mathbf{Z}) \xrightarrow{\text{lattice}} H^1(X, \mathscr{O}_{X,\mathrm{an}}) \longrightarrow H^1(X, \mathscr{O}^*_{X,\mathrm{an}}) \xrightarrow{c_1} H^2(X, \mathbf{Z}),$$

where c_1 is the first Chern class map. The quotient

$$\mathrm{Pic}^0_{\mathrm{an}}(X) := H^1(X, \mathscr{O}_{X,\mathrm{an}})/H^1(X, \mathbf{Z})$$

is a complex torus. It parametrizes isomorphism classes of holomorphic line bundles on X which are topologically trivial and there is an exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Pic}^0_{\mathrm{an}}(X) \longrightarrow \mathrm{Pic}_{\mathrm{an}}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0, \tag{1}$$

where $\mathrm{NS}(X) := \mathrm{Im}(c_1) \subset H^2(X, \mathbf{Z})$ is a finitely generated abelian group (the *Néron–Severi group* of X).

When X is a smooth projective complex variety, Serre's GAGA theorems show that the analytic Picard group of X (as defined above) and its algebraic Picard group $\mathrm{Pic}(X)$ (defined analogously) are isomorphic. We have therefore an exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0.$$

Example 8. Let $X = V/\Gamma$ be a complex torus. One can show that the exact sequence (1) is identical to the exact sequence

$$0 \longrightarrow \bar{V}^\vee/\Gamma^\vee \longrightarrow \mathrm{Pic}_{\mathrm{an}}(X) \longrightarrow \textstyle\bigwedge^2\Gamma^\vee \cap \bigwedge^{1,1} V^\vee \longrightarrow 0,$$

where $\bigwedge^2\Gamma^\vee \cap \bigwedge^{1,1} V^\vee$ is the abelian group of Hermitian forms on V whose imaginary part is integral on Γ, and $\Gamma^\vee \subset \bar{V}^\vee$ is the abelian group of $\mathbf{C}$-antilinear forms on V whose imaginary part is integral on Γ. The torus $\mathrm{Pic}^0(X) = \bar{V}^\vee/\Gamma^\vee$ is called the *dual torus* of X, often denoted by $\widehat{X}$.

The theory can also be developed over any field for smooth proper varieties (Grothendieck) but we will not explain how except in the case where X is an abelian variety. This includes the fact that $\mathrm{Pic}^0(X)$ is an abelian variety defined over $\mathbf{k}$. Note that it is not at all *a priori* obvious why the group $\mathrm{Pic}^0(X)$ should have the structure of a variety.

2.4. **Albanese varieties.** Let again X be a compact Kähler manifold. Hodge theory tells us that the quotient

$$\mathrm{Alb}(X) := H^0(X, \Omega^1_X)^\vee / H_1(X, \mathbf{Z})$$

is a complex torus (note that $H^0(X, \Omega^1_X) = H^{1,0}(X)$ and that $H_1(X, \mathbf{Z})$ should really be modulo torsion) called *the Albanese variety* of X. It is the dual torus of $\mathrm{Pic}^0(X)$.

Given any point $x_0 \in X$, we define a holomorphic map

$$\begin{aligned} \mathrm{alb}_X \colon X &\longrightarrow \mathrm{Alb}(X) \\ x &\longmapsto \Big(\alpha \mapsto \int_{x_0}^{x} \alpha\Big). \end{aligned}$$

In this formula, α is a holomorphic 1-form on X and the integral is taken over any path on X from x_0 to x; the ambiguity in the choice of the path is reflected by the fact that we take the quotient by $H_1(X, \mathbf{Z})$ to define $\mathrm{Alb}(X)$.

This map is called the *Albanese map* of X and it has the following universal property: any holomorphic map from X to a complex torus factors through alb_X.

3. Line bundles on an abelian variety

3.1. **The dual abelian variety.** Let A be an abelian variety defined over a field $\mathbf{k}$. In this section, we study the group

$$\mathrm{Pic}(A) := \{\text{algebraic line bundles on } A\}/\text{isomorphism}.$$

For any $x \in A(\mathbf{k})$, we denote by τ_x the translation $a \mapsto a - x$. The theorem of the square (Theorem 5) implies that the map

$$\begin{aligned} \varphi_L \colon A(\mathbf{k}) &\longrightarrow \mathrm{Pic}(A) \\ x &\longmapsto \tau_x^* L \otimes L^{-1} \end{aligned}$$

is a morphism of groups. We let $K(L)$ be its kernel.

Note that $\varphi_{L \otimes M} = \varphi_L + \varphi_M$ and $\varphi_{\mathscr{O}_A} = 0$. This allows us to give a first definition of the dual of A.

Definition 9. The (set of $\mathbf{k}$-points of the) dual abelian variety of an abelian variety A is the subgroup

$$\widehat{A}(\mathbf{k}) := \mathrm{Pic}^0(A) := \{L \in \mathrm{Pic}(A) \mid \varphi_L = 0\}$$

of $\mathrm{Pic}(A)$.

We have the following properties:

- the image of any φ_L is contained in $\mathrm{Pic}^0(A)$;
- when L is ample, φ_L is surjective with finite kernel.

The first item follows from the theorem of the square. As for the second item, the surjectivity is the key point and is hard to prove ([Mu, Theorem 1, p. 77]). The finiteness of the kernel comes from the fact that $L \otimes (-\mathbf{1}_A)^* L$, which is ample, is trivial on the kernel.

Example 10. In the case of an elliptic curve E (i.e., of an abelian variety of dimension 1) defined over an algebraically closed field $\mathbf{k}$, any line bundle L of degree 1 (hence ample) can be written, by Riemann–Roch, as $\mathscr{O}_E(x_0)$, and, by the theorem of the square,

$$\varphi_L(x) = \mathscr{O}_E((x_0 + x) - x_0) = \mathscr{O}_E(x - 0_E).$$

In particular, $\varphi_L = \varphi_{\mathscr{O}_E(0_E)}$ is independent of L (of degree 1) and it is injective. Any line bundle L of degree 0 can be written as the "difference" of two line bundles of degree 1, hence φ_L is zero. It follows that $\mathrm{Pic}^0(E)$ is the group of line bundles of degree 0, it is isomorphic to E by $\varphi_{\mathscr{O}_E(0_E)}$ (this puts an algebraic structure on $\mathrm{Pic}^0(E)$), and via this identification, for any line bundle L on E, the map φ_L is multiplication by $\deg(L)$.

In general, since A is projective, there is an ample line bundle L on A (for which φ_L has finite kernel $K(L)$, but is in general not injective) and we would like to define an algebraic structure on the quotient $\widehat{A} = A/K(L)$ (this requires of course to first define $K(L)$ as a subgroup scheme of A, not just as a set of $\mathbf{k}$-points). This method works but only in characteristic 0. In positive characteristic, one needs to take into account the scheme structure on the group scheme $K(L)$ (which may be non-reduced) and taking the quotient is more difficult. This construction of the dual abelian variety $\widehat{A}$ is described in [Mu].

3.2. **The Riemann–Roch theorem and the index.** Given an invertible sheaf L on a proper variety X of dimension g, one shows that there exists an integer (L^g) such that

$$\forall m \in \mathbf{Z} \qquad \chi(X, L^{\otimes m}) = (L^g)\frac{m^g}{g!} + O(m^{g-1}).$$

On an abelian variety, the situation is particularly simple.

Theorem 11 (Riemann–Roch)**.** *Let L be an invertible sheaf on an abelian variety A of dimension g. One has*

$$\chi(A, L) = \frac{(L^g)}{g!} \qquad \textit{and} \qquad \deg(\varphi_L) = \chi(A, L)^2.$$

Theorem 12 (Index of a line bundle)**.** *Let L be an invertible sheaf on an abelian variety A of dimension g such that $K(L)$ is finite. There exists a unique integer $i := i(L)$, called the* index *of L, such that $H^i(A, L) \neq 0$. The invertible sheaf L is ample if and only if $i(L) = 0$ and one has then $h^0(A, L) = \frac{(L^g)}{g!} > 0$.*

Example 13. Let L_1 be an invertible sheaf on an abelian variety A_1, let L_2 be an invertible sheaf on an abelian variety A_2, and set $L := L_1 \boxtimes L_2$, invertible sheaf on the abelian variety $A := A_1 \times_{\mathbf{k}} A_2$. Then,

$$\widehat{A} := \widehat{A_1} \times_{\mathbf{k}} \widehat{A_2} \ , \ \varphi_L = \varphi_{L_1} \times_{\mathbf{k}} \varphi_{L_2} \ , \ K(L) = K(L_1) \times_{\mathbf{k}} K(L_2).$$

If both $K(L_1)$ and $K(L_2)$ are finite, one has $i(L) = i(L_1) + i(L_2)$. In particular, if L_1 and L_2^{-1} are ample, $i(L) = \dim(A_2)$.

Definition 14. A *polarization* ℓ on an abelian variety A of dimension g is an ample line bundle L on A defined up to translation. It defines a morphism $\varphi_\ell \colon A \to \widehat{A}$. The polarization ℓ is said to be *principal* if φ_ℓ is an isomorphism (equivalently, when $(L^g) = g!$, or when $h^0(A, L) = 1$). In that case, a *theta divisor* on A is the unique element of the linear system $|L|$ (i.e., the divisor of zeroes of any non-zero section of L); it is well-defined up to translations in A.

Ample invertible sheaves (or polarizations) L such that $h^0(A, L) = 1$ (or equivalently $(L^g) = g!$) deserve special attention. They are called principal and $\varphi_L : A \to \widehat{A}$ is an isomorphism. A given abelian variety may not admit such a line bundle.

Remark 15. When $\mathbf{k} = \mathbf{C}$, recall from Example 8 that a line bundle L on a complex torus $X = V/\Gamma$ has a first Chern class $c_1(L) \in H^2(X, \mathbf{Z})$ which corresponds to a Hermitian form H_L on V whose imaginary part $\omega_L := \mathrm{Im}(H_L)$ is integral on Γ. For such a skew symmetric form, one can define the Pfaffian $\mathrm{Pf}(\omega_L)$. One has then

$$K(L) \text{ finite} \iff H_L \text{ non-degenerate},$$

the index $i(L)$ is the number of negative eigenvalues of H_L, and the integer $\chi(X, L)$ is the Pfaffian $\mathrm{Pf}(\omega_L)$. In particular $h^0(A, L) = 1$ if and only if the skew symmetric form ω_L is unimodular on Γ.

We end this section with an important theorem which, in the case of elliptic curves, follows from the Riemann–Roch theorem. In general, it is a not-too-difficult consequence of the theorem of the square (Theorem 5).

Theorem 16 (Lefschetz)**.** *Let L be an ample invertible sheaf on an abelian variety. The invertible sheaf $L^{\otimes 2}$ is generated by global sections and for any $m \geq 3$, the invertible sheaf $L^{\otimes m}$ is very ample.*

4. Jacobians of curves

We already defined in Section 2.2 the Jacobian $J(C)$ of a (smooth projective) complex curve C as

$$J(C) := H^{0,1}(C)/H^1(C, \mathbf{Z}).$$

The intersection form on $H^1(C, \mathbf{Z})$ is unimodular, and positive definite on $H^{0,1}(C)$ (Riemann's bilinear relations). By the discussion above, it defines a principal polarization θ_C on $J(C)$, and also a theta divisor $\Theta_C \subset J(C)$ (well-defined up to translation).

This can be achieved over any field and the theta divisor can be described geometrically as follows. The Jacobian $J(C) = \mathrm{Pic}^0(C)$ in that case "parametrizes" (isomorphism classes of) invertible sheaves on C of degree 0 (its construction in general is due to Weil). For any integer d, we let $J^d(C)$ parametrize (isomorphism classes of) invertible sheaves on C of degree d (it is a translated version of the Jacobian, not naturally a group any more). For $d \geq 0$, we also let $C^{(d)}$ be the d-th symmetric self-product of C (i.e., the quotient of C^d by the action of the symmetric group $\mathfrak{S}_d$).

Theorem 17 (Riemann, Kempf)**.** *Let C be a smooth projective curve of genus $g \geq 1$ defined over a field $\mathbf{k}$. Consider the Abel–Jacobi morphism*

$$\begin{aligned} \varphi_d \colon C^{(d)} &\longrightarrow J^d(C) \\ (x_1, \dots, x_d) &\longmapsto [\mathscr{O}_C(x_1 + \cdots + x_d)]. \end{aligned}$$

(a) *If $d \geq d$, the map φ_d is surjective.*
(b) *If $d = g - 1$, the map φ_d is birational onto a (translated) theta divisor*

$$\Theta := \{[L] \in J^{g-1}(C) \mid H^0(C, L) \neq 0\}.$$

(c) *If $d \le g-1$, the map φ_d is birational onto the subvariety*

$$W_d(C) := \{[L] \in J^d(C) \mid H^0(C,L) \neq 0\},$$

which has class $\theta_d := \theta_C^{g-d}/(g-d)!$ and singular locus

$$W_d^1(C) := \{[L] \in J^d(C) \mid h^0(C,L) \ge 2\}.$$

In (c), the class can be taken in $H^{2d}(J(C),\mathbf{Z})$ when $\mathbf{k} = \mathbf{C}$, and in the analogous étale cohomology group in general, but the equality is not valid in the Chow group when $1 \le d \le g-2$ (and not even modulo algebraic equivalence).

Corollary 18. *Let C be a smooth projective curve of genus $g \ge 1$ defined over a field $\mathbf{k}$, with (g-dimensional) Jacobian $J(C)$ and theta divisor Θ_C. The singular locus of Θ_C has dimension $g-3$ when C is hyperelliptic, and dimension $g-4$ otherwise.*

Negative dimensions means that the singular locus is empty. Note that every principally polarized abelian variety of dimension ≤ 3 is the Jacobian of a smooth projective curve (or a product of such). A general principally polarized abelian variety of any given dimension has smooth theta divisor.

Proposition 19 (Matsusaka)**.** *Let C be a smooth projective curve of genus $g \ge 1$ defined over a field $\mathbf{k}$, with (g-dimensional) Jacobian $J(C)$ and principal polarization θ_C. The class θ_1 is represented by a curve in $J(C)$.*

Conversely, if a principally polarized abelian variety (A,θ) contains a (possibly non-reduced or reducible) curve with class θ_1, it is isomorphic to the Jacobian of a smooth projective curve or a product of such.

The first part of the statement is a consequence of Theorem 17; the second part is known as Matsusaka's criterion.

One can also use Theorem 17 to prove the Torelli theorem for curves.

Proposition 20 (Torelli)**.** *Let C and C' be smooth projective curves. Any isomorphism between the principally polarized abelian varieties $(J(C),\theta_C)$ and $(J(C'),\theta_{C'})$ is induced (up to sign) by an isomorphism between C and C'.*

5. Prym varieties

There is another instance of principally polarized abelian varieties whose geometry one can describe. We work over a field of characteristic other than 2. Let $\pi : \widetilde{C} \to C$ be a double étale cover between smooth projective curves of genus $g(C) = g$ and $g(\widetilde{C}) = \tilde{g} = 2g-1$. The *norm morphism* $\mathrm{Nm}_\pi \colon J(\widetilde{C}) \to J(C)$ takes an invertible sheaf $\mathscr{O}_{\widetilde{C}}(\widetilde{D})$ to the invertible sheaf $\mathscr{O}_C(\pi_*\widetilde{D})$. Its kernel has two connected components and we let P (the *Prym variety* of the cover π) be the connected component of $0_{J(\widetilde{C})}$. It is an abelian variety of dimension $h := g-1$.

Theorem 21 (Wirtinger, Mumford)**.** *The canonical principal polarization on $J(\widetilde{C})$ induces twice a principal polarization ξ on P.*

One can even describe geometrically a theta divisor in P. As in the Jacobian case (see Theorem 17), it is better to look at certain translates of the abelian varieties involved. Consider

$$\mathrm{Nm}_\pi\colon J^{\tilde g-1}(\widetilde C)\to J^{2g-2}(C).$$

The inverse image of $[\omega_C]\in J^{2g-2}(C)$ has two connected components which are distinguished by the parity of $h^0(\widetilde C,\widetilde L)$. Let

$$P^*:=\{\widetilde L\in J^{\tilde g-1}(\widetilde C)\mid \mathrm{Nm}_\pi(\widetilde L)=\omega_C \text{ and } h^0(\widetilde C,\widetilde L)\text{ even}\}.$$

The intersection of the (canonical) theta divisor $\Theta_{\widetilde C}\subset J^{\tilde g-1}(\widetilde C)$ with P^* is therefore singular (Theorem 17); it is in fact 2Ξ, where

$$\Xi:=\{\widetilde L\in P^*\mid h^0(\widetilde C,\widetilde L)\ge 2\}$$

represents the principal polarization ξ.

This allowed Mumford to give a geometric description of the singularities of Ξ (very much in the spirit of Theorem 17). They are of two kinds:

- either $\mathrm{mult}_{\widetilde L}\Theta_{\widetilde C}\ge 4$, i.e., $h^0(\widetilde C,\widetilde L)\ge 4$ (the corresponding points are called *stable singularities* and occur whenever $h\ge 6$);
- or $\mathrm{mult}_{\widetilde L}\Theta_{\widetilde C}=2$, but $TC_{\Theta_{\widetilde C},\widetilde L}\supset T_{P^*,\widetilde L}$ (the corresponding points are called *exceptional singularities* and do not occur when C is general of given genus).

One should compare the following result with Corollary 18.

Corollary 22. *For any Prym variety (P,ξ) of dimension h, one has* $\dim(\mathrm{Sing}(\Xi))\ge h-6$.

And the next one with Proposition 19.

Proposition 23. *Let (P,ξ) be the h-dimensional Prym variety associated with the double étale cover $\widetilde C\to C$ and let σ be the associated involution on $\widetilde C$. For any $\tilde x_0\in\widetilde C(\mathbf{k})$, the curve $\{\tilde x+\tilde x_0-\sigma(\tilde x)-\sigma(\tilde x_0)\mid \tilde x\in\widetilde C\}$ in P has class $2\xi^{h-1}/(h-1)!$.*

The converse is not true: if a principally polarized abelian variety contains a curve with twice the minimal class, it is not necessarily a Prym variety, but the situation is well understood (Welters).

The Prym construction was extended by Beauville to (certain non-étale) double covers of singular (nodal) curves. Jacobians of curves are then (generalized) Prym varieties (associated with Wirtinger coverings) and any principally polarized abelian variety of dimension ≤ 5 is a (generalized) Prym variety.

One may wonder about the Torelli problem for Prym varieties (see Proposition 20): is a double étale cover determined by its Prym variety? There are known exceptions to this. One comes from the fact that in low dimensions, double étale covers of curves of genus g depend on more parameters than principally polarized abelian varieties of dimension $h=g-1$. But there are also exceptions in any dimensions that come from the "tetragonal construction" (Donagi): if a curve C has a g^1_4, this construction produces double étale covers of two other curves (also with a g^1_4) with the same Prym variety. There is also another construction when C has a g^2_5 (Verra).

The following result is usually referred to as “Generic Torelli theorem” (note that a general curve of genus ≥ 7 has no g^1_4 and no g^2_5).

Theorem 24 (Donagi–Smith). *A double étale cover of a general curve of genus ≥ 7 is determined by its Prym variety.*

One also knows that if $g(C) \geq 13$ and C has a g^1_4 with no double fiber, has no g^1_3, and is not bielliptic, the only other double covers with the same Prym of those obtained via the tetragonal construction. But the following question remains open.

Question 25. Is a double étale cover of a curve with no g^1_4 and no g^2_5 determined by its Prym variety?

Generalizations of the Prym construction have been proposed by Kanev, and Alexeev & al. recently used Kanev's theory to show that on any principally polarized abelian variety (A, θ) of dimension 6, $6\theta_1$ is the class of a curve in A.

6. Intermediate Jacobians

We want to generalize (over $\mathbf{C}$) the construction of Jacobians of curves to (smooth projective) varieties X of higher dimensions. Of course, $\mathrm{Pic}^0(X) = H^1(X, \mathscr{O}_X)/H^1(X, \mathbf{Z})$ is an abelian variety attached to X but it has no canonical polarization. When X has odd dimension $2m+1$, let us consider instead the Hodge decomposition

$$H^{2m+1}(X, \mathbf{C}) = H^{0,2m+1}(X) \oplus \cdots \oplus H^{m,m+1}(X) \oplus (\text{complex conjugate})$$

and set

$$J(X) := (H^{0,2m+1}(X) \oplus \cdots \oplus H^{m,m+1}(X))/H^{2m+1}(X, \mathbf{Z}).$$

This is a complex torus, but it is in general not algebraic; the intersection form takes different signs on the various factors $H^{2m+1-q,q}(X)$ and does not define a polarization. However, one case when the intersection form does define a principal polarization θ_X on $J(X)$ is when all factors but $H^{m,m+1}(X)$ are zero. This happens for example in the following cases:

- $m = 1$ and X is a Fano threefold (i.e., $-K_X$ is ample), e.g., when $X \subset \mathbf{P}^4_{\mathbf{C}}$ is a smooth cubic hypersurface: $J(X)$ is then a 5-dimensional principally polarized abelian variety;
- $X \subset \mathbf{P}^{2m+3}_{\mathbf{C}}$ is the smooth complete intersection of two quadrics: $J(X)$ is then the Jacobian of a hyperelliptic curve of genus $m+1$ (Reid);
- $X \subset \mathbf{P}^{2m+4}_{\mathbf{C}}$ is the smooth complete intersection of three quadrics: $J(X)$ is then the Prym variety of a double cover of a plane curve of degree $2m+5$ (Tjurin, Beauville);
- $X \subset \mathbf{P}^6_{\mathbf{C}}$ is a smooth cubic hypersurface: $J(X)$ is then a 21-dimensional principally polarized abelian variety.

It is in general difficult to say anything about the geometry of the principally polarized abelian variety $(J(X), \theta_X)$ (one usually needs ad hoc parametrizations of the theta divisor as in the Riemann–Kempf Theorem 17), although this is a very important problem in view of the following result.

Theorem 26 (Clemens–Griffiths). *Let X be a smooth projective complex threefold with $H^{0,3}(X) = 0$. If X is rational (i.e., birationally isomorphic to $\mathbf{P}^3_{\mathbf{C}}$), $(J(X), \theta_X)$ is a product of Jacobians of curves. In particular,* $\mathrm{codim}_{J(X)}(\mathrm{Sing}(\Theta_X)) \leq 4$.

In the list above, some intermediate Jacobians are Prym varieties. This happens more generally in the following situation. Let $X \to \mathbf{P}^2_{\mathbf{C}}$ be a quadric bundle of dimension $2m+1$, let $C \subset \mathbf{P}^2_{\mathbf{C}}$ be its discriminant curve (that parametrizes singular fibers), and let $\pi\colon \widetilde{C} \to C$ be the associated double cover (defined by the two families of $\mathbf{P}^m_{\mathbf{C}}$ contained in the singular fibers, which are generically corank-1 quadrics in a $\mathbf{P}^{2m}_{\mathbf{C}}$).

Theorem 27 (Mumford, Tjurin, Beauville)**.** *In this situation, one has* $H^{2m+1}(X,\mathbf{C}) = H^{m,m+1}(X) \oplus H^{m+1,m}(X)$ *and* $(J(X),\theta_X)$ *is isomorphic to the Prym variety of the covering* π*.*

One can use this to analyze the singularities of the theta divisor of $J(X)$ and use the Clemens–Griffiths criterion (Theorem 26) to prove irrationality (in dimension 3), as in the following result.

Corollary 28 (Clemens–Griffiths, Beauville)**.** *Let* $X \subset \mathbf{P}^4_{\mathbf{C}}$ *be a smooth cubic threefold with 5-dimensional intermediate Jacobian* $(J(X),\theta_X)$*.*

- *The theta divisor* Θ_X *of* $J(X)$ *has a unique singular point* a*.*
- *The projective tangent cone* $\mathbf{P}TC_{\Theta_X,a} \subset \mathbf{P}(T_{J(X),a})$ *is isomorphic to* $X \subset \mathbf{P}^4_{\mathbf{C}}$*.*
- *The threefold* X *is not rational.*
- *The principally polarized abelian variety* $J(X)$ *contains a surface with minimal class* $\theta_X^3/3!$*.*

Note that the second item implies a Torelli theorem for smooth cubic threefolds: if X and X' are two such threefolds, any isomorphism between $(J(X),\theta_X)$ and $(J(X'),\theta_{X'})$ is induced by an isomorphism between X and X' (compare with Theorem 20).

Sketch of partial proof. The threefold X contains a line L and projection from L induces a conic bundle $\mathrm{Bl}_L X \to \mathbf{P}^2_{\mathbf{C}}$ with discriminant curve a plane quintic $C \subset \mathbf{P}^2_{\mathbf{C}}$. The intermediate Jacobian of X is then the Prym variety of a double cover $\pi\colon \widetilde{C} \to C$. The singularities of the theta divisor can therefore be analyzed using Mumford's description (see Section 5). One finds that both stable and exceptional singularities give the unique point $\pi^* g^2_5$. This "proves" the first item. The third item follows from the Clemens–Griffiths criterion (Theorem 26). □

Theorem 27 has also been used in the case (already mentioned above) where $X \subset \mathbf{P}^{2m+4}_{\mathbf{C}}$ is the smooth complete intersection of three quadrics: $J(X)$ is then the Prym variety of a double cover of a plane curve of degree $2m+5$ and one can use this to deduce (as in the case of cubic threefolds) a Torelli theorem (Friedman–Smith, Debarre). But this says nothing about rationality in dimensions > 3 (the Clemens–Griffiths criterion does not apply).

7. Minimal cohomology classes

We are still over $\mathbf{C}$. If (A,θ) is a principally polarized abelian variety of dimension g, we defined in Theorem 17, for $1 \le d \le g$, the minimal (i.e., non-divisible) cohomology classes

$$\theta_d := \theta^{g-d}/(g-d)! \in H^{2g-2d}(A,\mathbf{Z}).$$

If (A,θ) is very general, the class of any subvariety of A is an integral multiple of a minimal class (Mattuck). But which (integral) multiples are represented by subvarieties?

Theorem 17 says that on the Jacobian of a smooth curve C, all minimal classes are represented: θ_d is represented by the subvariety $W_d(C) \subset J(C)$. On a Prym variety, $2\theta_1$ is the class of a curve (Proposition 23). On the intermediate Jacobian of a cubic threefold, θ_2 is the class of a surface.

Questions 29. (1) Given g and $1 \le d \le g$, what is the minimal (positive) number $v(g,d)$ such that any principally polarized abelian variety of dimension g contains a subvariety with class $v(g,d)\theta_d$?

(2) Given g and $1 \le d \le g$, what is the minimal (positive) number $c(g,d)$ such that any principally polarized abelian variety of dimension g contains an algebraic d-cycle with class $c(g,d)\theta_d$?

(3) If a principally polarized abelian variety (A,θ) of dimension g contains a subvariety with minimal class θ_d, with $1 \le d \le g-2$, is (A,θ) the Jacobian of a curve or the intermediate Jacobian of a cubic threefold?

Regarding (1) and (2), we obviously have $v(g,g) = v(g,g-1) = c(g,g) = c(g,g-1) = 1$, and $c(g,d) \mid v(g,d)$ for all d. Regarding (3), I proved that given g, Jacobians of curves and intermediate Jacobians of cubic threefolds form irreducible components of the locus of principally polarized abelian varieties of dimension g that contain a subvariety with minimal class.

Not much else is known:

- $v(2,1) = v(3,1) = c(2,1) = c(3,1) = 1$ (Jacobians);
- $v(4,1) = v(5,1) = v(5,2) = c(4,1) = c(5,1) = 2$ (Pryms);
- $v(g,1) \ge 3$ if $g \ge 6$ (Welters);
- $v(6,1) \in \{3,4,5,6\}$ (Alexeev & al.).

The most interesting (and the most difficult) is probably (2). Note that

$$c(g,d) > 1 \iff \begin{array}{c}\text{The integral Hodge conjecture does not hold on a very}\\ \text{general principally polarized abelian variety of dimension } g.\end{array}$$

But it is also related to recent work of Voisin on the (stable) rationality of threefolds as follows. Recall that a variety X is said to be *stably rational* if $X \times \mathbf{P}^m_{\mathbf{C}}$ is rational for some integer m. Stably rational but non-rational varieties are known to exist. Voisin proved that a stably rational variety X of dimension n has a *cohomological decomposition of the diagonal*

$$[\Delta_X] = [X \times \{x\}] + [Z] \in H^{2n}(X \times X, \mathbf{Z}), \tag{2}$$

where $\mathsf{pr}_1(\mathrm{Supp}(Z)) \subsetneq X$. Moreover, when X is a rationally connected (e.g., Fano) threefold, (2) implies that the minimal class θ_1 on the intermediate Jacobian $J(X)$ is algebraic (i.e., represented by an algebraic 1-cycle).

For a Fano 3-fold X, this implication

$$X \text{ stably rational } \Longrightarrow \theta_1 \text{ is the class of an algebraic 1-cycle}$$

for should be compared with the Clemens–Griffiths criterion (Theorem 26)

$$X \text{ rational } \Longrightarrow \theta_1 \text{ is the class of a curve.}$$

When $X \subset \mathbf{P}^4_{\mathbf{C}}$ is a smooth cubic threefold, we know (indirectly) that θ_1 is not the class of a curve, but we do not know whether it is the class of an algebraic 1-cycle. In fact, we do not know whether X is stably rational.

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