GUSHEL–MUKAI VARIETIES: INTERMEDIATE JACOBIANS

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To the memory of A.N. Tyurin

Abstract. We describe intermediate Jacobians of Gushel–Mukai varieties X of dimensions 3 or 5: if $A \subset \bigwedge^3 V_6$ is the Lagrangian subspace associated with X, we prove that the intermediate Jacobian of X is isomorphic to the Albanese variety of the canonical double covering of any of the two dual Eisenbud–Popescu–Walter surfaces $Y_A^{\geq 2}$ and $Y_{A^\perp}^{\geq 2}$. As an application, we describe the period maps for Gushel–Mukai threefolds and fivefolds.

1. Introduction

This article is an addition to the series [DK1, DK2, DK4, KP1] on the geometry of Gushel–Mukai varieties. For an introduction, we recommend the survey [D].

A smooth complex Gushel–Mukai (GM for short) variety of dimension $n \in \{2, 3, 4, 5, 6\}$ is a smooth dimensionally transverse intersection

$$X = \mathsf{CGr}(2, V_5) \cap \mathbf{P}(W) \cap Q, \tag{1}$$

where V_5 is a 5-dimensional complex vector space, $\mathsf{CGr}(2, V_5) \subset \mathbf{P}(\mathbf{C} \oplus \bigwedge^2 V_5)$ is the cone over the Grassmannian $\mathsf{Gr}(2, V_5)$ in its Plücker embedding, $W \subset \mathbf{C} \oplus \bigwedge^2 V_5$ is a linear subspace of dimension $n + 5$, and $Q \subset \mathbf{P}(W)$ is a quadratic hypersurface.

GM varieties of dimension 2 are Brill–Noether general K3 surfaces of genus 6. GM fourfolds and sixfolds are Fano varieties but they share some properties with K3 surfaces. For instance, their derived categories have a component of K3 type ([KP1, Proposition 2.6 and Proposition 2.9]) and their vanishing cohomology of middle dimension is isomorphic to a Tate twist of the primitive second cohomology of a certain hyperkähler fourfold associated with their K3 category ([DK2, Theorem 5.1]). This allowed us to describe the period map for GM fourfolds and sixfolds ([DK2, Proposition 5.27]).

GM threefolds and fivefolds behave differently. They are also Fano varieties, but the nontrivial components of their derived categories have some features of the derived category of a curve ([KP1, Proposition 2.9]) and the Hodge structure on their middle cohomology defines a 10-dimensional principally polarized abelian variety, the intermediate Jacobian of the GM variety ([DK2, Proposition 3.1]). The main goal of this article is to identify these Hodge structures; in other words, we will describe the intermediate Jacobians and period maps of GM threefolds and fivefolds.

The main object we use to study a GM variety X is its associated Lagrangian data set constructed in [DK1]. It is a triple $(V_6(X), V_5(X), A(X))$ (or (V_6, V_5, A) for short) that consists of a 6-dimensional vector space V_6, a hyperplane $V_5 \subset V_6$, and a subspace $A \subset \bigwedge^3 V_6$ which is Lagrangian with respect to the symplectic form given by exterior product,

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and contains no decomposable vectors (this means that the intersection $\mathbf{P}(A) \cap \mathsf{Gr}(3, V_6)$ is empty in $\mathbf{P}(\bigwedge^3 V_6)$). A GM variety can be reconstructed from its Lagrangian data set ([DK1, Theorem 3.6]). Moreover, Lagrangian data sets can be used to describe the moduli stack of smooth GM varieties and its coarse moduli space ([DK4]).

It is not a surprise then that many geometric properties of a GM variety can be described in terms of its Lagrangian data set, particularly in terms of the Eisenbud–Popescu–Walter (EPW for short) varieties

$$Y_A^{\geq 3} \subset Y_A^{\geq 2} \subset Y_A^{\geq 1} \subset Y_A^{\geq 0} = \mathbf{P}(V_6),$$

where $Y_A^{\geq 1}$ is a sextic hypersurface (called an EPW sextic) with singular locus $Y_A^{\geq 2}$, itself an integral surface with singular locus the finite set $Y_A^{\geq 3}$, and the dual EPW varieties

$$Y_{A^\perp}^{\geq 3} \subset Y_{A^\perp}^{\geq 2} \subset Y_{A^\perp}^{\geq 1} \subset Y_{A^\perp}^{\geq 0} = \mathbf{P}(V_6^\vee)$$

associated with the Lagrangian $A^\perp \subset \bigwedge^3 V_6^\vee$ (see Section 2.2 for the definitions).

Let A be a Lagrangian with no decomposable vectors (such as $A(X)$ and $A(X)^\perp$). O'Grady constructed a canonical double covering

$$\widetilde{Y}_A^{\geq 1} \longrightarrow Y_A^{\geq 1},$$

étale away from the surface $Y_A^{\geq 2}$, which is a hyperkähler fourfold (called a double EPW sextic) when $Y_A^{\geq 3}$ is empty (this holds for A general). When X is a GM variety of even dimension, the double EPW sextic $\widetilde{Y}_{A(X)}^{\geq 1}$ is the hyperkähler fourfold mentioned above whose primitive second cohomology is isomorphic to a Tate twist of the vanishing middle cohomology of X.

We also defined in [DK3, Theorem 5.2(2)] a canonical double covering

$$\widetilde{Y}_A^{\geq 2} \longrightarrow Y_A^{\geq 2},$$

étale away from the finite set $Y_A^{\geq 3}$, where $\widetilde{Y}_A^{\geq 2}$ is a surface (called a double EPW surface) which has an ordinary double point over each point of $Y_A^{\geq 3}$ and is smooth elsewhere. It has an Albanese variety $\mathrm{Alb}(\widetilde{Y}_A^{\geq 2})$ which can be defined as the Albanese variety of any desingularization.

The first main result of this paper is the following.

Theorem 1.1. *For any Lagrangian subspace $A \subset \bigwedge^3 V_6$ with no decomposable vectors, the Albanese variety $\mathrm{Alb}(\widetilde{Y}_A^{\geq 2})$ has a canonical principal polarization such that there is an isomorphism*

$$\mathrm{Alb}(\widetilde{Y}_A^{\geq 2}) \simeq \mathrm{Alb}(\widetilde{Y}_{A^\perp}^{\geq 2}) \tag{2}$$

of principally polarized abelian varieties.

If X is a smooth GM variety of dimension $n \in \{3, 5\}$, there is a canonical isomorphism

$$H_n(X, \mathbf{Z}) \simeq H_1(\widetilde{Y}_{A(X)}^{\geq 2}, \mathbf{Z}) \tag{3}$$

of polarized Hodge structures. It induces an isomorphism

$$\mathrm{Jac}(X) \simeq \mathrm{Alb}(\widetilde{Y}_{A(X)}^{\geq 2}) \tag{4}$$

of principally polarized abelian varieties.

We prove this theorem for GM threefolds X with $Y_{A(X)}^{\geq 3} = \varnothing$ in Theorem 4.4 and for GM fivefolds X with $Y_{A(X)}^{\geq 3} \neq \varnothing$ in Theorem 5.3. In particular, we use the natural principal polarization of the intermediate Jacobian $\mathrm{Jac}(X)$ to produce a principal polarization on $\mathrm{Alb}(\widetilde{Y}_{A(X)}^{\geq 2})$

and we deduce the isomorphism (2) from a birational isomorphism (a line transform) between two GM threefolds X and X' such that $A(X') = A(X)^{\perp}$ (such pairs are called period duals in [DK1]). The extension to arbitrary GM threefolds and fivefolds is given in Section 6.

Remark 1.2. We are not aware of any direct way (not involving GM varieties) of defining a principal polarization on the Albanese variety $\mathrm{Alb}(\widetilde{Y}^{\geq 2}_{A})$, nor of a direct proof of the isomorphism (2). On the other hand, the isomorphism (2) can be thought of as a Hodge-theoretic incarnation of the equivalence between the nontrivial components of derived categories of odd-dimensional GM varieties conjectured in [KP1, Conjecture 3.7] and proved in [KP2, Corollary 6.5]. It would be interesting to extract the principal polarization on $\mathrm{Alb}(\widetilde{Y}^{\geq 2}_{A})$ from the categorical data and to deduce the isomorphism (2) from the equivalence of categories.

Let X be a smooth GM variety of dimension $n \in \{3, 5\}$ and assume $Y^3_{A(X)} = \varnothing$. To prove the isomorphisms (3) and (4), it is natural to construct a subscheme or a cycle

$$Z \subset X \times \widetilde{Y}^{\geq 2}_{A(X)}$$

of codimension $\frac{n+1}{2}$ and use the Abel–Jacobi map $\mathsf{AJ}_Z \colon H_1(\widetilde{Y}^{\geq 2}_{A(X)}, \mathbf{Z}) \to H_n(X, \mathbf{Z})$. For this, one needs an interpretation of the double EPW surface $\widetilde{Y}^{\geq 2}_{A(X)}$ (or some other closely related surface) as a moduli space of sheaves or as a parameter space of cycles on X.

When X is a GM threefold, the most natural moduli space of sheaves to consider is the Hilbert scheme of conics on X. This scheme was thoroughly studied in [L] and [DIM1, Section 6]; in [DK5], we prove that it is isomorphic to the blow up of a point of the dual double EPW surface $\widetilde{Y}^{\geq 2}_{A(X)^{\perp}}$. Similarly, for a GM fivefold X, one could use the Hilbert scheme of quadric surfaces in X; we proved in [DK5] that it has a connected component isomorphic to a $\mathbf{P}^1$-bundle over $\widetilde{Y}^{\geq 2}_{A(X)^{\perp}}$. In the case of GM threefolds, it is claimed in [IM1, Section 5.1] and [IM2, Theorem 9] that the Clemens–Letizia degeneration method can be applied to prove that the Abel–Jacobi map given by the universal conic is an isomorphism; however, it is not clear whether this method would work in the case of GM fivefolds, so we need a different approach.

Another option in the case of a GM threefold X would be to use the moduli space $\mathscr{M}_X(2; 1, 5)$ of Gieseker semistable rank-2 torsion-free sheaves on X with $\mathrm{c}_1 = 1$, $\mathrm{c}_2 = 5$, and $\mathrm{c}_3 = 0$. This space was shown in [DIM1, Section 8] to be birational to the Hilbert scheme of conics on X', a line transform of X (see Section 4.5), hence to the double EPW surface $\widetilde{Y}^{\geq 2}_{A(X)}$; the natural correspondence is provided by the second Chern class of the universal sheaf on the product $X \times \mathscr{M}_X(2; 1, 5)$. We use a small modification of this construction in which the moduli space of sheaves is kept implicit. We explain it below.

If X is a GM threefold and $L_0 \subset X$ is a line, the corresponding (inverse) line transform $X' \dashrightarrow X$ takes a general conic $C' \subset X'$ to a rational quartic curve $C \subset X$ to which the line L_0 is bisecant (the corresponding rank-2 sheaf on X can then be obtained by Serre's construction applied to $C \cup L_0$; in particular, the curve $C \cup L_0$ represents the second Chern class of this sheaf). We consider the union $C \cup L_0$ as a quintic curve of arithmetic genus 1 on X containing the line L_0 and construct the correspondence Z as the closure of a family of such curves parameterized by an open subscheme of $\widetilde{Y}^{\geq 2}_{A(X)}$.

To prove that the Abel–Jacobi map AJ_Z associated with this family of curves is an isomorphism, we make the crucial observation that over the curve

$$Y^{\geq 2}_{A(X),V_5(X)} := Y^{\geq 2}_{A(X)} \cap \mathbf{P}(V_5(X)), \tag{5}$$

the double covering $\widetilde{Y}^{\geq 2}_{A(X)} \to Y^{\geq 2}_{A(X)}$ splits, and that over a general point y of one of the components of its preimage, there is a relation

$$Z_y + L_y = S_y \cap X \tag{6}$$

in the Chow group $\mathrm{CH}_1(X)$ of 1-cycles. Here Z_y is the fiber of the correspondence Z over y, L_y is a line on X, and S_y is a cubic surface scroll on $M_X := \mathsf{CGr}(2, V_5) \cap \mathbf{P}(W)$. Moreover, the curve (5) is birational to the Hilbert scheme $F_1(X)$ of lines on X and the line L_y in (6) comes from the universal family of lines over $F_1(X)$.

From these observations and from the vanishing of the odd cohomology of M_X, it follows that for X general, there is a morphism $\phi\colon F_1(X) \to \widetilde{Y}^{\geq 2}_{A(X)}$ such that the composition

$$H_1(F_1(X), \mathbf{Z}) \xrightarrow{\phi_*} H_1(\widetilde{Y}^{\geq 2}_{A(X)}, \mathbf{Z}) \xrightarrow{\mathsf{AJ}_Z} H_3(X, \mathbf{Z})$$

is the opposite of the Abel–Jacobi map defined by the universal family of lines. The latter map is surjective by an argument of Clemens–Tyurin, hence AJ_Z is surjective as well. It is not hard to check that the source and target of AJ_Z are free abelian groups of rank 20, hence AJ_Z is an isomorphism.

A similar argument works for GM fivefolds: rational quartic curves are replaced by rational quartic surface scrolls, reducible quintic curves by reducible quintic del Pezzo surfaces, the Hilbert scheme of lines by a component of the Hilbert scheme of planes, and a higher-dimensional analogue of the Clemens–Tyurin argument is applied.

For GM fivefolds X, the isomorphism (4) may be proved by a completely different topological argument. When X is general, we consider the double covering $\widetilde{Y}^{\geq 2}_{A(X),V_5(X)}$ (in contrast with the case of GM threefolds, this is a smooth curve of genus 161) of the curve (5) induced by the double covering $\widetilde{Y}^{\geq 2}_{A(X)} \to Y^{\geq 2}_{A(X)}$. Using classical monodromy arguments, we prove that its Jacobian has three simple factors: the Jacobian of the curve $Y^{\geq 2}_{A(X),V_5(X)}$ (of dimension 81), the Albanese variety of the surface $\widetilde{Y}^{\geq 2}_{A(X)}$ (of dimension 10), and a simple factor of dimension 70. The curve $\widetilde{Y}^{\geq 2}_{A(X),V_5(X)}$ parameterizes planes on X (see Section 2.5.1) and the corresponding Abel–Jacobi map

$$H_1(\widetilde{Y}^{\geq 2}_{A(X),V_5(X)}, \mathbf{Z}) \longrightarrow H_5(X, \mathbf{Z})$$

is surjective by a generalization of the Clemens–Tyurin argument. The induced surjective morphism

$$\mathrm{Jac}(\widetilde{Y}^{\geq 2}_{A(X),V_5(X)}) \longrightarrow \mathrm{Jac}(X)$$

therefore has connected kernel. The description of the simple factors implies that it has to be isogeneous to the product of the 81-dimensional and 70-dimensional factors. Therefore, $\mathrm{Jac}(X)$ is isomorphic to the remaining 10-dimensional factor $\mathrm{Alb}(\widetilde{Y}^{\geq 2}_{A(X)})$.

To complete the proof of Theorem 1.1 and to describe the period maps for GM varieties of dimension 3 or 5, we investigate the rational map

$$\mathbf{M}^{\mathrm{EPW}} = \mathsf{LGr}(\textstyle\bigwedge^3 V_6) /\!\!/ \mathrm{PGL}(V_6) \dashrightarrow \mathbf{A}_{10} \tag{7}$$

from the coarse moduli space of EPW sextics to the coarse moduli space of principally polarized abelian varieties of dimension 10 defined by $[A] \mapsto [\mathrm{Alb}(\widetilde{Y}_A^{\ge 2})]$ when A has no decomposable vectors and $Y_A^3 = \varnothing$. Let $\mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}} \subset \mathbf{M}^{\mathrm{EPW}}$ be the open subset parameterizing Lagrangians with no decomposable vectors and let $\mathbf{r}$ be the involution of $\mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}}$ defined by $[A] \mapsto [A^\perp]$ (see [O2]). We show in Proposition 6.2 that the map (7) extends to a regular morphism

$$\bar{\wp} \colon \mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}}/\mathbf{r} \longrightarrow \mathbf{A}_{10}$$

such that $\bar{\wp}([A])$ is the Albanese variety of (any desingularization of) the double EPW surface $\mathrm{Alb}(\widetilde{Y}_A^{\ge 2})$.

Let now $\mathbf{M}^{\mathrm{GM}}_n$ be the coarse moduli space of GM varieties of dimension n (see [DK4] and Section 2.3). We use the above result to prove the following.

Theorem 1.3. *For $n \in \{3, 5\}$, the period map $\wp_n \colon \mathbf{M}^{\mathrm{GM}}_n \to \mathbf{A}_{10}$ factors as the composition*

$$\wp_n \colon \mathbf{M}^{\mathrm{GM}}_n \longrightarrow \mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}} \longrightarrow \mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}}/\mathbf{r} \xrightarrow{\bar{\wp}} \mathbf{A}_{10},$$

where the first map is given by $[X] \mapsto [A(X)]$ and the second map is the canonical projection. In particular, $\wp_n([X]) = [\mathrm{Alb}(\widetilde{Y}^{\ge 2}_{A(X)})]$.

This factorization of the period map for GM threefolds was discussed in the introduction of [DIM1] (see also [DIM1, Remark 7.4]); moreover, it was conjectured there that the map $\bar{\wp}$ is generically injective (the computation in [DIM1, Theorem 5.1] shows that it has finite fibers).

The story of GM threefolds is very similar to the story of quartic double solids. The articles [W, V1] were an inspiration to us; in particular, we took the idea of using the Clemens–Tyurin argument from [W].

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2. Gushel–Mukai and Eisenbud–Popescu–Walter varieties

We work over the field of complex numbers. Given a subvariety X of a projective space, we denote by $F^k(X)$ the Hilbert scheme parameterizing linear spaces of dimension k in X.

2.1. Geometry of $\mathsf{Gr}(2,5)$.

Let V_5 be a 5-dimensional vector space. We denote by

$$\mathsf{Gr}(2, V_5) \subset \mathbf{P}(\textstyle\bigwedge^2 V_5)$$

the Grassmannian of 2-dimensional vector subspaces in V_5 in its Plücker embedding. It has codimension 3 and degree 5. We recall some standard facts about its geometry.

A subspace of V_5 of dimension k will usually be denoted by V_k or U_k.

Lemma 2.1. *We have the following isomorphisms:*

(a) $F^1(\mathsf{Gr}(2, V_5)) \simeq \mathsf{Fl}(1, 3; V_5)$*; the line corresponding to a flag $V_1 \subset V_3 \subset V_5$ is the set of all $[U_2] \in \mathsf{Gr}(2, V_5)$ such that $V_1 \subset U_2 \subset V_3$.*

(b) $F^2(\mathsf{Gr}(2, V_5)) = \mathsf{Fl}(1, 4; V_5) \sqcup \mathsf{Gr}(3, V_5)$*; the plane corresponding to a flag $V_1 \subset V_4 \subset V_5$ is the set of all $[U_2] \in \mathsf{Gr}(2, V_5)$ such that $V_1 \subset U_2 \subset V_4$, and the plane corresponding to a subspace $V_3 \subset V_5$ is the set of all $[U_2] \in \mathsf{Gr}(2, V_5)$ such that $U_2 \subset V_3$.*

(c) $F^3(\mathsf{Gr}(2, V_5)) \simeq \mathbf{P}(V_5)$*; the 3-space corresponding to a subspace $V_1 \subset V_5$ is the set of all $[U_2] \in \mathsf{Gr}(2, V_5)$ such that $V_1 \subset U_2$.*

(d) $F^4(\mathsf{Gr}(2, V_5)) = \varnothing$; *there are no* 4*-spaces on* $\mathsf{Gr}(2, V_5)$.

Planes on $\mathsf{Gr}(2, V_5)$ parameterized by the two components in Lemma 2.1(b) are traditionally known as σ-planes and τ-planes. We use the notation

$$F^2_\sigma(\mathsf{Gr}(2, V_5)) \simeq \mathsf{Fl}(1, 4; V_5)$$

for the connected component of $F^2(\mathsf{Gr}(2, V_5))$ parameterizing σ-planes.

If a finite morphism $\gamma\colon X \to \mathsf{Gr}(2, V_5)$ is compatible with the polarizations, it induces a morphism $F^2(\gamma)\colon F^2(X) \to F^2(\mathsf{Gr}(2, V_5))$ between Hilbert schemes and we denote by

$$F^2_\sigma(X) \subset F^2(X) \tag{8}$$

the preimage of $F^2_\sigma(\mathsf{Gr}(2, V_5))$.

We will need the following classical result.

Lemma 2.2. *Let* $V_2 \subset V_5$ *be a* 2*-dimensional subspace. We have the equality*

$$\mathsf{Gr}(2, V_5) \cap \mathbf{P}(V_2 \wedge V_5) = \mathrm{Cone}_{\mathbf{P}(\bigwedge^2 V_2)}(\mathbf{P}(V_2) \times \mathbf{P}(V_5/V_2))$$

in $\mathbf{P}(\bigwedge^2 V_5)$, *where the right side is the cone over the cubic scroll.*

Typically, an intersection $\mathsf{Gr}(2, V_5) \cap \mathbf{P}^5$ is 2-dimensional (and is a quintic del Pezzo surface). In the next lemma, we discuss some pathological intersections.

Lemma 2.3. *Assume* $\mathbf{P}^5 \subset \mathbf{P}(\bigwedge^2 V_5)$ *is a linear subspace such that* $\dim(\mathsf{Gr}(2, V_5) \cap \mathbf{P}^5) = 3$. *The only possible* 3*-dimensional component of* $\mathsf{Gr}(2, V_5) \cap \mathbf{P}^5$ *of even degree is a hyperplane section of some* $\mathsf{Gr}(2, V_4) \subset \mathsf{Gr}(2, V_5)$.

Proof. Write $\mathbf{P}^5 = \mathbf{P}(W_6)$, where $W_6 \subset \bigwedge^2 V_5$, and set $T := \mathbf{P}(W_6^\perp) \cap \mathsf{Gr}(2, V_5^\vee)$. This is a subscheme of $\mathbf{P}(W_6^\perp) \simeq \mathbf{P}^3$, hence $\dim(T) \le 3$. We discuss all the possibilities for $\dim(T)$ and check the claim in each case.

Assume first $\dim(T) \le 1$. For a general subspace $W_8 \subset \bigwedge^2 V_5$ containing W_6, we then have $\mathbf{P}(W_8^\perp) \cap \mathsf{Gr}(2, V_5^\vee) = \varnothing$, hence $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_8)$ is a smooth quintic del Pezzo fourfold ([DK1, Proposition 2.24]) not contained in a hyperplane. Its Picard group is $\mathbf{Z}$, hence for any subspace $W_7 \subset W_8$ containing W_6, its hyperplane section $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_7)$ is an irreducible threefold not contained in a hyperplane. Therefore, the hyperplane section $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_6)$ of this threefold has no 3-dimensional components.

Assume now $\dim(T) = 2$. Since $\mathsf{Gr}(2, V_5^\vee)$ is an intersection of quadrics, so is T, hence T is either an irreducible quadric surface or contains a plane.

If T contains a plane, we have, by Lemma 2.1(b), either $\mathbf{P}(\bigwedge^2 V_2^\perp) \subset T$, in which case $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_6)$ is a hyperplane section of $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(V_2 \wedge V_5)$, and by Lemma 2.2 it is an irreducible threefold of degree 3, or $\mathbf{P}(V_1^\perp \wedge V_4^\perp) \subset T$, in which case $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_6)$ is a hyperplane section of $\mathsf{Gr}(2, V_4) \cup \mathbf{P}(V_1 \wedge V_4)$, hence is the union of a hyperplane section of $\mathsf{Gr}(2, V_4)$ and of a linear 3-space.

If T is an irreducible quadric, we have $\mathbf{P}(W_6^\perp) \subset \mathbf{P}(\bigwedge^2 V_1^\perp)$ and $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_6)$ is the union of $\mathbf{P}(V_1 \wedge V_5)$ and two planes (or a double plane) corresponding to the intersection of $\mathsf{Gr}(2, V_5/V_1)$ with the line $\mathbf{P}^1$ given by the orthogonal of $W_6^\perp \subset \bigwedge^2 V_1^\perp$. Therefore, the only 3-dimensional component of $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_6)$ has degree 1.

Finally, assume $\dim(T) = 3$. By Lemma 2.1(c), we have $W_6^\perp = V_4^\perp \wedge V_5^\vee$ and the intersection $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_6) = \mathsf{Gr}(2, V_4)$ has dimension 4.

Therefore, the only case when $\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_6)$ has a 3-dimensional component of even degree is the case when T contains a σ-plane, and in this case, this component is a hyperplane section of $\mathsf{Gr}(2, V_4)$. □

We will also need the following standard locally free resolution for the cone $\mathsf{CGr}(2, V_5)$.

Lemma 2.4. *There is an exact sequence*

$$0 \to \mathscr{O}(-5) \to V_5^\vee \otimes \mathscr{O}(-3) \to V_5 \otimes \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_{\mathsf{CGr}(2,V_5)} \to 0$$

of coherent sheaves on $\mathbf{P}(\mathbf{C} \oplus \bigwedge^2 V_5)$.

2.2. **Eisenbud–Popescu–Walter varieties and their double coverings.** Let V_6 be a 6-dimensional (complex) vector space. We consider subspaces $A \subset \bigwedge^3 V_6$ that are Lagrangian for the symplectic form given by exterior product. Those that contain no decomposable vectors (that is, such that $\mathbf{P}(A) \cap \mathsf{Gr}(3, V_6) = \varnothing$) are parameterized by the complement

$$\mathsf{LGr}_{\mathrm{ndv}}(\textstyle\bigwedge^3 V_6) \subset \mathsf{LGr}(\textstyle\bigwedge^3 V_6) \tag{9}$$

of a hypersurface in the Lagrangian Grassmannian $\mathsf{LGr}(\bigwedge^3 V_6)$.

Given a Lagrangian subspace $A \subset \bigwedge^3 V_6$, one defines its EPW varieties; they form a chain

$$Y_A^{\ge 4} \subset Y_A^{\ge 3} \subset Y_A^{\ge 2} \subset Y_A^{\ge 1} \subset Y_A^{\ge 0} = \mathbf{P}(V_6)$$

of closed subschemes, where, set-theoretically, $Y_A^{\ge k}$ is the set of points $[v] \in \mathbf{P}(V_6)$ such that $\dim(A \cap (v \wedge \bigwedge^2 V_6)) \ge k$ (the scheme structure is defined in [DK3, (18)]).

The Lagrangian subspace $A \subset \bigwedge^3 V_6$ defines a Lagrangian subspace $A^\perp \subset \bigwedge^3 V_6^\vee$ hence, as above, dual EPW varieties

$$Y_{A^\perp}^{\ge 4} \subset Y_{A^\perp}^{\ge 3} \subset Y_{A^\perp}^{\ge 2} \subset Y_{A^\perp}^{\ge 1} \subset Y_{A^\perp}^{\ge 0} = \mathbf{P}(V_6^\vee).$$

Set-theoretically, the variety $Y_{A^\perp}^{\ge k}$ is the set of points $[V_5] \in \mathbf{P}(V_6^\vee) = \mathsf{Gr}(5, V_6)$ such that $\dim(A \cap \bigwedge^3 V_5) \ge k$. We also use the notation

$$Y_A^k = Y_A^{\ge k} \smallsetminus Y_A^{\ge k+1} \quad \text{and} \quad Y_{A^\perp}^k = Y_{A^\perp}^{\ge k} \smallsetminus Y_{A^\perp}^{\ge k+1}.$$

Assume for the rest of this section that A contains no decomposable vectors. A combination of results of O'Grady (see [DK1, Theorem B.2]) gives the following:

- $Y_A := Y_A^{\ge 1}$ is a normal sextic hypersurface (called an EPW sextic);
- $Y_A^{\ge 2} = \mathrm{Sing}(Y_A)$ is a normal integral surface of degree 40 (called an EPW surface);
- $Y_A^{\ge 3} = \mathrm{Sing}(Y_A^{\ge 2})$ is a finite scheme, empty when A is general;
- $Y_A^{\ge 4}$ is empty.

Note that the Lagrangian subspace $A^\perp$ also contains no decomposable vectors and analogous statements hold for dual EPW varieties.

EPW varieties have canonical double coverings. First, there is a double covering

$$\widetilde{Y}_A^{\ge 0} \longrightarrow Y_A^{\ge 0} = \mathbf{P}(V_6)$$

branched along the EPW sextic Y_A. Next, O'Grady constructed in [O3, Section 1.2] a canonical double covering

$$\widetilde{Y}_A \longrightarrow Y_A$$

étale away from $Y_A^{\geq 2}$; whenever $Y_A^{\geq 3} = \varnothing$, the scheme $\widetilde{Y}_A$ is a smooth hyperkähler fourfold (called a double EPW sextic). Finally, in [DK3, Theorem 5.2(2)], we constructed a canonical double covering

$$\pi_A \colon \widetilde{Y}_A^{\geq 2} \longrightarrow Y_A^{\geq 2} \tag{10}$$

étale away from $Y_A^{\geq 3}$, where $\widetilde{Y}_A^{\geq 2}$ is an integral normal surface (called a double EPW surface), and proved an isomorphism ([DK3, Theorem 5.2(2)])

$$\pi_{A*}\mathscr{O}_{\widetilde{Y}_A^{\geq 2}} \simeq \mathscr{O}_{Y_A^{\geq 2}} \oplus \omega_{Y_A^{\geq 2}}(-3). \tag{11}$$

The double coverings π_A and $\pi_{A^\perp}$ are the main characters of this article. We now prove some results about the surface $\widetilde{Y}_A^{\geq 2}$ that will be needed later on.

Proposition 2.5. *Let A be a Lagrangian subspace with no decomposable vectors and assume $Y_A^{\geq 3} = \varnothing$, so that $\widetilde{Y}_A^{\geq 2}$ and $Y_A^{\geq 2}$ are smooth connected projective surfaces. One has*

$$H^1(Y_A^{\geq 2}, \mathscr{O}_{Y_A^{\geq 2}}) = 0, \qquad H^1(\widetilde{Y}_A^{\geq 2}, \mathscr{O}_{\widetilde{Y}_A^{\geq 2}}) \simeq A, \tag{12}$$

and the abelian group $H_1(\widetilde{Y}_A^{\geq 2}, \mathbf{Z})$ is torsion-free of rank 20.

Proof. From (11) and Serre duality, we deduce that there are isomorphisms

$$\begin{aligned} H^1(\widetilde{Y}_A^{\geq 2}, \mathscr{O}_{\widetilde{Y}_A^{\geq 2}}) &\simeq H^1(Y_A^{\geq 2}, \mathscr{O}_{Y_A^{\geq 2}}) \oplus H^1(Y_A^{\geq 2}, \omega_{Y_A^{\geq 2}}(-3)) \\ &\simeq H^1(Y_A^{\geq 2}, \mathscr{O}_{Y_A^{\geq 2}}) \oplus H^1(Y_A^{\geq 2}, \mathscr{O}(3))^\vee. \end{aligned}$$

From the table in [DK2, Corollary B.5], we see that the first summand vanishes, whereas the second summand is isomorphic to A. This proves the statements (12) of the proposition.

To prove that $H_1(\widetilde{Y}_A^{\geq 2}, \mathbf{Z})$ is torsion-free, we use a degeneration argument. Let $S \subset \mathbf{P}^3$ be a smooth quartic surface containing no lines. Ferretti proved in [F, Proposition 4.1 and Corollary 4.2] that there is a smooth deformation of the surface $Y_A^{\geq 2}$ to the surface $\mathrm{Bit}(S) \subset \mathsf{Gr}(2,4)$ of bitangent lines to S.

Let $X \to \mathbf{P}^3$ be the double solid branched over S and let $F_1(X)$ be the variety of lines on X. There is a connected double étale cover $F_1(X) \to \mathrm{Bit}(S)$ whose associated order-2 line bundle on $\mathrm{Bit}(S)$ is $\omega_{\mathrm{Bit}(S)}(-3)$ ([W, Proposition (3.35)]). It then follows from (11) that the surface $F_1(X)$ is a smooth deformation of the surface $\widetilde{Y}_A^{\geq 2}$. They are therefore diffeomorphic. The statement that $H_1(\widetilde{Y}_A^{\geq 2}, \mathbf{Z})$ is torsion-free of rank 20 then follows from the analogous statement for $H_1(F_1(X), \mathbf{Z})$, proved in [W, Section 6, Proposition, p. 71]. □

2.3. **Gushel–Mukai varieties.** Let $n \in \{3,4,5,6\}$. As recalled in the introduction (see (1)), a Gushel–Mukai variety of dimension n is a dimensionally transverse intersection

$$X = \mathsf{CGr}(2, V_5) \cap \mathbf{P}(W) \cap Q.$$

It is the intersection in $\mathbf{P}(W)$ of the 6-dimensional space of quadrics $V_6(X) \subset \mathrm{Sym}^2(W^\vee)$ generated by the space

$$V_5(X) := V_5 = \textstyle\bigwedge^4 V_5^\vee \subset \mathrm{Sym}^2(\bigwedge^2 V_5^\vee)$$

of (the restrictions to W of) Plücker quadrics and the quadric Q. In particular, one can replace Q by any other quadric in the space $V_6(X) \smallsetminus V_5(X)$.

The intersection

$$M_X := \mathsf{CGr}(2, V_5) \cap \mathbf{P}(W) \tag{13}$$

is called the Grassmannian hull of X. There are two types of GM varieties:

- if M_X does not contain the vertex of the cone $\mathsf{CGr}(2, V_5)$, then $M_X \simeq \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W)$ is a linear section of $\mathsf{Gr}(2, V_5)$ and $X = M_X \cap Q$ is a quadratic section of M_X; these GM varieties are called ordinary;
- if M_X contains the vertex of the cone $\mathsf{CGr}(2, V_5)$, then M_X is a cone over the linear section $M'_X = \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W')$ of $\mathsf{Gr}(2, V_5)$, and $X \to M'_X$ is a double covering branched along a quadratic section $X' = M'_X \cap Q'$; these GM varieties are called special.

Note that X' is an ordinary GM variety of dimension $n - 1$; it is called the opposite variety of X.

With every GM variety X, we associated in [DK1, Section 3.2] a Lagrangian data set

$$(V_6(X), V_5(X), A(X)) \tag{14}$$

consisting of

- the 6-dimensional space $V_6(X)$ of quadrics containing X,
- the hyperplane $V_5(X) \subset V_6(X)$ of Plücker quadrics,
- a Lagrangian subspace $A(X) \subset \bigwedge^3 V_6(X)$.

The Lagrangian data sets of a GM variety and of its opposite GM variety coincide.

Many properties of X are related to properties of its Lagrangian data set. For instance, when X is smooth and $\dim(X) \geq 3$, the space $A(X)$ contains no decomposable vectors ([DK1, Theorem 3.16]) and $\dim(A(X) \cap \bigwedge^3 V_5(X)) \leq 3$.

Conversely, if (V_6, V_5, A) is a Lagrangian data set such that A contains no decomposable vectors and $\ell = \dim(A \cap \bigwedge^3 V_5) \leq 3$, there are exactly two smooth GM varieties X such that $(V_6(X), V_5(X), A(X)) = (V_6, V_5, A)$: one ordinary GM variety of dimension $5 - \ell$ and one special GM variety of dimension $6 - \ell$ ([DK1, Theorem 3.10]).

In [DK4], we upgraded the above constructions to a description of the moduli stacks $\mathfrak{M}_n^{\mathrm{GM}}$ of smooth GM varieties of dimension n and their coarse moduli spaces $\mathbf{M}_n^{\mathrm{GM}}$. In particular, we showed in [DK4, Theorem 5.15(a)] that the coarse moduli space of smooth GM varieties of dimension $n \geq 3$ is the quasiprojective GIT quotient

$$\mathbf{M}_n^{\mathrm{GM}} = \{(A, V_5) \in \mathsf{LGr}_{\mathrm{ndv}}(\textstyle\bigwedge^3 V_6) \times \mathbf{P}(V_6^\vee) \mid \dim(A \cap \bigwedge^3 V_5) \in \{5-n, 6-n\}\} /\!\!/ \mathrm{PGL}(V_6) \tag{15}$$

(see (9) for the notation). In particular, as explained in [DK4, Section 6.1], there is a map

$$\mathfrak{p}_n \colon \mathbf{M}_n^{\mathrm{GM}} \longrightarrow \mathsf{LGr}_{\mathrm{ndv}}(\textstyle\bigwedge^3 V_6) /\!\!/ \mathrm{PGL}(V_6) \tag{16}$$

whose fiber at $[A]$ parameterizes GM n-folds X such that $[A(X)] = [A]$. Explicitly, we have

$$\mathfrak{p}_n^{-1}([A]) = (Y_{A^\perp}^{\geq 5-n} \smallsetminus Y_{A^\perp}^{\geq 7-n}) /\!\!/ \mathrm{PGL}(V_6)_A, \tag{17}$$

where $\mathrm{PGL}(V_6)_A$ is the stabilizer of A in $\mathrm{PGL}(V_6)$, a finite (generically trivial) group ([DK1, Proposition B.9]). In the case $n \in \{4, 6\}$, we showed in [DK2, Proposition 5.27] (see also [DK4, Proposition 6.1]) that the map $\mathfrak{p}_n$ can be thought of as the period map for GM n-folds.

We recall from [DK2] the integral cohomology groups of GM threefolds and fivefolds and of their Grassmannian hulls.

Proposition 2.6. *Let X be a smooth GM variety of dimension $n \in \{3, 5\}$. The even cohomology $H^{\mathrm{even}}(X, \mathbf{Z})$ is pure Tate of ranks $(1,1,1,1)$ when $n = 3$, and $(1,1,2,2,1,1)$ when $n = 5$.*

The odd cohomology is

$$H^{\mathrm{odd}}(X, \mathbf{Z}) = H^n(X, \mathbf{Z}) \simeq \mathbf{Z}^{20}.$$

Furthermore, if X is ordinary, so that M_X is smooth, the even cohomology $H^{\mathrm{even}}(M_X, \mathbf{Z})$ is pure Tate of ranks $(1, 1, 2, 1, 1)$ when $n = 3$, and $(1, 1, 2, 2, 2, 1, 1)$ when $n = 5$, and the odd cohomology $H^{\mathrm{odd}}(M_X, \mathbf{Z})$ vanishes.

Proof. The first part follows from [DK2, Propositions 3.1 and 3.4]. The second part is a standard consequence of the Lefschetz Theorem. □

We will also need the following result.

Lemma 2.7. *A smooth GM fivefold contains no quadric of dimension* 3 *whose image in* $\mathsf{Gr}(2, V_5)$ *is a hyperplane section of some* $\mathsf{Gr}(2, V_4)$.

Proof. Let $Q \subset \mathsf{CGr}(2, V_5)$ be a 3-dimensional quadric contained in a smooth GM fivefold X. Then Q does not contain the vertex of the cone $\mathsf{CGr}(2, V_5)$ (because X does not), hence its projection from the vertex to $\mathsf{Gr}(2, V_5)$ is well defined. Assume it is a hyperplane section of some $\mathsf{Gr}(2, V_4)$. Then Q is a local complete intersection and its normal bundle splits as

$$N_{Q/\mathsf{CGr}(2,V_5)} \simeq \mathscr{U}^\vee \oplus \mathscr{O}(1)^{\oplus 2},$$

where $\mathscr{U}$ is the restriction of the tautological bundle of $\mathsf{Gr}(2, V_5)$. Any GM fivefold X is the intersection of a hyperplane and a quadric in $\mathsf{CGr}(2, V_5)$. If X contains Q, the differentials of the equations of X give a morphism

$$N_{Q/\mathsf{CGr}(2,V_5)} \to \mathscr{O}(1) \oplus \mathscr{O}(2).$$

Clearly, X is singular at any degeneracy point of that morphism. If X is smooth, this morphism is therefore surjective, hence its kernel is a vector bundle of rank 2. This is absurd, since a simple computation shows that its third Chern class is nonzero. Therefore, X cannot contain Q. □

2.4. **Linear spaces on quadrics containing GM varieties.** If $X \subset \mathbf{P}(W)$ is a GM variety and $V_6(X)$ is the space of quadrics in $\mathbf{P}(W)$ through X, we denote by

$$\mathscr{Q} \subset \mathbf{P}(W) \times \mathbf{P}(V_6(X)) \tag{18}$$

the total space of this family of quadrics and, for $v \in V_6(X)$ nonzero, by Q_v the corresponding quadric in $\mathbf{P}(W)$.

The Lagrangian data set associated with X can be used to describe the ranks of the family of quadrics (18): by [DK1, Proposition 3.13(b)], we have

$$\mathrm{Ker}(Q_v) = A(X) \cap (v \wedge \textstyle\bigwedge^2 V_6(X)) \qquad \text{for all } v \in V_6(X) \smallsetminus V_5(X). \tag{19}$$

In particular, $Y^k_{A(X)} \smallsetminus \mathbf{P}(V_5(X))$ is the locus of non-Plücker quadrics of corank k containing X.

In fact, the family of quadrics (18) itself can be reconstructed from the Lagrangian data set, which allows us to relate the double covering (10) to the coverings associated with the family of quadrics by [DK3, Theorem 3.1]. Note that (18) corresponds to an embedding

$$\mathscr{O}_{\mathbf{P}(V_6(X))}(-1) \hookrightarrow \mathrm{Sym}^2 W^\vee \otimes \mathscr{O}_{\mathbf{P}(V_6(X))}.$$

On $\mathbf{P}(V_6(X)) \smallsetminus \mathbf{P}(V_5(X))$, the line bundle $\mathscr{O}_{\mathbf{P}(V_6(X))}(-1)$ is trivial, hence double coverings of any quadratic degeneracy loci are well defined over that set by [DK3, Remark 3.2].

Lemma 2.8. *Let X be a smooth GM variety of dimension n and let*

$$(V_6, V_5, A) = (V_6(X), V_5(X), A(X))$$

be the corresponding Lagrangian data set. Let $k \in \{0, 1, 2\}$. Over $\mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$, the canonical double covering of the k-th degeneracy locus $Y_A^{\geq k} \smallsetminus \mathbf{P}(V_5)$ of the family of quadrics (18) coincides with $\widetilde{Y}_A^{\geq k} \times_{\mathbf{P}(V_6)} (\mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5))$.

Proof. Recall from [DK3, Theorem 5.2] that the double covering $\pi_A \colon \widetilde{Y}_A^{\geq k} \to Y_A^{\geq k}$ (see Section 2.2) is associated with the pair of Lagrangian subbundles

$$\mathscr{A}_1 = A \otimes \mathscr{O}, \quad \mathscr{A}_2 = \bigwedge^2 T_{\mathbf{P}(V_6)}(-3)$$

of $\bigwedge^3 V_6 \otimes \mathscr{O}$ over $\mathbf{P}(V_6)$. To identify the double coverings, we will show that the family of quadrics (18) can be related to this pair by the isotropic reduction procedure of [DK3, Section 4.2]. We will use freely the notation from [DK3].

Asssume first that X is ordinary. We restrict the above subbundles to $\mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$ and consider a third Lagrangian subbundle

$$\mathscr{A}_3 := \bigwedge^3 V_5 \otimes \mathscr{O}$$

of $\bigwedge^3 V_6 \otimes \mathscr{O}$ over $\mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$. Applying isotropic reduction with respect to the rank-2 subbundle

$$\mathscr{I} := \mathscr{A}_1 \cap \mathscr{A}_3 = (A \cap \bigwedge^3 V_5) \otimes \mathscr{O}$$

in the sense of [DK3, Section 4.2], we obtain three Lagrangian subbundles $\overline{\mathscr{A}}_1, \overline{\mathscr{A}}_2, \overline{\mathscr{A}}_3$ in a symplectic vector bundle $\overline{\mathscr{V}}$. Note that $\bigwedge^3 V_5/(A \cap \bigwedge^3 V_5) \simeq W^\vee$, hence

$$\overline{\mathscr{A}}_3 \simeq W^\vee \otimes \mathscr{O}$$

and that the maps [DK3, (23)] are isomorphisms over $\mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$ for the triple $\overline{\mathscr{A}}_1, \overline{\mathscr{A}}_2, \overline{\mathscr{A}}_3$ and the trivial line bundle $\mathscr{L} = \mathscr{O}$. Therefore, the construction [DK3, (24)] defines a family of quadratic forms on the trivial vector bundle $\overline{\mathscr{A}}_2^\vee \simeq W \otimes \mathscr{O}$ over $\mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$.

By [DK1, proof of Theorem 3.6 and Appendix C], this family of quadrics coincides with the restriction of $\mathscr{Q}$ to $\mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$. Applying [DK3, Proposition 4.5 and Proposition 4.7], we see that the associated double coverings coincide with $\widetilde{Y}_A^{\geq k} \times_{\mathbf{P}(V_6)} (\mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5))$.

Assume now that X is a special GM variety. Recall from [DK1, Lemma 2.33] that there is a canonical direct sum decomposition $W = W_0 \oplus W_1$, with $\dim(W_1) = 1$, such that for the family of quadrics $\mathbf{q} \colon V_6 \to \mathrm{Sym}^2 W^\vee$ defining X, we have

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1, \qquad \mathbf{q}_0 \colon V_6 \to \mathrm{Sym}^2 W_0^\vee, \quad \mathbf{q}_1 \colon V_6 \to V_6/V_5 \xrightarrow{\sim} \mathrm{Sym}^2 W_1^\vee,$$

and the family of quadrics $\mathbf{q}_0$ corresponds to the ordinary GM variety X_0 opposite to X. The family of quadrics $\mathbf{q}_1$ is nondegenerate over $\mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$, hence the families of quadrics $\mathbf{q}$ and $\mathbf{q}_0$ have the same degeneration loci and isomorphic cokernel sheaves. By [DK3, Theorem 3.2], they induce the same double covers of degeneration loci. We conclude by the argument of the first part of the proof. ☐

We will need the following consequence of the above lemma. Set

$$Y_{A,V_5}^{\geq k} := Y_A^{\geq k} \cap \mathbf{P}(V_5). \tag{20}$$

and consider the open surface

$$S_0 := Y_A^{\geq 2} \smallsetminus (Y_{A,V_5}^{\geq 2} \cup Y_A^3) \subset \mathbf{P}(V_6) \tag{21}$$

and the base change $\mathscr{Q}_0 := \mathscr{Q} \times_{\mathbf{P}(V_6)} S_0$ of the family of quadrics (18).

Corollary 2.9. *Let X be a smooth GM variety of odd dimension $n = 2s+1$ and let*

$$(V_6, V_5, A) = (V_6(X), V_5(X), A(X))$$

be the corresponding Lagrangian data set. Let $\Pi_0 \subset X$ be a linear subspace of dimension s. There is an isomorphism

$$F^{s+3}_{\Pi_0}(\mathscr{Q}_0/S_0) \simeq \widetilde{Y}^{\geq 2}_A \times_{\mathbf{P}(V_6)} S_0$$

of schemes over S_0, where the left side is the subscheme of the relative Hilbert scheme of linear spaces of dimension $s+3$ that contain the subspace Π_0.

Proof. First, [DK3, Proposition 3.10] (with $m = \dim(W) = n+5 = 2s+6$ and $k = 2$) proves that the Stein factorization of the natural morphism $F^{s+3}(\mathscr{Q}_0/S_0) \to S_0$ takes the form

$$F^{s+3}(\mathscr{Q}_0/S_0) \to \widetilde{Y}^{\geq 2}_A \times_{\mathbf{P}(V_6)} S_0 \xrightarrow{\pi_A} Y^{\geq 2}_A \times_{\mathbf{P}(V_6)} S_0 = S_0.$$

(we use [DK3, Lemma 3.9] to check normality of $F^{s+3}(\mathscr{Q}_0/S_0)$ and Lemma 2.8 to identify the second degeneracy locus and its double covering).

Furthermore, as the proof of [DK3, Proposition 3.10] shows, we have an isomorphism

$$F^{s+3}(\mathscr{Q}_0/S_0) \simeq F^{s+1}(\bar{\mathscr{Q}}_0/S_0), \tag{22}$$

where $\bar{\mathscr{Q}}_0 \to S_0$ is the family of nondegenerate $(2s+2)$-dimensional quadrics obtained from $\mathscr{Q}_0$ by passing to the quotients by the kernel spaces of quadrics. Moreover, for any $[v] \in S_0$, if $Q_v \subset \mathbf{P}(W)$ is the fiber of $\mathscr{Q}_0$ at $[v]$, we have

$$X = \mathsf{CGr}(2, V_5) \cap Q_v,$$

hence $X \cap \operatorname{Sing}(Q_v) = \varnothing$ (otherwise X would be singular). Therefore, the space Π_0 intersects none of these kernel spaces hence projects isomorphically onto a space $\bar{\Pi}_0$ in the fiber $\bar{Q}_v$ of $\bar{\mathscr{Q}}_0$ at $[v]$, so that a $(s+3)$-space in Q_v contains Π_0 if and only if the corresponding $(s+1)$-space in $\bar{Q}_0$ contains $\bar{\Pi}_0$. This proves that we also have an isomorphism

$$F^{s+3}_{\Pi_0}(\mathscr{Q}_0/S_0) \simeq F^{s+1}_{\bar{\Pi}_0}(\bar{\mathscr{Q}}_0/S_0)$$

of S_0-schemes. Finally, since the family of $(2s+2)$-dimensional quadrics $\bar{\mathscr{Q}}_0$ is everywhere nondegenerate, it follows from [KS, Lemma 2.12] that $F^{s+1}_{\bar{\Pi}_0}(\bar{\mathscr{Q}}_0/S_0)$ is isomorphic to the étale double covering of S_0 obtained from the Stein factorization of the map $F^{s+1}(\bar{\mathscr{Q}}_0/S_0) \to S_0$, which, because of the isomorphism (22) and the observation made at the beginning of the proof, is isomorphic to $\widetilde{Y}^{\geq 2}_A \times_{\mathbf{P}(V_6)} S_0 \to S_0$. □

2.5. **Linear spaces on ordinary GM threefolds and fivefolds.** In [DK2], we described the Hilbert schemes of linear spaces on a smooth GM variety X in terms of its Lagrangian data set (V_6, V_5, A) (see (14)), its EPW varieties, and their double covers. We focus here on the Hilbert scheme $F_1(X)$ of lines on an ordinary GM threefold and the Hilbert scheme $F^2_\sigma(X)$ of σ-planes on an ordinary GM fivefold (see (8) for the definitions). The description of [DK2] was given in terms of the first quadratic fibration

$$\rho_1 \colon \mathbf{P}_X(\mathscr{U}_X) \longrightarrow \mathbf{P}(V_5), \tag{23}$$

where $\mathscr{U}_X$ is the pullback to X of the tautological rank-2 vector bundle $\mathscr{U}$ on $\mathsf{Gr}(2, V_5)$ and the map ρ_1 is the pullback along the embedding $X \hookrightarrow \mathsf{Gr}(2, V_5)$ of the projection

$$\tilde{\rho}_1 \colon \mathbf{P}_{\mathsf{Gr}(2,V_5)}(\mathscr{U}) \simeq \mathsf{Fl}(1, 2; V_5) \longrightarrow \mathbf{P}(V_5).$$

For each $[v] \in \mathbf{P}(V_5)$, we have

$$\tilde{\rho}_1^{-1}([v]) = \mathbf{P}(v \wedge V_5) \simeq \mathbf{P}^3,$$

so that the fiber $\rho_1^{-1}([v])$ is a subscheme of $\mathbf{P}(v \wedge V_5)$.

The Hilbert schemes $F_1(X)$ of lines on X was identified in [DK2, Proposition 4.1] with the relative Hilbert scheme of lines of the map ρ_1 and the Hilbert scheme $F_\sigma^2(X)$ of σ-planes on X with the relative Hilbert schemes of planes of the map ρ_1. This defines maps

$$\sigma\colon F_1(X) \to \mathbf{P}(V_5) \qquad \text{and} \qquad \sigma\colon F_\sigma^2(X) \to \mathbf{P}(V_5). \tag{24}$$

To explain what these maps look like, we introduce some notation. We recall the notation (20) and set

$$\widetilde{Y}^{\geq 2}_{A,V_5} := \pi_A^{-1}(Y^{\geq 2}_{A,V_5}) \subset \widetilde{Y}^{\geq 2}_{A}$$

(see (10) for the notation). Note that $Y_{A,V_5} := Y^{\geq 1}_{A,V_5}$ is a sextic hypersurface, $Y^{\geq 2}_{A,V_5}$ is a Cohen–Macaulay curve ([DK1, Lemma B.6]), and $Y^{\geq 3}_{A,V_5}$ is a finite scheme.

2.5.1. *σ-planes on ordinary GM fivefolds.* Let $X = \mathsf{Gr}(2, V_5) \cap Q$ be an ordinary GM fivefold with Lagrangian data set (V_6, V_5, A). By (15), we have $\dim(A \cap \bigwedge^3 V_5) = 0$, hence

$$[V_5] \in \mathbf{P}(V_6^\vee) \smallsetminus Y_{A^\perp}.$$

By [DK1, Proposition 4.5], the fibers of the first quadratic fibration ρ_1 defined in (23) are

$$\rho_1^{-1}([v]) \text{ is } \begin{cases} \text{a smooth quadric in } \mathbf{P}(v \wedge V_5) & \text{if } [v] \in \mathbf{P}(V_5) \smallsetminus Y^{\geq 1}_{A,V_5}, \\ \text{a quadric of corank 1 in } \mathbf{P}(v \wedge V_5) & \text{if } [v] \in Y^{1}_{A,V_5}, \\ \text{the union of two planes in } \mathbf{P}(v \wedge V_5) & \text{if } [v] \in Y^{2}_{A,V_5}, \\ \text{a double plane in } \mathbf{P}(v \wedge V_5) & \text{if } [v] \in Y^{3}_{A,V_5}. \end{cases} \tag{25}$$

Using this, we proved in [DK2, Theorem 4.3(b)] and [DK3, Corollary 5.5] that there is an isomorphism

$$\tilde{\sigma}\colon F_\sigma^2(X) \xrightarrow{\ \sim\ } \widetilde{Y}^{\geq 2}_{A,V_5} \tag{26}$$

such that $\pi_A \circ \tilde{\sigma}\colon F_\sigma^2(X) \to Y^{\geq 2}_{A,V_5}$ is the second map σ from (24). This has the following simple consequence (compare with [N, Lemma 2.2]).

Lemma 2.10. *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors. For a general GM fivefold X such that $A(X) = A$, the Hilbert scheme $F_\sigma^2(X)$ of σ-planes contained in X is a smooth connected curve of genus* 161.

Proof. According to the description of the moduli space in (15), a general GM fivefold X with $A(X) = A$ corresponds to a general point $[V_5] \in \mathbf{P}(V_6^\vee) \smallsetminus Y_{A^\perp}$ and for such a $[V_5]$, the finite scheme $Y^{\geq 3}_{A,V_5}$ is empty, the curve $Y^{\geq 2}_{A,V_5}$ is smooth by Bertini's theorem, hence so is its étale (because $Y^{\geq 3}_{A,V_5} = \varnothing$) double cover $F_\sigma^2(X) \simeq \widetilde{Y}^{\geq 2}_{A,V_5}$. Since $K_{Y^{\geq 2}_A}$ is numerically equivalent to $3H$, where H is the hyperplane class (see (11) or [O1, (4.0.32)]), its genus is

$$g(Y^{\geq 2}_{A,V_5}) = 1 + \frac{1}{2}(K_{Y^{\geq 2}_A} \cdot H + H^2) = 1 + 2H^2 = 1 + 2\deg(Y^{\geq 2}_A) = 81. \tag{27}$$

The curve $\widetilde{Y}^{\geq 2}_{A,V_5}$ is ample on the integral surface $\widetilde{Y}^{2}_{A}$, hence connected, therefore its genus is $2g(Y^{\geq 2}_{A,V_5}) - 1 = 161$. □

2.5.2. *Lines on ordinary GM threefolds.* Let $X = \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W) \cap Q$ be a smooth ordinary GM threefold, where $W \subset \bigwedge^2 V_5$ has codimension 2. By (15), we have $\dim(A \cap \bigwedge^3 V_5) = 2$, that is,

$$[V_5] \in Y^2_{A^\perp},$$

and the line $\mathbf{P}(W^\perp) \subset \mathbf{P}(\bigwedge^2 V_5^\vee)$ is a pencil of skew-symmetric forms on V_5. Since X is smooth, these forms all have one-dimensional kernels ([DK1, Remark 2.25]) and these kernels form a smooth conic

$$\Sigma_1(X) = F^2_\sigma(M_X) \subset \mathbf{P}(V_5) \tag{28}$$

(see also [DIM1, Section 3.2]). By [DK1, Proposition 4.5], we have $Y^3_{A,V_5} \subset \Sigma_1(X) \subset Y_{A,V_5}$ and the fibers of the first quadratic fibration ρ_1 defined in (23) are

$$\rho_1^{-1}([v]) = \begin{cases} \text{two reduced points in } \mathbf{P}(v \wedge V_5), & \text{if } [v] \in \mathbf{P}(V_5) \smallsetminus Y^{\geq 1}_{A,V_5}, \\ \text{a double point in } \mathbf{P}(v \wedge V_5), & \text{if } [v] \in Y^1_{A,V_5} \smallsetminus \Sigma_1(X), \\ \text{the line } \mathbf{P}(W \cap (v \wedge V_5)), & \text{if } [v] \in Y^2_{A,V_5} \smallsetminus \Sigma_1(X), \\ \text{a smooth conic in } \mathbf{P}(v \wedge V_5), & \text{if } [v] \in \Sigma_1(X) \cap Y^1_{A,V_5}, \\ \text{the union of two lines in } \mathbf{P}(v \wedge V_5), & \text{if } [v] \in \Sigma_1(X) \cap Y^2_{A,V_5}, \\ \text{a double line in } \mathbf{P}(v \wedge V_5), & \text{if } [v] \in Y^3_{A,V_5}. \end{cases} \tag{29}$$

Using this, we proved the following result.

Proposition 2.11 ([DK2, Theorem 4.7])**.** *Let X be a smooth ordinary GM threefold. The morphism defined in* (24) *factors through*

$$\sigma\colon F_1(X) \longrightarrow Y^{\geq 2}_{A,V_5}. \tag{30}$$

It is an isomorphism over $Y^{\geq 2}_{A,V_5} \smallsetminus \Sigma_1(X)$ and a double cover over the points of $Y^{\geq 2}_{A,V_5} \cap \Sigma_1(X)$, branched over $Y^3_{A,V_5} \cap \Sigma_1(X)$.

In addition, elementary deformation theory implies that $F_1(X)$ has pure dimension 1, and local embedding dimension 2 at every singular point ([IP, Proposition 4.2.2], [KPS, Lemma 2.2.3]).

Lemma 2.12. *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors. For a general GM threefold X such that $A(X) = A$, the curve $Y^{\geq 2}_{A,V_5}$ has arithmetic genus* 81 *and the intersection $\Sigma_1(X) \cap Y^{\geq 2}_{A,V_5}$ is a finite scheme of length* 10 *contained in* $\mathrm{Sing}(Y^{\geq 2}_{A,V_5})$.

If also A is general, the curve $F_1(X)$ is smooth irreducible, the map $\sigma\colon F_1(X) \to Y^{\geq 2}_{A,V_5}$ is the normalization morphism, and $\mathrm{Sing}(Y^{\geq 2}_{A,V_5}) = \Sigma_1(X) \cap Y^{\geq 2}_{A,V_5}$.

Proof. For general $[V_5] \in Y^{\geq 2}_{A^\perp}$, the conic $\Sigma_1(X)$ is not contained in $Y^{\geq 2}_A$. Indeed, $\Sigma_1(X)$ can be identified with the fiber of the (partial) desingularization $q\colon \widehat{Y}_A \to Y_{A^\perp}$ ([DK1, Section B.2]) over the point $[V_5] \in Y^2_{A^\perp}$. If a general fiber is contained in the exceptional divisor $E = p^{-1}(Y^{\geq 2}_A)$ of the contraction $p\colon \widehat{Y}_A \to Y_A$, the exceptional divisor $E' = q^{-1}(Y^{\geq 2}_{A^\perp})$ of q is contained in E, hence $E' = E$ since E is irreducible; but this was shown to be false in the proof of [DK1, Lemma B.5].

The equivalence $E \underset{\mathrm{lin}}{\equiv} 24H' - 5E'$ established in the proof of [DK1, Lemma B.5] implies that the finite scheme $\Sigma_1(X) \cap Y^{\geq 2}_{A,V_5}$ has length 10. It is contained in $\mathrm{Sing}(Y^{\geq 2}_{A,V_5})$ because the finite birational map σ is $2:1$ over $\Sigma_1(X)$. The arithmetic genus of $Y^{\geq 2}_{A,V_5}$ was computed in (27).

When also A is general, X is a general GM threefold, hence $F_1(X)$ is a smooth irreducible curve of genus 71 ([M, Proposition 6.4], [IP, Theorem 4.2.7]), so that $Y^{\geq 2}_{A,V_5}$ is an integral curve which is smooth away from $\Sigma_1(X) \cap Y^{\geq 2}_{A,V_5}$ and $F_1(X)$ is its normalization. □

Lemma 2.13. *For any smooth GM threefold X, the curve $F_1(X)$ is connected.*

Proof. Consider a general deformation $\pi\colon \mathscr{X} \to B$ with central fiber X, parameterized by a smooth irreducible curve B. For any line $L \subset X$, there is an exact sequence

$$0 \to N_{L/X} \to N_{L/\mathscr{X}} \to \mathscr{O}_L \to 0.$$

It follows that $\chi(L, N_{L/\mathscr{X}}) = \chi(L, N_{L/X}) + 1 = 2$ hence, by deformation theory, every component of the Hilbert scheme $F_1(\mathscr{X})$ of lines contained in $\mathscr{X}$ has dimension at least 2 at the point $[L]$. Since $F_1(X)$ has pure dimension 1 at $[L]$, every component of $F_1(\mathscr{X})$ passing through $[L]$ dominates B. Deformations of L in $\mathscr{X}$ are contained in the fibers of π, hence every irreducible component of $F_1(\mathscr{X}/B)$ dominates B. Since the general fiber of $F_1(\mathscr{X}/B) \to B$ is a smooth irreducible curve (Lemma 2.12), Stein factorization implies that every fiber is connected. □

3. Topological preliminaries

In [T, Section 4.3], Tyurin gave a beautiful argument (which he attributed to Clemens) proving the surjectivity of the Abel–Jacobi map given by the universal line on a threefold. In this section, we recall this argument and prove the generalization on which our results about GM threefolds and fivefolds are based.

3.1. Abel–Jacobi maps.

We start by recalling a few properties of Abel–Jacobi maps.

Let X and Y be smooth proper varieties, let Z be a cycle of codimension c on $X \times Y$, and let k be an integer. The Abel–Jacobi map

$$\mathsf{AJ}_Z\colon H_k(Y, \mathbf{Z}) \longrightarrow H_{k+2c}(X, \mathbf{Z})$$

is defined as the composition

$$\begin{aligned} H_k(Y,\mathbf{Z}) &\xrightarrow{\sim} H^{2d_Y-k}(Y,\mathbf{Z}) \xrightarrow{p_Y^*} H^{2d_Y-k}(X\times Y,\mathbf{Z}) \\ &\xrightarrow{\cdot\smile[Z]} H^{2d_Y+2d_X-2c-k}(X\times Y,\mathbf{Z}) \xrightarrow{\sim} H_{k+2c}(X\times Y,\mathbf{Z}) \xrightarrow{p_{X*}} H_{k+2c}(X,\mathbf{Z}), \end{aligned}$$

where $d_X = \dim(X)$, $d_Y = \dim(Y)$, the isomorphisms are given by Poincaré duality, and the middle map is the cup-product with the cohomology class of the cycle Z.

We will use the following functoriality properties of the Abel–Jacobi map.

Lemma 3.1. *Let $i\colon X \to X'$ and $j\colon Y' \to Y$ be morphisms of smooth proper varieties.*

(a) *We have* $\mathsf{AJ}_Z \circ j_* = \mathsf{AJ}_{(\mathrm{Id}_X \times j)^*(Z)}$ *and* $i_* \circ \mathsf{AJ}_Z = \mathsf{AJ}_{(i\times \mathrm{Id}_Y)_*(Z)}$.

(b) *If Z' is a cycle on $X' \times Y$, one has* $\mathsf{AJ}_{(i\times\mathrm{Id}_Y)^*(Z')} = i^* \circ \mathsf{AJ}_{Z'}$.

(c) *If Z' is a cycle on $X \times Y'$, one has* $\mathsf{AJ}_{(\mathrm{Id}_X \times j)_*(Z')} = \mathsf{AJ}_{Z'} \circ j^*$.

Proof. The lemma follows from base change and the projection formula. □

3.2. **Generalized blow up decomposition.** We will need the following (co)homological result generalizing the formula for the (co)homology of a smooth blow up.

Lemma 3.2. *Let S be a smooth proper variety and let E be a rank-r vector bundle on S. Let $s \in H^0(S, E^\vee) = H^0(\mathbf{P}_S(E), \mathscr{O}(1))$ be a global section. Let $\widetilde{S} \subset \mathbf{P}_S(E)$ be the zero-locus of s considered as a section of $\mathscr{O}(1)$ and let $Z \subset S$ be the zero-locus of s considered as a section of $E^\vee$. Then $\widetilde{S}$ is a smooth hypersurface in $\mathbf{P}_S(E)$ if and only if Z is smooth of pure codimension r in S; in this case, there are direct sum decompositions*

$$H^\bullet(\widetilde{S}, \mathbf{Z}) = H^\bullet(S, \mathbf{Z}) \oplus H^{\bullet-2}(S, \mathbf{Z}) \oplus \cdots \oplus H^{\bullet-2(r-2)}(S, \mathbf{Z}) \oplus H^{\bullet-2(r-1)}(Z, \mathbf{Z})$$

and

$$H_\bullet(\widetilde{S}, \mathbf{Z}) = H_{\bullet-2(r-1)}(Z, \mathbf{Z}) \oplus H_{\bullet-2(r-2)}(S, \mathbf{Z}) \oplus \cdots \oplus H_{\bullet-2}(S, \mathbf{Z}) \oplus H_\bullet(S, \mathbf{Z}).$$

Proof. We have a commutative diagram

$$\begin{array}{ccccc} \mathbf{P}_Z(E) & \xhookrightarrow{\ i\ } & \widetilde{S} & \hookrightarrow & \mathbf{P}_S(E) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \\ Z & \xhookrightarrow{\ j\ } & S, & & \end{array}$$

where π is a $\mathbf{P}^{r-2}$-fibration away from Z and a $\mathbf{P}^{r-1}$-fibration over Z. In particular, $\widetilde{S}$ is a smooth hypersurface over $S \smallsetminus Z$. Therefore, for the first statement, we have to check that $\widetilde{S}$ is smooth of codimension 1 at a point $(x, e) \in \mathbf{P}_Z(E) \subset \widetilde{S}$ if and only if Z is smooth of codimension r at x. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{C} & \xrightarrow{\ e\ } & E_x & \longrightarrow & T_{\mathbf{P}_S(E),(x,e)} & \longrightarrow & T_{S,x} & \longrightarrow 0 \\ & & & & & {\scriptstyle 0}\searrow & \big\downarrow{\scriptstyle ds} & \swarrow{\scriptstyle (\text{dashed})} & & \\ & & & & & & \mathbf{C}, & & & \end{array}$$

where ds is the differential of s considered as a section of $\mathscr{O}(1)$. The restriction of the map ds to E_x is zero, hence it factors through the dashed arrow, which can be identified with the differential of s considered as a section of $E^\vee$ evaluated at $e \in E_x$. Thus, the vertical arrow is surjective at (x, e) for any $e \in E_x$ if and only if $ds \colon T_{S,x} \to E_x^\vee$ is surjective, that is, if and only if Z is smooth of codimension r at x.

Denote by $H \in H^2(\widetilde{S}, \mathbf{Z})$ the restriction to $\widetilde{S}$ of the relative hyperplane class of $\mathbf{P}_S(E)$. Then $\mathbf{P}_Z(E)$ has in $\widetilde{S}$ codimension $r - 1$ and

$$\mathrm{c}_{r-1}(N_{\mathbf{P}_Z(E)/\widetilde{S}}) = (-1)^{r-1}\big(i^*(H)^{r-1} + p^*j^*\mathrm{c}_1(E) \smile i^*(H)^{r-2} + \cdots + p^*j^*\mathrm{c}_{r-1}(E)\big).$$

Indeed, this follows from the standard exact sequence

$$0 \to N_{\mathbf{P}_Z(E)/\widetilde{S}} \to N_{\mathbf{P}_Z(E)/\mathbf{P}_S(E)} \to \mathscr{O}(1)|_{\mathbf{P}_Z(E)} \to 0$$

in view of the isomorphism $N_{\mathbf{P}_Z(E)/\mathbf{P}_S(E)} \simeq p^*(N_{Z/S}) \simeq p^*j^*(E^\vee)$ and the Whitney formula. In particular,

$$p_*\mathrm{c}_{r-1}(N_{\mathbf{P}_Z(E)/\widetilde{S}}) = (-1)^{r-1}. \tag{31}$$

We will now prove the direct sum decomposition of cohomology; the homological decomposition is proved analogously or follows from Poincaré duality. Consider the maps

$$\begin{aligned}\phi_k\colon H^\bullet(S,\mathbf{Z}) &\longrightarrow H^{\bullet+2k}(\widetilde{S},\mathbf{Z})\\ \xi &\longmapsto \pi^*(\xi)\smile H^k\end{aligned}$$

and

$$\begin{aligned}\phi_Z\colon H^\bullet(Z,\mathbf{Z}) &\longrightarrow H^{\bullet+2(r-1)}(\widetilde{S},\mathbf{Z})\\ \xi &\longmapsto i_*(p^*(\xi)).\end{aligned}$$

We claim that

$$H^\bullet(\widetilde{S},\mathbf{Z}) = \phi_0(H^\bullet(S,\mathbf{Z}))\oplus\phi_1(H^{\bullet-2}(S,\mathbf{Z}))\oplus\cdots\oplus\phi_{r-2}(H^{\bullet-2(r-2)}(S,\mathbf{Z}))\oplus\phi_Z(H^{\bullet-2(r-1)}(Z,\mathbf{Z})).$$

To prove that, we define maps

$$\begin{aligned}\psi_k\colon H^\bullet(\widetilde{S},\mathbf{Z}) &\longrightarrow H^{\bullet-2k}(S,\mathbf{Z})\\ \eta &\longmapsto \pi_*(\eta\smile H^{r-2-k})\end{aligned}$$

and

$$\begin{aligned}\psi_Z\colon H^\bullet(\widetilde{S},\mathbf{Z}) &\longrightarrow H^{\bullet-2(r-1)}(Z,\mathbf{Z})\\ \eta &\longmapsto p_*(i^*(\eta)).\end{aligned}$$

If $k\le l$, we have

$$\psi_l(\phi_k(\xi)) = \pi_*(\pi^*(\xi)\smile H^{r-2-l+k}) = \xi\smile\pi_*(H^{r-2-l+k}) = \delta_{k,l}\xi.$$

For $0\le k\le r-2$, we have

$$\psi_Z(\phi_k(\xi)) = p_*(i^*(\pi^*(\xi)\smile H^k)) = p_*(p^*(j^*(\xi))\smile i^*(H)^k) = j^*(\xi)\smile p_*(i^*(H^k)) = 0.$$

Using (31), we obtain

$$\psi_Z(\phi_Z(\xi)) = p_*(i^*(i_*(p^*(\xi)))) = p_*(p^*(\xi)\smile \mathrm{c}_{r-1}(N_{\mathbf{P}_Z(E)/\widetilde{S}})) = (-1)^{r-1}\xi.$$

If we define maps

$$\psi := (\psi_0\oplus\psi_1\oplus\cdots\oplus\psi_{r-2})\oplus\psi_Z \quad\text{and}\quad \phi := (\phi_0\oplus\phi_1\oplus\cdots\oplus\phi_{r-2})\oplus\phi_Z,$$

it follows that the map $\psi\circ\phi$ is lower triangular with ±1 on the diagonal, hence invertible, so that ϕ is injective and ψ is surjective.

The injectivity of ψ (hence the surjectivity of ϕ) follows from projective bundle formulas for the maps $\widetilde{S}\smallsetminus\mathbf{P}_Z(E)\to S\smallsetminus Z$ and $\mathbf{P}_Z(E)\to Z$, and excision. □

3.3. **The Clemens–Tyurin argument.** The following result is a generalization of [T, Section 4.3] (see also [W, Lemma (4.6)]); the original result is the case $m=1$.

Proposition 3.3. *Let Y be a smooth projective variety of dimension $2m+2$ and let $X\subset Y$ be a smooth hyperplane section of Y. Let $F_Y\subset F^m(Y)$ be a smooth closed subscheme of the Hilbert scheme of m-dimensional linear projective spaces in Y, let $F_X\subset F_Y$ be the closed subscheme parameterizing projective spaces contained in X, and let $\mathscr{L}_Y\subset F_Y\times Y$ and $\mathscr{L}_X\subset F_X\times X$ be the pullbacks of the corresponding universal families of projective spaces. Assume that*

(a) *F_Y is smooth;*
(b) *$\mathscr{L}_Y$ is dominant and generically finite over Y;*

(c) *F_X is smooth of pure codimension $m+1$ in F_Y;*
(d) $H_{2m+1}(Y,\mathbf{Z}) = H_{2m+3}(Y,\mathbf{Z}) = 0$.

Then the Abel–Jacobi map

$$\mathsf{AJ}_{\mathscr{L}_X}\colon H_1(F_X,\mathbf{Z}) \longrightarrow H_{2m+1}(X,\mathbf{Z})$$

is surjective.

Proof. Consider the incidence diagram

$$\begin{array}{ccc} & \mathscr{L}_Y & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ F_Y & & Y. \end{array}$$

Set $\widehat{X} := q^{-1}(X)$ and $\hat{p} := p|_{\widehat{X}}$, $\hat{q} := q|_{\widehat{X}}$, and consider the restricted diagram

$$\begin{array}{ccc} & \widehat{X} & \\ {\scriptstyle \hat{p}}\swarrow & & \searrow{\scriptstyle \hat{q}} \\ F_Y & & X. \end{array}$$

Since $\widehat{X} \subset \mathscr{L}_Y$ is a relative hyperplane section, the hypotheses of Lemma 3.2 are satisfied by assumptions (a) and (c) of the proposition, hence $\widehat{X}$ is smooth and there is a direct sum decomposition

$$H_{2m+1}(\widehat{X},\mathbf{Z}) \simeq H_1(F_X,\mathbf{Z}) \oplus H_3(F_Y,\mathbf{Z}) \oplus \cdots \oplus H_{2m+1}(F_Y,\mathbf{Z}).$$

The Abel–Jacobi map $\mathsf{AJ}_{\mathscr{L}_X}$ is the restriction of $\hat{q}_*$ to the summand $H_1(F_X,\mathbf{Z})$. Let i be the inclusion $X \hookrightarrow Y$. By Lemma 3.1, the restriction of $\hat{q}_*$ to the other summands $H_{2k+1}(F_Y,\mathbf{Z})$, for $k \in \{1,\dots,m\}$, factors through the map $i^*\colon H_{2m+3}(Y,\mathbf{Z}) \to H_{2m+1}(X,\mathbf{Z})$, which vanishes by assumption (d). The surjectivity of $\mathsf{AJ}_{\mathscr{L}_X}$ therefore follows from the surjectivity of

$$\hat{q}_*\colon H_{2m+1}(\widehat{X},\mathbf{Z}) \longrightarrow H_{2m+1}(X,\mathbf{Z}).$$

By assumption (b), the map $\hat{q}$ is dominant and generically finite hence, by [BM, Lemma 7.15], the image of $\hat{q}_*$ contains the vanishing cycles, that is, the kernel of the map

$$i_*\colon H_{2m+1}(X,\mathbf{Z}) \to H_{2m+1}(Y,\mathbf{Z}).$$

By (d), the target is zero, so this proves the surjectivity of $\mathsf{AJ}_{\mathscr{L}_X}$. □

3.4. **Intermediate Jacobians and their endomorphisms.** Let X be a smooth projective variety of dimension $2m-1$. We consider the middle cohomology $H^{2m-1}(X,\mathbf{Z})$ with its natural Hodge structure of weight $2m-1$. The complex torus

$$\mathrm{Jac}(X) = H^{2m-1}(X,\mathbf{C})/(F^m H^{2m-1}(X,\mathbf{C}) + H^{2m-1}(X,\mathbf{Z})),$$

where $F^m H^{2m-1}(X,\mathbf{C}) \subset H^{2m-1}(X,\mathbf{C})$ comes from the Hodge filtration, is called the (Griffiths) intermediate Jacobian of X (see [BL, Chapter 4]). Poincaré duality induces a hermitian form on $H^{2m-1}(X,\mathbf{C})$ which is not necessarily positive definite but defines (in the terminology of [BL, Chapter 2]) a canonical *nondegenerate* line bundle on $\mathrm{Jac}(X)$. If moreover

$$F^{m+1} H^{2m-1}(X,\mathbf{C}) = 0, \tag{32}$$

the hermitian form is positive definite, the line bundle is ample, and it defines a principal polarization on the abelian variety $\mathrm{Jac}(X)$.

More generally, a polarized rational Hodge structure of odd weight defines an isogeny class of complex tori which, under a vanishing assumption analogous to (32), becomes an isogeny class of abelian varieties.

By [DK2, Proposition 3.1], the condition (32) holds for any GM variety X of dimension 3 or 5 and $\mathrm{Jac}(X)$ is a principally polarized abelian variety of dimension 10.

Let now M be a smooth projective variety of dimension $2m$. For a smooth hypersurface $\jmath\colon X \hookrightarrow M$, we denote by

$$H^{2m-1}(X,\mathbf{Q})_{\mathrm{van}} := \mathrm{Ker}\big(\jmath_*\colon H^{2m-1}(X,\mathbf{Q}) \to H^{2m+1}(M,\mathbf{Q})\big)$$

the *vanishing cohomology*. By [V2, Proposition 2.27], there is an orthogonal direct sum decomposition

$$H^{2m-1}(X,\mathbf{Q}) = H^{2m-1}(X,\mathbf{Q})_{\mathrm{van}} \oplus \jmath^* H^{2m-1}(M,\mathbf{Q}). \tag{33}$$

In particular, the Hodge structure $H^{2m-1}(X,\mathbf{Q})_{\mathrm{van}}$ acquires a polarization from Poincaré duality on X and we denote by

$$\mathrm{Jac}(X)_{\mathrm{van}} \subset \mathrm{Jac}(X)$$

the corresponding isogeny class of nondegenerate complex tori.

We will say that the endomorphism ring of a complex torus T is trivial if any endomorphism of T is the multiplication by an integer. If $T \neq 0$, this means that the endomorphism ring $\mathrm{End}(T)$ is isomorphic to $\mathbf{Z}$ or, equivalently, that the rational endomorphism ring $\mathrm{End}(T) \otimes \mathbf{Q}$ is isomorphic to $\mathbf{Q}$; so we can extend this terminology to isogeny classes of complex tori.

If T is a nondegenerate and nonzero complex torus with trivial endomorphism ring, it is indecomposable with Picard number 1 ([BL, Propositions 1.7.3 and 4.3.7]).

The next result is an old statement made by Severi and proved in [CvG, Theorem (1.1)]).

Proposition 3.4. *Let M be a smooth projective variety of dimension $2m$. If $\jmath\colon X \hookrightarrow M$ is a very general hyperplane section, the endomorphism ring of $\mathrm{Jac}(X)_{\mathrm{van}}$ is trivial.*

Proof. Assume first that there is a rational map $f\colon M \dashrightarrow \mathbf{P}^1$, which we resolve by blowing up a smooth codimension-2 subvariety to obtain a morphism $\tilde{f}\colon \widetilde{M} \to M \overset{f}{\dashrightarrow} \mathbf{P}^1$ with critical values $t_1,\dots,t_r \in \mathbf{P}^1$, and that the strict transform of X is the fiber over $0 \in \mathbf{P}^1 \smallsetminus \{t_1,\dots,t_r\}$. Let $\tilde{\jmath}\colon X \hookrightarrow \widetilde{M}$ be the embedding and let

$$\rho\colon \pi_1(\mathbf{P}^1 \smallsetminus \{t_1,\dots,t_r\},0) \longrightarrow \mathrm{Sp}(H^{2m-1}(X,\mathbf{Q}))$$

be the monodromy representation.

Assume moreover that the only singularities of the fibers of $\tilde{f}$ are nodes. As explained in [V2, Sections 2.2.2 and 2.3.1], one can then attach to each singular point of a fiber a (noncanonically defined) vanishing cycle in $H^{2m-1}(X,\mathbf{Q})_{\mathrm{van}}$ and the vanishing cycles span the vector space $H^{2m-1}(X,\mathbf{Q})_{\mathrm{van}}$ ([V2, Lemma 2.26]; this reference deals with the case where each singular fiber has a single node but the proofs extend to the general case).

When $f\colon M \dashrightarrow \mathbf{P}^1$ is a Lefschetz pencil, each singular fiber X_{t_i} has a single node, to which correspond a vanishing cycle

$$\delta_i \in H^{2m-1}(X,\mathbf{Q})_{\mathrm{van}}$$

and an element γ_i of $\pi_1(\mathbf{P}^1 \smallsetminus \{t_1, \dots, t_r\}, 0)$ that acts on $H^{2m-1}(X, \mathbf{Q})$, via the monodromy representation, as the transvection

$$T_{\delta_i} \colon x \longmapsto x - (x \cdot \delta_i)\delta_i$$

([V2, Theorem 3.16]). The main results of the Picard–Lefschetz theory are:

- the vanishing cycles $\delta_1, \dots, \delta_r$ are in the same orbit for the monodromy representation ([V2, Corollary 3.24]);
- the monodromy representation is absolutely irreducible ([V2, Theorem 3.27] or [PS2, Lemma 3.13]).

It follows that the monodromy is "big": the Zariski closure of its image is the full symplectic group $\mathrm{Sp}(H^{2m-1}(X, \mathbf{C})_{\mathrm{van}})$ ([PS1, Lemma 4]). As in the proof of [PS1, Theorem 17], for $t \in \mathbf{P}^1$ very general, any endomorphism of $\mathrm{Jac}(X_t)_{\mathrm{van}}$ intertwines every element of the monodromy group, hence every element of the symplectic group. It must therefore be a multiple of the identity: the endomorphism ring of $\mathrm{Jac}(X_t)_{\mathrm{van}}$ is trivial. □

Corollary 3.5. *If M is a smooth projective variety of dimension $2m$ with $H^{2m-1}(M, \mathbf{Q}) = 0$ and $\jmath \colon X \hookrightarrow M$ is a very general hyperplane section, the endomorphism ring of* $\mathrm{Jac}(X)$ *is trivial.*

These results apply to intermediate Jacobians of GM threefolds and fivefolds.

Corollary 3.6. *Let X be a very general GM variety of dimension $n \in \{3, 5\}$. We have*

$$\mathrm{End}(\mathrm{Jac}(X)) \simeq \mathbf{Z}. \tag{34}$$

In particular, the Picard number of $\mathrm{Jac}(X)$ *is* 1.

Proof. We may assume that X is ordinary; it is then a very ample divisor in its Grassmannian hull M_X, which is the entire Grassmannian $\mathsf{Gr}(2, V_5)$ when $n = 5$ or the fixed smooth fourfold $\mathsf{Gr}(2, V_5) \cap \mathbf{P}^7 \subset \mathbf{P}(\bigwedge^2 V_5)$ when $n = 3$. We have $H^n(M_X, \mathbf{Q}) = 0$ in both cases by Proposition 2.6, hence $\mathrm{Jac}(X)_{\mathrm{van}} = \mathrm{Jac}(X)$, so Corollary 3.5 implies the claim. □

The isomorphism (34) was also proved in [DIM1, Corollary 5.3].

4. Intermediate Jacobians of GM threefolds

In this section, we study the intermediate Jacobians of GM threefolds. The main result (Theorem 4.4) is stated at the end of Section 4.1 and its proof takes up the rest of Section 4.

4.1. **Family of curves.** Let X be an arbitrary smooth GM threefold. Its associated Lagrangian subspace $A \subset \bigwedge^3 V_6$ has no decomposable vectors (Section 2.3). Let $Y_A^{\geq 2} \subset \mathbf{P}(V_6)$ be the corresponding EPW surface and let

$$\pi_A \colon \widetilde{Y}_A^{\geq 2} \to Y_A^{\geq 2}$$

be the double covering from (10), which is connected and étale away from the finite set Y_A^3.

In this section, we construct a subvariety

$$Z \subset X \times \widetilde{Y}_A^{\geq 2}$$

such that the map $Z \to \widetilde{Y}_A^{\geq 2}$ is, over a dense open subset of $\widetilde{Y}_A^{\geq 2}$, a family of quintic curves of arithmetic genus 1 containing a fixed line $L_0 \subset X$. We then check that the associated Abel–Jacobi map gives an isomorphism between $\mathrm{Alb}(\widetilde{Y}_A^{\geq 2})$ and $\mathrm{Jac}(X)$.

We start by choosing a line $L_0 \subset X$. It is for the moment arbitrary, but we will impose some restrictions in Section 4.2. We consider the open surface $S_0 \subset Y_A^{\geq 2}$ defined by (21) and the family of quadrics $\mathscr{Q}_0 \to S_0$ obtained by base change to S_0 of the family (18). We denote by $F^4(\mathscr{Q}_0/S_0) \to S_0$ the relative Hilbert scheme of linear 4-spaces in the fibers of $\mathscr{Q}_0 \to S_0$ and by $F^4_{L_0}(\mathscr{Q}_0/S_0) \to S_0$ the subscheme parameterizing those 4-spaces which contain the line L_0. Applying Corollary 2.9, we obtain an isomorphism

$$F^4_{L_0}(\mathscr{Q}_0/S_0) \simeq \widetilde{S}_0 := \widetilde{Y}_A^2 \times_{Y_A^2} S_0 \tag{35}$$

of schemes over S_0. In particular, the canonical map $F^4_{L_0}(\mathscr{Q}_0/S_0) \to S_0$ is the double étale covering $\pi\colon \widetilde{S}_0 \to S_0$ induced by the double covering π_A. Note that $\widetilde{S}_0$ is a smooth surface. Let

$$\widetilde{\mathscr{Q}}_0 := \mathscr{Q}_0 \times_{S_0} \widetilde{S}_0$$

be the base change of the family of quadrics $\mathscr{Q}_0 \to S_0$ along π. We have a canonical map

$$\widetilde{S}_0 \to F^4_{L_0}(\mathscr{Q}_0/S_0) \times_{S_0} \widetilde{S}_0 \hookrightarrow F^4(\mathscr{Q}_0/S_0) \times_{S_0} \widetilde{S}_0 \simeq F^4(\widetilde{\mathscr{Q}}_0/\widetilde{S}_0),$$

where the first map is the product of the isomorphism (35) with the identity map. By construction, it is a section of the projection $F^4(\widetilde{\mathscr{Q}}_0/\widetilde{S}_0) \to \widetilde{S}_0$.

Let $\mathscr{P}^4 \subset \widetilde{\mathscr{Q}}_0 \subset \mathbf{P}(W) \times \widetilde{S}_0$ be the pullback of the universal family of projective 4-spaces over $F^4(\widetilde{\mathscr{Q}}_0/\widetilde{S}_0)$ along the section $\widetilde{S}_0 \to F^4(\widetilde{\mathscr{Q}}_0/\widetilde{S}_0)$ constructed above. Set

$$Z_0 := \mathscr{P}^4 \cap (M_X \times \widetilde{S}_0), \tag{36}$$

where the Grassmannian hull $M_X \subset \mathbf{P}(W)$ was defined in (13) and the intersection is taken inside $\mathbf{P}(W) \times \widetilde{S}_0$.

Proposition 4.1. *The map $Z_0 \to \widetilde{S}_0$ is a flat family of local complete intersection curves in X of degree 5 and arithmetic genus 1 containing L_0. In particular, $Z_0 \subset X \times \widetilde{S}_0$.*

Proof. Let $y \in \widetilde{S}_0$ and set $[v] = \pi(y) \in \mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$. The fiber of Z_0 over y is

$$Z_{0,y} = M_X \cap \mathscr{P}^4_y = \mathsf{CGr}(2, V_5) \cap \mathscr{P}^4_y.$$

The cone $\mathsf{CGr}(2, V_5) \subset \mathbf{P}(\mathbf{C} \oplus \bigwedge^2 V_5)$ has codimension 3 and degree 5. Therefore, the intersection $\mathsf{CGr}(2, V_5) \cap \mathscr{P}^4_y$ has dimension at least 1 and degree at most 5 (and if the dimension is 1, the degree is 5). Furthermore, $\mathscr{P}^4_y \subset Q_v$, hence

$$Z_{0,y} \subset M_X \cap Q_v = X.$$

Since X contains no surfaces of degrees less than 10 ([DK2, Corollary 3.5]), $Z_{0,y}$ is a local complete intersection curve in X of degree 5. This also proves the inclusion $Z_0 \subset X \times \widetilde{S}_0$.

Since the curve $Z_{0,y}$ is a dimensionally transverse linear section of $\mathsf{CGr}(2, V_5)$ of codimension 6, the resolution of Lemma 2.4 restricts on $\mathscr{P}^4_y$ to a resolution

$$0 \to \mathscr{O}_{\mathscr{P}^4_y}(-5) \to \mathscr{O}_{\mathscr{P}^4_y}(-3)^{\oplus 5} \to \mathscr{O}_{\mathscr{P}^4_y}(-2)^{\oplus 5} \to \mathscr{O}_{\mathscr{P}^4_y} \to \mathscr{O}_{Z_{0,y}} \to 0.$$

It follows that $h^0(Z_{0,y}, \mathscr{O}_{Z_{0,y}}) = h^1(Z_{0,y}, \mathscr{O}_{Z_{0,y}}) = 1$, hence $Z_{0,y}$ is a connected curve of arithmetic genus 1, with Hilbert polynomial $h_{Z_{0,y}}(t) = 5t$. Since the Hilbert polynomial does not depend on y, the family of curves Z_0 is flat over $\widetilde{S}_0$. Finally, $L_0 \subset M_X$ and $L_0 \subset \mathscr{P}^4_y$ by construction, hence $L_0 \subset Z_{0,y}$. □

We now extend the family of curves $Z_0 \to \widetilde{S}_0$ to a family defined over the entire surface $\widetilde{Y}^{\geq 2}_A$. We will need the following construction.

Definition 4.2. Let $\mathscr{Z} \subset \mathscr{X} \times \mathscr{S}$ be an $\mathscr{S}$-flat family of subschemes in a projective variety $\mathscr{X}$. Let $\varphi \colon \mathscr{S} \to \mathrm{Hilb}(\mathscr{X})$ be the induced morphism. Let $\mathscr{S} \subset \overline{\mathscr{S}}$ be a (partial) compactification of $\mathscr{S}$. Then φ can be considered as a rational map $\overline{\mathscr{S}} \dashrightarrow \mathrm{Hilb}(\mathscr{X})$. Let $\widetilde{\mathscr{S}} \subset \overline{\mathscr{S}} \times \mathscr{X}$ be the graph of φ and let $\tilde{\varphi} \colon \widetilde{\mathscr{S}} \to \mathrm{Hilb}(\mathscr{X})$ be the projection. Let $\widetilde{\mathscr{Z}} \subset \mathscr{X} \times \widetilde{\mathscr{S}}$ be the pullback of the universal subscheme in $\mathscr{X} \times \mathrm{Hilb}(\mathscr{X})$ and let

$$\overline{\mathscr{Z}} \subset \mathscr{X} \times \overline{\mathscr{S}}$$

be the scheme-theoretic image of $\widetilde{\mathscr{Z}}$ by the morphism $\mathscr{X} \times \widetilde{\mathscr{S}} \to \mathscr{X} \times \overline{\mathscr{S}}$. We will call the subscheme $\overline{\mathscr{Z}}$ the Hilbert closure of $\mathscr{Z}$ with respect to the embedding $\mathscr{S} \subset \overline{\mathscr{S}}$.

We apply this construction to the subscheme $Z_0 \subset X \times \widetilde{S}_0$ and the embedding $\widetilde{S}_0 \subset \widetilde{Y}^{\geq 2}_A$.

Lemma 4.3. *Let*

$$Z \subset X \times \widetilde{Y}^{\geq 2}_A \tag{37}$$

be the Hilbert closure of the subscheme $Z_0 \subset X \times \widetilde{S}_0$ with respect to the embedding $\widetilde{S}_0 \subset \widetilde{Y}^{\geq 2}_A$. Away from a finite subscheme of $\widetilde{Y}^{\geq 2}_A$, the scheme Z is a flat family of curves of degree 5 and arithmetic genus 1 containing the line L_0. Moreover, we have

$$Z \times_{\widetilde{Y}^{\geq 2}_A} \widetilde{S}_0 = Z_0$$

as subschemes of $X \times \widetilde{S}_0$.

Proof. Since the surface $\widetilde{Y}^{\geq 2}_A$ is normal, the rational map $\widetilde{Y}^{\geq 2}_A \dashrightarrow \mathrm{Hilb}(X)$ defined by the subscheme Z_0 extends regularly to all codimension-1 points of $\widetilde{Y}^{\geq 2}_A$. The nonflat locus of the morphism $Z \to \widetilde{Y}^{\geq 2}_A$ is therefore supported in codimension 2, hence is a finite subscheme. All the remaining properties of Z are clear from the construction of the Hilbert closure. □

By Proposition 4.1, every irreducible component of Z_0 has dimension 3. By definition of the Hilbert closure, the same is true for Z.

The main result of this section is the following (recall from Propositions 2.5 and 2.6 that the abelian groups $H_1(\widetilde{Y}^{\geq 2}_{A(X)}, \mathbf{Z})$ and $H_3(X, \mathbf{Z})$ are both torsion-free of rank 20).

Theorem 4.4. *For any Lagrangian subspace $A \subset \bigwedge^3 V_6$ such that A has no decomposable vectors and $Y^{\geq 3}_A = \varnothing$, the abelian variety $\mathrm{Alb}(\widetilde{Y}^{\geq 2}_A)$ has a canonical principal polarization. If moreover $Y^{\geq 3}_{A^\perp} = \varnothing$, there is an isomorphism*

$$\mathrm{Alb}(\widetilde{Y}^{\geq 2}_A) \simeq \mathrm{Alb}(\widetilde{Y}^{\geq 2}_{A^\perp}) \tag{38}$$

of principally polarized abelian varieties.

Furthermore, if X is a smooth GM threefold such that $A(X) = A$, and if $L_0 \subset X$ is any line and $Z \subset X \times \widetilde{Y}^{\geq 2}_{A(X)}$ is the subscheme defined in Lemma 4.3, the Abel–Jacobi map

$$\mathsf{AJ}_Z \colon H_1(\widetilde{Y}^{\geq 2}_A, \mathbf{Z}) \longrightarrow H_3(X, \mathbf{Z}) \tag{39}$$

is an isomorphism of integral Hodge structures which induces an isomorphism

$$\mathrm{Alb}(\widetilde{Y}^{\ge 2}_A) \xrightarrow{\sim} \mathrm{Jac}(X) \tag{40}$$

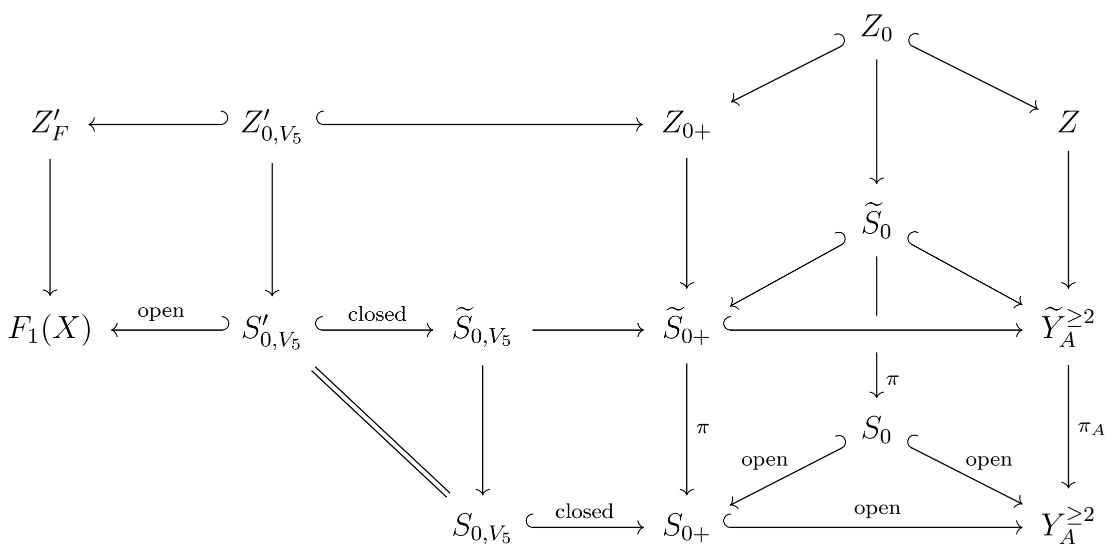
of principally polarized abelian varieties.

If A is very general, the principal polarization of $\mathrm{Alb}(\widetilde{Y}^{\ge 2}_A)$ is unique by Corollary 3.6.

The proof of this theorem takes up the rest of the section.

4.2. **The boundary of the family.** To prove Theorem 4.4, we study, over the boundary $\widetilde{Y}^{\ge 2}_A \smallsetminus \widetilde{S}_0$, the family of curves Z constructed in Lemma 4.3. This boundary consists of the curve $\widetilde{Y}^{\ge 2}_{A,V_5}$ and the finite set Y^3_A. As we will see in the proof of Proposition 4.15, finite sets are not important for the Abel–Jacobi map, so we will concentrate on a dense open subset (denoted by S_{0,V_5} and defined in Definition 4.7) of the curve $\widetilde{Y}^{\ge 2}_{A,V_5}$. We will construct a diagram

(41)



where $S_{0+} = S_0 \cup S_{0,V_5}$ and all squares are cartesian. The lower vertical arrows are double coverings (étale except for the right one, which is only étale away from $Y^{\ge 3}_A$). We want to emphasize that the schemes Z_{0+} and Z are *different* over the boundary $\widetilde{S}_{0+} \smallsetminus \widetilde{S}_0 \subset \widetilde{Y}^{\ge 2}_A \smallsetminus \widetilde{S}_0$, and their relation will be crucial for the rest of the proof. In fact, the map $Z'_{0,V_5} \to S'_{0,V_5}$ is a flat family of surfaces in M_X, while the map $Z \to \widetilde{Y}^{\ge 2}_A$ is a family of curves in X.

To construct the diagram, we need to impose some restrictions on X and L_0. First, we will assume from now on that X is ordinary. To explain the restriction imposed on L_0, we will need the following definition (the map $\sigma\colon F_1(X) \to Y^{\ge 2}_{A,V_5}$ and the conic $\Sigma_1(X) \subset \mathbf{P}(V_5)$ were defined in (30) and (28)).

Definition 4.5. A line L on X is nice if $\sigma([L]) \notin \Sigma_1(X)$.

We will use the following simple observation.

Lemma 4.6. *If $L \subset X$ is a nice line and $[v] := \sigma([L])$, one has*

$$\mathbf{P}(W) \cap \mathbf{P}(v \wedge V_5) = L.$$

In particular, the subspace $\mathbf{P}(W)$ is transverse to $\mathbf{P}(v \wedge V_5) \subset \mathbf{P}(\bigwedge^2 V_5)$.

Proof. By definition of a nice line, we have $[v] \in Y^2_{A,V_5} \smallsetminus \Sigma_1(X)$, hence (29) applies. □

From now on, we will assume that L_0 is a nice line and set

$$[v_0] := \sigma([L_0]) \in \mathbf{P}(V_5).$$

Definition 4.7. Denote by S_{0,V_5} the dense open complement in the curve $Y^{\geq 2}_{A,V_5}$ of the finite set $Y^{\geq 2}_{A,V_5} \cap \Sigma_1(X)$ and the finite subset of $Y^{\geq 2}_{A,V_5}$ corresponding to lines intersecting L_0 (including the line L_0 itself). Each point of the curve S_{0,V_5} thus corresponds to a nice line on X. Set

$$S_{0+} := Y^{\geq 2}_A \smallsetminus (Y^{\geq 2}_{A,V_5} \smallsetminus S_{0,V_5}) = S_0 \cup S_{0,V_5} \subset Y^{\geq 2}_A.$$

This is a smooth open subscheme of $Y^{\geq 2}_A$ containing S_0 and with finite complement.

Lemma 4.8. *The double covering $\pi_A \colon \widetilde{Y}^{\geq 2}_A \to Y^{\geq 2}_A$ splits over the curve $Y^{\geq 2}_{A,V_5}$.*

Proof. As we saw in the proof of Corollary 2.9, the double covering $\pi \colon \widetilde{S}_0 \to S_0$ induced by π_A agrees with the relative Hilbert scheme $F^2_{\bar{L}_0}(\bar{\mathscr{Q}}_0/S_0) \to S_0$ of planes containing the projection $\bar{L}_0$ of the line L_0, where the family of quadrics $\bar{\mathscr{Q}}_0 \to S_0$ is obtained from the family (18) by restricting to S_0 and passing to the quotients with respect to the 2-dimensional kernel spaces of quadrics. We will prove that this identification also holds over S_{0+}.

Denote by $\mathscr{Q}_{0+} \to S_{0+}$ the restriction of the family of quadrics (18) to S_{0+}. For $[v] \in S_{0,V_5}$, the quadric Q_v is the Plücker quadric

$$Q_v = \mathbf{P}(W) \cap \mathrm{Cone}_{\mathbf{P}(v \wedge V_5)}(\mathsf{Gr}(2, V_5/\mathbf{C}v)).$$

By Definition 4.7, the line L_v corresponding to the point $[v] \in S_{0,V_5} \subset Y^{\geq 2}_{A,V_5}$ is nice hence, by Lemma 4.6, the space $\mathbf{P}(W)$ intersects $\mathbf{P}(v \wedge V_5)$ transversely along the line L_v so that

$$Q_v = \mathrm{Cone}_{L_v}(\mathsf{Gr}(2, V_5/\mathbf{C}v)). \tag{42}$$

In particular, Q_v has corank 2 and its vertex does not meet L_0 (by Definition 4.7). Therefore, by passing to the quotients with respect to the kernel spaces of quadrics, we obtain, as in the proof of Corollary 2.9, a family $\bar{\mathscr{Q}}_{0+} \to S_{0+}$ of nondegenerate 4-dimensional quadrics over S_{0+} and conclude that the Hilbert scheme $F^2_{\bar{L}_0}(\bar{\mathscr{Q}}_{0+}/S_{0+})$ of planes in its fibers containing $\bar{L}_0$ is isomorphic to $F^4_{L_0}(\mathscr{Q}_{0+}/S_{0+})$ and gives an étale double covering of S_{0+}. Over the dense open subset $S_0 \subset S_{0+}$, this covering is induced by π_A, hence the same is true over S_{0+}, that is,

$$F^4_{L_0}(\mathscr{Q}_{0+}/S_{0+}) \simeq F^2_{\bar{L}_0}(\bar{\mathscr{Q}}_{0+}/S_{0+}) \simeq \widetilde{S}_{0+} := \widetilde{Y}^2_A \times_{Y^2_A} S_{0+}. \tag{43}$$

Therefore, to prove that the covering $\pi_A \colon \widetilde{Y}^{\geq 2}_A \to Y^{\geq 2}_A$ splits over $Y^{\geq 2}_{A,V_5}$, it is enough to check that the covering $F^2_{\bar{L}_0}(\bar{\mathscr{Q}}_{0+}/S_{0+}) \to S_{0+}$ splits over S_{0,V_5}. We do that by constructing a section of this covering over S_{0,V_5} as follows: for $[v] \in S_{0,V_5}$, consider the plane

$$\mathbf{P}(v_0 \wedge (V_5/\mathbf{C}v)) \subset \mathsf{Gr}(2, V_5/\mathbf{C}v) = \bar{Q}_v \tag{44}$$

(note that $[v_0] \neq [v]$ for $[v] \in S_{0,V_5}$ by Definition 4.7). The line $\bar{L}_0$ is contained in this plane because $L_0 = \mathbf{P}(W) \cap \mathbf{P}(v_0 \wedge V_5)$ by Lemma 4.6. Therefore, we obtain a regular map

$$S_{0,V_5} \longrightarrow F^2_{\bar{L}_0}(\bar{\mathscr{Q}}_0/S_0) = F^4_{L_0}(\mathscr{Q}_{0+}/S_{0+}) \simeq \widetilde{S}_{0+} \tag{45}$$

which gives the required section. □

Remark 4.9. The map (45) is the restriction of the map

$$\begin{array}{rcl} \mathbf{P}(V_5) \smallsetminus \{[v_0]\} & \longrightarrow & F^4_{L_0}(\mathscr{Q}/\mathbf{P}(V_6)) \\ [v] & \longmapsto & \mathbf{P}\big(W \cap ((\mathbf{C}v_0 \oplus \mathbf{C}v) \wedge V_5)\big), \end{array}$$

hence it is well defined even if the curve $Y^{\geq 2}_{A,V_5}$ is not reduced.

We still denote by π the double covering $\widetilde{S}_{0+} \to S_{0+}$ constructed above. Note that $\widetilde{S}_{0+}$ is a smooth irreducible open surface in $\widetilde{Y}^{\geq 2}_A$ containing $\widetilde{S}_0$ as an open subscheme. Let

$$\widetilde{\mathscr{Q}}_{0+} = \mathscr{Q}_{0+} \times_{S_{0+}} \widetilde{S}_{0+} \simeq \mathscr{Q} \times_{\mathbf{P}(V_6)} \widetilde{S}_{0+}$$

be the base change of the family $\mathscr{Q}_{0+} \to S_{0+}$ along π. The isomorphism (43) induces a section

$$\widetilde{S}_{0+} \longrightarrow F^4_{L_0}(\widetilde{\mathscr{Q}}_{0+}/\widetilde{S}_{0+})$$

of its relative Hilbert scheme of 4-spaces and we denote by

$$\mathscr{P}^4_+ \subset \widetilde{\mathscr{Q}}_{0+} \subset \mathbf{P}(W) \times \widetilde{S}_{0+}$$

the corresponding family of projective 4-spaces over $\widetilde{S}_{0+}$, which agrees by construction with the family $\mathscr{P}^4$ over $\widetilde{S}_0 \subset \widetilde{S}_{0+}$. We set

$$Z_{0+} := \mathscr{P}^4_+ \cap (M_X \times \widetilde{S}_{0+}). \tag{46}$$

This defines the middle column of the diagram (41). We denote by $Z_{0+,y} = \mathscr{P}^4_{+y} \cap M_X \subset M_X$ the fiber of Z_{0+} over a point $y \in \widetilde{S}_{0+}$.

Lemma 4.10. *We have $Z_{0+} \times_{\widetilde{S}_{0+}} \widetilde{S}_0 = Z_0$ and, for general points y of every irreducible component of $\widetilde{S}_{0+} \smallsetminus \widetilde{S}_0$, we have $Z_y \subset Z_{0+,y}$, where Z_y is the fiber of the scheme Z defined in* (37)*.*

Proof. The equality follows from the fact that the family of 4-spaces $\mathscr{P}^4_+$ agrees with $\mathscr{P}^4$ over $\widetilde{S}_0$. Let y be a general point of an irreducible component of $\widetilde{S}_{0+} \smallsetminus \widetilde{S}_0$. By continuity, we obtain $Z_y \subset \mathscr{P}^4_{+,y}$. Since $Z_y \subset X \subset M_X$, we also get $Z_y \subset Z_{0+,y}$. ∎

We denote by

$$S'_{0,V_5} \subset \widetilde{S}_{0+} \tag{47}$$

the image of the map (45) and by

$$Z'_{0,V_5} := Z_{0,+} \times_{\widetilde{S}_{0+}} S'_{0,V_5} \subset M_X \times S'_{0,V_5}$$

the restriction of the family (46) to the curve S'_{0,V_5}.

Proposition 4.11. *The map $Z'_{0,V_5} \to S'_{0,V_5}$ is a flat family of surfaces which are isomorphic to hyperplane sections of a cubic scroll $\mathbf{P}^1 \times \mathbf{P}^2$.*

We will give a more detailed description of the fibers of Z'_{0,V_5} in Lemma 4.13.

Proof. Let $y \in \widetilde{S}'_{0,V_5}$ and set again $[v] = \pi(y) \in \mathbf{P}(V_5)$. The proof of Lemma 4.8 shows that the quadric Q_v has the form (42). By (44), the point $y \in S'_{0,V_5}$ corresponds to the plane

$$\mathbf{P}(v_0 \wedge (V_5/\mathbf{C}v)) \subset \mathsf{Gr}(2, V_5/\mathbf{C}v) = \bar{Q}_v$$

and its preimage in Q_v is the 4-space

$$\mathscr{P}^4_{+y} = \mathbf{P}(((v \wedge V_5) \cap W) \oplus (v_0 \wedge (V_5/\mathbf{C}v))) = \mathbf{P}(V_2 \wedge V_5) \cap \mathbf{P}(W),$$

where $V_2 \subset V_5$ is the subspace spanned by v_0 and v (they are linearly independent by Definition 4.7). Furthermore, by Lemma 2.2, the fiber of Z'_{0,V_5} at y can be written as

$$Z'_y := M_X \cap \mathscr{P}^4_{+y} = \mathrm{Cone}_{\mathbf{P}(\bigwedge^2 V_2)}\big(\mathbf{P}(V_2) \times \mathbf{P}(V_5/V_2)\big) \cap \mathbf{P}(W),$$

so it is a linear section of a cone over the 3-dimensional cubic scroll.

The vertex $[\bigwedge^2 V_2]$ of the cone does not belong to $\mathbf{P}(W)$. Indeed, since both L_0 and L_v are nice lines (in the sense of Definition 4.5), we have by Lemma 4.6

$$\mathbf{P}(W) \cap \mathbf{P}(v_0 \wedge V_5) = L_0 \quad \text{and} \quad \mathbf{P}(W) \cap \mathbf{P}(v \wedge V_5) = L_v. \tag{48}$$

But $[\bigwedge^2 V_2] \in \mathbf{P}(v_0 \wedge V_5) \cap \mathbf{P}(v \wedge V_5)$, so if it also belongs to $\mathbf{P}(W)$, we get $L_0 \cap L_v \neq \varnothing$, which contradicts Definition 4.7.

Since W has codimension 2 in $\bigwedge^2 V_5$ and is transverse to $V_2 \wedge V_5$ by Lemma 4.6, it follows that Z'_y is isomorphic to a hyperplane section of $\mathbf{P}(V_2) \times \mathbf{P}(V_5/V_2)$. It is easy to see that its Hilbert polynomial is $h_{Z'_y}(t) = (t+1)(\frac{3}{2}t + 1)$. Since it does not depend on y, the family of surfaces Z'_{0,V_5} is flat over S'_{0,V_5}. ☐

Remark 4.12. As in Remark 4.9, the family Z'_{0,V_5} is the restriction of the family of surfaces

$$Z'_v = \mathrm{Cone}_{[v_0 \wedge v]}\big(\mathbf{P}(\mathbf{C}v_0 \oplus \mathbf{C}v) \times \mathbf{P}(V_5/(\mathbf{C}v_0 \oplus \mathbf{C}v))\big) \cap \mathbf{P}(W)$$

over $\mathbf{P}(V_5) \smallsetminus \{[v_0]\}$. When $[v] \in \mathbf{P}(V_5) \smallsetminus \mathbf{P}(V_3)$, where $V_3 \subset V_5$ is defined by $L_0 = \mathbf{P}(v_0 \wedge V_3)$, these surfaces are hyperplane sections of the cubic scroll $\mathbf{P}^1 \times \mathbf{P}^2$. This proves flatness of the family Z' even when the curve S'_{0,V_5} is not reduced.

Propositions 4.1 and 4.11 show that the components of Z_0 and Z'_{0,V_5} all have dimension 3. They are components of the scheme Z_{0+}, which has other 3-dimensional components over the curve $S''_{0,V_5} = \widetilde{S}_{0+} \smallsetminus (\widetilde{S}_0 \cup S'_{0,V_5})$, but we will not need this fact.

Finally, to construct the left column of (41), recall that the curve S'_{0,V_5} is by definition isomorphic to the open subscheme $S_{0,V_5} \subset Y^{\geq 2}_{A,V_5} \smallsetminus \Sigma_1(X)$, which via the map $\sigma\colon F_1(X) \to Y^{\geq 2}_{A,V_5}$ is identified with an open subscheme in the Hilbert scheme of lines $F_1(X)$ (see Proposition 2.11). Applying to Z'_{0,V_5} the construction of Hilbert closure from Definition 4.2, we obtain a subscheme

$$Z'_F \subset M_X \times F_1(X) \tag{49}$$

such that $Z'_F \times_{F_1(X)} S'_{0,V_5} \simeq Z'_{0,V_5}$. Note that Z'_F may be not flat over the singular locus of the curve $F_1(X)$.

4.3. **A relation between the subschemes.** Let X be a smooth ordinary GM threefold and let L_0 be a nice line on X. In (37) and (49), we constructed subschemes $Z \subset X \times \widetilde{Y}^{\geq 2}_A$ and $Z'_F \subset M_X \times F_1(X)$. The proof of Theorem 4.4 is based on a relation between the schemes

$$Z \cap (X \times S'_{0,V_5}) \qquad \text{and} \qquad Z'_F \cap (X \times S'_{0,V_5}),$$

where the curve S'_{0,V_5} defined in (47) is considered as a subscheme of $\widetilde{Y}^{\geq 2}_A$ and $F_1(X)$.

To prove such a relation, we will assume that the Hilbert scheme of lines $F_1(X)$ is a smooth curve (by Lemma 2.13, it is then irreducible). This assumption implies that the open curves $Y^2_{A,V_5} \smallsetminus \Sigma_1(X)$ and S'_{0,V_5} are also smooth and irreducible. The next lemma sharpens the results of Proposition 4.11 under this assumption.

Lemma 4.13. *Assume that $F_1(X)$ is smooth. For a general point y in S'_{0,V_5}, the fiber Z'_y of $Z'_{0,V_5} \to S'_{0,V_5}$ is a smooth cubic surface scroll and the lines L_0 and $L_{\pi(y)}$ are distinct fibers of the ruling of this scroll.*

Proof. We saw at the end of the proof of Proposition 4.11 that Z'_y is a hyperplane section of $\mathbf{P}(V_2) \times \mathbf{P}(V_5/V_2)$. There are two kinds of such hyperplane sections:

(a) smooth cubic scrolls with projection to $\mathbf{P}^1$ induced by $\mathbf{P}(V_2) \times \mathbf{P}(V_5/V_2) \to \mathbf{P}(V_2)$,
(b) unions $(\mathbf{P}(V_2) \times \mathbf{P}(V'_2)) \cup (\{[v']\} \times \mathbf{P}(V_5/V_2))$, where $V'_2 \subset V_5/V_2$ and $[v'] \in \mathbf{P}(V_2)$.

In case (b), the σ-plane $\{[v']\} \times \mathbf{P}(V_5/V_2)$ is contained in $Z'_y \subset M_X$ hence, $[v'] \in \Sigma_1(X)$ by (28) and $[v'] \neq [v_0]$. Since V_2 is the subspace of V_5 spanned by v_0 and a vector v such that $[v] = \pi(y)$, and since $[v'] \in \mathbf{P}(V_2) \smallsetminus \{[v_0]\}$, case (b) holds only if

$$[v] \in \mathrm{pr}_{v_0}(\Sigma_1(X)) \cap \mathrm{pr}_{v_0}(Y^{\geq 2}_{A,V_5}) \subset \mathbf{P}(V_5/\mathbf{C}v_0),$$

where $\mathrm{pr}_{v_0} \colon \mathbf{P}(V_5) \dashrightarrow \mathbf{P}(V_5/\mathbf{C}v_0)$ is the projection from v_0. Since $Y^{\geq 2}_{A,V_5}$ is an integral curve of degree 40, its image by pr_{v_0} is contained in the image of the conic $\Sigma_1(X)$ only if the line connecting v_0 with a general point of $\Sigma_1(X)$ intersects $Y^{\geq 2}_{A,V_5}$ in 20 points. But the surface $Y^{\geq 2}_A$ is an intersection of hypersurfaces of degree 6 by [DK2, (33)], hence the same is true for its hyperplane section $Y^{\geq 2}_{A,V_5}$, and the curve $Y^{\geq 2}_{A,V_5}$ would then contain the cone $\mathrm{Cone}_{[v_0]}(\Sigma_1(X))$, which is absurd. Therefore, for y general in $\widetilde{S}'_{0,V_5}$, we are in case (a).

By (48), the lines L_0 and L_v are contained in fibers of the map $\mathbf{P}(V_2) \times \mathbf{P}(V_5/V_2) \to \mathbf{P}(V_2)$. In case (a), they are therefore rulings of the scroll. ☐

Since the curve S'_{0,V_5} is isomorphic to a dense open subscheme of the smooth curve $F_1(X)$, the locally closed embedding $S'_{0,V_5} \hookrightarrow \widetilde{Y}^{\geq 2}_A$ extends to a regular map

$$\phi \colon F_1(X) \longrightarrow \widetilde{Y}^{\geq 2}_A. \qquad (50)$$

We combine all these maps into the commutative diagram

$$\begin{array}{ccccccc}
 & & & X \times S'_{0,V_5} & & & \\
 & & \overset{\text{open}}{\swarrow} & \downarrow & \overset{\text{loc. closed}}{\searrow} & & \\
\mathscr{L}_1(X) & \hookrightarrow & X \times F_1(X) & \xrightarrow{\quad\phi\quad} & X \times \widetilde{Y}^{\geq 2}_A & \hookleftarrow & Z \\
 & & \downarrow i & \downarrow i & \downarrow i & & \\
 & & & M_X \times S'_{0,V_5} & & & \\
 & & \overset{\text{open}}{\swarrow} & & \overset{\text{loc. closed}}{\searrow} & & \\
Z'_F & \hookrightarrow & M_X \times F_1(X) & \xrightarrow{\quad\phi\quad} & M_X \times \widetilde{Y}^{\geq 2}_A, & &
\end{array} \qquad (51)$$

where $\mathscr{L}_1(X) \subset X \times F_1(X)$ is the universal family of lines, $i \colon X \hookrightarrow M_X$ is the embedding, and the schemes Z and Z'_F are defined by (37) and (49) respectively. We make the following key observation.

Proposition 4.14. *Assume that $F_1(X)$ is smooth. There is a dense open subscheme $U \subset F_1(X)$ such that*

$$Z'_F \cap (X \times U) = \big(\phi^{-1}(Z) \cap (X \times U)\big) + \big(\mathscr{L}_1(X) \cap (X \times U)\big) \qquad (52)$$

as cycles.

Proof. Since we only need an equality over a dense open subset of $F_1(X)$, we may base change both sides along the open embedding $S'_{0,V_5} \hookrightarrow F_1(X)$. The left side can then be rewritten as

$$Z'_F \cap (X \times S'_{0,V_5}) = Z'_{0,V_5} \cap (X \times S'_{0,V_5}).$$

By Proposition 4.11 and Lemma 4.13, the morphism $Z'_{0,V_5} \to S'_{0,V_5}$ is a flat family of surfaces whose general fiber is a smooth cubic surface scroll. Since X contains no surfaces of degrees less than 10 ([DK2, Corollary 3.5]), it contains no components of any fiber of Z'_{0,V_5}. Therefore, the morphism

$$Z'_{0,V_5} \times_{M_X} X \to S'_{0,V_5}$$

is a flat family of curves whose fiber over a general point $y \in S'_{0,V_5}$ is the intersection

$$C_y := Z'_y \cap Q_0$$

of the smooth cubic surface scroll Z'_y with any non-Plücker quadric Q_0 containing X. Such an intersection is a connected curve of degree 6 and arithmetic genus 2. Since the lines L_0 and $L_{\pi(y)}$ are contained both in the scroll Z'_y and the quadric Q_0, they are components of C_y.

To describe the remaining component, we denote by e the class of the exceptional section L_e of the scroll Z'_y and by f the class of a fiber of its ruling. We have

$$e^2 = -1, \qquad e \cdot f = 1, \qquad f^2 = 0.$$

The hyperplane class is equal to $e + 2f$ hence, the class of C_y in Z'_y is $2e + 4f$. As we observed above, the lines L_0 and $L_{\pi(y)}$ are fibers of the ruling hence,

$$C'_y := C_y - L_0 - L_{\pi(y)}$$

is an effective divisor on Z'_y with class $2e + 2f$.

If C'_y contains a line, the class of this line is either f (the class of a fiber of the ruling), or e (the class of the exceptional section L_e of Z'_y). If it is f, the residual components have class $2e + f$, and since $(2e + f) \cdot e = -1$, the section L_e is in both cases a component of C'_y hence, a line on X. The line L_e is in the finite set of lines on X intersecting L_0, and $L_{\pi(y)}$ is in the finite set of lines on X intersecting a line that intersects L_0. It follows that for y general, the curve C'_y contains no lines.

By Lemma 4.10, the curve Z_y is contained in the surface Z'_y for general $y \in S'_{0,V_5}$. Therefore, by Lemma 4.3, for general $y \in S'_{0,V_5}$, the sextic curve C_y contains the quintic curve Z_y, hence $C_y = Z_y + L'_y$, where L'_y is a line. Since Z_y contains L_0 and C'_y contains no lines, the line L'_y must be $L_{\pi(y)}$. Thus,

$$C_y = Z_y + L_{\pi(y)}.$$

Since this holds for y general in S'_{0,V_5}, it follows that (52) holds over a dense open subscheme $U \subset S'_{0,V_5} \subset F_1(X)$. □

4.4. **Abel–Jacobi maps.** Let X be a smooth GM threefold with associated Lagrangian A. Assume that $Y^3_A = \varnothing$, so that $\widetilde{Y}^{\ge 2}_A$ is a smooth surface, and that the Hilbert scheme of lines $F_1(X)$ is smooth. The subscheme $Z \subset X \times \widetilde{Y}^{\ge 2}_A$ was constructed in Lemma 4.3. Consider the universal line $\mathscr{L}_1(X) \subset X \times F_1(X)$, the Abel–Jacobi maps

$$\mathsf{AJ}_Z \colon H_1(\widetilde{Y}^{\ge 2}_A, \mathbf{Z}) \to H_3(X, \mathbf{Z}) \quad \text{and} \quad \mathsf{AJ}_{\mathscr{L}_1(X)} \colon H_1(F_1(X), \mathbf{Z}) \to H_3(X, \mathbf{Z}),$$

and the map $\phi\colon F_1(X) \to \widetilde{Y}^{\ge 2}_A$ defined in (50).

Proposition 4.15. *Let X be a smooth ordinary GM threefold such that $F_1(X)$ is smooth and let L_0 be a nice line on X. The composition of maps*

$$H_1(F_1(X), \mathbf{Z}) \xrightarrow{\phi_*} H_1(\widetilde{Y}_A^{\geq 2}, \mathbf{Z}) \xrightarrow{\mathsf{AJ}_Z} H_3(X, \mathbf{Z})$$

is equal to the map $-\mathsf{AJ}_{\mathscr{L}_1(X)}$.

Proof. By Lemma 3.1(a), it is enough to check that the Abel–Jacobi map given by the image of $[Z]$ with respect to the pullback map

$$(\mathrm{Id}_X \times \phi)^* \colon H^4(X \times \widetilde{Y}_A^{\geq 2}, \mathbf{Z}) \longrightarrow H^4(X \times F_1(X), \mathbf{Z})$$

is equal to $-\mathsf{AJ}_{\mathscr{L}_1(X)}$. Equality (52) implies that there is a cycle Z''_D supported on $X \times D$, where $D = F_1(X) \smallsetminus U \overset{\delta}{\hookrightarrow} F_1(X)$ is a finite subscheme of $F_1(X)$, such that

$$(\mathrm{Id}_X \times \phi)^*([Z]) + [\mathscr{L}_1(X)] = (i \times \mathrm{Id}_{F_1(X)})^*([Z'_F]) + (\mathrm{Id}_X \times \delta)_*([Z''_D]).$$

Let us show that the Abel–Jacobi map defined by the right side of this equality is zero.

By Lemma 3.1(b), the Abel–Jacobi map corresponding to the cycle $(i \times \mathrm{Id}_{F_1(X)})^*([Z'_F])$ is equal to the composition

$$H_1(F_1(X), \mathbf{Z}) \xrightarrow{\mathsf{AJ}_{Z'_F}} H_5(M_X, \mathbf{Z}) \xrightarrow{i^*} H_3(X, \mathbf{Z}).$$

Since $H_5(M_X, \mathbf{Z}) = 0$ by Proposition 2.6, this map vanishes. Similarly, the Abel–Jacobi map corresponding to the cycle $(\mathrm{Id}_X \times \delta)_*([Z''_D])$ is equal to the composition

$$H_1(F_1(X), \mathbf{Z}) \xrightarrow{\delta^*} H_{-1}(D, \mathbf{Z}) \xrightarrow{\mathsf{AJ}_{Z''_D}} H_3(X, \mathbf{Z}),$$

hence vanishes as well. This completes the proof of the proposition. □

The above proposition relates the Abel–Jacobi maps AJ_Z and $\mathsf{AJ}_{\mathscr{L}_1(X)}$. The next lemma uses the Clemens–Tyurin argument (Section 3.3) to show that the latter is surjective.

Lemma 4.16. *Let X be a general GM threefold and let $\mathscr{L}_1(X) \to F_1(X)$ be the universal family of lines contained in X. The Abel–Jacobi map*

$$\mathsf{AJ}_{\mathscr{L}_1(X)} \colon H_1(F_1(X), \mathbf{Z}) \longrightarrow H_3(X, \mathbf{Z})$$

is surjective.

Proof. Let Y be a general GM fourfold and let $X \subset Y$ be a general hyperplane section, so that X is a general GM threefold. Let $F_Y = F_1(Y)$ be the Hilbert scheme of lines contained in Y. We check that the assumptions of Proposition 3.3 (with $m = 1$) are satisfied.

Assumption (a) holds since $F_1(Y)$ is a smooth irreducible threefold by [DK2, Proposition 5.3]. Furthermore, the map $q \colon \mathscr{L}_1(Y) \to Y$ is generically finite of degree 6 ([DK2, Lemma 5.6]) hence, (b) holds as well. Next, $F_1(X)$ is a smooth curve by Lemma 2.12 hence (c) holds. Finally,

$$H_3(Y, \mathbf{Z}) = H_5(Y, \mathbf{Z}) = 0$$

by [DK2, Proposition 3.1] hence, (d) holds.

Applying Proposition 3.3, we deduce the surjectivity of $\mathsf{AJ}_{\mathscr{L}_1(X)}$. □

Combining the above results, we can now prove Theorem 4.4.

Proof of Theorem 4.4. Let $A \subset \bigwedge^3 V_6$ be a general Lagrangian subspace. As recalled in Section 2.3, any hyperplane $V_5 \subset V_6$ corresponding to a point of $Y^{\geq 2}_{A^\perp} \subset \mathbf{P}(V_6^\vee)$ defines a smooth GM threefold X with $A(X) = A$. We choose a general such $[V_5]$, so that X is a general GM threefold. We also choose a nice line $L_0 \subset X$. A combination of Proposition 4.15 and Lemma 4.16 proves that the map AJ_Z is surjective. By Propositions 2.5 and 2.6, its source and target are free abelian groups of rank 20. Therefore, the Abel–Jacobi map (39) is an isomorphism.

Since the Abel–Jacobi map is defined by the cohomology class of an algebraic cycle, it preserves the Hodge structures hence, it induces an isomorphism of the corresponding abelian varieties: the Albanese variety of $\widetilde{Y}^{\geq 2}_A$ and the intermediate Jacobian of X.

Since the scheme $Z \subset X \times \widetilde{Y}^{\geq 2}_A$ is defined in Section 4.1 for all X and since this definition works in families, it follows by continuity that the map (40) is an isomorphism for any A such that $Y^3_A = \varnothing$ (so that the surface $\widetilde{Y}^{\geq 2}_A$ is smooth) and any smooth X such that $A(X) = A$.

Since $\mathrm{Jac}(X)$ carries a canonical principal polarization, the isomorphism (40) of the theorem defines a principal polarization on the abelian variety $\mathrm{Alb}(\widetilde{Y}^{\geq 2}_A)$. Since $Y^{\geq 2}_{A^\perp}$ is connected and the isomorphism (40) depends continuously on X, this principal polarization is independent of the choice of $[V_5] \in Y^{\geq 2}_{A^\perp}$. This defines a canonical principal polarization on the abelian variety $\mathrm{Alb}(\widetilde{Y}^{\geq 2}_A)$ and the isomorphism (40) is compatible with it by construction.

It remains to construct an isomorphism (38). Choose a point

$$(V_1, V_5) \in (Y^2_A \times Y^2_{A^\perp}) \cap \mathsf{Fl}(1, 5; V_6)$$

such that $A \cap (V_1 \wedge \bigwedge^2 V_5) = 0$ (it exists by [DK1, Lemma B.5]). Let X and X' be the smooth ordinary GM threefolds corresponding to the Lagrangian data sets (V_6, V_5, A) and $(V_6^\vee, V_1^\perp, A^\perp)$ respectively. We will prove in Proposition 4.17 that there is a diagram

$$\begin{array}{ccc} \widehat{X} & \overset{\psi}{\dashrightarrow} & \widehat{X}' \\ \downarrow & & \downarrow \\ X & & X', \end{array}$$

where both vertical morphisms are blow ups of smooth rational curves and ψ is a flop. By [FW, Proposition 3.1], the morphism $H^3(\widehat{X}; \mathbf{Z}) \to H^3(\widehat{X'}; \mathbf{Z})$ induced by the correspondence defined by the graph of ψ is an isomorphism of polarized Hodge structures. It induces in particular an isomorphism $\mathrm{Jac}(\widehat{X}) \xrightarrow{\sim} \mathrm{Jac}(\widehat{X}')$ of principally polarized abelian varieties between intermediate Jacobians. Therefore, there is a chain of isomorphisms

$$\mathrm{Alb}(\widetilde{Y}^{\geq 2}_A) \simeq \mathrm{Jac}(X) \simeq \mathrm{Jac}(\widehat{X}) \xrightarrow{\ \sim\ } \mathrm{Jac}(\widehat{X}') \simeq \mathrm{Jac}(X') \simeq \mathrm{Alb}(\widetilde{Y}^{\geq 2}_{A^\perp})$$

of principally polarized abelian varieties. This concludes the proof of the theorem. □

4.5. **The line transform.** In this section, we revisit the birational isomorphism of [DK1, Proposition 4.19] and identify it with an elementary transformation along a line, a birational transformation between GM threefolds defined in [IP, Proposition 4.3.1] and [DIM1, Section 7.2] (this relation was mentioned without proof in [DK1, Section 4.6]).

Proposition 4.17. *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors. Consider subspaces $V_1 \subset V_5 \subset V_6$ such that*

$$[V_1] \in Y^2_A, \quad [V_5] \in Y^2_{A^\perp}, \quad A \cap (V_1 \wedge \textstyle\bigwedge^2 V_5) = 0. \tag{53}$$

Let X and X' be the smooth ordinary GM threefolds corresponding to the Lagrangian data sets (V_6, V_5, A) and $(V_6^\vee, V_1^\perp, A^\perp)$, and let $L_0 \subset X$ and $L'_0 \subset X'$ be the lines corresponding to the points $[V_1]$ of Y^2_{A,V_5} and $[V_5^\perp]$ of $Y^2_{A^\perp,V_1^\perp}$ via the maps (30).

There is a diagram of birational maps

$$\begin{array}{ccccccc} & & \mathrm{Bl}_{L_0}(X) & \overset{\psi}{\dashrightarrow} & \mathrm{Bl}_{L'_0}(X') & & \\ & {\scriptstyle\beta}\swarrow & & {\scriptstyle\rho_X}\searrow \quad \swarrow{\scriptstyle\rho_{X'}} & & \searrow{\scriptstyle\beta'} & \\ X & & \overset{\varpi}{\dashrightarrow} & \bar{X} & \overset{\varpi'}{\dashleftarrow} & & X', \end{array} \tag{54}$$

where β and β' are the respective blow ups of X and X' along the lines L_0 and L'_0, the birational maps ϖ and ϖ' are the respective linear projections of X from L_0, and of X' from L'_0, the morphisms ρ_X and $\rho_{X'}$ are small crepant extremal birational contractions, the variety $\bar{X}$ is a normal, Gorenstein, cubic hypersurface in $\mathsf{Gr}(2, V_5/V_1) = \mathsf{Gr}(2, V_1^\perp/V_5^\perp)$ with terminal singularities, and the map $\psi = \rho_{X'}^{-1} \circ \rho_X$ is a flop.

The proposition says that the birational isomorphism $\varpi'^{-1} \circ \varpi \colon X \dashrightarrow X'$ is the "elementary rational map with center along the line L_0" in the sense of [IP, (4.1.1)] (or elementary transformation along the line L_0) if the GM threefold X and the line L_0 are sufficiently general. We use the proposition and the fact that the elementary transformation is defined for any X and L_0 to specify what the moduli point of the result of the elementary transformation is in general (we use the description of the coarse moduli space $\mathbf{M}_3^{\mathrm{GM}}$ of GM threefolds given in (15)).

Corollary 4.18. *Let X be a smooth GM threefold with moduli point $([A], [V_5]) \in \mathbf{M}_3^{\mathrm{GM}}$ and let $L \subset X$ be any line. The moduli point of the result X'_L of the elementary transformation of X along the line L is $([A^\perp], \sigma(L)) \in \mathbf{M}_3^{\mathrm{GM}}$.*

Proof. Let $\mathfrak{M}_3^{\mathrm{GM}}$ be the moduli stack of smooth GM threefolds (see [DK4]), let $\mathscr{X} \to \mathfrak{M}_3^{\mathrm{GM}}$ be the universal family of threefolds over it, and let $F_1(\mathscr{X}/\mathfrak{M}_3^{\mathrm{GM}})$ be the relative Hilbert scheme of lines. As we already mentioned, by [IP, Section 4.1], the elementary transformation is defined for any line contained in any GM threefold X. Moreover, this transformation can be performed in a family for a family of lines and will produce a family of GM threefolds. This defines a morphism

$$\begin{array}{rcl} F_1(\mathscr{X}/\mathfrak{M}_3^{\mathrm{GM}}) & \longrightarrow & \mathbf{M}_3^{\mathrm{GM}} \\ ([X], [L]) & \longmapsto & [X'_L]. \end{array}$$

By [DK4, Theorem 5.11] and [DK2, Theorem 4.7], the left side is irreducible and birational to

$$\{(A, V_5, V_1) \in \mathsf{LGr}_{\mathrm{ndv}}(\textstyle\bigwedge^3 V_6) \times \mathbf{P}(V_6^\vee) \times \mathbf{P}(V_6) \mid [V_5] \in Y^{\geq 2}_{A^\perp},\ [V_1] \in Y^{\geq 2}_A\} /\!\!/ \mathrm{PGL}(V_6).$$

By Proposition 4.17, over the dense open subset of triples (A, V_5, V_1) satisfying condition (53), this map coincides with the projection

$$(A, V_5, V_1) \mapsto (A^\perp, V_1) \in \mathbf{M}_3^{\mathrm{GM}}.$$

Since the target is separated ([DK4, Theorem 5.15]), by continuity, the two maps coincide everywhere. $\square$

To prove Proposition 4.17, we start with some preliminaries. First, subspaces V_1 and V_5 satisfying the conditions (53) exist: this follows from [DK1, Lemma B.5] (where $\widehat{Y}_A$ is defined in [DK1, (39)]). More exactly, for $[V_5]$ general in Y^2_A (so that X is a general GM threefold associated with the fixed Lagrangian A), these conditions will be satisfied for a general $[V_1] \in Y^2_{A,V_5}$, corresponding to a general line $L_0 \subset X$. As explained in the proof of [DK1, Theorem 4.20], the conditions (53) are equivalent to

$$[V_1] \in Y^2_{A,V_5} \smallsetminus \Sigma_1(X) \qquad \text{and} \qquad [V_5^\perp] \in Y^2_{A^\perp, V_1^\perp} \smallsetminus \Sigma_1(X'),$$

hence the lines L_0 and L_0' are nice and the assumptions of [DK1, Proposition 4.19] are satisfied. It was explained in the proof of that proposition that these lines can be written as

$$L_0 = \mathbf{P}(V_1 \wedge V_3), \qquad L_0' = \mathbf{P}(V_5^\perp \wedge V_3^\perp) \tag{55}$$

for the same subspace V_3 such that $V_1 \subset V_3 \subset V_5$.

Following [DK1, Section 4.4], we introduce the second quadratic fibration

$$\rho_2 \colon \mathbf{P}_X(V_5/\mathscr{U}_X) \longrightarrow \mathsf{Gr}(3, V_5),$$

and analogously for X', and study the diagram [DK1, (4.29)]

$$\begin{array}{ccccccc} & \widetilde{X} & & & & \widetilde{X}' & \\ \swarrow^{f} & & \searrow^{\tilde\rho_2} & & \swarrow^{\tilde\rho_2'} & & \searrow^{f'} \\ X & & & \mathsf{Gr}(2, V_5/V_1) & & & X', \end{array} \tag{56}$$

where $\tilde\rho_2$ is obtained from ρ_2 by restriction to $\mathsf{Gr}(2, V_5/V_1) \subset \mathsf{Gr}(3, V_5)$ and analogously for $\tilde\rho_2'$. The next lemma is a refinement of [DK1, Lemma 4.18].

Lemma 4.19. *The scheme $\widetilde{X}$ has two irreducible components,* $\mathrm{Bl}_{L_0}(X)$ *and* $f^{-1}(L_0)$*. They are both smooth of dimension* 3 *and meet transversely along the exceptional divisor of* $\mathrm{Bl}_{L_0}(X)$*. Moreover, the map* $\tilde\rho_2 \colon \widetilde{X} \to \mathsf{Gr}(2, V_5/V_1)$ *is induced by the linear projection from the line* L_0*.*

Proof. We defined in [DK1, Section 4.4] the EPW quartic hypersurface $Z_A \subset \mathsf{Gr}(3, V_6)$. For any subspace $V_1 \subset V_5$, we denote by

$$Z_{A,V_5} \subset \mathsf{Gr}(3, V_5), \quad Z_{A,V_1,V_5} \subset \mathsf{Gr}(2, V_5/V_1),$$

the subschemes obtained by intersecting Z_A with the subvarieties $\mathsf{Gr}(3, V_5)$ and $\mathsf{Gr}(2, V_5/V_1)$ of $\mathsf{Gr}(3, V_6)$.

Consider the commutative diagram

$$\begin{array}{ccccccc} \mathbf{P}_X(V_5/\mathscr{U}_X) & \longrightarrow & \mathbf{P}_{M_X}(V_5/\mathscr{U}_{M_X}) & \longrightarrow & \mathbf{P}_{\mathsf{Gr}(2,V_5)}(V_5/\mathscr{U}) & = & \mathsf{Fl}(2,3;V_5) \\ & \searrow^{\rho_2} & & \searrow & \downarrow & & \\ & & Z_{A,V_5} & \hookrightarrow & \mathsf{Gr}(3, V_5), & & \end{array}$$

where M_X was defined in (13) and the vertical arrow is the canonical projection, which induces the diagonal arrows (the left diagonal arrow factors through Z_{A,V_5} by [DK1, Proposition 4.10]). Pulling this diagram back by the inclusion $\mathsf{Gr}(2, V_5/V_1) \subset \mathsf{Gr}(3, V_5)$, we obtain the diagram

$$\begin{array}{ccccc} \widetilde{X} & \longrightarrow & \mathrm{Bl}_{L_0}(M_X) & \longrightarrow & \mathrm{Bl}_{\mathbf{P}(V_1 \wedge V_5)}(\mathsf{Gr}(2, V_5)) \\ & \searrow^{\tilde\rho_2} & & \searrow & \downarrow \\ & & Z_{A,V_1,V_5} & \hookrightarrow & \mathsf{Gr}(2, V_5/V_1). \end{array}$$

Indeed, $\mathsf{Gr}(2, V_5/V_1) \subset \mathsf{Gr}(3, V_5)$ is the zero-locus of the section of the vector bundle $V_5/\mathscr{U}_3$ corresponding to V_1, and by [K, Lemma 2.1], the zero-locus of the corresponding section on $\mathbf{P}_{\mathsf{Gr}(2,V_5)}(V_5/\mathscr{U})$ is the blow up of $\mathsf{Gr}(2, V_5)$ along the zero-locus of induced section of $V_5/\mathscr{U}$, which is equal to the locus $\mathbf{P}(V_1 \wedge V_5)$ of 2-dimensional subspaces in V_5 containing V_1. Note also that the map

$$\mathrm{Bl}_{\mathbf{P}(V_1 \wedge V_5)}(\mathsf{Gr}(2, V_5)) \to \mathsf{Gr}(2, V_5/V_1)$$

is induced by the linear projection $V_5 \to V_5/V_1$, or equivalently by the linear projection $\mathbf{P}(\bigwedge^2 V_5) \dashrightarrow \mathbf{P}(\bigwedge^2(V_5/V_1))$ from $\mathbf{P}(V_1 \wedge V_5)$.

Furthermore, $M_X \subset \mathsf{Gr}(2, V_5)$ is the linear section by the subspace $\mathbf{P}(W) \subset \mathbf{P}(\bigwedge^2 V_5)$ which is transverse to $\mathbf{P}(V_1 \wedge V_5)$ by Lemma 4.6, because the line L_0 is nice; the pullback of $\mathbf{P}_{M_X}(V_5/\mathscr{U}_{M_X})$ is therefore $\mathrm{Bl}_{L_0}(M_X)$. Moreover, the map $\mathrm{Bl}_{L_0}(M_X) \to \mathsf{Gr}(2, V_5/V_1)$ is induced by the linear projection from $\mathbf{P}(W \cap (V_1 \wedge V_5)) = L_0$.

To prove the lemma, it remains to note that

$$\mathrm{Bl}_{L_0}(M_X) \times_{M_X} X = \mathrm{Bl}_{L_0}(X) \cup f^{-1}(L_0),$$

because the first component is the strict transform of X and the second component is the exceptional divisor of $\mathrm{Bl}_{L_0}(M_X) \to M_X$. □

It was proved in [DK1, Lemma 4.18 and Proposition 4.19] that

- $\tilde{\rho}_2$ maps $f^{-1}(L_0)$ birationally onto the Schubert hyperplane divisor $\mathsf{D} \subset \mathsf{Gr}(2, V_5/V_1)$ parameterizing subspaces intersecting V_3/V_1, where V_3 was defined in (55);
- $\tilde{\rho}_2$ maps $\mathrm{Bl}_{L_0}(X)$ birationally onto a cubic hypersurface $\bar{X} \subset \mathsf{Gr}(2, V_5/V_1)$;
- the image of $\tilde{\rho}_2$ is the quartic hypersurface Z_{A,V_1,V_5}; it is therefore equal to $\bar{X} \cup \mathsf{D}$.

We denote by $\rho_X \colon \mathrm{Bl}_{L_0}(X) \to \bar{X}$ the (birational) restriction of $\tilde{\rho}_2$ and we define $\rho_{X'}$ similarly.

Proof of Proposition 4.17. We have already constructed the left part of the diagram (54). The right part is constructed analogously. It remains to prove that ψ is a flop.

As explained in [IP, Lemma 4.1.1 and Corollary 4.3.2], ρ_X is a flopping contraction. The same is true for $\rho_{X'}$, so ψ is either a flop or an isomorphism. If ψ is an isomorphism, the maps β and β' are the contractions of the same extremal ray, hence $X \simeq X'$. Let us show that this is impossible. We can perform this construction on a fixed X (that is, with A and V_5 fixed) but with $[V_1]$ varying in the curve Y^2_{A,V_5}. Locally, the map ψ will remain an isomorphism and the threefolds X' obtained by the construction all isomorphic to X. But this is impossible, since their moduli points describe the curve

$$Y^2_{A,V_5} /\!\!/ \mathrm{PGL}(V_6)_A \subset \mathfrak{p}_3^{-1}([A^\perp]) \subset \mathbf{M}_3^{\mathrm{GM}}$$

(recall that the map $\mathfrak{p}_n$ was defined in (16) and that the group $\mathrm{PGL}(V_6)_A$ is finite). It follows that ψ is a flop and the proof of the proposition is complete. □

Remark 4.20. One can analyze the situation further and describe the maps ρ_X and $\rho_{X'}$: the intersections with $\mathsf{Gr}(2, V_5/V_1)$ of the subscheme $\Sigma_2(X) \subset \mathsf{Gr}(3, V_5)$ defined in [DK1, (4.2)] and of the analogous subscheme $\Sigma_2(X') \subset \mathsf{Gr}(2, V_6/V_1)$ are cubic surfaces $\Sigma_{2,V_1}(X)$ and $\Sigma_{2,V_5}(X')$ such that

$$\bar{X} \cap \mathsf{D} = \Sigma_{2,V_1}(X) \cup \Sigma_{2,V_5}(X')$$

and the morphisms ρ_X and $\rho_{X'}$ are the blow ups of $\bar{X}$ along the Weil divisors $\Sigma_{2,V_1}(X)$ and $\Sigma_{2,V_5}(X')$, respectively.

5. Intermediate Jacobians of GM fivefolds

In this section, we perform, for GM fivefolds, a construction analogous to the one of Section 4. The curves in the construction are replaced by surfaces: lines by planes, elliptic quintic curves by quintic del Pezzo surfaces, and rational quartic curves by rational quartic surface scrolls. In Section 5.5, we give an alternative proof of the main result.

5.1. **Family of surfaces.** Given an arbitrary GM fivefold X with associated Lagrangian A, we begin by choosing an arbitrary σ-plane $\Pi_0 \subset X$ (that is, a point of $F^2_\sigma(X)$; see (8)). We consider the open surface $S_0 \subset Y_A^{\ge 2}$ defined by (21) and the family of quadrics $\mathscr{Q}_0 \to S_0$ obtained by restricting to S_0 the universal family (18) of 8-dimensional quadrics containing X. We denote by $F^5(\mathscr{Q}_0/S_0) \to S_0$ the relative Hilbert scheme of linear 5-spaces in the fibers of $\mathscr{Q}_0 \to S_0$ and by $F^5_{\Pi_0}(\mathscr{Q}_0/S_0) \to S_0$ the subscheme parameterizing those 5-spaces which contain the plane Π_0. Applying Corollary 2.9, we obtain an isomorphism

$$F^5_{\Pi_0}(\mathscr{Q}_0/S_0) \simeq \widetilde{S}_0 := \widetilde{Y}^2_A \times_{Y^2_A} S_0 \tag{57}$$

of schemes over S_0. In particular, the canonical map $F^5_{\Pi_0}(\mathscr{Q}_0/S_0) \to S_0$ is the double étale covering $\pi\colon \widetilde{S}_0 \to S_0$ induced by the double covering π_A.

Note that $\widetilde{S}_0$ is a smooth surface. Let

$$\widetilde{\mathscr{Q}}_0 := \mathscr{Q}_0 \times_{S_0} \widetilde{S}_0$$

be the base change of the family of quadrics $\mathscr{Q}_0 \to S_0$ along π. We have a canonical map

$$\widetilde{S}_0 \to F^5_{\Pi_0}(\mathscr{Q}_0/S_0) \times_{S_0} \widetilde{S}_0 \hookrightarrow F^5(\mathscr{Q}_0/S_0) \times_{S_0} \widetilde{S}_0 \simeq F^5(\widetilde{\mathscr{Q}}_0/\widetilde{S}_0),$$

where the first map is the product of the isomorphism (57) with the identity map. By construction, it is a section of the projection $F^5(\widetilde{\mathscr{Q}}_0/\widetilde{S}_0) \to \widetilde{S}_0$.

Let $\mathscr{P}^5 \subset \widetilde{\mathscr{Q}}_0 \subset \mathbf{P}(W) \times \widetilde{S}_0$ be the pullback of the universal family of projective 5-spaces over $F^5(\widetilde{\mathscr{Q}}_0/\widetilde{S}_0)$ along this section. Set

$$Z_0 := \mathscr{P}^5 \cap (M_X \times \widetilde{S}_0), \tag{58}$$

where the Grassmannian hull $M_X \subset \mathbf{P}(W)$ was defined in (13).

Proposition 5.1. *The map $Z_0 \to \widetilde{S}_0$ is a flat family of surfaces in X containing Π_0 with Hilbert polynomial*

$$h(t) = \frac{5}{2}(t^2 + t) + 1. \tag{59}$$

In particular, $Z_0 \subset X \times \widetilde{S}_0$.

Proof. Let $y \in \widetilde{S}_0$ and set $[v] := \pi(y) \in \mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$. The fiber of Z_0 over y is

$$Z_{0,y} := M_X \cap \mathscr{P}^5_y = \mathsf{CGr}(2, V_5) \cap \mathscr{P}^5_y.$$

Since the cone $\mathsf{CGr}(2, V_5) \subset \mathbf{P}(\mathbf{C} \oplus \bigwedge^2 V_5)$ has codimension 3 and degree 5, the intersection $\mathsf{CGr}(2, V_5) \cap \mathscr{P}^5_y$ has dimension at least 2 and degree at most 5 (and if the dimension is 2, the degree is 5). Furthermore, $\mathscr{P}^5_y \subset Q_v$ hence,

$$Z_{0,y} \subset M_X \cap Q_v = X.$$

Since X contains no divisors of degrees less than 10, we have $\dim(Z_{0,y}) \le 3$ and, if $\dim(Z_{0,y}) = 3$, any irreducible 3-dimensional component $Z_{0,y}$ has even degree ([DK2, Corollary 3.5]). By

Lemma 2.3, its image in $\mathsf{Gr}(2, V_5)$ must be a hyperplane section of $\mathsf{Gr}(2, V_4)$ and Lemma 2.7 gives a contradiction. Therefore $Z_{0,y}$ is a surface. This argument also proves the inclusion

$$Z_0 \subset X \times \widetilde{S}_0.$$

Since the surface $Z_{0,y}$ is a dimensionally transverse linear section of $\mathsf{CGr}(2, V_5)$, it follows from Lemma 2.4 that we have a resolution

$$0 \to \mathscr{O}_{\mathscr{P}^5_y}(-5) \to \mathscr{O}_{\mathscr{P}^5_y}(-3)^{\oplus 5} \to \mathscr{O}_{\mathscr{P}^5_y}(-2)^{\oplus 5} \to \mathscr{O}_{\mathscr{P}^5_y} \to \mathscr{O}_{Z_{0,y}} \to 0.$$

It follows that the Hilbert polynomial of $Z_{0,y}$ is given by (59). Since it is independent of y, the family of surfaces Z_0 is flat over $\widetilde{S}_0$. Finally, $\Pi_0 \subset M_X$ and $\Pi_0 \subset \mathscr{P}^5_y$ by construction hence, $\Pi_0 \subset Z_{0,y}$. □

Applying to the family $Z_0 \to \widetilde{S}_0$ the construction of Hilbert closure from Definition 4.2, we obtain the following result.

Lemma 5.2. *There is a subscheme*

(60) $$Z \subset X \times \widetilde{Y}^{\geq 2}_A$$

such that away from a finite subset of the surface $\widetilde{Y}^{\geq 2}_A$*, the map* $Z \to \widetilde{Y}^{\geq 2}_A$ *is a flat family of surfaces with Hilbert polynomial* (59) *containing the plane* Π_0*. Moreover, we have*

$$Z \times_{\widetilde{Y}^{\geq 2}_A} \widetilde{S}_0 = Z_0$$

as subschemes of $X \times \widetilde{S}_0$.

By Proposition 5.1, the schemes Z_0 and Z have pure dimension 4. The map $Z \to \widetilde{Y}^{\geq 2}_A$ may be nonflat over a finite subscheme.

The main result of this section is the following (recall from Propositions 2.5 and 2.6 that the abelian groups $H_1(\widetilde{Y}^{\geq 2}_{A(X)}, \mathbf{Z})$ and $H_5(X, \mathbf{Z})$ are both torsion-free of rank 20 and from Theorem 4.4 that the abelian variety $\mathrm{Alb}(\widetilde{Y}^{\geq 2}_{A(X)})$ is endowed with a canonical principal polarization).

Theorem 5.3. *Let X be a smooth GM fivefold and let Π_0 be a σ-plane contained in X. Let $Z \subset X \times \widetilde{Y}^{\geq 2}_{A(X)}$ be the subscheme defined by* (60)*. If $Y^{\geq 3}_{A(X)} = \varnothing$, the Abel–Jacobi map*

$$\mathsf{AJ}_Z \colon H_1(\widetilde{Y}^{\geq 2}_{A(X)}, \mathbf{Z}) \longrightarrow H_5(X, \mathbf{Z})$$

is an isomorphism of integral Hodge structures. It induces an isomorphism

(61) $$\mathrm{Alb}(\widetilde{Y}^{\geq 2}_{A(X)}) \xrightarrow{\ \sim\ } \mathrm{Jac}(X)$$

of principally polarized abelian varieties from the Albanese variety of the surface $\widetilde{Y}^{\geq 2}_{A(X)}$ to the intermediate Jacobian of X.

Moreover, the principal polarization of $\mathrm{Alb}(\widetilde{Y}^{\geq 2}_A)$ is unique for A very general.

The proof of this theorem takes up Sections 5.2–5.4.

5.2. **The boundary of the family.** Let X be a smooth GM fivefold. To prove Theorem 5.3, we study the family of surfaces Z described in Lemma 5.2 over the boundary $\widetilde{Y}^{\ge 2}_A \smallsetminus \widetilde{S}_0$. We assume that X is general (in particular it is ordinary), so that the curve $Y^{\ge 2}_{A,V_5}$ is smooth. In particular, this means that Y^3_A is empty, hence $Y^{\ge 2}_A = Y^2_A$, and $W = \bigwedge^2 V_5$.

Consider the Hilbert scheme $F^2_\sigma(X)$ of σ-planes on X; we identify it with the smooth connected curve $\widetilde{Y}^{\ge 2}_{A,V_5}$ via the isomorphism (26). For $y \in \widetilde{Y}^{\ge 2}_{A,V_5}$, we denote by $\Pi_y \subset X$ the corresponding plane.

We denote by $y_0 \in \widetilde{Y}^{\ge 2}_{A,V_5}$ the point such that $\Pi_{y_0} = \Pi_0$ is the plane chosen in Section 5.1. Set $[v_0] := \pi_A(y_0)$ and denote by Π'_0 the plane corresponding to the other point in $\pi_A^{-1}([v_0])$. By Lemma 2.1(b), we have, for appropriate hyperplanes $V_4, V'_4 \subset V_5$,

$$\Pi_0 = \mathbf{P}(v_0 \wedge V_4) \quad \text{and} \quad \Pi'_0 = \mathbf{P}(v_0 \wedge V'_4). \tag{62}$$

We set

$$Y^{\ge 2}_{A,V_4} := Y^{\ge 2}_A \cap \mathbf{P}(V_4) \quad \text{and} \quad Y^{\ge 2}_{A,V'_4} := Y^{\ge 2}_A \cap \mathbf{P}(V'_4).$$

Both are hyperplane sections of the smooth curve Y^2_{A,V_5} hence, are finite (note that the induced double coverings of these sets parameterize σ-planes on X that intersect Π_0 and Π'_0, respectively).

Denote by $\widetilde{\mathscr{Q}}$ the pullback of the family of quadrics $\mathscr{Q}$ along the map $\widetilde{Y}^{\ge 2}_A \to Y^{\ge 2}_A$. A section $\widetilde{S}_0 \to F^5_{\Pi_0}(\widetilde{\mathscr{Q}}_0/\widetilde{S}_0)$ was constructed in Section 5.1. Since the surface $\widetilde{Y}^{\ge 2}_A$ is smooth and the Hilbert scheme $F^5_{\Pi_0}(\widetilde{\mathscr{Q}}/\widetilde{Y}^{\ge 2}_A)$ is proper over $\widetilde{Y}^{\ge 2}_A$, this section extends to an open subset

$$\widetilde{S}_{0+} \subset \pi_A^{-1}(Y^{\ge 2}_A \smallsetminus (Y^{\ge 2}_{A,V_4} \cup Y^{\ge 2}_{A,V'_4})) \subset \widetilde{Y}^{\ge 2}_A \tag{63}$$

which contains a general point of the curve $\widetilde{S}_{0,V_5} := \widetilde{S}_{0+} \smallsetminus \widetilde{S}_0 \subset \widetilde{Y}^{\ge 2}_{A,V_5}$. We denote by

$$\mathscr{P}^5_+ \subset \mathbf{P}(\textstyle\bigwedge^2 V_5) \times \widetilde{S}_{0+}$$

the corresponding family of 5-spaces and, for $y \in \widetilde{S}_{0+}$, by $\mathscr{P}^5_{+y} \subset \mathbf{P}(\bigwedge^2 V_5)$ the corresponding 5-space. By definition, we have $\Pi_0 \subset \mathscr{P}^5_{+y}$ for each $y \in \widetilde{S}_{0,V_5}$.

Lemma 5.4. *For each point $y \in \widetilde{S}_{0,V_5}$, we have*

$$\mathscr{P}^5_{+y} = \langle \Pi_y, \Pi_0 \rangle. \tag{64}$$

Proof. Set $[v] = \pi_A(y) \in Y^{\ge 2}_{A,V_5}$ and let $W_6 \subset \bigwedge^2 V_5$ be the 6-dimensional subspace corresponding to the 5-space $\mathscr{P}^5_y \subset \mathbf{P}(\bigwedge^2 V_5)$. By definition, we have

$$\mathbf{P}(W_6) \subset Q_v = \mathrm{Cone}_{\mathbf{P}(v\wedge V_5)}(\mathsf{Gr}(2, V_5/v)).$$

Since $\mathsf{Gr}(2, V_5/v)$ is a smooth 4-dimensional quadric, the maximal dimension of a linear subspace that it contains is 2, hence the subspace

$$W_y := W_6 \cap (v \wedge V_5)$$

is at least 3-dimensional. We claim that $\mathbf{P}(W_y) \subset X$.

Let $\{w_i\}$ be a basis of W_y, let q_v be an equation of Q_v, and consider a line $\langle [v], [v'] \rangle \subset \mathbf{P}(V_6)$ tangent to $Y^{\ge 2}_A$ at $[v]$, with $[v'] \in \mathbf{P}(V_6) \smallsetminus \mathbf{P}(V_5)$. Let $\mathrm{Spec}(\mathbf{C}[\epsilon]/\epsilon^2) \to Y^{\ge 2}_A$ be the corresponding morphism that takes the closed point to $[v]$. Since $\widetilde{S}_{0+}$ is étale over $Y^{\ge 2}_A$, the morphism can be lifted to a morphism

$$\mathrm{Spec}(\mathbf{C}[\epsilon]/\epsilon^2) \longrightarrow \widetilde{S}_{0+}$$

that takes the closed point to y. This implies that there are vectors w'_i in $\bigwedge^2 V_5$ such that the subspace in $\bigwedge^2 V_5$ generated by $w_i + \epsilon w'_i$ is isotropic for the quadratic form $q_v + \epsilon q_{v'}$, where $q_{v'}$ is an equation of $Q_{v'}$. We have therefore

$$0 = (q_v + \epsilon q_{v'})(w_i + \epsilon w'_i, w_j + \epsilon w'_j) = \epsilon(q_v(w_i, w'_j) + q_v(w_j, w'_i) + q_{v'}(w_i, w_j)).$$

Note that $q_v(w_i, w'_j) = q_v(w_j, w'_i) = 0$, since $w_i, w_j \in v \wedge V_5 = \mathrm{Ker}(q_v)$ for all i, j. It follows that $q_{v'}(w_i, w_j) = 0$ for all i, j hence, W_y is isotropic for $q_{v'}$, that is, $\mathbf{P}(W_y) \subset Q_{v'}$. Since $\mathbf{P}(W_y) \subset \mathbf{P}(v \wedge V_5) \subset \mathsf{Gr}(2, V_5)$, we conclude that

$$\mathbf{P}(W_y) \subset \mathsf{Gr}(2, V_5) \cap Q_{v'} = X,$$

thus proving the claim.

Since $\mathbf{P}(W_y) \subset \mathbf{P}(v \wedge V_5) \cap X$ and X contains no projective 3-spaces ([DK2, Theorem 4.2]), it follows that $\dim(W_y) = 3$ and $\mathbf{P}(W_y)$ is a σ-plane on X. Moreover, the induced map

$$\begin{array}{rcl} \widetilde{S}_{0,V_5} & \longrightarrow & F^2_\sigma(X) \\ y & \longmapsto & \mathbf{P}(W_y) \end{array}$$

is a $Y^{\geq 2}_{A,V_5}$-morphism. But $F^2_\sigma(X) \simeq \widetilde{Y}^{\geq 2}_{A,V_5}$, while $\widetilde{S}_{0,V_5}$ is an open subscheme in $\widetilde{Y}^{\geq 2}_{A,V_5}$, and $\widetilde{Y}^{\geq 2}_{A,V_5}$ is a connected étale covering of $Y^{\geq 2}_{A,V_5}$. Therefore, replacing if necessary the isomorphism (26) by its composition with the involution of the double covering, we may assume that the above map $\widetilde{S}_{0,V_5} \to F^2_\sigma(X)$ coincides with the embedding $\widetilde{S}_{0,V_5} \hookrightarrow \widetilde{Y}^{\geq 2}_{A,V_5}$, hence $\mathbf{P}(W_y) = \Pi_y$ for all $y \in \widetilde{S}_{0,V_5}$. This means that

$$\Pi_y \subset \mathscr{P}^5_{+y}$$

for all $y \in \widetilde{S}_{0,V_5}$. Since $\Pi_0 \subset \mathscr{P}^5_{+y}$ by definition, the right side of (64) is contained in the left side. Finally, since $\Pi_0 \cap \Pi_y = \varnothing$ by definition of $\widetilde{S}_{0+}$ (since planes intersecting Π_0 are parameterized by the double cover of the subscheme $Y^{\geq 2}_{A,V_4}$), the inclusion is an equality. □

The family $\mathscr{P}^5_+$ of projective 5-spaces over $\widetilde{S}_{0+}$ agrees by construction with the family $\mathscr{P}^5$ over $\widetilde{S}_0 \subset \widetilde{S}_{0+}$. We set

$$Z_{0+} := \mathscr{P}^5_+ \cap (M_X \times \widetilde{S}_{0+}). \tag{65}$$

Comparing this with (58), we obtain

$$Z_{0+} \times_{\widetilde{S}_{0+}} \widetilde{S}_0 = Z_0.$$

We denote by

$$Z_{0,V_5} := Z_{0,+} \times_{\widetilde{S}_{0+}} \widetilde{S}_{0,V_5} \subset M_X \times \widetilde{S}_{0,V_5}$$

the restriction of the family (65) to the curve $\widetilde{S}_{0,V_5} = \widetilde{S}_{0+} \smallsetminus \widetilde{S}_0$.

Proposition 5.5. *Let X be a general GM fivefold. The map $Z_{0,V_5} \to \widetilde{S}_{0,V_5}$ is a flat family of cubic scrolls $\mathbf{P}^1 \times \mathbf{P}^2$.*

Proof. Let $y \in \widetilde{S}_{0,V_5}$ and set again $[v] := \pi(y) \in \mathbf{P}(V_5)$. By (64), the 5-space $\mathscr{P}^5_{+y}$ is the linear span of the planes Π_0 and Π_y, that is, a hyperplane in $\mathbf{P}(V_2 \wedge V_5)$, where $V_2 \subset V_5$ is the subspace spanned by v_0 and v, hence $\mathsf{Gr}(2, V_5) \cap \mathscr{P}^5_{+y}$ is a hyperplane section of the cone $\mathrm{Cone}_{\mathbf{P}(\wedge^2 V_5)}(\mathbf{P}(V_2) \times \mathbf{P}(V_5/V_2))$ (see Lemma 2.2).

The vertex $[\bigwedge^2 V_2] = [v_0 \wedge v]$ of the cone does not belong to $\mathscr{P}^5_{+y}$: if it did, Π_0 and Π_y would intersect at the point $[v_0 \wedge v]$ and this would contradict the definition of S_{0+}. Therefore, the fiber of Z_{0,V_5} over y is isomorphic to the 3-dimensional cubic scroll $\mathbf{P}(V_2) \times \mathbf{P}(V_5/V_2)$. $\square$

Propositions 5.1 and 5.5 show that the components of Z_0 and Z_{0,V_5} all have dimension 4. They are components of the scheme Z_{0+}. We consider the Hilbert closure

$$Z_F \subset M_X \times F^2_\sigma(X) \tag{66}$$

of Z_{0,V_5} in $M_X \times F^2_\sigma(X)$, constructed as in Definition 4.2.

5.3. **A relation between the subschemes.** Let X be a general GM fivefold. In (60) and (66), we constructed subschemes $Z \subset X \times \widetilde{Y}^{\geq 2}_A$ and $Z_F \subset M_X \times F^2_\sigma(X)$. The proof of Theorem 5.3 is based on a relation between the schemes $Z \cap (X \times \widetilde{S}_{0,V_5})$ and $Z_F \cap (X \times \widetilde{S}_{0,V_5})$, where the curve $\widetilde{S}_{0,V_5}$ is considered as a subscheme of both the surface $\widetilde{Y}^{\geq 2}_A$ and the curve $F^2_\sigma(X)$.

Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & & X \times \widetilde{S}_{0,V_5} & & & \\
 & & \overset{\text{open}}{\swarrow} & & \overset{\text{loc. closed}}{\searrow} & & \\
\mathscr{L}^2_\sigma(X) & \hookrightarrow & X \times F^2_\sigma(X) & \xrightarrow{\quad\tilde\sigma\quad} & X \times \widetilde{Y}^{\geq 2}_A & \hookleftarrow & Z \\
 & & \downarrow{\scriptstyle i} & \downarrow{\scriptstyle i} & \downarrow{\scriptstyle i} & & \\
 & & & M_X \times \widetilde{S}_{0,V_5} & & & \\
 & & \overset{\text{open}}{\swarrow} & & \overset{\text{loc. closed}}{\searrow} & & \\
Z_F & \hookrightarrow & M_X \times F^2_\sigma(X) & \xrightarrow{\quad\tilde\sigma\quad} & M_X \times \widetilde{Y}^{\geq 2}_A, & &
\end{array} \tag{67}$$

where $\mathscr{L}^2_\sigma(X) \subset X \times F^2_\sigma(X)$ is the universal family of σ-planes and $i\colon X \hookrightarrow M_X$ is the embedding.

Proposition 5.6. *We have an equality*

$$Z_F \cap (X \times \widetilde{S}_{0,V_5}) = (\tilde\sigma^{-1}(Z) \cap (X \times \widetilde{S}_{0,V_5})) + (\mathscr{L}^2_\sigma(X) \cap (X \times \widetilde{S}_{0,V_5})) \tag{68}$$

of cycles.

Proof. The left side of (68) can be rewritten as

$$Z_F \cap (X \times \widetilde{S}_{0,V_5}) = Z_{0,V_5} \cap (X \times \widetilde{S}_{0,V_5}) = Z_{0,V_5} \times_{M_X} X.$$

By Proposition 5.5, the morphism $Z_{0,V_5} \to \widetilde{S}_{0,V_5}$ is a flat family of smooth 3-dimensional cubic scrolls. Since X contains no such threefolds ([DK2, Corollary 3.5]), it contains no fibers of Z_{0,V_5}. Therefore, the morphism

$$Z_{0,V_5} \times_{M_X} X \longrightarrow \widetilde{S}_{0,V_5}$$

is a flat family of surfaces whose fiber over $y \in \widetilde{S}_{0,V_5}$ is the intersection

$$S_y := Z_{0,V_5,y} \cap Q_0$$

of a smooth 3-dimensional cubic scroll $Z_{0,V_5,y} \simeq \mathbf{P}^2 \times \mathbf{P}^1$ with any non-Plücker quadric Q_0 containing X. Such an intersection is a surface of class $2f_2 + 2f_1$ in $Z_{0,V_5,y}$, where f_i is the preimage of the hyperplane class on $\mathbf{P}^i$ under the projection $Z_{0,V_5,y} \simeq \mathbf{P}^2 \times \mathbf{P}^1 \to \mathbf{P}^i$.

By (64) and (46), the planes Π_0 and Π_y are contained both in the scroll $Z_{0,V_5,y}$ and the quadric Q_0, hence they are components of S_y, each of class f_1. Therefore,

$$S_y = \Pi_0 + \Pi_y + S'_y,$$

where $S'_y \subset Z_{0,V_5,y}$ is a surface of class $2f_2$, that is, the product of a conic in $\mathbf{P}^2$ with $\mathbf{P}^1$. In particular, it has degree 4 and contains no planes. Since $Z_y \subset Z_{0,V_5,y}$ is a surface of degree 5 that contains Π_0, we have $Z_y = S'_y + \Pi_0$ for all $y \in \widetilde{S}_{0,V_5}$; this proves (68). □

5.4. **Abel–Jacobi maps.** Let X be a smooth GM fivefold with associated Lagrangian A. Assume $Y^3_A = \varnothing$, so that $\widetilde{Y}^{\geq 2}_A$ is a smooth surface and the curve $\widetilde{Y}^{\geq 2}_{A,V_5}$, hence also the Hilbert scheme $F^2_\sigma(X)$ of σ-planes in X, is a smooth curve. Consider the Abel–Jacobi maps

$$\mathsf{AJ}_Z \colon H_1(\widetilde{Y}^{\geq 2}_A, \mathbf{Z}) \to H_5(X, \mathbf{Z}) \quad \text{and} \quad \mathsf{AJ}_{\mathscr{L}^2_\sigma(X)} \colon H_1(F^2_\sigma(X), \mathbf{Z}) \to H_5(X, \mathbf{Z})$$

and the isomorphism $\tilde{\sigma} \colon F^2_\sigma(X) \xrightarrow{\sim} \widetilde{Y}^{\geq 2}_A$ from (26).

Proposition 5.7. *The composition of maps*

$$H_1(F^2_\sigma(X), \mathbf{Z}) \xrightarrow{\tilde{\sigma}_*} H_1(\widetilde{Y}^{\geq 2}_A, \mathbf{Z}) \xrightarrow{\mathsf{AJ}_Z} H_5(X, \mathbf{Z})$$

is equal to the map $-\mathsf{AJ}_{\mathscr{L}^2_\sigma(X)}$.

Proof. Analogous to the proof of Proposition 4.15. □

The above proposition connects the Abel–Jacobi maps AJ_Z and $\mathsf{AJ}_{\mathscr{L}^2_\sigma(X)}$. The next lemma uses the Clemens–Tyurin argument (Section 3.3) to show that the latter is surjective.

Lemma 5.8. *Let X be a general GM fivefold. The Abel–Jacobi map*

$$\mathsf{AJ}_{\mathscr{L}^2_\sigma(X)} \colon H_1(F^2_\sigma(X), \mathbf{Z}) \longrightarrow H_5(X, \mathbf{Z})$$

is surjective.

Proof. Let Y be a general GM sixfold and let $X \subset Y$ be a general hyperplane section, so that X is a general GM fivefold. Take $F_Y = F^2_\sigma(Y)$, the Hilbert scheme of σ-planes contained in Y. We check that the assumptions of Proposition 3.3 (with $m = 2$) are satisfied.

Assumption (a) holds since $F^2_\sigma(Y)$ is a smooth irreducible fourfold by [DK2, Corollary 5.13]. Furthermore, the map $q \colon \mathscr{L}^2_\sigma(Y) \to Y$ is generically finite of degree 12 ([DK2, Lemma 5.15]), hence (b) holds as well. Next, $F^2_\sigma(X)$ is a smooth curve by Lemma 2.10, hence (c) holds. Finally,

$$H_5(Y, \mathbf{Z}) = H_7(Y, \mathbf{Z}) = 0$$

by [DK2, Proposition 3.1], hence (d) holds.

Applying Proposition 3.3, we deduce the surjectivity of $\mathsf{AJ}_{\mathscr{L}^2_\sigma(X)}$. □

Combining the above results, we can now prove Theorem 5.3.

Proof of Theorem 5.3. Assume first that the GM fivefold X is general. A combination of Proposition 5.7 and Lemma 5.8 proves that the map AJ_Z is surjective. By Propositions 2.5 and 2.6, its source and target are free abelian groups of rank 20. Therefore, the Abel–Jacobi map is an isomorphism.

Since the Abel–Jacobi map is defined by the cohomology class of an algebraic cycle, it preserves the Hodge structures hence, induces an isomorphism of the corresponding abelian varieties: the Albanese variety of $\widetilde{Y}^{\geq 2}_{A(X)}$ and the intermediate Jacobian of X.

Since the scheme $Z \subset X \times \widetilde{Y}^{\geq 2}_{A}$ was defined in Section 5.1 for all X and since this definition works in families, these two statements follow by continuity for any X such that $Y^3_{A(X)} = \varnothing$.

It remains to prove that the isomorphism (61) respects the principal polarizations. For X very general, the Picard number of $\mathrm{Jac}(X)$ is 1 by Corollary 3.6, hence any two principal polarizations on $\mathrm{Jac}(X)$ coincide. This proves the claim for very general X, then for any smooth X such that $\widetilde{Y}^{\geq 2}_{A(X)}$ is also smooth by continuity. □

5.5. **Simplicity argument.** In this section we give an alternative argument for the isomorphism $\mathrm{Jac}(X) \simeq \mathrm{Alb}(\widetilde{Y}^{\geq 2}_{A(X),V_5(X)})$ for a smooth GM fivefold X, based on a simplicity result of independent interest, analogous to the one proved in Proposition 3.4.

Let S be a smooth connected projective surface and let $\jmath\colon C \hookrightarrow S$ be a smooth (irreducible) ample curve. By the Lefschetz theorem, the morphism $\jmath_*\colon H_1(C, \mathbf{Z}) \to H_1(S, \mathbf{Z})$ is surjective hence, the induced morphsim

$$\mathrm{Jac}(C) \simeq \mathrm{Alb}(C) \longrightarrow \mathrm{Alb}(S)$$

is surjective with connected kernel. We denote this kernel by $K(C, S)$.

Consider now a connected double étale cover $\pi\colon \widetilde{S} \to S$, and set $\widetilde{C} := \pi^{-1}(C)$, a smooth ample curve on $\widetilde{S}$.

Lemma 5.9. *There is a commutative diagram*

$$\begin{array}{ccccccccc}
 & & K(\widetilde{C}, C) & \to & P(\widetilde{C}, C) & \to & P(\widetilde{S}, S) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K(\widetilde{C}, \widetilde{S}) & \longrightarrow & \mathrm{Jac}(\widetilde{C}) & \longrightarrow & \mathrm{Alb}(\widetilde{S}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K(C, S) & \longrightarrow & \mathrm{Jac}(C) & \longrightarrow & \mathrm{Alb}(S) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array} \tag{69}$$

where $K(\widetilde{C}, C)$, $P(\widetilde{C}, C)$ (the Prym variety of the double cover $\widetilde{C} \to C$), and $P(\widetilde{S}, S)$ are the neutral components of the respective kernels of the vertical maps induced by π.

Proof. The surjectivity of the maps $\mathrm{Jac}(\widetilde{C}) \to \mathrm{Jac}(C)$ and $\mathrm{Alb}(\widetilde{S}) \to \mathrm{Alb}(S)$ is obvious. The only thing we have to prove is the surjectivity of the map $K(\widetilde{C}, \widetilde{S}) \to K(C, S)$ or, equivalently, the surjectivity of the map $P(\widetilde{C}, C) \to P(\widetilde{S}, S)$. On the level of cotangent spaces, the surjectivity of this second map corresponds to the injectivity of the restriction $H^1(S, \eta) \to H^1(C, \eta|_C)$, where η is the line bundle of order 2 on S corresponding to the double étale covering π. Its kernel is controlled by $H^1(S, \eta(-C))$, which vanishes by Kodaira vanishing and Serre duality because $\eta(C)$ is ample on S. This proves the injectivity of the morphism $H^1(S, \eta) \to H^1(C, \eta|_C)$ hence, the lemma. □

The next statement is the main result of this section. The notation was defined in Section 3.4.

Theorem 5.10. *Let $S \subset \mathbf{P}^N$ be a smooth connected projective surface and let $\pi \colon \widetilde{S} \to S$ be a connected double étale cover. Let $H \subset \mathbf{P}^N$ be a very general hyperplane and set $C := S \cap H$. With the notation above, the endomorphism ring of the abelian variety $K(\widetilde{C}, C)$ is trivial.*

Proof. Choose a Lefschetz pencil $f \colon S \dashrightarrow \mathbf{P}^1$ of hyperplane sections of S. The connected double étale cover $\pi \colon \widetilde{S} \to S$ induces for each $t \in \mathbf{P}^1$ a connected double étale cover $\pi_t \colon \widetilde{C}_t \to C_t$ between fibers. Denote by $\jmath_t \colon C_t \hookrightarrow S$ and $\tilde{\jmath}_t \colon \widetilde{C}_t \hookrightarrow \widetilde{S}$ the embeddings.

For $t \in \mathbf{P}^1 \smallsetminus \{t_1, \dots, t_r\}$, the involution τ of $\widetilde{S}$ attached to π acts on each summand of the direct sum decomposition

$$H^1(\widetilde{C}_t, \mathbf{Q}) = H^1(\widetilde{C}_t, \mathbf{Q})_{\mathrm{van}} \oplus \tilde{\jmath}_t^* H^1(\widetilde{S}, \mathbf{Q})$$

from (33) and the τ-invariant subspaces are

$$H^1(C_t, \mathbf{Q}) = H^1(C_t, \mathbf{Q})_{\mathrm{van}} \oplus \jmath_t^* H^1(S, \mathbf{Q}).$$

The abelian variety $K(C_t, S)$ is obtained from the Hodge structure of $H^1(C_t, \mathbf{Q})_{\mathrm{van}}$, hence its endomorphism ring is trivial by Proposition 3.4. Therefore, to study the neutral component $K(\widetilde{C}_t, C_t)$ of the kernel of the surjection

$$K(\widetilde{C}_t, \widetilde{S}) \longrightarrow K(C_t, S),$$

we need to study the rational Hodge structure on the τ-antiinvariant subspace $H^1(\widetilde{C}_t, \mathbf{Q})^-_{\mathrm{van}}$.

For each $i \in \{1, \dots, r\}$, the curve $\widetilde{C}_{t_i}$ has two nodes over the node of C_{t_i} hence, there are two disjoint vanishing cycles δ'_i and $\delta''_i = \tau^*(\delta'_i)$. Since the vanishing cycles span the vector space $H^1(\widetilde{C}_t, \mathbf{Q})_{\mathrm{van}}$, the cycles $\delta'_1 - \delta''_1, \dots, \delta'_r - \delta''_r$ span the antiinvariant subspace $H^1(\widetilde{C}_t, \mathbf{Q})^-_{\mathrm{van}}$. The image of the monodromy representation

$$\tilde{\rho} \colon \pi_1(\mathbf{P}^1 \smallsetminus \{t_1, \dots, t_r\}, t) \longrightarrow \mathrm{Sp}(H^1(\widetilde{C}_t, \mathbf{Q}))$$

consists of automorphisms that are τ-equivariant and, reasoning as in the proof of [V2, Proposition 3.23], we see that, up to changing signs, the classes $\delta'_1 - \delta''_1, \dots, \delta'_r - \delta''_r$ are all in the same monodromy orbit. Moreover, as in the proof of Proposition 3.4, there is for each $i \in \{1, \dots, r\}$ an element of $\pi_1(\mathbf{P}^1 \smallsetminus \{t_1, \dots, t_r\}, 0)$ that acts on $H^1(\widetilde{C}_t, \mathbf{Q})$ by

$$T_i := T_{\delta''_i} \circ T_{\delta'_i} \colon x \longmapsto x - (x \cdot \delta'_i)\delta'_i - (x \cdot \delta''_i)\delta''_i.$$

If x is τ-antiinvariant, we have

$$(x \cdot \delta''_i) = (x \cdot \tau^*(\delta'_i)) = (\tau^*(x) \cdot \delta'_i) = -(x \cdot \delta'_i),$$

hence

$$T_i(x) = x - (x \cdot \delta'_i)(\delta'_i - \delta''_i) = x - \frac{1}{2}(x \cdot (\delta'_i - \delta''_i))(\delta'_i - \delta''_i).$$

One then deduces from that and [PS1, Lemma 4] that the monodromy action on $H^1(\widetilde{C}_t, \mathbf{Q})^-_{\mathrm{van}}$ is big; it follows that the Zariski closure of the monodromy group for $K(\widetilde{C}_t, C_t)$ is the full symplectic group. As in the proof of [PS1, Theorem 17], for $t \in \mathbf{P}^1$ very general, any endomorphism of $K(\widetilde{C}_t, C_t)$ intertwines every element of the monodromy group, hence every element of the symplectic group. It must therefore be a multiple of the identity.

The endomorphism ring of the abelian variety $K(\widetilde{C}_t, C_t)$ is therefore trivial. ∎

We now apply the theorem to GM fivefolds. Let X be a general GM fivefold with Lagrangian data set (V_6, V_5, A). Our starting point is again the surjectivity, proved in Lemma 5.8, of the Abel–Jacobi map

$$\mathsf{AJ}_{\mathscr{L}^2_\sigma(X)} \colon H_1(F^2_\sigma(X), \mathbf{Z}) \longrightarrow H_5(X, \mathbf{Z})$$

associated with the Hilbert scheme $F^2_\sigma(X)$ that parametrizes σ-planes contained in X. This Hilbert scheme is isomorphic to the smooth curve $\widetilde{Y}^{\geq 2}_{A,V_5}$ (Lemma 2.10) defined as the inverse image by the double cover

$$\pi_A \colon \widetilde{Y}^{\geq 2}_A \longrightarrow Y^{\geq 2}_A$$

of the hyperplane section $Y^{\geq 2}_{A,V_5} = Y^{\geq 2}_A \cap \mathbf{P}(V_5)$. The surjectivity of the map $\mathsf{AJ}_{\mathscr{L}^2_\sigma(X)}$ is therefore equivalent to the connectedness of the kernel of the induced surjective morphism

$$\Phi\colon \operatorname{Jac}(\widetilde{Y}^{\geq 2}_{A,V_5}) \longrightarrow \operatorname{Jac}(X) \tag{70}$$

between Jacobians. By Lemma 2.10 and Proposition 2.6, the dimension of this kernel is

$$g(\widetilde{Y}^{\geq 2}_{A,V_5}) - \dim(\operatorname{Jac}(X)) = 161 - 10 = 151.$$

Corollary 5.11. *Let X be a smooth GM fivefold with Lagrangian data set (V_6, V_5, A). Assume that the surface $\widetilde{Y}^{\geq 2}_A$ and the curve $\widetilde{Y}^{\geq 2}_{A,V_5}$ are smooth. The morphism Φ from (70) then factors as*

$$\Phi\colon \operatorname{Jac}(\widetilde{Y}^{\geq 2}_{A,V_5}) \twoheadrightarrow \operatorname{Alb}(\widetilde{Y}^{\geq 2}_A) \xrightarrow{\ \sim\ } \operatorname{Jac}(X),$$

where the first arrow is the Albanese map $\operatorname{Jac}(\widetilde{Y}^{\geq 2}_{A,V_5}) = \operatorname{Alb}(\widetilde{Y}^{\geq 2}_{A,V_5}) \to \operatorname{Alb}(\widetilde{Y}^{\geq 2}_A)$.

Proof. Since $\operatorname{Alb}(Y^{\geq 2}_A) = 0$ (Proposition 2.5), the diagram (69) reads

$$\begin{array}{ccccccccc}
 & & K_{V_5} & \longrightarrow & P_{A,V_5} & \longrightarrow & \operatorname{Alb}(\widetilde{Y}^{\geq 2}_A) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \widetilde{K}_{V_5} & \longrightarrow & \operatorname{Jac}(\widetilde{Y}^{\geq 2}_{A,V_5}) & \to & \operatorname{Alb}(\widetilde{Y}^{\geq 2}_A) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Jac}(Y^{\geq 2}_{A,V_5}) & = & \operatorname{Jac}(Y^{\geq 2}_{A,V_5}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0, & & & &
\end{array} \tag{71}$$

where $K_{V_5} := K(\widetilde{Y}^{\geq 2}_{A,V_5}, Y^{\geq 2}_{A,V_5})$, $\widetilde{K}_{V_5} := K(\widetilde{Y}^{\geq 2}_{A,V_5}, \widetilde{Y}^{\geq 2}_A)$, and P_{A,V_5} is the Prym variety of the double covering $\widetilde{Y}^{\geq 2}_{A,V_5} \to Y^{\geq 2}_{A,V_5}$. The genus of the curve $Y^{\geq 2}_{A,V_5}$ is 81 by (27), hence the dimension of the variety P_{A,V_5} is 80. Also, the dimension of $\operatorname{Alb}(\widetilde{Y}^{\geq 2}_A)$ is 10 by Proposition 2.5. Therefore, $\dim(K_{V_5}) = 70$.

When V_5 is a very general hyperplane in V_6, the abelian varieties $\operatorname{Jac}(Y^{\geq 2}_{A,V_5})$ and K_{V_5} are simple by Proposition 3.4 and Theorem 5.10, hence they are the only two simple factors of the abelian variety $\widetilde{K}_{V_5}$. Since $\operatorname{Jac}(X)$ has dimension 10, the abelian variety $\widetilde{K}_{V_5}$ must therefore be contained in the kernel of Φ.

In other words, the composition $\widetilde{K}_{V_5} \hookrightarrow \operatorname{Jac}(\widetilde{Y}^{\geq 2}_{A,V_5}) \stackrel{\Phi}{\twoheadrightarrow} \operatorname{Jac}(X)$ vanishes for V_5 very general. By continuity, it vanishes for all hyperplanes V_5 such that $\widetilde{Y}^{\geq 2}_{A,V_5}$ is smooth. The kernel of Φ, being connected of dimension 151, must then be equal to $\widetilde{K}_{V_5}$, which implies the corollary. $\square$

Note that this argument cannot be applied to GM threefolds, because the corresponding hyperplane sections $Y^{\geq 2}_{A,V_5}$ are very far from being general.

6. Period maps

In this section, we prove Theorems 1.1 and 1.3. We will use the description (15) of the coarse moduli space $\mathbf{M}^{\mathrm{GM}}_n$ of smooth GM varieties of dimension n. Let

$$\mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}} = \mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6) /\!\!/ \mathrm{PGL}(V_6). \tag{72}$$

be the coarse quasiprojective moduli space of EPW sextics defined by Lagrangian subspaces A with no decomposable vectors (see (9) for the definition). This is an open subset of the coarse moduli space $\mathbf{M}^{\mathrm{EPW}}$ of EPW sextics defined in (7). We denote by $\mathbf{r}$ the involution of $\mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}}$ defined by $\mathbf{r}([A]) = ([A^\perp])$.

Lemma 6.1. *There exists a regular morphism* $\bar{\wp}\colon \mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}}/\mathbf{r} \to \mathbf{A}_{10}$ *such that for* $n \in \{3,5\}$*, the composition*

$$\mathbf{M}^{\mathrm{GM}}_n \xrightarrow{\mathfrak{p}_n} \mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}} \longrightarrow \mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}}/\mathbf{r} \xrightarrow{\bar{\wp}} \mathbf{A}_{10} \tag{73}$$

is equal to the period map $\wp_n\colon \mathbf{M}^{\mathrm{GM}}_n \to \mathbf{A}_{10}$.

Proof. Set

$$\mathscr{Y}^{5-n}_{\mathscr{A}^\perp} := \{(A, V_5) \in \mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6) \times \mathbf{P}(V_6^\vee) \mid \dim(A \cap \bigwedge^3 V_5) = 5 - n\},$$

so that $\mathscr{Y}^{5-n}_{\mathscr{A}^\perp} /\!\!/ \mathrm{PGL}(V_6)$ is an open subscheme of $\mathbf{M}^{\mathrm{GM}}_n$. The projection

$$\mathrm{pr}\colon \mathscr{Y}^{5-n}_{\mathscr{A}^\perp} \to \mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6)$$

is a smooth surjective morphism. We show that the map

$$\begin{aligned} \tilde{\wp}_n\colon \mathscr{Y}^{5-n}_{\mathscr{A}^\perp} &\longrightarrow \mathbf{A}_{10} \\ (A, V_5) &\longmapsto [\mathrm{Jac}(X_{A,V_5})]. \end{aligned}$$

factors through pr. By smooth descent, it is enough for this to show that the two maps

$$\mathscr{Y}^{5-n}_{\mathscr{A}^\perp} \times_{\mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6)} \mathscr{Y}^{5-n}_{\mathscr{A}^\perp} \rightrightarrows \mathbf{A}_{10}$$

defined as compositions of the projections to the factors with the map $\tilde{\wp}_n$ are equal. Since the fiber product is a smooth variety, it is enough to verify the equality pointwise on an open subset. For $n = 3$, it holds by Theorem 4.4 over the open subset of $\mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6)$ corresponding to Lagrangians A such that $Y^3_A = \varnothing$. Therefore, for $n = 3$, the map $\tilde{\wp}_n$ factors as the composition

$$\mathscr{Y}^{5-n}_{\mathscr{A}^\perp} \xrightarrow{\mathrm{pr}} \mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6) \longrightarrow \mathbf{A}_{10}.$$

For $n = 5$, the above composition is equal to the map $\tilde{\wp}_5$ over the same open subset of $\mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6)$ by Theorem 5.3. This proves the factorization in both cases.

The maps $\tilde{\wp}_n$ are $\mathrm{PGL}(V_6)$-invariant, hence so is the map

$$\mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6) \longrightarrow \mathbf{A}_{10}$$

constructed via factorization. Therefore, it factors through a regular map

$$\mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}} = \mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6) /\!\!/ \mathrm{PGL}(V_6) \longrightarrow \mathbf{A}_{10}.$$

Similarly, this map is $\mathbf{r}$-invariant by Theorem 4.4, hence we obtain a regular morphism

$$\bar{\wp}\colon \mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}}/\mathbf{r} \longrightarrow \mathbf{A}_{10}.$$

The composition (73) agrees with the period map by construction. ☐

Recall that we denoted by $\mathrm{Alb}(\widetilde{Y}_A^{\geq 2})$ the Albanese variety of (any desingularization of) the double EPW surface $\widetilde{Y}_A^{\geq 2}$.

Proposition 6.2. *For any Lagrangian $[A] \in \mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6)$, we have $\bar{\wp}([A]) = [\mathrm{Alb}(\widetilde{Y}_A^{\geq 2})]$.*

Proof. If $Y_A^3 = \varnothing$, the equality holds by Theorems 4.4 and 5.3, and Lemma 6.1. Assume $Y_A^3 \neq \varnothing$.

Consider a neighborhood $U \subset \mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6)$ of $[A]$ such that the determinant of the tautological bundle $\mathscr{A}$ is trivial over U. Consider a universal family $\mathscr{Y}_{\mathscr{A}}^{\geq 2} \to U$ of EPW surfaces over it and the composition

$$p\colon \widetilde{\mathscr{Y}}_{\mathscr{A}}^{\geq 2} \xrightarrow{\ \pi\ } \mathscr{Y}_{\mathscr{A}}^{\geq 2} \longrightarrow \mathsf{LGr}'_{\mathrm{ndv}}(\textstyle\bigwedge^3 V_6),$$

where π is the double covering constructed as in the proof of [DK3, Theorem 5.2(2)]. The argument of [DK3, Theorem 5.2(2)] proves that $\widetilde{\mathscr{Y}}_{\mathscr{A}}^{\geq 2}$ is a smooth variety.

Let $C \subset U$ be a general smooth affine curve passing through the point $[A]$, so that the base change

$$\widetilde{\mathscr{Y}}_{C}^{\geq 2} := \widetilde{\mathscr{Y}}_{\mathscr{A}}^{\geq 2} \times_U C$$

is a smooth threefold (upon shrinking C, it is enough for this that the tangent vector to C be away from the finite union of hyperplanes in the tangent space to U at $[A]$) and the morphism $p_C\colon \widetilde{\mathscr{Y}}_{C}^{\geq 2} \to C$ is smooth over the complement of the central point $[A] \in C$. The central fiber is the surface $\widetilde{Y}_A^{\geq 2}$, which is smooth away from a finite number of ordinary double points ([DK3, Theorem 5.2(2)]).

Consider a double covering $\widetilde{C} \to C$ branched at $[A]$. The base change

$$\widetilde{\mathscr{Y}}_{\widetilde{C}}^{\geq 2} := \widetilde{\mathscr{Y}}_{C}^{\geq 2} \times_C \widetilde{C}$$

is a threefold with finitely many ordinary double points in the central fiber. By [At], there is an analytic simultaneous resolution $\widetilde{\mathscr{Y}}_{\widetilde{C}}^{\geq 2\prime} \to \widetilde{\mathscr{Y}}_{\widetilde{C}}^{\geq 2}$ such that the composition

$$p'\colon (\widetilde{\mathscr{Y}}_{\widetilde{C}}^{\geq 2})' \longrightarrow \widetilde{\mathscr{Y}}_{\widetilde{C}}^{\geq 2} \longrightarrow \widetilde{C}$$

is smooth with central fiber isomorphic to the (smooth) blow up $\widetilde{Y}_A^{\geq 2\prime} \to \widetilde{Y}_A^{\geq 2}$ of the singular points of $\widetilde{Y}_A^{\geq 2}$. The sheaf $R^1 p'_* \underline{\mathbf{Z}}$ is locally constant and its stalk at $[A]$ is isomorphic to $H^1(\widetilde{Y}_A^{\geq 2\prime}, \mathbf{Z})$.

By Theorems 4.4 and 5.3, away from the point $[A]$, this sheaf carries a variation of Hodge structure that comes from the middle cohomology of a family of smooth projective varieties of odd dimension, hence it has a canonical principal polarization. Since the sheaf $R^1 p_* \underline{\mathbf{Z}}$ is locally constant on the whole $\widetilde{C}$, this polarization extends across this point. In particular, the natural Hodge structure on the stalk $H^1(\widetilde{Y}_A^{\geq 2\prime}, \mathbf{Z})$ at $[A]$ has a principal polarization, hence provides a principal polarization on the Albanese variety $\mathrm{Alb}(\widetilde{Y}_A^{\geq 2\prime})$ and the map $\widetilde{C} \to \mathbf{A}_{10}$ defined by the above variation takes the point $[A]$ to $[\mathrm{Alb}(\widetilde{Y}_A^{\geq 2\prime})]$. Since this map agrees on $\widetilde{C} \smallsetminus \{[A]\}$ with the composition

$$\widetilde{C} \longrightarrow C \longrightarrow U \longrightarrow \mathbf{M}_{\mathrm{ndv}}^{\mathrm{EPW}} \longrightarrow \mathbf{M}_{\mathrm{ndv}}^{\mathrm{EPW}}/\mathbf{r} \xrightarrow{\ \bar{\wp}\ } \mathbf{A}_{10},$$

it agrees everywhere, hence $\bar{\wp}([A]) = [\mathrm{Alb}(\widetilde{Y}_A^{\geq 2\prime})]$. ☐

We now use these results to prove Theorem 1.3.

Proof of Theorem 1.3. The factorization of the period map is proved in Lemma 6.1 and the equality $\wp_n([X]) = [\mathrm{Alb}(\widetilde{Y}^{\geq 2}_{A(X)})]$ follows from this factorization and Proposition 6.2. □

Remark 6.3. Consider the natural action of $\mathrm{PGL}(V_6)$ on $\mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6) \times \mathbf{P}(V_6^\vee)$, linearized as in [DK4, Section 5.4]. For each $n \in \{3,4,5,6\}$, there is by [DK4, Theorem 5.15] a canonical embedding

$$\mathbf{M}_n^{\mathrm{GM}} \subset (\mathsf{LGr}_{\mathrm{ndv}}(\textstyle\bigwedge^3 V_6) \times \mathbf{P}(V_6^\vee)) /\!\!/ \mathrm{PGL}(V_6) \tag{74}$$

and $(\mathsf{LGr}_{\mathrm{ndv}}(\bigwedge^3 V_6) \times \mathbf{P}(V_6^\vee)) /\!\!/ \mathrm{PGL}(V_6) \to \mathbf{M}^{\mathrm{EPW}}_{\mathrm{ndv}}$ is generically a $\mathbf{P}^5$-fibration (the fiber over any point $[A]$ is isomorphic to $\mathbf{P}(V_6^\vee) /\!\!/ \mathrm{PGL}(V_6)_A$). The inclusion (74) is an open embedding for $n = 5$, a closed embedding for $n = 3$, and

$$(\mathsf{LGr}_{\mathrm{ndv}}(\textstyle\bigwedge^3 V_6) \times \mathbf{P}(V_6^\vee)) /\!\!/ \mathrm{PGL}(V_6) = \mathbf{M}_5^{\mathrm{GM}} \sqcup \mathbf{M}_3^{\mathrm{GM}}$$

by (17). This property is reminiscent of the Satake compactification.

We can also complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The map $\bar{\wp}$ defines a principal polarization on $\mathrm{Alb}(\widetilde{Y}^{\geq 2}_A)$ for each Lagrangian A with no decomposable vectors; this proves the first part of the theorem.

Furthermore, the isomorphism (3) for A with $Y^3_A = \varnothing$ was established in Theorems 4.4 and 5.3. For A with $Y^3_A \neq \varnothing$, the proof of Proposition 6.2 gives an isomorphism

$$H_n(X, \mathbf{Z}) \simeq H_1((\widetilde{Y}^{\geq 2}_{A(X)})', \mathbf{Z}).$$

Since the only singularities of $\widetilde{Y}^{\geq 2}_A$ are ordinary double points, there is a canonical isomorphism $H_1((\widetilde{Y}^{\geq 2}_A)', \mathbf{Z}) \xrightarrow{\sim} H_1(\widetilde{Y}^{\geq 2}_A, \mathbf{Z})$. This proves the second part of the theorem.

Finally, Theorem 1.3 implies that for $n \in \{3,5\}$, we have an isomorphism

$$\mathrm{Jac}(X) \simeq \mathrm{Alb}(\widetilde{Y}^{\geq 2}_A)$$

of principally polarized abelian varieties for all smooth GM varieties X of dimension n; this proves the last part of the theorem. □

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