GUSHEL–MUKAI VARIETIES: CLASSIFICATION AND BIRATIONALITIES

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ABSTRACT. We perform a systematic study of Gushel–Mukai varieties—quadratic sections of linear sections of cones over the Grassmannian $\mathsf{Gr}(2,5)$. This class of varieties includes Clifford general curves of genus 6, Brill–Noether general polarized K3 surfaces of genus 6, prime Fano threefolds of genus 6, and their higher-dimensional analogues.

We establish an intrinsic characterization of normal Gushel–Mukai varieties in terms of their excess conormal sheaves, which leads to a new proof of the classification theorem of Gushel and Mukai. We give a description of isomorphism classes of Gushel–Mukai varieties and their automorphism groups in terms of linear algebraic data naturally associated to these varieties.

We carefully develop the relation between Gushel–Mukai varieties and Eisenbud–Popescu–Walter sextics introduced earlier by Iliev–Manivel and O'Grady. We describe explicitly all Gushel–Mukai varieties whose associated EPW sextics are isomorphic or dual (we call them period partners or dual varieties respectively). Finally, we show that in dimension 3 and higher, period partners/dual varieties are always birationally isomorphic.

1. INTRODUCTION

This paper is devoted to the investigation of the geometry of a class of varieties which we call Gushel–Mukai varieties, or GM varieties for short, which are *dimensionally transverse intersections of a cone over the Grassmannian* $\mathsf{Gr}(2,5)$ *with a linear space and a quadratic hypersurface,* all defined over a field $\mathbf{k}$ of characteristic zero.

Smooth GM varieties are isomorphic either to quadratic sections of a linear section of $\mathsf{Gr}(2,5) \subset \mathbf{P}^9$ (*ordinary* GM varieties) or to double covers of a linear section of $\mathsf{Gr}(2,5)$ branched along a quadratic section (*special* GM varieties). Their dimension is at most 6.

GM varieties first appeared in the classification of complex Fano threefolds: Gushel showed in [G] that any smooth prime complex Fano threefold of degree 10 is a GM variety. Mukai later extended in [Mu1] Gushel's results to higher-dimensional smooth Fano varieties of coindex 3, degree 10, and Picard number 1, to Brill–Noether general polarized K3 surfaces of degree 10, and to Clifford general curves of genus 6 (see Section 2.3 for the definitions).

In Section 2, we give an intrinsic characterization of normal GM varieties in terms of their twisted excess conormal sheaf (Theorem 2.3). In particular, we show that the (rational) map from a GM variety to $\mathsf{Gr}(2,5)$ is given by the sections of this rank-2 sheaf, hence is canonical. Developing this observation, we define *GM data* (Definition 2.5) as sets of linear algebraic objects required to define a GM variety. We show that there is a functorial bijection between the set of all normal polarized GM varieties and a suitable set of GM data (Theorem 2.9).

We also describe the automorphism group of a normal polarized GM variety in terms of the associated GM data (Corollary 2.10).

Using this intrinsic characterization of GM varieties, we prove an extension of Mukai's classification results (Theorem 2.14): any normal locally factorial complex projective variety X of dimension $n \geq 3$ with terminal singularities and $\mathrm{codim}(\mathrm{Sing}(X)) \geq 4$, together with an ample Cartier divisor class H such that $K_X \underset{\mathrm{lin}}{\equiv} -(n-2)H$, $H^n = 10$, and $\mathrm{Pic}(X) = \mathbf{Z}H$, is a GM variety.
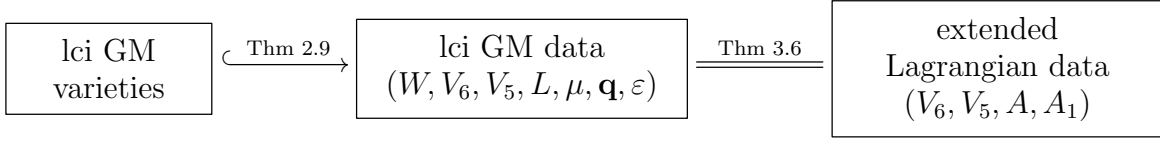
In contrast with the smooth case, a singular GM variety may have arbitrarily high dimension; besides the two smooth types discussed above (ordinary and special), there are also quadric bundles of arbitrary dimensions over a linear section of $\mathsf{Gr}(2,5)$, whose discriminant locus is a quadratic section. However, if one restricts to locally complete intersection (lci) GM varieties, one gets the ordinary and special types only. In particular, their dimension is again at most 6 and the theory becomes very much parallel to that of smooth GM varieties.

We end Section 2 by studying geometric properties of GM varieties. We define the *Grassmannian hull* M_X of a GM variety X as the intersection of the cone over $\mathsf{Gr}(2,5)$ in which X sits with the linear span of X, so that X is a quadratic section of M_X. We show that M_X is always smooth when $\dim(X) \geq 3$ and $\mathrm{codim}(\mathrm{Sing}(X)) \geq 4$ (Proposition 2.20) and discuss its singularities for smooth GM curves and surfaces. We also study the special features of lci GM varieties. We explain the relations between the two types (ordinary and special) and we construct a (birational) involution on the class of lci GM varieties which exchanges ordinary and special varieties.
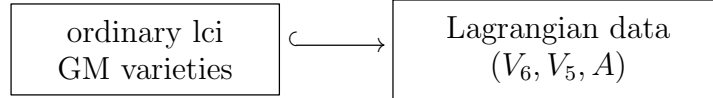
In Section 3, we discuss a relation between complex GM varieties and Eisenbud–Popescu–Walter (EPW) sextics ([EPW], [O1]). An EPW sextic is a hypersurface of degree 6 in the projectivization $\mathbf{P}(V_6)$ of a 6-dimensional vector space V_6 which depends on the choice of a Lagrangian subspace $A \subset \bigwedge^3 V_6$. EPW sextics have many wonderful properties which were thoroughly investigated by O'Grady. A relation between GM varieties and EPW sextics was discovered by Iliev–Manivel in [IM]. They showed that if V_6 is the space of quadrics through an ordinary 5-dimensional GM variety X in the projective embedding given by H, one can construct a Lagrangian subspace $A(X) \subset \bigwedge^3 V_6$; conversely, given a general Lagrangian subspace $A \subset \bigwedge^3 V_6$ and a general hyperplane $V_5 \subset V_6$, one can reconstruct a 5-dimensional GM variety X_{A,V_5}.

We suggest an extension of the Iliev–Manivel construction which works for all lci GM varieties (of both types—ordinary and special—and in all dimensions). Formally, this is done by associating with an lci GM data another set which we call an extended Lagrangian data. This is a quadruple (V_6, V_5, A, A_1), where $A \subset \bigwedge^3 V_6$ is a Lagrangian subspace and A_1, which encodes the type of the GM data, is a Lagrangian subspace in a fixed 2-dimensional symplectic space. In Theorem 3.6, we show that when the base field $\mathbf{k}$ is quadratically closed ($\mathbf{k} = \mathbf{k}^2$), there is a bijection between the set of isomorphism classes of lci GM data and the set of isomorphism classes of extended Lagrangian data.

Concatenating this construction with the construction in Section 2, we obtain a diagram

$$\boxed{\begin{array}{c} \text{lci GM} \\ \text{varieties} \end{array}} \xhookrightarrow{\text{Thm 2.9}} \boxed{\begin{array}{c} \text{lci GM data} \\ (W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon) \end{array}} \xlongequal{\text{Thm 3.6}} \boxed{\begin{array}{c} \text{extended} \\ \text{Lagrangian data} \\ (V_6, V_5, A, A_1) \end{array}}$$

where the objects in the boxes are taken modulo isomorphisms. Restricting ourselves to ordinary GM varieties simplifies the picture: A_1 becomes redundant, so we can restrict to triples (V_6, V_5, A) which we simply call Lagrangian data, the bijection now works over arbitrary fields, and the diagram above simplifies to an *inclusion*

$$\boxed{\begin{array}{c} \text{ordinary lci} \\ \text{GM varieties} \end{array}} \xhookrightarrow{} \boxed{\begin{array}{c} \text{Lagrangian data} \\ (V_6, V_5, A) \end{array}}$$

where the objects in the boxes are again taken modulo isomorphisms.

The correspondence between lci GM varieties and extended Lagrangian data has many remarkable properties. One of the most impressive is a criterion for the smoothness of the GM variety X_{A,A_1,V_5} constructed from an extended Lagrangian data (V_6, V_5, A, A_1), which shows in particular that if $\dim(X_{A,A_1,V_5}) \geq 3$, smoothness is equivalent to an explicit property of the Lagrangian subspace A and does not depend on the hyperplane $V_5 \subset V_6$ (Theorem 3.14).

We spend some time discussing the structure of lci GM varieties associated with the same Lagrangian subspace $A \subset \bigwedge^3 V_6$, but with different choices of A_1 and hyperplane $V_5 \subset V_6$. As it was already mentioned, A_1 just encodes the type of the GM variety, and switching the type of A_1 results in switching the type of the GM variety. We show that for fixed (A, A_1), the dimension of X_{A,A_1,V_5} only depends on which stratum of the EPW stratification of the space $\mathbf{P}(V_6^\vee)$ the hyperplane V_5 belongs to (extending again the results of Iliev and Manivel).

If $\dim(X_{A,A_1,V_5}) = \dim(X_{A,A_1',V_5'})$, we say that the GM varieties X_{A,A_1,V_5} and $X_{A,A_1',V_5'}$ are *period partners*. One of the main results of the paper is that smooth period partners of dimension ≥ 3 are birationally isomorphic (Corollary 4.16). In particular, no smooth GM varieties of dimension ≥ 3 are birationally rigid (Corollary 4.17).

We also introduce a notion of duality for GM varieties. Given a Lagrangian subspace $A \subset \bigwedge^3 V_6$, its orthogonal complement $A^\perp \subset \bigwedge^3 V_6^\vee$ is also Lagrangian; the choice of a line $V_1 \subset V_6$ (which can be considered as a hyperplane in $V_6^\vee$) and an arbitrary choice of A_1' allow us to construct a GM variety $X_{A^\perp,A_1',V_1}$. If $\dim(X_{A_0^\perp,A_1',V_1}) = \dim(X_{A,A_1,V_5})$, we say that the GM varieties $X_{A^\perp,A_1',V_1}$ and X_{A,A_1,V_5} are dual. Our second main result is that smooth dual GM varieties of dimension ≥ 3 are birationally isomorphic (Theorem 4.20).

The birational isomorphisms relating period partners are generalizations of conic transformations and those relating dual varieties are generalizations of line transformations (see [DIM]). In relation with birationalities for dual varieties, we define another interesting geometric object associated with GM varieties—a special hypersurface of degree 4 in $\mathsf{Gr}(3,6)$. As the EPW sextic, this hypersurface is defined as a special Lagrangian intersection locus and they play similar roles. We call it the EPW quartic and suspect that it may have interesting geometry. These quartics are investigated in more details in [IKKR].

For the reader's convenience, we gathered some of the material we need in the main body of this article in three appendices. In Appendix A, we introduce excess conormal sheaves for

intersections of quadrics, discuss how they change under simple geometric operations (taking cones and hyperplane and quadratic sections), and compute them for the Grassmannians $\mathsf{Gr}(2,5)$ and $\mathsf{Gr}(2,6)$. In Appendix B, we recall the definition of EPW sextics and their main properties. Most of the results there are extracted from a series of papers of O'Grady. In Appendix C, we discuss the classical relation between the Lagrangian geometry of a symplectic vector space and the projective duality of quadrics. We also define a notion of isotropic reduction in this context which is very useful for the description of the various quadratic fibrations associated with Gushel–Mukai varieties.

One of the motivations for this research was an attempt to construct a moduli space for GM varieties. The unified and extended constructions we introduce here should be important ingredients to attack the moduli problem. We plan to do this in a forthcoming paper.

In another companion paper, we show that period partners of even dimensions have isomorphic primitive middle Hodge structures and describe this Hodge structure in terms of the associated EPW sextic.

In a joint work of the second author with Alexander Perry, we discuss derived categorical aspects of GM varieties.

1.1. **Notation and conventions.** All schemes are separated of finite type over a field $\mathbf{k}$. For simplicity, we will always assume that $\mathbf{k}$ is a subfield of $\mathbf{C}$, although many of our results remain valid in positive characteristics. A $\mathbf{k}$-variety is an integral scheme (separated of finite type) over $\mathbf{k}$. We usually abbreviate "local(ly) complete intersection" as "lci".

We write $\underset{\mathrm{lin}}{\equiv}$ for linear equivalence of divisors.

Given a vector space V, we denote by $V^\vee$ the dual space and, given a vector subspace $U \subset V$, we denote by $U^\perp \subset V^\vee$ its orthogonal complement, i.e., the space of linear functions on V vanishing on U.

Given a vector space V, we denote by $\mathbf{P}(V)$ its projectivization, i.e., the variety of lines in V. Given a line $V_1 \subset V$, we write $[V_1]$, or even V_1, for the corresponding point of $\mathbf{P}(V)$. More generally, we denote by $\mathsf{Gr}(k, V)$ the Grassmannian of k-dimensional subspaces in V and, given a subspace $V_k \subset V$, we write $[V_k]$, or even V_k, for the corresponding point of the Grassmannian. Analogously, we denote by $\mathsf{Fl}(k_1, \ldots, k_r; V)$ the flag variety of chains of subspaces in V of dimensions $k_1 < \cdots < k_r$. Finally, given a vector bundle $\mathscr{E}$ on a scheme X, we denote by $\mathbf{P}_X(\mathscr{E})$ its projectivization, i.e., the scheme of lines in fibers of $\mathscr{E}$ over X.

Let W be a vector space and let $X \subset \mathbf{P}(W)$ be a subscheme. For a point $x \in X$, we write $\mathbf{T}_{X,x}$ for the embedded tangent space of X at x. For any vector space K, we denote by $\mathsf{C}_K X \subset \mathbf{P}(W \oplus K)$ the cone over X with vertex $\mathbf{P}(K)$ and by $\mathsf{C}_K^\circ X$ the punctured cone, i.e., $\mathsf{C}_K^\circ X = \mathsf{C}_K X \smallsetminus \mathbf{P}(K)$.

Given closed subschemes $X_1, X_2 \subset \mathbf{P}(W)$, we say that the intersection $X_1 \cap X_2$ is dimensionally transverse if $\mathrm{Tor}_{>0}(\mathscr{O}_{X_1}, \mathscr{O}_{X_2}) = 0$ (this condition is also known as Tor-independence). When X_1 and X_2 are both Cohen–Macaulay, this condition is equivalent to

$$\mathrm{codim}_x(X_1 \cap X_2) = \mathrm{codim}_x(X_1) + \mathrm{codim}_x(X_2)$$

for any closed point $x \in X_1 \cap X_2$ (where the codimensions are in $\mathbf{P}(W)$). When $X_2 \subset \mathbf{P}(W)$ is a hypersurface, the intersection $X_1 \cap X_2$ is dimensionally transverse if and only if no associated point of X_1 is contained in X_2.

1.2. **Acknowledgements.** This article owns much to the ideas and work of Kieran O'Grady, Atanas Iliev, and Laurent Manivel. We would also like to thank Olivier Benoist, Alex Perry, and Yuri Prokhorov for their help and many interesting discussions.

2. Geometry of Gushel–Mukai varieties

2.1. **Intrinsic characterization of GM varieties.** In the introduction, we defined a GM variety as a dimensionally transverse intersection of a cone over $\mathsf{Gr}(2,5)$ with a linear subspace and a quadratic hypersurface.

More precisely, let V_5 be a **k**-vector space of dimension 5 and consider the Plücker embedding $\mathsf{Gr}(2, V_5) \subset \mathbf{P}(\bigwedge^2 V_5)$. Let K be an arbitrary **k**-vector space. Consider the cone

$$\mathsf{C_K Gr}(2, V_5) \subset \mathbf{P}(\textstyle\bigwedge^2 V_5 \oplus K)$$

with vertex $\mathbf{P}(K)$ and choose a vector subspace $W \subset \bigwedge^2 V_5 \oplus K$ and a subscheme $Q \subset \mathbf{P}(W)$ defined by one quadratic equation (possibly zero).

Definition 2.1. The scheme

$$(1) \qquad\qquad X = \mathsf{C_K Gr}(2, V_5) \cap \mathbf{P}(W) \cap Q.$$

is called a **GM intersection**. A GM intersection X is a **GM variety** if X is integral and

$$(2) \qquad\qquad \dim(X) = \dim(W) - 5 \geq 1.$$

When X is a GM variety, Q is a quadratic hypersurface and $\mathsf{C_K Gr}(2, V_5) \cap \mathbf{P}(W)$ and Q are Cohen–Macaulay subschemes of $\mathbf{P}(W)$, hence their intersection X is dimensionally transverse. In particular, a GM variety is always Cohen–Macaulay.

A GM variety X has a canonical polarization, the restriction H of the hyperplane class on $\mathbf{P}(W)$; we will call (X, H) a **polarized GM variety**.

Definition 2.2. An **isomorphism of polarized GM varieties** between (X, H) and (X', H') is a pair (ϕ, ψ), where $\phi\colon X \xrightarrow{\sim} X'$ is an isomorphism of algebraic varieties and $\psi\colon \mathscr{O}_X(H) \xrightarrow{\sim} \phi^* \mathscr{O}_{X'}(H')$ an isomorphism of line bundles. We consider the set of all polarized GM varieties as a groupoid, with this notion of isomorphism.

We denote by $\mathrm{Aut}(X, H)$ the group of automorphisms of a polarized GM variety and by $\mathrm{Aut}_H(X)$ its image in $\mathrm{Aut}(X)$. There is an exact sequence

$$(3) \qquad\qquad 1 \to \mathbb{G}_\mathrm{m} \to \mathrm{Aut}(X, H) \to \mathrm{Aut}_H(X) \to 1,$$

where the subgroup $\mathbb{G}_\mathrm{m} \subset \mathrm{Aut}(X, H)$ acts trivially on X and by dilations on $\mathscr{O}_X(H)$.

The definition of a GM variety is not intrinsic. The following theorem gives an intrinsic characterization, at least for normal varieties. A key ingredient is the excess conormal sheaf, which is defined over **k**, for any variety which is an intersection of quadrics (see Appendix A).

Recall that a coherent sheaf $\mathscr{F}$ is **simple** if $\mathrm{Hom}(\mathscr{F}, \mathscr{F}) = \mathbf{k}$.

Theorem 2.3. *A normal polarized projective variety (X, H) of dimension $n \geq 1$ is a polarized GM variety if and only if all the following conditions hold:*

(1) $H^n = 10$ and $K_X = -(n - 2)H$; in particular, X is Gorenstein;
(2) H is very ample and the vector space $W := H^0(X, \mathscr{O}_X(H))^\vee$ has dimension $n + 5$;

(3) X is an intersection of quadrics in $\mathbf{P}(W)$ and the space $V_6 := H^0(\mathbf{P}(W), \mathscr{I}_X(2)) \subset \mathsf{S}^2 W^\vee$ of quadrics through X has dimension 6;

(4) the twisted excess conormal sheaf $\mathscr{U}_X := \mathscr{E}\mathscr{N}_X^\vee(2H)$ of X in $\mathbf{P}(W)$ is simple.

Proof. We first prove that conditions (a)–(d) are satisfied by normal GM varieties. Let X be such a variety, defined by (1), with $\dim(X) = n$ and $\dim(W) = n + 5$.

(a) Since the degree of $\mathsf{C}_\mathrm{K}\mathsf{Gr}(2, V_5)$ is 5 and the degree of Q is 2, the dimensional transversality implies that the degree of X is 10. Let $\dim(K) = k$. The canonical class of $\mathsf{C}_\mathrm{K}\mathsf{Gr}(2, V_5)$ is $-(5+k)H$. On the other hand, the codimension of W in $\bigwedge^2 V_5 \oplus K$ is $(10+k)-(n+5) = 5+k-n$, hence the canonical class of X is

$$K_X = (-(5+k) + (5+k-n) + 2)H = -(n-2)H.$$

(b) The very ampleness of H is clear. By (2), it is enough to show $H^0(X, \mathscr{O}_X(H)) = W^\vee$. We use the resolution

$$0 \to \mathscr{O}(-5) \to V_5^\vee \otimes \mathscr{O}(-3) \to V_5 \otimes \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_{\mathsf{C}_\mathrm{K}\mathsf{Gr}(2,V_5)} \to 0$$

of the cone $\mathsf{C}_\mathrm{K}\mathsf{Gr}(2, V_5)$ in $\mathbf{P}(\bigwedge^2 V_5 \oplus K)$. Restricting it to $\mathbf{P}(W)$, tensoring with the resolution $0 \to \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_Q \to 0$ of the quadric Q, and using the dimensional transversality of the intersection, we obtain the resolution

(4) $0 \to \mathscr{O}(-7) \to (V_5^\vee \oplus \mathbf{k}) \otimes \mathscr{O}(-5) \to (V_5 \otimes \mathscr{O}(-4)) \oplus (V_5^\vee \otimes \mathscr{O}(-3)) \to$

$$\to (V_5 \oplus \mathbf{k}) \otimes \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_X \to 0$$

of X in $\mathbf{P}(W)$. Twisting it by $\mathscr{O}(1)$, we get $H^0(X, \mathscr{O}_X(H)) = H^0(\mathbf{P}(W), \mathscr{O}_{\mathbf{P}(W)}(1)) = W^\vee$ (since $\dim(W) = n + 5 \geq 6$, the only other term that could contribute is the term $\mathscr{O}(-6)$ at the very beginning; the contribution is non-zero only for $n = 1$, but it actually contributes to $H^1(X, \mathscr{O}_X(H))$).

(c) The resolution (4) implies that X is an intersection of quadrics. Furthermore, twisting the resolution by $\mathscr{O}(2)$, we see that $H^0(\mathbf{P}(W), \mathscr{I}_X(2H)) = V_5 \oplus \mathbf{k}$ is 6-dimensional.

(d) To prove that $\mathscr{U}_X$ is simple, we may assume $\mathbf{k} = \mathbf{C}$.

Assume $n = 1$. By (a), we have $\deg(K_X) = 10$, so X is a smooth curve of genus 6. Moreover, by (b) and (c), its canonical class $K_X = H$ is very ample and its canonical model is an intersection of quadrics, hence X is neither hyperelliptic, nor trigonal, nor a plane quintic. Mukai constructs in [Mu2, Section 5] a stable vector bundle of rank 2 on X, proves that it embeds X into the cone $\mathsf{CGr}(2, 5)$ over $\mathsf{Gr}(2, 5)$ with vertex a point, and that X is an intersection $\mathsf{CGr}(2, 5) \cap \mathbf{P}^5 \cap Q$. A combination of Proposition A.7 and Lemmas A.4, A.5 and A.6 then shows that Mukai's bundle is isomorphic to the twisted excess conormal bundle $\mathscr{U}_X$, which is therefore stable, hence simple.

We finish the proof by induction on n: assume $n \geq 2$ and let $X' \subset X$ be a dimensionally transverse normal hyperplane section of X, so that X' is a GM variety of dimension $n-1$, and $\mathscr{U}_{X'}$ is simple by the induction hypothesis. Applying the functor $\mathrm{Hom}(\mathscr{U}_X, -)$ to the exact sequence

$$0 \to \mathscr{U}_X(-H) \to \mathscr{U}_X \to \mathscr{U}_X|_{X'} \to 0$$

and using pullback-pushforward adjunction and the isomorphism $\mathscr{U}_{X'} \simeq \mathscr{U}_X|_{X'}$ (Lemma A.5), we obtain an exact sequence

$$0 \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H)) \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X) \to \mathrm{Hom}(\mathscr{U}_{X'}, \mathscr{U}_{X'}).$$

If $\dim(\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X)) > 1$, the simplicity of $\mathscr{U}_{X'}$ implies $\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H)) \neq 0$. On the other hand, an analogous argument produces an exact sequence

$$0 \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-2H)) \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H)) \to \mathrm{Hom}(\mathscr{U}_{X'}, \mathscr{U}_{X'}(-H)) = 0,$$

which implies $\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-2H)) \neq 0$. Iterating the argument, we see that for all $k \geq 0$, we have $\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-kH)) \neq 0$. This is possible only if $\mathscr{U}_X$ has zero-dimensional torsion. But it has not since, by (32), it is a subsheaf of $V_6 \otimes \mathscr{O}_X$ and X is integral.

We now prove that conditions (a)–(d) are also sufficient.

Let X be a normal projective variety with a Cartier divisor H which satisfies properties (a)–(d) of Theorem 2.3. Consider the spaces W and V_6 of respective dimensions $n+5$ and 6, defined by conditions (b) and (c). Since V_6 is a space of quadrics in W, it comes with a map

$$\mathbf{q}\colon V_6 \to \mathsf{S}^2 W^\vee.$$

Consider the twisted excess conormal sheaf $\mathscr{U}_X$. Its restriction $\mathscr{U}_{X_{\mathrm{sm}}}$ to the smooth locus X_{sm} of X is locally free of rank 2. Since the rank of the conormal sheaf is 4, we compute, using the exact sequence (32),

$$c_1(\mathscr{U}_{X_{\mathrm{sm}}}) = -c_1(\mathscr{N}^\vee_{X_{\mathrm{sm}}/\mathbf{P}(W)}(2H)) = -8H - c_1(\mathscr{N}^\vee_{X_{\mathrm{sm}}/\mathbf{P}(W)}) =$$
$$= -8H - K_{\mathbf{P}(W)}|_X + K_X = -8H + (n+5)H - (n-2)H = -H.$$

Hence there is a unique 1-dimensional vector space L such that

$$(5) \qquad\qquad L \otimes \bigwedge{}^2 \mathscr{U}^\vee_{X_{\mathrm{sm}}} = \mathscr{O}_{X_{\mathrm{sm}}}(H).$$

Dualizing the embedding $\mathscr{U}_{X_{\mathrm{sm}}} \hookrightarrow V_6 \otimes \mathscr{O}_{X_{\mathrm{sm}}}$ and taking its wedge square, we get a surjection $\bigwedge^2 V_6^\vee \otimes \mathscr{O}_{X_{\mathrm{sm}}} \twoheadrightarrow \bigwedge^2 \mathscr{U}^\vee_{X_{\mathrm{sm}}}$. Taking into account (5), we obtain a linear map

$$L \otimes \bigwedge{}^2 V_6^\vee \to H^0(X_{\mathrm{sm}}, \mathscr{O}_{X_{\mathrm{sm}}}(H)) = H^0(X, \mathscr{O}_X(H)) = W^\vee$$

(we use the normality of X to identify sections on X and on X_{sm}) and by duality a linear map $\mu\colon L \otimes W \to \bigwedge^2 V_6$. This map can be factored through an injection $\bar{\mu}\colon L \otimes W \hookrightarrow \bigwedge^2 V_6 \oplus K$ for some vector space K. The maps

$$X_{\mathrm{sm}} \hookrightarrow \mathbf{P}(W) \simeq \mathbf{P}(L \otimes W) \hookrightarrow \mathbf{P}(\bigwedge{}^2 V_6 \oplus K) \dashrightarrow \mathbf{P}(\bigwedge{}^2 V_6)$$

and

$$X_{\mathrm{sm}} \to \mathsf{Gr}(2, V_6) \hookrightarrow \mathbf{P}(\bigwedge{}^2 V_6),$$

where the map $X_{\mathrm{sm}} \to \mathsf{Gr}(2, V_6)$ is induced by the embedding $\mathscr{U}_{X_{\mathrm{sm}}} \hookrightarrow V_6 \otimes \mathscr{O}_{X_{\mathrm{sm}}}$, are given by the same linear system, hence they agree. This shows $X \subset \mathsf{C}_K \mathsf{Gr}(2, V_6)$ and $X_{\mathrm{sm}} \subset \mathsf{C}_K^\circ \mathsf{Gr}(2, V_6)$, so by Proposition A.3, we have a commutative diagram

$$0 \to (V_6 \otimes \mathscr{U}_{X_{\mathrm{sm}}})/\mathsf{S}^2 \mathscr{U}_{X_{\mathrm{sm}}} \longrightarrow \bigwedge{}^2 V_6 \otimes \mathscr{O}_{X_{\mathrm{sm}}} \longrightarrow \det(V_6) \otimes \bar{\mu}^* \mathscr{N}^\vee_{\mathsf{Gr}(2,V_6)}(2) \to 0$$
$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$
$$0 \to L^{-2} \otimes \det(V_6) \otimes \mathscr{U}_{X_{\mathrm{sm}}} \to L^{-2} \otimes \det(V_6) \otimes V_6 \otimes \mathscr{O}_{X_{\mathrm{sm}}} \to L^{-2} \otimes \det(V_6) \otimes \mathscr{N}^\vee_{X_{\mathrm{sm}}}(2) \to 0,$$

where the top row comes from the pullback to X_{sm} of the excess conormal sequence of $\mathsf{Gr}(2, V_6)$ (see Proposition A.8) and we twisted everything by $\det(V_6)$. The left vertical arrow induces a morphism $\lambda'\colon V_6 \otimes \mathscr{U}_{X_{\mathrm{sm}}} \to L^{-2} \otimes \det(V_6) \otimes \mathscr{U}_{X_{\mathrm{sm}}}$.

By condition (d), the sheaf $\mathscr{U}_X$ is simple, hence $\mathscr{U}_{X_{\mathrm{sm}}}$ is simple as well: $\mathscr{U}_X$ is torsion-free, hence any endomorphism of $\mathscr{U}_{X_{\mathrm{sm}}}$ extends to an endomorphism of $\mathscr{U}_X$. Therefore, λ' is given by a linear form

$$\lambda\colon V_6 \to L^{-2} \otimes \det(V_6).$$

Since λ' vanishes on $\mathsf{S}^2\mathscr{U}_{X_{\mathrm{sm}}}$, the image of $\mathscr{U}_{X_{\mathrm{sm}}}$ in $V_6 \otimes \mathscr{O}_{X_{\mathrm{sm}}}$ (via the sequence (32)) is contained in $\mathrm{Ker}(\lambda) \otimes \mathscr{O}_{X_{\mathrm{sm}}}$. Moreover, the middle vertical map in the diagram above is given by $v_1 \wedge v_2 \mapsto \lambda(v_1)v_2 - \lambda(v_2)v_1$.

Let us show that the form λ is non-zero. If $\lambda = 0$, the middle vertical map in the diagram is zero, which means that all the quadrics cutting out $\mathsf{C}_K\mathsf{Gr}(2, V_6)$ contain $\mathbf{P}(W)$, i.e., $\mathbf{P}(W) \subset \mathsf{C}_K\mathsf{Gr}(2, V_6)$. In other words, $\mathbf{P}(W)$ is a cone over $\mathbf{P}(W') \subset \mathsf{Gr}(2, V_6)$, with vertex a subspace of K. The map $X_{\mathrm{sm}} \to \mathsf{Gr}(2, V_6)$ induced by $\mathscr{U}_{X_{\mathrm{sm}}}$ therefore factors through $\mathbf{P}(W')$, i.e., the vector bundle $\mathscr{U}_{X_{\mathrm{sm}}}$ is a pullback from $\mathbf{P}(W')$ of the restriction of the tautological bundle of $\mathsf{Gr}(2, V_6)$ to $\mathbf{P}(W')$. We show that this is impossible.

There are two types of linear spaces on $\mathsf{Gr}(2, V_6)$: the first type corresponds to 2-dimensional subspaces containing a given vector and the second type to those contained in a given 3-subspace $V_3 \subset V_6$. If W' is of the first type, the restriction of the tautological bundle to $\mathbf{P}(W')$ is isomorphic to $\mathscr{O} \oplus \mathscr{O}(-1)$, hence $\mathscr{U}_{X_{\mathrm{sm}}} \simeq \mathscr{O}_{X_{\mathrm{sm}}} \oplus \mathscr{O}_{X_{\mathrm{sm}}}(-H)$. In particular, it is not simple, which is a contradiction. If W' is of the second type, the embedding $\mathscr{U}_X \to V_6 \otimes \mathscr{O}_X$ factors through a subbundle $V_3 \otimes \mathscr{O}_X \subset V_6 \otimes \mathscr{O}_X$. Recall that V_6 is the space of quadrics passing through X in $\mathbf{P}(W)$. Consider the scheme-theoretic intersection M of the quadrics corresponding to the vector subspace $V_3 \subset V_6$. Since the embedding of the excess conormal sheaf factors through $V_3 \otimes \mathscr{O}_X$, the variety X is the complete intersection of M with the 3 quadrics corresponding to the quotient space V_6/V_3. But the degree of X is then divisible by 8, which contradicts the fact that it is 10 by (a).

Thus $\lambda \neq 0$ and $V_5 := \mathrm{Ker}(\lambda)$ is a hyperplane in V_6. It fits in the exact sequence

$$0 \to V_5 \to V_6 \xrightarrow{\lambda} L^{-2} \otimes \det(V_6) \to 0,$$

which induces a canonical isomorphism

$$\varepsilon\colon \det(V_5) \xrightarrow{\sim} L^{\otimes 2}.$$

Moreover, the composition $\mathscr{U}_X \hookrightarrow V_6 \otimes \mathscr{O}_X \xrightarrow{\lambda} L^{\otimes 2} \otimes \det(V_6) \otimes \mathscr{O}_X$ vanishes on X_{sm}, hence on X as well. This shows that the embedding $\mathscr{U}_X \hookrightarrow V_6 \otimes \mathscr{O}_X$ factors through $V_5 \otimes \mathscr{O}_X$.

We now replace V_6 with V_5 and repeat the above argument. We get a linear map

$$\mu\colon L \otimes W \to \textstyle\bigwedge^2 V_5$$

and an embedding $\bar{\mu}\colon L \otimes W \hookrightarrow \bigwedge^2 V_5 \oplus K$ which induce embeddings $X \subset \mathsf{C}_K\mathsf{Gr}(2, V_5)$ and $X_{\mathrm{sm}} \subset \mathsf{C}_K^\circ\mathsf{Gr}(2, V_5)$. The commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{U}_{X_{\mathrm{sm}}} & \longrightarrow & V_5 \otimes \mathscr{O}_{X_{\mathrm{sm}}} & \longrightarrow & \bar{\mu}^* \mathscr{N}^\vee_{\mathsf{Gr}(2,V_5)} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{U}_{X_{\mathrm{sm}}} & \longrightarrow & V_6 \otimes \mathscr{O}_{X_{\mathrm{sm}}} & \longrightarrow & \mathscr{N}^\vee_{X_{\mathrm{sm}}} & \longrightarrow & 0
\end{array}
$$

of Proposition A.3 (we use ε to cancel out the twists by $\det(V_5^\vee)$ and by L^2 in the top and bottom rows) then shows that inside the space V_6 of quadrics cutting out X in $\mathbf{P}(W)$, the hyperplane V_5 is the space of quadratic equations of $\mathsf{Gr}(2, V_5)$, i.e., of Plücker quadrics.

As the Plücker quadrics cut out the cone $\mathsf{C}_{\mathrm{K}}\mathsf{Gr}(2, V_5)$ in $\mathbf{P}(\bigwedge^2 V_5 \oplus K)$, they cut out $\mathsf{C}_{\mathrm{K}}\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W)$ in $\mathbf{P}(W)$. Since X is the intersection of 6 quadrics by condition (c), we finally obtain

$$(6) \qquad\qquad X = \mathsf{C}_{\mathrm{K}}\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W) \cap Q,$$

where Q is any non-Plücker quadric (corresponding to a point in $V_6 \smallsetminus V_5$), so X is a GM variety. $\qquad\qquad\square$

A consequence of Theorem 2.3 is that being a GM variety is a geometric property. Recall that the base field $\mathbf{k}$ is by our assumption a subfield of $\mathbf{C}$. We denote by $X_{\mathbf{C}}$ the extension of scalars from $\mathbf{k}$ to $\mathbf{C}$, and by $H_{\mathbf{C}}$ the induced polarization of $X_{\mathbf{C}}$.

Corollary 2.4. *A normal polarized variety (X, H) is a polarized GM variety if and only if $(X_{\mathbf{C}}, H_{\mathbf{C}})$ is.*

Proof. One direction is clear from the definition, so we only have to check that if $(X_{\mathbf{C}}, H_{\mathbf{C}})$ is GM, so is (X, H). We show that if the conditions of Theorem 2.3 are satisfied for $(X_{\mathbf{C}}, H_{\mathbf{C}})$, they are satisfied for (X, H) as well. The only property for which this is not completely obvious is the equality $K_X = -(n-2)H$, but it follows from the injectivity of the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathbf{C}})$, which holds since X is integral and projective. $\qquad\square$

2.2. **GM data.** In the course of the proof of Theorem 2.3, we associated with any normal polarized GM variety vector spaces W, V_6, V_5, L, and maps $\mathbf{q}$, μ, ε. We axiomatize these as follows.

Definition 2.5. A GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ of dimension n (over $\mathbf{k}$) consists of

- a $\mathbf{k}$-vector space W of dimension $n + 5$,
- a $\mathbf{k}$-vector space V_6 of dimension 6,
- a hyperplane $V_5 \subset V_6$,
- a 1-dimensional $\mathbf{k}$-vector space L,
- a $\mathbf{k}$-linear map $\mu \colon L \otimes W \to \bigwedge^2 V_5$,
- a $\mathbf{k}$-linear map $\mathbf{q} \colon V_6 \to \mathsf{S}^2 W^{\vee}$,
- a $\mathbf{k}$-linear isomorphism $\varepsilon \colon \det(V_5) \to L^{\otimes 2}$,

such that the following diagram commutes

$$(7) \qquad \begin{array}{ccc} V_5 & \lhook\joinrel\longrightarrow & V_6 \\ {\scriptstyle \varepsilon}\downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \mathbf{q}} \\ L^{\otimes 2} \otimes \bigwedge^4 V_5^{\vee} & \xrightarrow{\ \mathsf{S}^2 \mu^{\vee}\ } & \mathsf{S}^2 W^{\vee}. \end{array}$$

In other words, $\mathbf{q}(v)(w_1, w_2) = \varepsilon(v \wedge \mu(w_1) \wedge \mu(w_2))$ for all $v \in V_5$ and $w_1, w_2 \in W$.

Definition 2.6. An isomorphism of GM data between GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ and $(W', V_6', V_5', L', \mu', \mathbf{q}', \varepsilon')$ is a triple of $\mathbf{k}$-linear isomorphisms $\varphi_W \colon W \to W'$, $\varphi_V \colon V_6 \to V_6'$, and $\varphi_L \colon L \to L'$, such that

$$\varphi_V(V_5) = V_5', \qquad \varepsilon' \circ \textstyle\bigwedge^5 \varphi_V = \varphi_L^{\otimes 2} \circ \varepsilon,$$

and the following diagrams commute

$$
\begin{array}{ccc}
V_6 & \xrightarrow{\ \mathbf{q}\ } & \mathsf{S}^2 W^\vee \\
\varphi_V \downarrow & & \uparrow \mathsf{S}^2 \varphi_W^\vee \\
V_6' & \xrightarrow{\ \mathbf{q}'\ } & \mathsf{S}^2 W'^\vee
\end{array}
\qquad\qquad
\begin{array}{ccc}
L \otimes W & \xrightarrow{\ \mu\ } & \textstyle\bigwedge^2 V_5 \\
\varphi_L \otimes \varphi_W \downarrow & & \downarrow \bigwedge^2 \varphi_V \\
L' \otimes W' & \xrightarrow{\ \mu'\ } & \textstyle\bigwedge^2 V_5'.
\end{array}
$$

In particular, the automorphism group of a GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ is the subgroup of $\mathrm{GL}(W) \times \mathrm{GL}(V_6) \times \mathbb{G}_\mathrm{m}$ of elements (g_W, g_V, g_L) such that

$$
g_V(V_5) = V_5, \qquad \det(g_V|_{V_5}) = g_L^2, \qquad (\mathsf{S}^2 g_W^\vee) \circ \mathbf{q} \circ g_V = \mathbf{q}, \qquad (\textstyle\bigwedge^2 g_V) \circ \mu = \mu \circ (g_L \otimes g_W).
$$

We consider the set of all polarized GM data as a groupoid, with this notion of isomorphism.

Lemma 2.7. *The collection $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ associated by Theorem 2.3 with a normal polarized GM variety (X, H) is a GM data. The association*

$$
(X, H) \longmapsto (W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)
$$

is a functor from the groupoid of polarized GM varieties to the groupoid of GM data, i.e., any isomorphism of polarized GM varieties induces an isomorphism of the associated GM data. Moreover, this association works in families.

Proof. In the course of the proof of Theorem (2.3), we showed that the subspace $V_5 \subset V_6$ of quadrics through X cuts out $\mathsf{C}_K \mathrm{Gr}(2, V_5) \cap \mathbf{P}(W)$ in $\mathbf{P}(W)$. Thus these quadrics are restrictions to $\mathbf{P}(W)$ of the Plücker quadrics. This is equivalent to the commutativity of (7), so the constructed data is a GM data.

Assume (ϕ, ψ) is an isomorphism of polarized GM varieties between (X, H) and (X', H'). We denote by $(W', V_6', V_5', L', \mu', \mathbf{q}', \varepsilon')$ the GM data for X'. The isomorphism ψ induces an isomorphism between W and W', and also between V_6 and V_6'. Moreover, it induces an isomorphism between the excess conormal sheaves of X and X', and an isomorphism between L and L', compatible with the isomorphisms (5). These isomorphisms are compatible with the hyperplanes V_5 and with the maps $\mathbf{q}$, μ, and ε, hence provide an isomorphism of the associated GM data. Moreover, the composition of isomorphisms of polarized GM varieties corresponds to the composition of the corresponding isomorphisms of GM data. This proves the functoriality.

Finally, given a smooth family $\mathscr{X} \to S$ of normal projective varieties with a Cartier divisor H on $\mathscr{X}$, such that each fiber of the family satisfies the conditions of Theorem 2.3, the same construction provides vector bundles $\mathscr{W}$, $\mathscr{V}_6$, $\mathscr{V}_5$, $\mathscr{L}$ on S (in the definition of the first two, one should replace global sections with pushforwards to S) and maps $\mathbf{q} \colon \mathscr{V}_6 \to \mathsf{S}^2 \mathscr{W}^\vee$, $\mu \colon \mathscr{L} \otimes \mathscr{W} \to \bigwedge^2 \mathscr{V}_5$, and $\varepsilon \colon \det(\mathscr{V}_5) \to \mathscr{L}^{\otimes 2}$. This shows that the association works in families. $\square$

Let $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ be a GM data. For each vector $v \in V_6$, we have a quadratic form $\mathbf{q}(v) \in \mathsf{S}^2 W^\vee$ and we denote by $Q(v) \subset \mathbf{P}(W)$ the subscheme it defines (a quadratic hypersurface when $\mathbf{q}(v) \neq 0$).

Lemma 2.8. *If $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ is a GM data of dimension n,*

$$
(8) \qquad\qquad X = X(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon) := \bigcap_{v \in V_6} Q(v)
$$

is a GM intersection of dimension $\geq n$. It is a GM variety if and only if X is integral of dimension n.

Proof. Lift the map $\mu \colon L \otimes W \to \bigwedge^2 V_5$ to an embedding $\bar{\mu} \colon L \otimes W \hookrightarrow \bigwedge^2 V_5 \oplus K$ for some K (for example, one can take $K = \mathrm{Ker}(\mu)$). It follows from the commutativity of (7) that for $v \in V_5$, the quadrics $Q(v)$ are the restrictions to $\mathbf{P}(W) = \mathbf{P}(L \otimes W)$ of the cones (with vertex $\mathbf{P}(K)$) over the Plücker quadrics in $\mathbf{P}(\bigwedge^2 V_5)$ with respect to the embedding $\bar{\mu}$. Since the intersection of the Plücker quadrics is $\mathsf{Gr}(2, V_5)$, this implies

$$(9) \qquad \bigcap_{v \in V_5} Q(v) = \mathsf{C}_{\mathrm{K}}\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W)$$

and, for any $v \in V_6 \smallsetminus V_5$, we have

$$X = \mathsf{C}_{\mathrm{K}}\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W) \cap Q(v).$$

This is a GM intersection, which is a GM variety if and only X is integral of dimension n. $\square$

Theorem 2.9. *The constructions of Lemmas 2.7 and 2.8 are mutually inverse and define an equivalence of groupoids between*

- *the groupoid of all normal polarized GM varieties (X, H) of dimension n and*
- *the groupoid of all GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ of dimension n such that the GM intersection $X(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ is n-dimensional and normal.*

This equivalence also works in families.

Proof. Let (X, H) be a GM variety and let $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ be the associated GM data. We saw during the proof of Theorem 2.3 that one can we can write X as in (6). Comparing with (9), we obtain $X(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon) \simeq X$. Conversely, start with a GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ and set $X := X(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$. Applying to X the construction of Theorem 2.3, one gets the original GM data back. Functoriality and the fact that the equivalence works in families are clear from the construction. $\square$

Recall that $\mathrm{Aut}(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ is a subgroup of $\mathrm{GL}(W) \times \mathrm{GL}(V_6) \times \mathbb{G}_{\mathrm{m}}$.

Corollary 2.10. *Let (X, H) be a normal polarized GM variety, with associated GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$. There is an isomorphism*

$$\mathrm{Aut}(X, H) \simeq \mathrm{Aut}(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$$

of algebraic groups and an exact sequence

$$1 \to \mathbb{G}_{\mathrm{m}} \to \mathrm{Aut}(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon) \to \mathrm{Aut}_H(X) \to 1.$$

Proof. Follows from Theorem 2.9 and (3). $\square$

2.3. **Mildly singular GM varieties.** In this section, we give a simpler intrinsic characterization of mildly singular GM varieties which will allow us to extend the Gushel–Mukai classification.

We say that a smooth projective curve X of genus 6 is $\mathsf{Clifford\ general}$ if $X_{\mathbf{C}}$ is neither hyperelliptic, nor trigonal, nor a plane quintic.

Proposition 2.11. *A smooth projective curve is a GM curve if and only if it is a Clifford general curve of genus 6. Equivalently, its canonical model is an intersection of 6 quadrics in $\mathbf{P}^5$.*

Proof. This follows from Theorem 2.3 combined with the Enriques–Babbage theorem. $\square$

If X is a GM variety, any integral hyperplane section of X is also a GM variety. We are going to show that the converse is also true under some additional assumptions. We start with the following result.

Lemma 2.12. *If (X, H) is a normal polarized GM variety, it is projectively normal, i.e., the canonical maps*
$$\mathsf{S}^m H^0(X, \mathscr{O}_X(H)) \to H^0(X, \mathscr{O}_X(mH))$$
are surjective for all $m \geq 0$.

Proof. Take any integer m and consider the twist

$$0 \to \mathscr{O}_{\mathbf{P}(W)}(m-7) \to \mathscr{O}_{\mathbf{P}(W)}(m-5)^{\oplus 6} \to \mathscr{O}_{\mathbf{P}(W)}(m-4)^{\oplus 5} \oplus \mathscr{O}_{\mathbf{P}(W)}(m-3)^{\oplus 5} \to$$
$$\to \mathscr{O}_{\mathbf{P}(W)}(m-2)^{\oplus 6} \to \mathscr{O}_{\mathbf{P}(W)}(m) \to \mathscr{O}_X(mH) \to 0$$

by $\mathscr{O}(m)$ of the resolution (4). We want to show that the map on global sections induced by the rightmost map is surjective. This holds because, $\mathbf{P}(W)$ being a projective space of dimension $n + 4 \geq 5$, the spaces $H^1(\mathbf{P}(W), \mathscr{O}_{\mathbf{P}(W)}(m-2))$, $H^2(\mathbf{P}(W), \mathscr{O}_{\mathbf{P}(W)}(m-3))$, $H^2(\mathbf{P}(W), \mathscr{O}_{\mathbf{P}(W)}(m-4))$, $H^3(\mathbf{P}(W), \mathscr{O}_{\mathbf{P}(W)}(m-5))$, $H^4(\mathbf{P}(W), \mathscr{O}_{\mathbf{P}(W)}(m-7))$ all vanish. $\square$

Proposition 2.13. *Let (X, H) be a normal polarized variety of dimension $n \geq 2$ such that $K_X = -(n-2)H$ and $H^1(X, \mathscr{O}_X) = 0$. If there is a hypersurface $X' \subset X$ in the linear system $|H|$ such that $(X', H|_{X'})$ is a normal polarized GM variety, (X, H) is also a polarized GM variety.*

Proof. Let us verify the conditions of Theorem 2.3. Condition (a) is true by our assumptions, since $H^n = H \cdot H^{n-1} = (H|_{X'})^{n-1} = 10$. Furthermore, X' is projectively normal by Lemma 2.12. Using [I, Lemma (2.9)], we conclude that X is also projectively normal. Since H is ample, it is very ample by [Mum, p. 38]. The exact sequence
$$0 \to \mathscr{O}_X \to \mathscr{O}_X(H) \to \mathscr{O}_{X'}(H) \to 0$$
with the assumption $H^1(X, \mathscr{O}_X) = 0$ imply
$$h^0(X, \mathscr{O}_X(H)) = h^0(X', \mathscr{O}_{X'}(H)) + 1 = n + 5.$$
This proves (b), and [I, Lemma (2.10)] proves (c). Finally, since $\mathscr{U}_{X'}$ is simple (because X' is a GM variety), the argument from the proof of Theorem 2.3(d) proves that $\mathscr{U}_X$ is also simple. $\square$

Following Mukai ([Mu3, Definition 3.8] (with a misprint in the English translation), [JK, Definition 10.1], [GLT, Definition 1.3]), we say that a complex polarized K3 surface (S, H) is Brill–Noether general if
$$h^0(S, D)h^0(S, H - D) < h^0(S, H)$$
for all divisors D on S not linearly equivalent to 0 or H. When $H^2 = 10$, this is equivalent, by [JK, Proposition 10.5], to the fact that $|H|$ contains a Clifford general smooth curve (by [GL], all smooth curves in $|H|$ are then Clifford general).

Theorem 2.14. *Let X be a normal locally factorial complex projective variety of dimension $n \geq 1$, with terminal singularities and $\mathrm{codim}(\mathrm{Sing}(X)) \geq 4$, together with an ample Cartier divisor H such that $K_X \underset{\mathrm{lin}}{\equiv} -(n-2)H$ and $H^n = 10$. If we assume that*

- *when $n \geq 3$, we have $\mathrm{Pic}(X) = \mathbf{Z}H$;*
- *when $n = 2$, the surface X is a Brill–Noether general K3 surface;*
- *when $n = 1$, the genus-6 curve X is Clifford general;*

then X is a GM variety.

To prove the theorem, we verify that a general hyperplane section of X satisfies the same conditions, then use induction on n.

Lemma 2.15. *Let X be a normal complex projective variety of dimension $n \geq 3$ with terminal singularities, $\mathrm{codim}(\mathrm{Sing}(X)) \geq 4$, and with an ample Cartier divisor H such that $H^n = 10$ and $K_X \underset{\mathrm{lin}}{\equiv} -(n-2)H$.*

The linear system $|H|$ is very ample and a general $X' \in |H|$ satisfies the same conditions: X' is normal with terminal singularities, $\mathrm{codim}(\mathrm{Sing}(X')) \geq 4$ and, if $H' := H|_{X'}$, we have $(H')^{n-1} = 10$ and $K_{X'} \underset{\mathrm{lin}}{\equiv} -(n-3)H'$.

Proof. By [A], the linear system $|H|$ is non-empty and by [M, Remark 2.6], a general $X' \in |H|$ is normal with terminal singularities. Moreover, $(H')^{n-1} = 10$ and, by the adjunction formula,

$$K_{X'} = (K_X + H)|_{X'} \underset{\mathrm{lin}}{\equiv} -(n-3)H|_{X'}.$$

Since X has terminal singularities, we have $H^1(X, \mathscr{O}_X) = 0$ by Kawamata–Viehweg vanishing, hence the linear series $|H'|$ is just the restriction of $|H|$ to X' and the base loci of $|H|$ and $|H'|$ are the same. Taking successive linear sections, we arrive at a linear section Y of dimension 3 with terminal singularities, $K_Y \underset{\mathrm{lin}}{\equiv} -H_Y$, and $H_Y^3 = 10$.

Assume some $x \in \mathrm{Sing}(X)$ is in the base locus of $|H|$. That point x is then on X', and, since a Cartier divisor is necessarily singular at a singular point of the ambient variety, it is also singular on X'. Repeating that argument, we see that x is singular on the threefold Y, and still a base point of $|H_Y| = |H||_Y$. Again, all divisors in $|H_Y|$ are singular at x, hence we are in one of the two cases described in [M, Theorem 2.9]. Since $H_Y^3 \neq 10$ in both of these cases, we get a contradiction. Therefore, $\mathrm{Sing}(X) \cap \mathrm{Base}(|H|) = \varnothing$ hence, by Bertini's theorem, $\mathrm{Sing}(X')$ still has codimension at least 4 in X' and eventually, Y is smooth.

If $\mathrm{Pic}(Y) = \mathbf{Z}H_Y$, the pair (Y, H_Y) is projectively normal by [IP, Corollary 4.1.13].

If not, we use the Mori–Mukai classification of smooth Fano threefolds Y with Picard number ≥ 2 ([MM], [IP, Chapter 12]) and see that there is only one family with anticanonical degree 10: Y must be a divisor of bidegree $(3,1)$ in $\mathbf{P}^3 \times \mathbf{P}^1$ and the pair (Y, H_Y) is again projectively normal.

A repeated use of [I, Lemma 2.9] then implies that in all cases, (X, H) is projectively normal, hence H is very ample by [Mum, p. 38]. $\qquad\square$

Lemma 2.16. *Let (X, H) be a polarized complex variety of dimension $n \geq 2$ which satisfies the hypotheses of Theorem 2.14. A general element of $|H|$ then satisfies the same properties.*

Proof. Assume first $n > 3$. By Lemma 2.15, we need only to prove that a general $X' \in |H|$ is locally factorial and that $\mathrm{Pic}(X')$ is generated by $H' := H|_{X'}$. Let $\mathrm{Cl}(X)$ be the group of Weil divisors on X, modulo linear equivalence. Since $|H|$ is very ample (Lemma 2.15), restriction of divisors induces, by [RT, Theorem 1], an isomorphism

$$\mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Cl}(X').$$

Since X is normal and locally factorial, the canonical inclusion $\mathbf{Z}H = \mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X)$ is an isomorphism. It follows that the canonical inclusion $\mathrm{Pic}(X') \hookrightarrow \mathrm{Cl}(X')$ is also an isomorphism, i.e., X' is locally factorial and $\mathrm{Pic}(X')$ is generated by H'.

When $n = 2$, X is a smooth Brill–Noether general K3 surface with a polarization H of degree 10, hence a general element of $|H|$ is a smooth Clifford general curve of genus 6.

When $n = 3$, X is a smooth Fano threefold with $\mathrm{Pic}(X) \simeq \mathbf{Z}$. By [IP, Corollary 4.1.13], it is an intersection of quadrics. Any smooth hyperplane section S of X is a degree-10 smooth K3 surface which is still an intersection of quadrics. A general hyperplane section of S is still an intersection of quadrics, hence is a Clifford general curve. This proves that S is Brill–Noether general. $\qquad\square$

We can now prove Theorem 2.14.

Proof of Theorem 2.14. We use induction on n. The case $n = 1$ was proved in Proposition 2.11, so we assume $n \geq 2$. A general hyperplane section X' of X has the same properties by Lemma 2.16, hence is a GM variety by the induction hypothesis. On the other hand, we have $H^1(X, \mathscr{O}_X) = 0$: this follows from Kodaira vanishing for $n \geq 3$, and from the fact that X is a K3 surface for $n = 2$. By Proposition 2.13, we conclude that X is a GM variety. $\qquad\square$

Remark 2.17. If we relax the conditions on the Picard group or on the singularities of X, the conclusion of Theorem 2.14 may be false. Indeed, a general divisor of bidegree $(3, 1)$ in $\mathbf{P}^3 \times \mathbf{P}^1$ is an example of a smooth Fano threefold of coindex 3 and degree 10 which is not an intersection of quadrics. Further counterexamples can be found in [PCS]: one is the threefold T_7 in Theorem 1.6 of that paper, which can be defined as the anticanonical image of the blow up of a quartic double solid in a line ([PCS, Example 1.11]). It has one ordinary double point hence has terminal Gorenstein singularities, but is not locally factorial, because it is the image of a non-trivial small contraction. Its Picard number is 1 and $|-K_{T_7}|$ is very ample of degree 10 and projective dimension 7, but T_7 is not an intersection of quadrics. In particular, the conclusion of Theorem 2.14 is definitely false for it.

2.4. **Grassmannian hulls.** We fix some more terminology and notation. Given a normal polarized GM variety (X, H) of dimension n, we consider the GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ constructed in Lemma 2.7 and define the Grassmannian hull of X by

$$M_X := \bigcap_{v \in V_5} Q(v) = \mathsf{C}_K \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W),$$

where the second equality is (9). Since $\dim(X) = n$, this intersection is dimensionally transverse and M_X is a variety of degree 5 and dimension $n + 1$, defined over the field $\mathbf{k}$. Moreover, by Lemma 2.8, we have

$$X = M_X \cap Q(v)$$

for any $v \in V_6 \smallsetminus V_5$. Since X is irreducible, so is M_X.

When μ is not injective, the Grassmannian hull is a cone. It is convenient to consider also the dimensionally transverse intersection

$$M'_X := \mathsf{Gr}(2, V_5) \cap \mathbf{P}(\mu(W)).$$

In terms of M'_X, the structure of a GM variety can be described as follows.

Lemma 2.18. *Let X be a GM variety with associated GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$. Set $K := \mathrm{Ker}(\mu)$ and $k := \dim(K)$, choose $v \in V_6 \smallsetminus V_5$ and set $Q := Q(v)$.*

If $k = 0$, we have $X = M'_X \cap Q$.

If $k > 0$, let $\tilde{X}$ be the blow up of X with center $\mathbf{P}(K) \cap Q \subset X$. The map μ then induces a regular map $\tilde{X} \xrightarrow{\tilde{\mu}} M'_X$ which is generically a $(k-1)$-dimensional quadric bundle if $\mathbf{P}(K) \not\subset Q$, or a $\mathbf{P}^{k-1}$-bundle if $\mathbf{P}(K) \subset Q$.

Proof. The first part is evident, since $M'_X = M_X$ when $K = 0$. For the second part, note that the blow up of the cone $M_X = \mathsf{C}_K(M'_X)$ at its vertex $\mathbf{P}(K)$ is a $\mathbf{P}^k$-bundle over M'_X. The blow up of X along $X \cap \mathbf{P}(K) = \mathbf{P}(K) \cap Q(v)$ is the same as the proper preimage of X. In particular, it comes with a regular map $\tilde{X} \to M'_X$.

If $\mathbf{P}(K) \not\subset Q$, the proper preimage coincides with the total preimage, hence its fibers are intersections $\mathbf{P}^k \cap Q$, i.e., generically, they are quadrics of dimension $k - 1$. If $\mathbf{P}(K) \subset Q$, the general fiber of the total preimage of X is a quadric $Q \cap \mathbf{P}^k$ which contains the hyperplane $\mathbf{P}(K) = \mathbf{P}^{k-1}$. Therefore, the general fiber of the proper preimage is the residual hyperplane. $\square$

Later on, we will give more details for the case $k = 1$ and $\mathbf{P}(K) \not\subset Q$.

In this article, we are mostly interested in the case of smooth GM varieties. However, for many questions, the condition $\mathrm{codim}(\mathrm{Sing}(X)) \geq 4$ is almost as good as smoothness. Keeping in mind possible applications of singular GM varieties, we keep the above assumption as long as possible.

Lemma 2.19. *Let $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ be a GM data of dimension n. If the associated GM intersection* (8) *is a GM variety X of dimension n with $\mathrm{codim}(\mathrm{Sing}(X)) \geq 4$, we have*

$$\dim(\mu(W)) \geq \min\{7, n+4\}.$$

Proof. As in the proof of Lemma 2.8, we can write X as in (1) with $K = \mathrm{Ker}(\mu)$. Then, $\mathbf{P}(K)$ is contained in the singular locus of $\mathsf{C}_K \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W)$, hence $\mathbf{P}(K) \cap Q(v)$ is contained in the singular locus of X. It has dimension at least $(n+5) - r - 2 = n + 3 - r$, where $r := \mathrm{rank}(\mu)$. Thus the condition $\mathrm{codim}(\mathrm{Sing}(X)) \geq 4$ implies that either $n + 3 - r < 0$, or $n + 3 - r \leq n - 4$. The first is equivalent to $r \geq n + 4$ and the second to $r \geq 7$. $\square$

Proposition 2.20. *Let X be a GM variety of dimension n with $\mathrm{codim}(\mathrm{Sing}(X)) \geq 4$. If $n \geq 3$ or $\mathrm{Ker}(\mu) \neq 0$, the variety M'_X is smooth. Otherwise (if $n \leq 2$ and $\mathrm{Ker}(\mu) = 0$), M'_X is equal to M_X and has finitely many rational double points.*

Proof. We begin with a useful observation. Let $W_0 \subset \bigwedge^2 V_5$ be a linear subspace. Assume $M' := \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_0)$ is a dimensionally transverse intersection. Viewing elements of the orthogonal complement $W_0^\perp \subset \bigwedge^2 V_5^\vee$ as skew-symmetric forms on V_5, one has (see, e.g., [PV, Corollary 1.6])

$$(10) \qquad \mathrm{Sing}(M') = \mathbf{P}(W_0) \cap \bigcup_{\omega \in W_0^\perp \smallsetminus \{0\}} \mathsf{Gr}(2, \mathrm{Ker}(\omega)).$$

If $\dim(W_0) = 10$, then $M' = \mathsf{Gr}(2, V_5)$ is smooth. If $\dim(W_0) = 9$, either a generator ω of $W_0^\perp$ has rank 2 and $\mathrm{Sing}(M')$ is a 2-plane, or else M' is smooth. If $\dim(W_0) = 8$, either some $\omega \in W_0^\perp$ has rank 2, in which case $\mathsf{Gr}(2, \mathrm{Ker}(\omega))$ is a 2-plane contained in the hyperplane $\omega^\perp$, whose intersection with $\mathbf{P}(W_0)$ therefore contains a line along which M' is singular, or else M' is smooth. Finally, if $\dim(W_0) \leq 7$, we have $\dim(M') \leq 3$. It follows that in all cases, either M' is smooth or $\mathrm{codim}(\mathrm{Sing}(M')) \leq 3$.

Now let X be a GM variety such that M'_X is singular. By the above observation, we have $\mathrm{codim}(\mathrm{Sing}(M'_X)) \leq 3$. Since $X = \mathsf{C}_{\mathrm{Ker}(\mu)} M'_X \cap Q(v)$ and $\mathrm{codim}(\mathrm{Sing}(X)) \geq 4$, the only possibility is that $Q(v)$ does not intersect the cone over $\mathrm{Sing}(M'_X)$. As $Q(v)$ is an ample hypersurface, this is only possible if $\mathrm{Ker}(\mu) = 0$ and $\mathrm{Sing}(M'_X)$ is finite. In this case, $M'_X = M_X$ has dimension 2 or 3, hence $n \leq 2$, so we are in the second case of the proposition.

It remains to show that in that case, the only singularities of M_X are rational double points. If $n = 1$, we have $\dim(W) = 6$ and $M_X = \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W)$ is a normal surface (it is lci with finite singular locus). The restriction map $H^0(\mathbf{P}(W), \mathscr{O}_{\mathbf{P}(W)}(1)) \to H^0(M_X, \mathscr{O}_{M_X}(1))$ is bijective, since its composition with the injection $H^0(M_X, \mathscr{O}_{M_X}(1)) \to H^0(X, \mathscr{O}_X(1))$ is. It follows that M_X is linearly normal. If it is a cone with vertex w, it is contained in $\mathsf{Gr}(2, V_5) \cap \mathbf{T}_{\mathsf{Gr}(2,V_5),w}$ (intersection with a tangent space), which is a 4-dimensional cone with vertex w and span $\mathbf{T}_{\mathsf{Gr}(2,V_5),w} \simeq \mathbf{P}^6$. Since M_X spans the 5-plane $\mathbf{P}(W)$, it is a hyperplane section of this cone, but this contradicts the fact that M_X is a surface.

So the quintic surface M_X is not a cone, hence it is a del Pezzo surface ([D, Proposition 8.1.6 and Definition 8.1.5]). This implies that the singularities of M_X are rational double points ([D, Theorem 8.1.11]).

When $n = 2$, the singular points of the threefold M_X are rational double points, since a general hyperplane section through such a point has this property, by the case $n = 1$.   □

Remark 2.21. Given a *singular* linear section M' of $\mathsf{Gr}(2, V_5)$ of dimension $n+1 \geq 4$, one checks that a general quadratic section of M' is singular in codimension 3 and its singularities are canonical but not terminal.

The following elementary result is useful for checking that Grassmannian hulls are smooth dimensionally transverse intersections.

Proposition 2.22. *Let $W_0 \subset \bigwedge^2 V_5$ be a linear subspace. Set $M := \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_0)$ and $M^\sharp := \mathsf{Gr}(2, V_5^\vee) \cap \mathbf{P}(W_0^\perp)$. Then M is a smooth dimensionally transverse intersection if and only if $M^\sharp$ has the same property. In that case, if moreover both M and $M^\sharp$ are non-empty, one has $\dim(M) + \dim(M^\sharp) = 2$.*

Proof. The scheme M is a smooth dimensionally transverse intersection at a point $w \in M$ if and only if $\mathbf{T}_{\mathsf{Gr}(2,V_5),w}$ and $\mathbf{P}(W_0)$ span $\mathbf{P}(\bigwedge^2 V_5)$. This is not the case exactly when there exists a hyperplane $H \subset \mathbf{P}(\bigwedge^2 V_5)$ containing both $\mathbf{T}_{\mathsf{Gr}(2,V_5),w}$ and $\mathbf{P}(W_0)$.

The pair (w, H) then belongs to the incidence variety of $\mathsf{Gr}(2, V_5)$, whose projective dual is $\mathsf{Gr}(2, V_5^\vee)$. It follows that the point $H^\perp \in \mathbf{P}(\bigwedge^2 V_5^\vee)$ is in $\mathsf{Gr}(2, V_5^\vee)$ and that the hyperplane $w^\perp$ contains $\mathbf{T}_{\mathsf{Gr}(2,V_5),H^\perp}$. Since $w \in \mathbf{P}(W_0)$, we also have $w^\perp \supset \mathbf{P}(W_0^\perp)$, so that $M^\sharp = \mathsf{Gr}(2, V_5^\vee) \cap \mathbf{P}(W_0^\perp)$ is not a smooth dimensionally transverse intersection at $H^\perp$.   □

Remark 2.23. Let X be a GM variety of dimension $n \geq 3$ with $\mathrm{codim}(\mathrm{Sing}(X)) \geq 4$. By Lemma 2.19, we have $\dim(\mu(W)) \geq 7$, hence $\dim(\mu(W)^\perp) \leq 3$. By Proposition 2.20, M'_X is smooth of dimension $6 - \mathrm{codim}(\mu(W)) \geq 3$. Proposition 2.22 therefore says that $(M'_X)^\sharp$ is empty (i.e., $\mu(W)^\perp \subset \bigwedge^2 V_5^\vee$ contains no forms of rank 2).

For smooth GM curves, $(M'_X)^\sharp$ is never empty. For smooth GM surfaces, $(M'_X)^\sharp$ may be non-empty. We finish this subsection with a discussion of these two cases.

Let X be a smooth GM curve. As we will explain in more details in Section 2.5, either M'_X is a smooth elliptic quintic curve in $\mathbf{P}(\mu(W)) = \mathbf{P}^4$, there is a double cover $X \to M'_X$

(so that X is bielliptic), M_X is the cone over M'_X, and $(M'_X)^\sharp$ is isomorphic to M'_X, or else $M_X = M'_X$ is a quintic del Pezzo surface in $\mathbf{P}(\mu(W)) = \mathbf{P}^5$.

Mukai remarked ([Mu2, Theorem 5.1.(2)]) that $(M'_X)^\sharp$ has a simple geometric interpretation as the set of all g^1_4 on X. This allows us to recover a classical result without appealing to the description of del Pezzo quintics as blow ups of the projective plane (compare with [S-B1, Proposition 4]).

Proposition 2.24. *Let X be a smooth complex GM curve. The Grassmannian hull M_X is smooth, i.e., X is the intersection of a smooth quintic del Pezzo surface with a quadric, if and only if X has exactly 5 distinct g^1_4.*

Proof. As we just explained, X is the intersection of a smooth quintic del Pezzo surface with a quadric if and only if M'_X is a smooth (quintic) surface. By Proposition 2.22, this is the case if and only if $(M'_X)^\sharp$ is a smooth scheme of dimension 0. On the other hand, the scheme $W^1_4(X)$ parametrizing g^1_4 on X is either infinite, in which case X is bielliptic, or it has length 5. The proposition therefore follows from Mukai's remark ([Mu2, Theorem 5.1.(2)]) that there is a bijection between the sets $(M'_X)^\sharp$ and $W^1_4(X)$.   □

Let (X, H) be a smooth GM surface. As in the case of curves, M'_X may be singular. By Proposition 2.20, this may only happen when μ is injective. By Proposition 2.22, $M_X = M'_X$ is then a singular threefold and $(M'_X)^\sharp$ is non-empty. This has strong consequences on X.

More precisely, one can show that $(M'_X)^\sharp$ is non-empty if and only if the Gushel bundle can be written over $\mathbf{C}$ as an extension

$$0 \to \mathscr{O}_{X_\mathbf{C}}(-h_2) \to \mathscr{U}_{X_\mathbf{C}} \to \mathscr{O}_{X_\mathbf{C}}(-h_1) \to 0,$$

where h_1 and h_2 are divisors on $X_\mathbf{C}$ such that

$$h_1^2 = 0, \qquad h_1 h_2 = 4, \qquad h_2^2 = 2, \qquad h_1 + h_2 = H$$

(compare with [GLT, Lemma 1.5]). In particular, the Picard number of $X_\mathbf{C}$ must be > 1.

2.5. Locally complete intersection GM varieties.
In the rest of the paper, we mostly deal with GM varieties which are lci. This condition can be described in terms of GM data.

Definition 2.25. A GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ is called lci if

- either $\mathrm{Ker}(\mu) = 0$, in which case the data is called ordinary;
- or $\dim(\mathrm{Ker}(\mu)) = 1$ and the point of $\mathbf{P}(W)$ corresponding to the kernel subspace $\mathrm{Ker}(\mu) \subset W$ does not lie on the quadric $Q(v)$ for any $v \in V_6 \smallsetminus V_5$, in which case the data is called special.

Proposition 2.26. *A normal GM variety is lci if and only if its canonically associated GM data is lci.*

Proof. Since $\mathsf{Gr}(2, V_5)$ is not a complete intersection, the non-lci locus of the cone $\mathsf{C}_K \mathsf{Gr}(2, V_5)$ is precisely its vertex $\mathbf{P}(K)$. Since a GM variety X is a dimensionally transverse intersection (1), it is lci if and only if it does not contain points of $\mathbf{P}(K)$, i.e., if and only if $X \cap \mathbf{P}(K) = \mathbf{P}(\mathrm{Ker}(\mu)) \cap Q(v)$ is empty. The latter condition can be rephrased as $\dim(\mathrm{Ker}(\mu)) \le 1$ and, in case of equality, the corresponding point is not in $Q(v)$.   □

The next lemma will usually be applied to smooth GM varieties, but as usual we prove it under a weaker assumption. In this form, it is a partial converse to Theorem 2.14.

Lemma 2.27. *Let (X, H) be an lci polarized complex GM variety of dimension $n \geq 3$. If* $\mathrm{codim}(\mathrm{Sing}(X)) \geq 4$, *the variety X is locally factorial and* $\mathrm{Pic}(X) = \mathbf{Z}H$. *In particular, the polarization H is the unique GM polarization on X, hence* $\mathrm{Aut}_H(X) = \mathrm{Aut}(X)$.

Proof. By [SGA2, Exp. XI, cor. 3.14], X is locally factorial. It remains to show that $\mathrm{Pic}(X) = \mathbf{Z}H$. Assume first $n = 3$, so that X is smooth. By Theorem 2.3, X is a Fano variety of degree 10 and is an intersection of quadrics. As we already mentioned in the proof of Lemma 2.15, the only Fano threefold of degree 10 and Picard number greater than 1 is not an intersection of quadrics. This proves the claim for $n = 3$. When $n > 3$, we proceed by induction: a general hyperplane section X' of X satisfies the same properties and restriction induces an injection $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(X')$ ([SGA2, Exp. XII, cor. 3.6]). $\square$

For any lci GM variety X, the Gushel sheaf $\mathscr{U}_X$ is locally free and defines a regular map
$$X \to \mathsf{Gr}(2, V_5)$$
which we call the Gushel map. It coincides with the map defined in Lemma 2.18 (the lci assumption means $X \cap \mathbf{P}(K) = \varnothing$, hence $\tilde{X} = X$). Moreover,

- either $\mu\colon W \to \bigwedge^2 V_5$ is injective, the Gushel map is a closed embedding, we have $M_X = M'_X$, and (1) can be rewritten as
$$X = \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W) \cap Q(v),$$
so that X is a quadratic section of M_X. These GM varieties will be called ordinary. For these varieties,
$$\dim(X) = \dim(W) - 5 \leq \dim(\textstyle\bigwedge^2 V_5) - 5 = 5.$$

- or else $\mathrm{Ker}(\mu)$ has dimension 1, the Gushel map induces a double covering $X \twoheadrightarrow M'_X$, and $M_X = \mathsf{C}M'_X$. These GM varieties will be called special. For these varieties,
$$\dim(X) = \dim(W) - 5 \leq \dim(\textstyle\bigwedge^2 V_5) + 1 - 5 = 6.$$

A special GM variety comes with a canonical involution—the involution of the double covering $X \twoheadrightarrow M'_X$.

We make a useful observation. Let $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ be an lci GM data, set

(11) $W_1 := \mathrm{Ker}(\mu) \subset W, \qquad W_0 := W/W_1,$

and let $\mu_0\colon W_0 \hookrightarrow \bigwedge^2 V_5$ be the embedding induced by μ.

Proposition 2.28. *For any lci GM data* $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$, *there is a unique splitting of the canonical exact sequence* $0 \to W_1 \to W \to W_0 \to 0$, *i.e., a direct sum decomposition*

(12) $W = W_0 \oplus W_1,$

such that the map $\mathbf{q}$ decomposes as

(13) $\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1, \qquad \text{where} \quad \mathbf{q}_0\colon V_6 \to \mathsf{S}^2 W_0^\vee \quad and \quad \mathbf{q}_1\colon V_6 \to \mathsf{S}^2 W_1^\vee.$

Proof. If the data is ordinary, we have $W_1 = 0$, $W_0 = W$ and there is nothing to prove. If W_1 is non-zero, it is 1-dimensional. Let $w_1 \in W_1$ be any non-zero vector and choose an arbitrary $v \in V_6 \smallsetminus V_5$. Since the data is lci, the quadric $Q(v)$ does not pass through the point $w_1 \in W_1$, hence the linear form $\mathbf{q}(v)(w_1, -) \in W^\vee$ does not vanish on w_1 and gives a decomposition

$W = W_0 \oplus W_1$ such that $\mathbf{q}(v) \in \mathsf{S}^2 W_0^\vee \oplus \mathsf{S}^2 W_1^\vee \subset \mathsf{S}^2 W^\vee$. By (7), we have $\mathbf{q}(V_5) \subset \mathsf{S}^2 W_0^\vee$, hence the decomposition of W does not depend on the choice of v and the image of $\mathbf{q}$ is contained in $\mathsf{S}^2 W_0^\vee \oplus \mathsf{S}^2 W_1^\vee \subset \mathsf{S}^2 W^\vee$. We let $\mathbf{q}_0$ and $\mathbf{q}_1$ be the summands of $\mathbf{q}$ corresponding to this direct sum decomposition. $\qquad\square$

Remark 2.29. One could prove the same result as in Proposition 2.28, replacing the lci condition by the assumption that, for $v \notin V_5$, the restriction to $W_1 := \mathrm{Ker}(\mu) \subset W$ of the quadratic form $\mathbf{q}(v)$ is non-degenerate (this assumption does not depend on the choice of v). The proof is the same—W_0 is defined as the orthogonal of W_1 in W with respect to $\mathbf{q}(v)$.

Corollary 2.30. *Let (X, H) be a normal lci polarized GM variety. The kernel of the canonical morphism $\mathrm{Aut}_H(X) \to \mathrm{PGL}(V_6)$ is trivial if X is ordinary, and is generated by the canonical involution of X if X is special.*

Proof. By Corollary 2.10, the group $\mathrm{Aut}_H(X)$ is a quotient of the group

$$\mathrm{Aut}(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon) \subset \mathrm{GL}(W) \times \mathrm{GL}(V_6) \times \mathbb{G}_{\mathrm{m}}$$

of automorphisms of its associated GM data. Such an automorphism $g = (g_W, g_V, g_L)$ maps to the identity in $\mathrm{PGL}(V_6)$ if its component g_V is scalar. Its actions on V_5, on $\bigwedge^2 V_5$, and hence on $W_0 \subset \bigwedge^2 V_5$, are then also scalar. For ordinary varieties $X \subset \mathbf{P}(W_0)$, the action of such a g on X is therefore trivial.

For special varieties, since the direct sum decomposition $W = W_0 \oplus W_1$ is canonical, it is preserved by g_W, and since W_1 is 1-dimensional, g_W acts by multiplication on each summand, by scalars t_0 and t_1. Then g acts by multiplication by t_0^{-2} on $\mathsf{S}^2 W_0^\vee$ and by t_1^{-2} on $\mathsf{S}^2 W_1$. Since the map $\mathbf{q}\colon V_6 \to \mathsf{S}^2 W_0^\vee \oplus \mathsf{S}^2 W_1^\vee$ commutes with the action of g, and the action of g on V_6 is scalar, we get $t_0^{-2} = t_1^{-2}$. Thus $t_1 = \pm t_0$, hence either g acts on $\mathbf{P}(W)$ identically, or by a reflection with respect to $\mathbf{P}(W_0)$, i.e., by the canonical involution of X. $\qquad\square$

For a special GM variety X of dimension $n \geq 2$, the branch locus X' of the Gushel map $X \twoheadrightarrow M_X'$ is the dimensionally transverse intersection of M_X' with a quadric, so it is an ordinary GM variety. This leads to an important birational operation on the set of all GM varieties: interchanging ordinary and special varieties. We formulate this operation on the level of GM data.

Lemma 2.31. *If $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ is a special lci GM data, $(W_0, V_6, V_5, L, \mu_0, \mathbf{q}_0, \varepsilon)$ is an ordinary lci GM data. Conversely, let $(W_0, V_6, V_5, L, \mu_0, \mathbf{q}_0, \varepsilon)$ be an ordinary lci GM data. Let W_1 be a 1-dimensional vector space, choose an isomorphism $V_6/V_5 \simeq S^2 W_1^\vee$, and let $\mathbf{q}_1$ be the composition $V_6 \to V_6/V_5 \to S^2 W_1^\vee$. Then $(W_0 \oplus W_1, V_6, V_5, L, \mu_0, \mathbf{q}_0 + \mathbf{q}_1, \varepsilon)$ is a special lci GM data. If the field $\mathbf{k}$ is quadratically closed, i.e., if $\mathbf{k} = \mathbf{k}^2$, these operations define a bijection between the sets of isomorphism classes of special and ordinary GM data respectively.*

Proof. The first part is obvious. For the second part, note that different choices of an isomorphism $V_6/V_5 \simeq S^2 W_1^\vee$ produce isomorphic GM data (via the isomorphism defined by $g_V = \mathsf{id}_V$, $g_W = \mathsf{id}_{W_0} + \sqrt{t}\,\mathsf{id}_{W_1}$, $g_L = \mathsf{id}_L$ for an appropriate $t \in \mathbf{k}$). $\qquad\square$

Remark 2.32. If $\mathbf{k} \neq \mathbf{k}^2$, there are two non-isomorphic (over $\mathbf{k}$) special varieties corresponding to a given ordinary variety. They become isomorphic over $\mathbf{k}(\sqrt{-1})$.

When $\mathbf{k} = \mathbf{k}^2$, this bijection can be interpreted as a bijection between isomorphism classes of special and ordinary lci GM varieties. We will denote by X_{ord} the ordinary lci GM

intersection associated with a given special lci GM intersection X and by X_{spe} the special lci GM intersection associated with a given ordinary lci GM intersection X. We define the **opposite** X_{opp} of an lci GM intersection X by

$$(14) \qquad\qquad X_{\mathrm{opp}} := \begin{cases} X_{\mathrm{ord}} & \text{if } X \text{ is special,} \\ X_{\mathrm{spe}} & \text{if } X \text{ is ordinary.} \end{cases}$$

3. GM varieties, Lagrangian data, and EPW sextics

Eisenbud–Popescu–Walter (EPW) sextics are special hypersurfaces of degree 6 in $\mathbf{P}(V_6)$ which can be constructed from Lagrangian subspaces $A \subset \bigwedge^3 V_6$. The definition and the main facts about EPW sextics can be found in Appendix B. A relation between GM varieties and EPW sextics was found in [IM]. In this section, we develop the approach of Iliev and Manivel and extend their construction to include both ordinary and special varieties.

3.1. **The discriminant locus.** The easiest way to relate an EPW sextic to a GM variety is via the discriminant locus. Let (X, H) be a normal polarized GM variety of dimension n as in Definition 2.1. By Theorem 2.3, the space V_6 of quadrics in $\mathbf{P}(W)$ containing X is 6-dimensional. We define $\widetilde{\mathrm{Disc}}(X)$ as the subscheme of $\mathbf{P}(V_6)$ of *singular* quadrics containing X. It is either $\mathbf{P}(V_6)$ or a hypersurface of degree $\dim(W) = n + 5$, and in the latter case, the multiplicity in $\widetilde{\mathrm{Disc}}(X)$ of the hyperplane $\mathbf{P}(V_5)$ of (restrictions of) Plücker quadrics is at least the corank of a general such quadric, which is at least $\dim(W) - 6 = n - 1$. We define the **discriminant locus** $\mathrm{Disc}(X)$ as follows: if $\widetilde{\mathrm{Disc}}(X) = \mathbf{P}(V_6)$, we set $\mathrm{Disc}(X) = \mathbf{P}(V_6)$; otherwise (and we will see in Corollary 3.17 that this is almost always the case), we set

$$(15) \qquad\qquad \mathrm{Disc}(X) := \widetilde{\mathrm{Disc}}(X) - (n-1)\mathbf{P}(V_5).$$

This is a sextic hypersurface in $\mathbf{P}(V_6)$.

Theorem 3.1 (Iliev–Manivel)**.** *Let X be a general (smooth and ordinary) complex GM variety of dimension $n \in \{3, 4, 5\}$. The discriminant locus $\mathrm{Disc}(X) \subset \mathbf{P}(V_6)$ is an EPW sextic.*

The result is proved in [IM, Proposition 2.1 and Lemma 2.2] for general GM fivefolds by explicitly constructing a Lagrangian $A \subset \bigwedge^3 V_6$ such that $\mathrm{Disc}(X) = Y_A$ (see also [O2, Proposition 2.18]). This construction extends to general GM varieties of dimensions 3, 4, or 5 ([IM, Proposition 2.4]). In the next section, we present a version of this construction which works better in families and allows us to treat at the same time both ordinary and special GM varieties. In particular, we give in Proposition 3.16 an extension of Theorem 3.1.

For the time being, we examine the discriminant locus of GM curves and relate the discriminant locus of a GM variety to that of its opposite as defined in (14).

Proposition 3.2. *Let X be a smooth GM curve. The discriminant locus $\mathrm{Disc}(X)$ is a reduced sextic hypersurface and,*

- *if X is ordinary, $\mathrm{Disc}(X)$ is geometrically integral and normal;*
- *if X is special (i.e., bielliptic), $\mathrm{Disc}(X)$ is the union of the Plücker hyperplane $\mathbf{P}(V_5)$ and a geometrically integral quintic hypersurface.*

Proof. We may assume $\mathbf{k} = \mathbf{C}$. A local calculation shows that the Zariski tangent space to $\widetilde{\mathrm{Disc}}(X)$ at a point corresponding to a quadric Q containing X has dimension greater than 4 if and only if

a) either Q has corank 1 but its vertex is on X;

b) or else Q has corank ≥ 2 (hence rank ≤ 4).

Assume first that X is ordinary. In case a), Q must be a Plücker quadric, because otherwise, X would be equal to $M_X \cap Q$ and would be singular at the vertex of Q. But corank-1 quadrics with vertex at $w \in X$ are in one-to-one correspondence with smooth quadrics containing the projection of M_X from w. This projection is a del Pezzo surface of degree 4, hence is the base locus of a pencil of quadrics in $\mathbf{P}^4$. Therefore, these quadrics form an open subset in a ruled surface with base X.

We now examine case b). The locus of quadrics of rank ≤ 4 containing X was studied extensively in [AH]. It contains five 2-planes (one for each g_4^1 on X, counted with multiplicities; [AH, (5.10)]) of Plücker quadrics and one more two-dimensional component, which comes from the singular points of the theta divisor of the Jacobian of X, i.e., from the g_5^1 on X.

All in all, this proves that the singular locus of $\mathrm{Disc}(X)$ has dimension 2. It follows that $\mathrm{Disc}(X)$ is an integral normal sextic hypersurface.

Assume now that X is bielliptic. In case a), Q must again be a Plücker quadric. It is then also singular at the vertex of M_X, hence cannot be of corank 1. So case a) is impossible.

Plücker quadrics are all singular at the vertex and have general corank 1, hence $\mathbf{P}(V_5)$ is a reduced component of $\mathrm{Disc}(X)$; write $\mathrm{Disc}(X) = \mathbf{P}(V_5) + D$, where D is a quintic hypersurface not containing $\mathbf{P}(V_5)$. Then $\mathrm{Sing}(\mathrm{Disc}(X)) \cap \mathbf{P}(V_5) = \mathbf{P}(V_5) \cap D$ corresponds to singular quadrics containing the quintic elliptic curve M_X', and this was proved in [AH, p. 170] to be an integral quintic hypersurface. It follows that D is also an integral quintic hypersurface. $\square$

Lemma 3.3. *Let X be a special lci GM variety with associated ordinary variety X_{ord}. Then $\mathrm{Disc}(X) = \mathrm{Disc}(X_{\mathrm{ord}})$.*

Proof. Let $(V_6, V_5, L, W, \mu, \mathbf{q}, \varepsilon)$ be the GM data associated with X. Let $W = W_0 \oplus W_1$ be the canonical direct sum decomposition of Proposition 2.28 and let $\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1$ be the decomposition of $\mathbf{q}$. By Lemma 2.31, the ordinary GM variety X_{ord} is determined by $\mathbf{q}_0 \colon V_6 \to \mathsf{S}^2 W_0^\vee$. In particular, the rank of $\mathbf{q}(v)$ is the sum of the ranks of $\mathbf{q}_0(v)$ and $\mathbf{q}_1(v)$. By Lemma 2.31, the map $\mathbf{q}_1$ vanishes on V_5 and induces an isomorphism $V_6/V_5 \simeq \mathsf{S}^2 W_1^\vee$. Therefore, the rank of $\mathbf{q}(v)$ equals the rank of $\mathbf{q}_0(v)$ for Plücker quadrics, and increases by 1 for non-Plücker quadrics, which means that the corank of Plücker quadrics increases by 1, and the corank of non-Plücker quadrics stays the same. This shows the lemma. $\square$

3.2. **Extended construction.** The content of this subsection is the construction, slightly modified from [IM], of a bijection between the set of lci GM data and the set of what we call (extended) Lagrangian data.

Consider a 6-dimensional vector space V_6 and endow the space $\bigwedge^3 V_6$ with the canonical $\det(V_6)$-valued symplectic form given by wedge product.

Definition 3.4. A Lagrangian data is a collection (V_6, V_5, A), where

- V_6 is a $\mathbf{k}$-vector space of dimension 6,

- $V_5 \subset V_6$ is a hyperplane,
- $A \subset \bigwedge^3 V_6$ is a Lagrangian subspace.

An isomorphism of Lagrangian data between (V_6, V_5, A) and (V_6', V_5', A') is a linear isomorphism $\varphi \colon V_6 \xrightarrow{\sim} V_6'$ such that $\varphi(V_5) = V_5'$ and $(\bigwedge^3 \varphi)(A) = A'$.

A natural extension of the Iliev–Manivel construction gives a bijection between the set of isomorphism classes of Lagrangian data and the set of isomorphism classes of ordinary GM data. Using the bijection of Lemma 2.31, one can also use Lagrangian data to parameterize all special GM data. However, to deal simultaneously with ordinary and special GM varieties, one needs a generalization of the Iliev–Manivel construction. We suggest such a generalization; it uses an additional Lagrangian subspace which encodes the type of a GM variety.

To lighten the notation (and for forward compatibility), we set

$$(16) \qquad\qquad L := (V_6/V_5)^\vee$$

and endow the space $\mathbf{k} \oplus L$ with the canonical L-valued symplectic form. Note that the group $\mathrm{GL}(L) \simeq \mathbb{G}_m$ acts naturally on the Lagrangian Grassmannian $\mathsf{LGr}(\mathbf{k} \oplus L) \simeq \mathbf{P}^1$.

Definition 3.5. An **extended Lagrangian data** is a collection (V_6, V_5, A, A_1), where (V_6, V_5, A) is a Lagrangian data and $A_1 \subset \mathbf{k} \oplus L$ is a Lagrangian subspace.

An **isomorphism of extended Lagrangian data** between (V_6, V_5, A, A_1) and (V_6', V_5', A', A_1') is an isomorphism between (V_6, V_5, A) and (V_6', V_5', A') and an element $t \in \mathbb{G}_m$ such that $t(A_1) = A_1'$.

The $\mathrm{GL}(L) \simeq \mathbb{G}_m$-action on $\mathsf{LGr}(\mathbf{k} \oplus L) \simeq \mathbf{P}^1$ has three orbits. To simplify the notation, we choose a coordinate on this line such that

- the subspace $A_1 = \mathbf{k} \subset \mathbf{k} \oplus L$ corresponds to the point $\{\infty\} \in \mathbf{P}^1$;
- the subspace $A_1 = L \subset \mathbf{k} \oplus L$ corresponds to the point $\{0\} \in \mathbf{P}^1$.

The points $\{0\}$ and $\{\infty\}$ are two of the $\mathrm{GL}(L)$-orbits and the point $\{1\} \in \mathbf{P}^1$ is in the third orbit. To simplify the notation, we will sometimes write $A_1 = \infty$, $A_1 = 0$, and $A_1 = 1$ instead of $A_1 = \mathbf{k}$, $A_1 = L$, and $A_1 \neq \mathbf{k}, L$, respectively.

Theorem 3.6. *If $\mathbf{k}$ is quadratically closed, there is a natural bijection (defined in the proof) between the set of isomorphism classes of extended Lagrangian data with $A_1 \neq \infty$ and the set of isomorphism classes of lci GM data.*

For any field $\mathbf{k}$, it induces a bijection between the set of isomorphism classes of Lagrangian data, the set of isomorphism classes of ordinary lci GM data.

Remark 3.7. If $\mathbf{k}$ is not quadratically closed, we still have a bijection between ordinary GM data and Lagrangian data, but only a two-to-one correspondence between special GM data and Lagrangian data.

The construction of GM data can be also applied to Lagrangian data with $A_1 = \infty$; however, the GM data obtained from this is not lci, and the lci condition is crucial for the inverse construction, since we need the canonical decomposition $W = W_0 \oplus W_1$ of Proposition 2.28.

Before describing the construction, we need one more piece of notation. The canonical projection

$$\lambda \colon V_6 \to V_6/V_5 = L^\vee$$

defines, by the Leibniz rule, for all $p \geq 1$, maps $\lambda_p \colon \bigwedge^p V_6 \to \bigwedge^{p-1} V_5 \otimes (V_6/V_5)$. They fit into exact sequences

$$0 \to \bigwedge^p V_5 \hookrightarrow \bigwedge^p V_6 \xrightarrow{\lambda_p} \bigwedge^{p-1} V_5 \otimes L^\vee \to 0.$$

Proof of Theorem 3.6. Let (V_6, V_5, A, A_1) be an extended Lagrangian data. We use the notation (16) and we choose an arbitrary isomorphism

$$\varepsilon \colon \det(V_5) \xrightarrow{\sim} L^2.$$

Consider the maps

$$A \lhook\joinrel\longrightarrow \bigwedge^3 V_6 \xrightarrow{\lambda_3} \bigwedge^2 V_5 \otimes L^\vee,$$

$$A_1 \lhook\joinrel\longrightarrow \mathbf{k} \oplus L \xrightarrow{\mathrm{pr}_1} \mathbf{k}.$$

Let W_0 and W_1 be their images, so that we have canonical factorizations

$$A \longrightarrow\!\!\!\!\!\rightarrow W_0 \overset{\mu_0}{\lhook\joinrel\longrightarrow} \bigwedge^2 V_5 \otimes L^\vee,$$

$$A_1 \longrightarrow\!\!\!\!\!\rightarrow W_1 \overset{\mu_1}{\lhook\joinrel\longrightarrow} \mathbf{k},$$

(defining the maps μ_0 and μ_1). We set

$$W = W_0 \oplus W_1 \quad , \quad \mu = \mu_0 \oplus \mu_1.$$

Note that by definition

$$\mathrm{Ker}(A \twoheadrightarrow W_0) = A \cap \bigwedge^3 V_5 \quad , \quad \mathrm{Ker}(A_1 \twoheadrightarrow W_1) = A_1 \cap L.$$

It remains to define the map $\mathbf{q} \colon V_6 \to \mathsf{S}^2 W^\vee$. For this, we first define

$$\tilde{\mathbf{q}}_0 \colon V_6 \otimes \mathsf{S}^2 A \to \mathbf{k}, \qquad \tilde{\mathbf{q}}_0(v)(\xi_1, \xi_2) := -\varepsilon(\lambda_4(v \wedge \xi_1) \wedge \lambda_3(\xi_2)),$$

$$\tilde{\mathbf{q}}_1 \colon V_6 \otimes \mathsf{S}^2 A_1 \to \mathbf{k}, \qquad \tilde{\mathbf{q}}_1(v)(x_1, x_1', x_2, x_2') := \lambda(v) x_1 x_2'.$$

It is not immediately clear that $\tilde{\mathbf{q}}_0(v)$ and $\tilde{\mathbf{q}}_1(v)$ are symmetric in their arguments; we will show later that this follows from the Lagrangian property of A and A_1, so the above definition makes sense.

For each $v \in V_6$ and $i \in \{0, 1\}$, the kernel of the quadratic form $\tilde{\mathbf{q}}_i(v)$ contains the kernel of the projections $A \twoheadrightarrow W_0$ and $A_1 \twoheadrightarrow W_1$. Indeed, if $\xi_2 \in A \cap \bigwedge^3 V_5$, we have $\lambda_3(\xi_2) = 0$, hence $\tilde{\mathbf{q}}_0(v)(\xi_1, \xi_2) = 0$ for any $\xi_1 \in A$. Analogously, if $(x_1, x_1') \in L$, i.e., $x_1 = 0$, then $\tilde{\mathbf{q}}_1(v)(x_1, x_1', x_2, x_2') = 0$ for any $(x_2, x_2') \in A_1$. This means that $\tilde{\mathbf{q}}_i(v)$, considered as a map $\mathsf{S}^2 A \to \mathbf{k}$ or $\mathsf{S}^2 A_1 \to \mathbf{k}$, factors through $\mathsf{S}^2 W_i$ and thus defines a quadratic form $\mathbf{q}_i(v) \in \mathsf{S}^2 W_i^\vee$. We finally define

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1 \colon V_6 \to \mathsf{S}^2 W_0^\vee \oplus \mathsf{S}^2 W_1^\vee \subset \mathsf{S}^2 W^\vee.$$

Let us show that $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ is a GM data. It only remains to check that $\mathbf{q}$ is symmetric (a fact which we already used a couple of times) and that the relation (7) holds.

For the symmetry of $\mathbf{q}$, we use the Lagrangian-quadratic correspondence (see Appendix C). We set $\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1 := \bigwedge^3 V_6 \oplus (\mathbf{k} \oplus L)$, a $\mathbf{Z}/2$-graded vector space endowed with the L-valued symplectic form

(17) $$\omega((\xi, x, x'), (\eta, y, y')) = \varepsilon(\lambda_6(\xi \wedge \eta)) + yx' - xy'.$$

Take $v \notin V_5$, so that $\lambda(v) \neq 0$, and consider the $\mathbf{Z}/2$-graded Lagrangian direct sum decomposition

(18) $$\mathbb{V} = \left(\bigwedge^3 V_5 \oplus L\right) \oplus \left((v \wedge \bigwedge^2 V_5) \oplus \mathbf{k}\right).$$

Since $\hat{A} := A \oplus A_1 \subset \mathbb{V}$ is another $\mathbf{Z}/2$-graded Lagrangian subspace, the corresponding quadratic form evaluated on elements $(\xi_1, x_1, x_1'), (\xi_2, x_2, x_2') \in \hat{A}$ is $\omega(\mathsf{pr}_1(\xi_1, x_1, x_1'), \mathsf{pr}_2(\xi_2, x_2, x_2'))$ (its symmetry follows from the Lagrangian property of $\hat{A}$; see the proof of Proposition C.1). For the decomposition (18), the projections are given by

$$
\begin{array}{rll}
\mathsf{pr}_1(\xi_1, x_1, x_1') & = & (\lambda(v)^{-1}\lambda_4(v \wedge \xi_1), x_1') \quad \in (\textstyle\bigwedge^3 V_5) \oplus L, \\
\mathsf{pr}_2(\xi_2, x_2, x_2') & = & (\lambda(v)^{-1}v \wedge \lambda_3(\xi_2), x_2) \quad \in (v \wedge \textstyle\bigwedge^2 V_5) \oplus \mathbf{k}.
\end{array}
$$

Substituting these into (17), we obtain the form $\tilde{\mathbf{q}}_0 + \tilde{\mathbf{q}}_1$ up to a rescaling by $\lambda(v)$. It follows that for $v \notin V_5$, the forms $\tilde{\mathbf{q}}_i(v)$ (and $\mathbf{q}_i(v)$) are symmetric. By continuity, the same is true for $v \in V_5$. On the other hand, for $v \in V_5$ (so that $\lambda(v) = 0$), we have $\tilde{\mathbf{q}}_1(v) = 0$ and

$$
\tilde{\mathbf{q}}_0(v)(\xi_1, \xi_2) = -\varepsilon(\lambda_4(v \wedge \xi_1) \wedge \lambda_3(\xi_2)) = \varepsilon(v \wedge \lambda_3(\xi_1) \wedge \lambda_3(\xi_2)).
$$

By definition of μ, this implies

$$
\mathbf{q}(v)(w_1, w_2) = \mathbf{q}_0(v)(w_1, w_2) = \varepsilon(v \wedge \mu_0(w_1) \wedge \mu_0(w_2)),
$$

which proves (7) (and gives another proof of the symmetry of $\mathbf{q}(v)$ for $v \in V_5$).

Let us show also that this GM data is lci. We have $W_1 = 0$ if $A_1 = L$, and $W_1 = A_1$ otherwise. But the quadratic form $\tilde{\mathbf{q}}_1$ on A_1 is non-trivial unless $A_1 = L$ or $A_1 = \mathbf{k}$. So the only case when $W_1 \neq 0$ and $\mathbf{q}_1 = 0$ simultaneously (the non-lci case) is the case $A_1 = \mathbf{k}$, which is excluded from our consideration.

Finally, if we rescale ε by $t \in \mathbf{k}^\times$, the map $\mathbf{q}_0$ will be also rescaled by t while the other data will not change. But the action of the element $(\varphi_V, \varphi_W, \varphi_L) \in \mathrm{GL}(V_6) \times \mathrm{GL}(W) \times \mathrm{GL}(L)$ defined by $\varphi_V = t\mathrm{id}_V$, $\varphi_W = \mathrm{id}_{W_0} + \sqrt{t}\,\mathrm{id}_{W_1}$, $\varphi_L = t^2\mathrm{id}_L$ precisely realizes such a rescaling (this is the only place where we use the assumption $\mathbf{k} = \mathbf{k}^2$; note however that it is unnecessary if $W_1 = 0$, i.e., for ordinary GM data). This means that different choices of ε produce isomorphic GM data.

We now explain the inverse construction. Let $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ be a GM data. Let $W = W_0 \oplus W_1$ and $\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1$ be the canonical direct sum decompositions of Proposition 2.28. Choose an arbitrary embedding

$$
\mu_1 \colon W_1 \hookrightarrow \mathbf{k}
$$

and consider the maps

$$
(19) \qquad V_5 \otimes W \otimes L \xrightarrow{\;f_1\;} \textstyle\bigwedge^3 V_5 \oplus L \oplus (V_6 \otimes W \otimes L) \xrightarrow{\;f_2\;} W^\vee \otimes L
$$

$$
\downarrow{\scriptstyle f_3}
$$

$$
\textstyle\bigwedge^3 V_6 \oplus (\mathbf{k} \oplus L) = \mathbb{V}
$$

defined by

$$
\begin{array}{rll}
f_1(v \otimes w \otimes l) & = & (-v \wedge \mu_0(l \otimes w), 0, v \otimes w \otimes l), \\
f_2(\xi, x', v \otimes w \otimes l)(w') & = & \varepsilon(\xi \wedge \mu_0(w')) + \mu_1(w') \otimes x' + \mathbf{q}(v)(w, w') \otimes l, \\
f_3(\xi, x', v \otimes w \otimes l) & = & (\xi + v \wedge \mu_0(l \otimes w), l \otimes \lambda(v)\mu_1(w), x'),
\end{array}
$$

(note that $\lambda(v) \in V_6/V_5 = L^\vee$, so $l \otimes \lambda(v) \in \mathbf{k}$) and define

$$
\hat{A} := f_3(\mathrm{Ker}(f_2)) \subset \mathbb{V}.
$$

We will check below that $\hat{A} \subset \mathbb{V}$ is a graded Lagrangian subspace.

We introduce gradings on the terms of (19) as follows. On the leftmost and the rightmost terms, the grading is induced by the direct sum decomposition $W = W_0 \oplus W_1$; in the middle column, we let $\bigwedge^3 V_5 \oplus V_6 \otimes W_0 \otimes L$ and $\bigwedge^3 V_6$ be the even parts, while $L \oplus (V_6 \otimes W_1 \otimes L)$ and $\mathbf{k} \oplus L$ are the odd parts. Then the gradings are preserved by all the maps. In particular, $\hat{A}$ is a direct sum $\hat{A} = A \oplus A_1$ with $A \subset \mathbb{V}_0 = \bigwedge^3 V_6$ and $A_1 \subset \mathbb{V}_1 = \mathbf{k} \oplus L$.

Note that $f_2 \circ f_1 = 0$ and $f_3 \circ f_1 = 0$. The second equality is obvious and the first follows from $\mathbf{q}(v)(w, w') = \varepsilon(v \wedge \mu(w) \wedge \mu(w'))$ for all $v \in V_5$, $w, w' \in W$, a reformulation of (7). These equalities imply

$$\hat{A} = f_3\big(\mathrm{Ker}(\mathrm{Coker}(f_1) \xrightarrow{f_2} W^\vee)\big).$$

Since the third component of f_1 is the natural embedding $V_5 \otimes W \otimes L \to V_6 \otimes W \otimes L$, choosing a vector $v_0 \in V_6 \smallsetminus V_5$ splits this map, and we can rewrite $\hat{A}$ as the vector space fitting into the exact sequence

(20) $$0 \to \hat{A} \to (\textstyle\bigwedge^3 V_5 \oplus L) \oplus (v_0 \otimes W \otimes L) \to W^\vee \otimes L \to 0,$$

which coincides with the exact sequence (46). Moreover, the embedding of $\hat{A}$ into $\mathbb{V}$ induced by f_3 coincides with the embedding discussed in the line after (46). So Lemma C.2 proves that $\hat{A}$ is Lagrangian.

Finally, we check how the constructed extended Lagrangian data depends on the choice of μ_1. The even part (with respect to the $\mathbf{Z}/2$-grading) of (19) does not depend on this choice, hence the same is true for A. On the other hand, a simple computation shows that $A_1 = L$ (i.e., corresponds to the point $0 \in \mathbf{P}^1$) if and only if $W_1 = 0$; moreover, $A_1 = \mathbf{k}$ (i.e., corresponds to the point $\infty \in \mathbf{P}^1$) if and only if $W_1 \neq 0$ and $\mathbf{q}_1 = 0$ (which is the non-lci case); otherwise, A_1 corresponds to a point of $\mathbf{P}^1 \smallsetminus \{0, \infty\}$. From this, it is clear that different choices of μ_1 produce isomorphic extended Lagrangian data.

The two constructions we explained are mutually inverse—this follows from Lemma C.2.

We now prove the second part of the theorem. First, assume that (V_6, V_5, A) is a Lagrangian data. We extend it by taking $A_1 = L$ and apply the above construction. One easily checks that the produced GM data has $W_1 = 0$, i.e., is ordinary. Moreover, a rescaling of ε by $t \in \mathbb{G}_\mathrm{m}$ results in the rescaling of $\mathbf{q} = \mathbf{q}_0$ by the same t, i.e., gives an isomorphic GM data (the isomorphism is given by $\varphi_V = t$, $\varphi_W = 1$, and $\varphi_L = t^2$).

Conversely, assume that $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ is an ordinary GM data, i.e., $W_1 = 0$. We apply the above construction (the map μ_1 has to be zero) and forget the odd part A_1 (which, as we observed above, is equal to L in this case) of the constructed Lagrangian. It is clear that we obtain a bijection between the set of isomorphism classes of Lagrangian data and the set ordinary GM data. $\qquad\square$

Remark 3.8. One can generalize this result by weakening the lci assumption for GM data as in Remark 2.29, but one then has to modify further the definition of Lagrangian data. To be more precise, fix a vector space K, consider the symplectic space $\mathbb{V}_1(K) := K \oplus (K^\vee \otimes L)$ with the canonical L-valued symplectic form (with the isomorphism relation for Lagrangian subspaces $A_1 \subset \mathbb{V}_1(K)$ induced by the natural action of the group $\mathrm{GL}(K)$ on $\mathbb{V}_1(K)$) and define K-Lagrangian data as quadruples (V_6, V_5, A, A_1), where $A \in \mathsf{LGr}(\mathbb{V}_0)$ and $A_1 \in \mathsf{LGr}(\mathbb{V}_1(K))$. Then, there is a bijection between the set of isomorphism classes of K-Lagrangian data such that $A_1 \cap K = 0$ and the set of isomorphism classes of GM data such that $\mathbf{q}(v)|_{\mathrm{Ker}(\mu)}$ is non-degenerate for (all) $v \notin V_5$, and $\dim(\mathrm{Ker}(\mu)) \leq \dim(K)$.

As explained in the introduction, the constructions of Theorems 2.3 and 3.6 can be combined: given a normal lci polarized GM variety (X, H), there is a canonical GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$ (Theorem 2.3), which is lci by Proposition 2.26; Theorem 3.6 then applies and gives an extended Lagrangian data (V_6, V_5, A, A_1).

To emphasize the dependence of the data on the original variety, we will sometimes write $W(X)$, $V_6(X)$, $V_5(X)$, $A(X)$, $A_1(X)$, and so on. All these spaces are well defined except perhaps for $A_1(X)$, which is defined only up to the $\mathbb{G}_{\mathrm{m}}$-action. In particular, the EPW sextic $Y_{A(X)} \subset \mathbf{P}(V_6(X))$ is well defined.

Conversely, let (V_6, V_5, A, A_1) be an extended Lagrangian data. We apply Theorem 3.6 and get a GM data, and we denote by X_{A,A_1,V_5} the corresponding GM intersection (usually the vector space V_6 will be fixed, so we exclude it from the notation). If $\mathbf{k} = \mathbf{k}^2$, the GM data is well defined up to isomorphism, hence the GM intersection X_{A,A_1,V_5} is also well defined up to isomorphism.

3.3. Geometric information from Lagrangian data.
The construction of a GM intersection X_{A,A_1,V_5} is quite involved. However, some of its geometrical properties can be read off directly from A, A_1, and V_5. The next proposition explains this.

Proposition 3.9. *Let $X = X_{A,A_1,V_5}$ be the GM intersection associated with an lci Lagrangian data (V_6, V_5, A, A_1) (in particular, $A_1 \neq \infty$).*

(a) *We have*

$$W(X)_0 = A/(A \cap \textstyle\bigwedge^3 V_5), \qquad\qquad W(X)_1 = A_1/(A_1 \cap L),$$
$$W(X)_0^\perp = A \cap \textstyle\bigwedge^3 V_5, \qquad\qquad W(X)_1^\perp = A_1 \cap L.$$

(b) *For $v \in V_6 \smallsetminus V_5$, we have*

$$(21) \qquad\qquad \mathrm{Ker}(\mathbf{q}(v)) = A \cap (v \wedge \textstyle\bigwedge^2 V_5).$$

(c) *If X is a GM variety,*

$$(22) \qquad\qquad \dim(X) = \begin{cases} 5 - \dim(A \cap \bigwedge^3 V_5), & \text{if } A_1 = 0, \\ 6 - \dim(A \cap \bigwedge^3 V_5), & \text{if } A_1 = 1. \end{cases}$$

Proof. (a) The first line is just the definition of W_0 and W_1 in the proof of Theorem 3.6. For the second line, note that $W(X) = W(X)_0 \oplus W(X)_1$ is the image of the Lagrangian subspace $\hat{A} = A \oplus A_1$ under its projection to the second summand of the Lagrangian direct sum decomposition (18). Hence its annihilator is the intersection of $\hat{A}$ with the first summand.

(b) By Proposition C.1, the kernel of the quadratic form $\tilde{\mathbf{q}}(v)$ on $\hat{A}$ is the direct sum $\big(\hat{A} \cap (\bigwedge^3 V_5 \oplus L)\big) \oplus \big(\hat{A} \cap ((v \wedge \bigwedge^2 V_5) \oplus \mathbf{k})\big)$. The quadratic form $\mathbf{q}(v)$ is induced by taking the quotient with respect to the first summand, hence its kernel is the second summand. Since $A_1 \neq \mathbf{k}$ by the lci assumption, it is equal to the right hand side of (21).

(c) If X is a GM variety, we have $\dim(X) = \dim(W(X)) - 5$. On the other hand, by part (a), we have

$$\dim(W(X)_0) = 10 - \dim(A \cap \textstyle\bigwedge^3 V_5), \qquad \dim(W(X)_1) = 1 - \dim(A_1 \cap L).$$

Combining these, we get (22). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following lemma explains how the GM intersection X_{A,A_1,V_5} depends on A_1 (for fixed A and V_5). Recall that we identified $\mathsf{LGr}(\mathbb{V}_1)$ with $\mathbf{P}^1$, so that the subspaces $\mathbf{k}$ and L of $\mathbb{V}_1$ correspond to the points ∞ and 0 in $\mathbf{P}^1$ respectively. We denote by $X_{A,0,V_5}$, X_{A,∞,V_5} and $X_{A,1,V_5}$ the GM intersections corresponding to the choices $A_1 = L$, $A_1 = \mathbf{k}$, and $A_1 \neq L, \mathbf{k}$, respectively.

Lemma 3.10. *Let $A \in \mathsf{LGr}(\bigwedge^3 V_6)$ and assume that $X := X_{A,0,V_5}$, with the choice $A_1 = L$, is a GM variety. Then X is ordinary, $X_{A,1,V_5} \simeq X_{\mathrm{spe}}$ is the double covering of M_X branched along X (hence is a special GM variety) and X_{A,∞,V_5} is the cone over X with vertex a point.*

Proof. If $A_1 = L$, we have $W_1 = 0$, hence X is ordinary. If $A_1 \neq L, \mathbf{k}$, the space W_1 is 1-dimensional and $\mathbf{q}_1$ induces an isomorphism $V_6/V_5 \simeq \mathsf{S}^2 W_1^\vee$. By Lemma 2.31, we have $X_{A,1,V_5} = X_{\mathrm{spe}}$.

Finally, if $A_1 = \mathbf{k}$, the space W_1 is 1-dimensional, but $\mathbf{q}_1 = 0$. Therefore X_{A,∞,V_5} is cut out in the cone over M_X by one equation $\mathbf{q}_0(v) = 0$ (where v is as above) and thus it is the cone over X. $\qquad\square$

In the next proposition, we discuss the relation between the Lagrangian data of a GM variety X and that of a hyperplane section. Such a hyperplane section is given by a linear function on the space $W(X) = W(X)_0 \oplus W(X)_1$. Moreover, by Proposition 3.9(a), we have an identification $W(X)_0^\vee = \bigwedge^3 V_5 / (A \cap \bigwedge^3 V_5)$.

Proposition 3.11. *Let X be a normal lci GM variety and let (V_6, V_5, A) be the corresponding Lagrangian data. Let $X' \subset X$ be a hyperplane section of X which is also normal and lci and let $\eta_0 \in \bigwedge^3 V_5$ be a lift of the even part of the equation of X'. The Lagrangian data of X' is then isomorphic to $(V_6, V_5, A(X'))$ and*

    (a) *if X' has the same type as X, then $A(X') = (A \cap \eta_0^\perp) \oplus \mathbf{k}\eta_0$;*
    (b) *if X is special and X' is ordinary, then $A(X') = (A \cap \eta^\perp) \oplus \mathbf{k}\eta$ for some $\eta \in A \oplus \mathbf{k}\eta_0$;*
    (c) *if X is special and $X' = X_{\mathrm{ord}}$, then $A(X') = A$.*

In particular, $\dim(A \cap A(X')) = 9$, unless X is special and $X' = X_{\mathrm{ord}}$.

Proof. Let us show $A \cap \eta_0^\perp \subset A(X')$. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A(X') & \longrightarrow & \bigwedge^3 V_5 \oplus W(X')_0 & \xrightarrow{(\mu_0,\mathbf{q}(v_0))} & W(X')_0^\vee & \longrightarrow & 0 \\
 & & & & \Big\downarrow{\scriptstyle\alpha} & & \Big\uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & \bigwedge^3 V_5 \oplus W(X)_0 & \xrightarrow{(\mu_0,\mathbf{q}(v_0))} & W(X)_0^\vee & \longrightarrow & 0,
\end{array}
$$

where the rows are the exact sequences (20) defining A and $A(X')$ respectively, $v_0 \in V_6 \setminus V_5$ is a fixed vector, and we omit for simplicity the L-factors. The image of $A \cap \eta_0^\perp$ under the left bottom arrow is contained in $\bigwedge^3 V_5 \oplus (W(X)_0 \cap \eta_0^\perp)$, hence in the image of α. So, by the commutativity of the diagram, it is in the kernel of the top right arrow. This proves $A \cap \eta_0^\perp \subset A(X')$ and in particular $\dim(A(X') \cap A) \geq 9$.

Note that any Lagrangian subspace A' containing $A \cap \eta_0^\perp$ can be written as $(A \cap \eta^\perp) \oplus \mathbf{k}\eta$ for some $\eta \in A \oplus \mathbf{k}\eta_0$. This proves (b). Furthermore, if X' has the same type as X then the odd part of the equation of X' is zero, hence $\eta_0 \in W(X')_0^\perp = A(X') \cap \bigwedge^3 V_5 \subset A(X')$. Thus $\eta_0 \subset A(X')$, so we should have $\eta = \eta_0$, and (a) follows. Finally, (c) was already proved in Lemma 3.10. $\qquad\square$

One can also give a criterion for two GM varieties to be isomorphic in terms of the associated data. Consider the group

$$(23) \qquad \mathrm{PGL}(V_6)_{A,V_5} := \{g \in \mathrm{PGL}(V_6) \mid (\textstyle\bigwedge^3 g)(A) = A, \ g(V_5) = V_5\}$$

of automorphisms of $\mathbf{P}(V_6)$ stabilizing A and V_5.

Proposition 3.12. *Assume* **k** *is quadratically closed. Let (X, H) and (X', H') be normal lci polarized GM varieties, with corresponding Lagrangian data (V_6, V_5, A, A_1) and (V_6', V_5', A', A_1').*

(a) *Any isomorphism $\phi\colon (X, H) \overset{\sim}{\longrightarrow} (X', H')$ induces an isomorphism $V_6 \overset{\sim}{\longrightarrow} V_6'$ which takes V_5 to V_5' and A to A'.*

(b) *Conversely, if X and X' have the same type, every isomorphism $V_6 \overset{\sim}{\longrightarrow} V_6'$ that takes V_5 to V_5' and A to A' is induced by an isomorphism between (X, H) and (X', H').*

(c') *If (X, H) is ordinary, $\mathrm{Aut}_H(X) \simeq \mathrm{PGL}(V_6)_{A,V_5}$.*

(c'') *If (X, H) is special, there is an exact sequence*

$$0 \to \mathbf{Z}/2 \to \mathrm{Aut}_H(X) \to \mathrm{PGL}(V_6)_{A,V_5} \to 1.$$

Proof. (a) By Theorem 2.9, ϕ induces an isomorphism $(\varphi_V, \varphi_W, \varphi_L)$ between the GM data corresponding to (X, H) and (X', H'). The isomorphism $\varphi_V\colon V_6 \overset{\sim}{\longrightarrow} V_6'$ takes V_5 to V_5' and it follows from Theorem 3.6 that the isomorphism $\bigwedge^3 \varphi_V\colon \bigwedge^3 V_6 \overset{\sim}{\longrightarrow} \bigwedge^3 V_6'$ maps the Lagrangian subspace A onto A',

(b) Assume conversely that X and X' have the same type and that there is an isomorphism $\varphi_V\colon V_6 \overset{\sim}{\longrightarrow} V_6'$ such that $\varphi_V(V_5) = V_5'$ and $(\bigwedge^3 \varphi_V)(A) = A'$. Since A_1 and A_1' are in the same $\mathbb{G}_\mathrm{m}$-orbit by assumption, there is, by Theorem 3.6, an isomorphism between the GM data for X and X'. By Theorem 2.9, it is induced by an isomorphism between the polarized GM varieties (X, H) and (X', H').

This implies that the image of the morphism $\mathrm{Aut}_H(X) \to \mathrm{PGL}(V_6)$ is $\mathrm{PGL}(V_6)_{A,V_5}$ and (c') and (c'') then follow from Corollary 2.30. $\qquad\square$

3.4. **Strongly smooth GM varieties.** We say that a subspace $A \subset \bigwedge^3 V_6$ contains no decomposable vectors if (see Section B.1)

$$\mathbf{P}(A) \cap \mathsf{Gr}(3, V_6) = \varnothing.$$

The crucial Theorem 3.14 below shows among other things that when a Lagrangian subspace A contains no decomposable vectors, the GM intersections X_{A,A_1,V_5} defined at the end of Section 3.2 are smooth GM varieties for all choices of $A_1 \neq \mathbf{k}$ and all choices of hyperplanes $V_5 \subset V_6$ (see Remark 3.15).

Recall that for any GM intersection X, we set $M_X = \mathsf{C}_K\,\mathsf{Gr}(2, V_5) \cap \mathbf{P}(W)$ and $M_X' = \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_0)$ (Section 2.4).

Definition 3.13. A GM variety X is strongly smooth if both intersections X and M_X' are dimensionally transverse and smooth.

By Proposition 2.20, any smooth GM variety is strongly smooth except perhaps if it is an ordinary GM surface or an ordinary GM curve. Note also that X is strongly smooth if and only if X_{opp} is strongly smooth (unless X is a special GM curve, in which case $X_{\mathrm{opp}} = X_{\mathrm{ord}}$ is just not defined).

Theorem 3.14. *Assume* $\mathbf{k} = \mathbf{C}$*. Let* (V_6, V_5, A, A_1) *be an extended Lagrangian data with* $A_1 \neq \infty$ *and let* $X := X_{A,A_1,V_5}$ *be the corresponding GM intersection. The following conditions are equivalent:*

   (i) *the GM intersection* X *is a strongly smooth GM variety;*
   (ii) *we have* $\dim(A \cap \bigwedge^3 V_5) + \dim(A_1 \cap L) \leq 5$*, the decomposable vectors in* A *are all in* $\bigwedge^3 V_5$*, and* $\mathbf{P}(A \cap \bigwedge^3 V_5) \cap \mathsf{Gr}(3, V_5)$ *is a smooth dimensionally transverse intersection in* $\mathbf{P}(\bigwedge^3 V_5)$*.*

In particular, if $\dim(A \cap \bigwedge^3 V_5) \leq 3$*, the GM intersection* X *is a smooth GM variety if and only if* A *contains no decomposable vectors.*

Proof. Assume that condition (ii) holds. By Proposition 3.9, we have $W_0^\perp = A \cap \bigwedge^3 V_5$, and $\mathbf{P}(W_0^\perp) \cap \mathsf{Gr}(3, V_5)$ is a smooth dimensionally transverse intersection. By Proposition 2.22, so is $M_X' := \mathbf{P}(W_0) \cap \mathsf{Gr}(2, V_5)$. Assume now, by contradiction, that, for $v \in V_6 \smallsetminus V_5$, the GM intersection $X_{A,A_1,V_5} = \mathsf{C}_{\mathrm{Ker}(\mu)} M_X' \cap Q(v)$ is not dimensionally transverse or is singular. In both cases, there exists $w \in \mathsf{C}_{\mathrm{Ker}(\mu)} M_X' \cap Q(v)$ such that the tangent space $\mathbf{T}_{Q(v),w}$ contains $\mathbf{T}_{M_X',w}$.

As $A_1 \neq \mathbf{k}$, we know by the proof of Lemma 3.10 that w cannot be the vertex of the cone. Since M_X' is smooth, $\mathbf{T}_{M_X',w}$ is therefore the intersection of the tangent spaces to Plücker quadrics, hence $\mathbf{T}_{Q(v),w}$ coincides with $\mathbf{T}_{Q(v'),w}$ for some $v' \in V_5$. Therefore, for some $t \in \mathbf{k}$, the quadric $Q(v + tv')$ is singular at w. So, replacing v with $v + tv'$, we may assume that w is in the kernel of $\mathbf{q}(v)$. By (21), this kernel is $A \cap (v \wedge \bigwedge^2 V_5)$; therefore, $v \wedge \mu(w) \in A$. Since $\mu(w) \in M_X'$, we have $\mu(w) \in \mathsf{Gr}(2, V_5)$, so $v \wedge \mu(w)$ is a decomposable vector in A which is not in $\bigwedge^3 V_5$, contradicting (ii). Therefore, X_{A,A_1,V_5} is a smooth GM variety (its dimension is ≥ 1 because of the condition $\dim(A \cap \bigwedge^3 V_5) + \dim(A_1 \cap L) \leq 5$).

Conversely, assume that condition (i) holds. Since $\dim(X) \geq 1$, we have $\dim(A \cap \bigwedge^3 V_5) + \dim(A_1 \cap L) \leq 5$ by Proposition 3.9. Assume, again by contradiction, that A contains a decomposable vector not in $\bigwedge^3 V_5$. This vector can be written as $v \wedge v_1 \wedge v_2$, where $v \in V_6 \smallsetminus V_5$ and $v_1, v_2 \in V_5$. As this vector is both in A and in $v \wedge \bigwedge^2 V_5$, it is by (21) in the kernel of $\mathbf{q}(v)$. On the other hand, $v_1 \wedge v_2 \in \mathsf{Gr}(2, V_5)$ and, as $\lambda_3(v \wedge v_1 \wedge v_2) = \lambda(v)v_1 \wedge v_2$, it is also in $\mathbf{P}(W_0)$. So $v_1 \wedge v_2$ is in M_X' and is a singular point of $Q(v)$. Hence it is a singular point of $X = \mathsf{C}_{\mathrm{Ker}(\mu)} M_X' \cap Q(v)$, contradicting (i). Finally, $\mathbf{P}(A \cap \bigwedge^3 V_5) \cap \mathsf{Gr}(3, V_5)$ is a smooth dimensionally transverse intersection by Proposition 2.22, since $M_X' = \mathbf{P}((A \cap \bigwedge^3 V_5)^\perp) \cap \mathsf{Gr}(2, V_5)$ is such by definition of strong smoothness.

When $\dim(X_{A,A_1,V_5}) \geq 3$, smoothness is equivalent to strong smoothness by Proposition 2.20 and $\dim(A \cap \bigwedge^3 V_5) \leq 3$ by (22), hence the intersection $\mathbf{P}(A \cap \bigwedge^3 V_5) \cap \mathsf{Gr}(3, V_5)$ is dimensionally transverse if and only if it is empty. Condition (ii) is therefore equivalent to the absence of decomposable vectors in A. $\qquad\square$

Remark 3.15. Let X be a strongly smooth complex GM variety, with associated Lagrangian subspace $A = A(X)$ and EPW sextics $Y_A \subset \mathbf{P}(V_6)$ and $Y_{A^\perp} \subset \mathbf{P}(V_6^\vee)$ (see Appendix B). Considering V_5 as a point of $\mathbf{P}(V_6^\vee)$, we have $V_5 \in Y_{A^\perp}^\ell$, where $\ell := \dim(A \cap \bigwedge^3 V_5)$ (see (34)). Formula (22) then reads

$$\ell = \begin{cases} 5 - \dim(X) & \text{if } X \text{ is ordinary;} \\ 6 - \dim(X) & \text{if } X \text{ is special.} \end{cases}$$

Theorem 3.14 implies the following.

- If $\ell \le 3$ (i.e., $\dim(X) \ge 3$ or X is an ordinary GM surface), A contains no decomposable vectors. By Theorem B.2, both Y_A and $Y_{A^\perp}$ are integral sextic hypersurfaces and the condition $\dim(A \cap \bigwedge^3 V_5') \le 3$ holds for all hyperplanes $V_5' \subset V_6$.
- If $\ell = 4$ (i.e., X is an ordinary GM curve or a special GM surface), A contains exactly 5 decomposable vectors and they are all in $\mathbf{P}(\bigwedge^3 V_5)$.
- If $\ell = 5$ (i.e., X is a special (bielliptic) curve), the decomposable vectors in $\mathbf{P}(A)$ form a smooth elliptic quintic curve contained in $\mathbf{P}(\bigwedge^3 V_5)$.

For a GM variety X, we defined in Section 3.1 the discriminant hypersurface $\mathrm{Disc}(X) \subset \mathbf{P}(V_6)$, which is either a sextic hypersurface or the whole space. We now give a unified proof of a generalization of Theorem 3.1 of Iliev and Manivel, proving that the schemes $\mathrm{Disc}(X)$ and $Y_{A(X)}$ are equal.

Proposition 3.16. *Assume* $\mathbf{k} = \mathbf{C}$. *Let* (V_6, V_5, A, A_1) *be a Lagrangian data with* $A_1 \ne \mathbf{k}$ *such that the corresponding GM intersection* X_{A,A_1,V_5} *is an (lci) GM variety. If either* Y_A *or* $\mathrm{Disc}(X_{A,A_1,V_5})$ *is a reduced hypersurface, we have* $Y_A = \mathrm{Disc}(X_{A,A_1,V_5})$.

Proof. We have $\mathrm{Ker}(\mathbf{q}(v)) = A \cap (v \wedge \bigwedge^2 V_5)$ by (21), hence, by definition of Y_A (see Definition B.1), Y_A and $\mathrm{Disc}(X_{A,A_1,V_5})$ coincide as sets on the complement of $\mathbf{P}(V_5)$. On the other hand, each of Y_A and $\mathrm{Disc}(X_{A,A_1,V_5})$ is either a sextic hypersurface or $\mathbf{P}(V_6)$. So, if one is a reduced hypersurface, so is the other, and they are equal. $\square$

The following result was proved in [O3, Proposition 2.18] when X is a smooth ordinary fivefold (it is then automatically strongly smooth).

Corollary 3.17. *Let* X *be a strongly smooth complex GM variety. The subschemes* $Y_{A(X)}$ *and* $\mathrm{Disc}(X)$ *of* $\mathbf{P}(V_6)$ *are equal and they are reduced sextic hypersurfaces. They are also integral and normal, unless* X *is a bielliptic curve, in which case they are union of a hyperplane and an integral quintic hypersurface.*

Proof. As in Remark 3.15, let us set $\ell := \dim(A(X) \cap \bigwedge^3 V_5)$ (this is $5 - \dim(X)$ if X is ordinary, and $6 - \dim(X)$ if X is special).

If $\ell \le 3$, we saw in Remark 3.15 that $Y_{A(X)}$ is an integral normal sextic hypersurface whose singularities are described in Theorem B.2. By Proposition 3.16, it is equal to $\mathrm{Disc}(X)$.

If $\ell = 4$ and X is an ordinary GM curve, $\mathrm{Disc}(X)$ is a reduced hypersurface (Proposition 3.2). By Proposition 3.16 again, it is equal to $Y_{A(X)_0}$. If X is a special GM surface then, by Lemma 3.3, its discriminant locus equals $\mathrm{Disc}(X_{\mathrm{ord}})$ which is an ordinary GM curve, and the previous argument applies.

Finally, if $\ell = 5$, then X is a special (i.e., bielliptic) GM curve, and the statements follow from Proposition 3.2 in the same way. $\square$

Remark 3.18. Let X be a strongly smooth complex GM curve, with Plücker hyperplane $V_5 := V_5(X) \subset V_6$ and associated Lagrangian $A = A(X)_0$.

When X *is ordinary,* V_5 *is the only point of* $Y_{A^\perp}^{\ge 4}$. Indeed, $\mathbf{P}(A)$ contains exactly 5 decomposable vectors $\bigwedge^3 V_{3,1}, \dots, \bigwedge^3 V_{3,5}$, which are all in $\bigwedge^3 V_5$ (Remark 3.15). Moreover, by [AH, (5.10)] or [O6, (2.3.2)], we have $V_5 = V_{3,i} + V_{3,j}$ for all $i \ne j$. Assume that $V_5' \subset V_6$ is another hyperplane in $Y_{A^\perp}^{\ge 4}$, so that $\dim(A \cap \bigwedge^3 V_5') \ge 4$. The points of the scheme $M' := \mathbf{P}(A \cap \bigwedge^3 V_5') \cap \mathsf{Gr}(3, V_5')$ must be among $V_{3,1}, \dots, V_{3,5}$. This implies $V_5' = V_5$, unless M' is a single

point, say $w := V_{3,1}$, with multiplicity 5. Since M' is a dimensionally transverse intersection, one checks, using its Koszul resolution in $\mathsf{Gr}(3, V_5')$ and Kodaira vanishing, that its linear span is $\mathbf{P}(A \cap \bigwedge^3 V_5')$. This implies $\mathbf{P}(A \cap \bigwedge^3 V_5') \subset \mathbf{T}_{\mathsf{Gr}(3, V_5'), w}$. But the 3-plane $\mathbf{P}(A \cap \bigwedge^3 V_5')$ must then meet the 4-dimensional Schubert cycle $\mathsf{Gr}(3, V_5') \cap \mathbf{T}_{\mathsf{Gr}(3, V_5'), w} = \mathbf{C}(\mathbf{P}^2 \times \mathbf{P}^1)$ along a curve, which consists of decomposable vectors in $\mathbf{P}(A)$. This is a contradiction.

When X is special (i.e., bielliptic), V_5 is the only point of $Y_{A^\perp}^{\geq 5}$. Indeed, the set Θ_A of decomposable vectors in $\mathbf{P}(A)$ is a smooth elliptic curve contained in $\bigwedge^3 V_5$ (Remark 3.15). If $V_5' \subset V_6$ is another point of $Y_{A^\perp}^{\geq 5}$, the scheme $\mathbf{P}(A \cap \bigwedge^3 V_5') \cap \mathsf{Gr}(3, V_5')$ has everywhere dimension ≥ 1 and is contained in Θ_A. These two schemes are thus equal, hence contained in $\mathsf{Gr}(3, V_5) \cap \mathsf{Gr}(3, V_5') = \mathsf{Gr}(3, V_5 \cap V_5')$. Since the linear span of Θ_A has dimension 4, this implies $V_5 = V_5'$.

By Proposition 3.12(b), the curve X is determined (up to isomorphism) by $A(X)$ and the hyperplane $V_5 \subset V_6$. In both cases, it is therefore uniquely determined by $A(X)$. In fact, X is also uniquely determined (up to isomorphism) by the sextic $Y_{A(X)}$. When X is special, this is because $\mathbf{P}(V_5)$ is the only degree-1 component of $Y_{A(X)}$ (Propositions 3.2 and 3.16). When X is ordinary, we saw during the proof of Proposition 3.2 that the singular locus of $Y_{A(X)}$ is the union of the image S_Θ of the singular locus of the theta divisor of $J(X)$ (one checks that it is an integral surface of degree 30), another irreducible surface of degree 10, and 5 planes. Moreover, the intersection of S_Θ with each of these 5 planes is a plane sextic curve whose normalization is X (this follows from [AH, footnote 1, p. 174]).

We can restate the results of Proposition 3.12 in terms of the EPW sextic $Y_{A(X)}$. The group $\mathrm{PGL}(V_6)_{A, V_5}$ was defined in (23); we define also

$$(24) \qquad \mathrm{PGL}(V_6)_{Y_A, V_5} := \{g \in \mathrm{PGL}(V_6) \mid g(Y_A) = Y_A \,, g(V_5) = V_5\}.$$

Proposition 3.19. *Let (X, H) and (X', H') be normal lci complex polarized GM varieties, with corresponding Lagrangian data (V_6, V_5, A) and (V_6', V_5', A').*

(a) *Any isomorphism $\phi\colon (X, H) \xrightarrow{\sim} (X', H')$ induces an isomorphism $V_6 \xrightarrow{\sim} V_6'$ which takes V_5 to V_5' and Y_A to $Y_{A'}$.*

(b) *If X is smooth and X and X' have same type and same dimension ≥ 3, every isomorphism $\varphi\colon V_6 \xrightarrow{\sim} V_6'$ that takes V_5 to V_5' and Y_A to $Y_{A'}$ is induced by an isomorphism between (X, H) and (X', H').*

(c) *If X is smooth of dimension ≥ 3, we have $\mathrm{PGL}(V_6)_{A, V_5} = \mathrm{PGL}(V_6)_{Y_A, V_5}$ and the group $\mathrm{Aut}(X)$ is finite.*

(d) *If X has dimension ≥ 3, the group $\mathrm{Aut}(X)$ is trivial if X is very general ordinary and $\mathrm{Aut}(X) \simeq \mathbf{Z}/2$ if X is very general special.*

Proof. (a) follows from Proposition 3.12 and the definition of Y_A.

(b) By Theorem 3.14, A contains no decomposable vectors and by Proposition B.8(b), we have $(\bigwedge^3 \varphi)(A) = A'$. Item (b) then follows from Proposition 3.12.

(c) The equality $\mathrm{PGL}(V_6)_{A, V_5} = \mathrm{PGL}(V_6)_{Y_A, V_5}$ is proved by the argument in part (b). Furthermore, $\mathrm{Aut}_H(X) = \mathrm{Aut}(X)$ when X is smooth and $\dim(X) \geq 3$ (Lemma 2.27). Therefore, by Proposition 3.12(c), to show that $\mathrm{Aut}(X)$ is finite, it is enough to show that $\mathrm{PGL}(V_6)_{Y_A, V_5}$ is finite. But the latter is a subgroup of the group $\mathrm{PGL}(V_6)_A$ defined in (42) and that group is finite by Proposition B.9(a).

(d) It is enough to show that $\mathrm{PGL}(V_6)_A = \mathrm{Aut}(Y_A)$ is trivial for a very general Lagrangian subspace $A \subset \bigwedge^3 V_6$. This is Proposition B.9(b). $\qquad\square$

Lengthy computations also show directly that for any smooth complex GM variety X of dimension ≥ 3, one has $H^0(X, T_X) = 0$ (these computations were done in [DIM, Theorem 3.4] for ordinary GM threefolds and in [DIM2, Proposition 4.1] for GM fourfolds). This implies that the automorphism group of X, being discrete, is finite, as asserted in the proposition.

3.5. **Period partners.** As we saw in the previous section, many properties of a GM variety depend only on $A(X)$—the even part of the corresponding Lagrangian subspace. More evidence will come further on.

Definition 3.20. Normal lci GM varieties X_1 and X_2 are period partners if $\dim(X_1) = \dim(X_2)$ and there exists an isomorphism $\varphi \colon V_6(X_1) \xrightarrow{\sim} V_6(X_2)$ such that $(\bigwedge^3\varphi)(A(X_1)) = A(X_2)$.

Remark 3.21. The last condition in the definition implies $\varphi(Y_{A(X_1)}) = Y_{A(X_2)}$. Conversely, assume that the lci GM varieties X_1 and X_2 are such that $\dim(X_1) = \dim(X_2) \geq 3$, that there exists an isomorphism $\varphi \colon V_6(X_1) \xrightarrow{\sim} V_6(X_2)$ such that $\varphi(Y_{A(X_1)}) = Y_{A(X_2)}$, and that X_1 is smooth. Proposition B.8 then shows that X_1 and X_2 are period partners and that X_2 is also smooth by Theorem 3.14.

Period partners may have different type—an ordinary GM variety may be a period partner of a special GM variety.

Remark 3.22. The name "period partners" suggests a relation with the period map. In a forthcoming paper, we will show that indeed, smooth GM varieties of the same dimension are period partners if and only if they are in the same fiber of an appropriately defined period map from the moduli space of GM varieties to an appropriate period domain.

A major difficulty in realizing the program of the remark is the construction of the moduli space of GM varieties. While this space is not available, we can formulate a naive description of the fiber of the period map as follows. The dual EPW stratification was defined in (37) and the group $\mathrm{PGL}(V_6)_A$ is defined in (42).

Theorem 3.23. *Let X be a smooth complex GM variety of dimension $n \geq 3$, with corresponding Lagrangian data (V_6, V_5, A). There is a natural bijection (defined in the proof) between the set of isomorphism classes of period partners of X and the set $\left(Y_{A^\perp}^{5-n} \sqcup Y_{A^\perp}^{6-n}\right)/\mathrm{PGL}(V_6)_A$.*

This theorem gives a precise answer to the question asked at the end of [IM, Section 4.5].

Proof. By Proposition 3.19, an lci GM variety X' is a period partner of X if and only if there is an isomorphism $\varphi \colon V_6(X') \xrightarrow{\sim} V_6$ such that $(\bigwedge^3\varphi)(A(X')) = A$, and $\varphi(V_5(X')) \in Y_{A^\perp}^{5-n}$ if X' is ordinary, or $\varphi(V_5(X')) \in Y_{A^\perp}^{6-n}$ if X' is special (by (22), this ensures the equality of dimensions). Since such an X' is automatically a smooth GM variety by Theorem 3.14, and the isomorphism φ is defined up to the action of $\mathrm{PGL}(V_6)_A$, this proves the lemma. $\qquad\square$

In Section 4, we show that over $\mathbf{C}$, period partners are always birationally isomorphic. In a companion paper, we show that they have the same primitive Hodge structure in the middle cohomology. Finally, in a joint work of the second author with Alexander Perry, we show that they have the same non-trivial piece of the derived category (this is a generalization of the Hodge structure statement).

3.6. **Duality.** We introduce here a notion of duality for GM varieties. It is similar in flavor to the notion of period partnership and we will see later that dual varieties share many geometric properties, as do period partners.

Definition 3.24. Normal lci GM varieties X and X' are called dual if $\dim(X) = \dim(X')$ and there exists an isomorphism $\varphi \colon V_6(X) \overset{\sim}{\longrightarrow} V_6(X')^\vee$ such that $(\bigwedge^3 \varphi)(A(X)) = A(X')^\perp$.

The last condition in the definition implies $\varphi(Y_{A(X)}) = Y_{A(X')^\perp}$. Conversely, as in Remark 3.21, if $\dim(X) = \dim(X') \geq 3$, if there exists an isomorphism $\varphi \colon V_6(X) \overset{\sim}{\longrightarrow} V_6(X')^\vee$ such that $\varphi(Y_{A(X)}) = Y_{A(X')^\perp}$, and if X is smooth, X' is also smooth and dual to X.

Duality is a symmetric relation. Moreover, if X' and X'' are both dual to X, they are period partners. Analogously, if X' is dual to X and X'' is a period partner of X' then X'' is dual to X.

One can give a description of all possible duals of a given GM variety analogous to the description of the set of all period partners.

Theorem 3.25. *Let X be a smooth complex GM variety of dimension $n \geq 3$, with corresponding Lagrangian data (V_6, V_5, A). There is a natural bijection between the set of isomorphism classes of its dual varieties and the set $\left(Y_A^{5-n} \sqcup Y_A^{6-n} \right) / \operatorname{PGL}(V_6)_A$.*

Proof. The proof is identical to that of Theorem 3.23.   □

Let X and X' be dual smooth GM varieties. Set $V_6 := V_6(X)$ and choose an isomorphism $V_6(X') \simeq V_6^\vee$. The hyperplane $V_5 := V_5(X) \subset V_6$ gives a line $V_5^\perp \subset V_6^\vee$ and conversely, the hyperplane $V_5' := V_5(X') \subset V_6^\vee$ gives a line $V_5'^\perp \subset V_6$. Since V_6 parameterizes quadrics cutting out X and $V_6^\vee$ those cutting out X', these lines define quadrics

$$Q(X) \subset \mathbf{P}(W(X')) \subset \mathbf{P}(\textstyle\bigwedge^2 V_5' \oplus \mathbf{k}) \quad \text{and} \quad Q(X') \subset \mathbf{P}(W(X)) \subset \mathbf{P}(\textstyle\bigwedge^2 V_5 \oplus \mathbf{k}).$$

Consider also the quadrics

$$Q_0(X') := Q(X') \cap \mathbf{P}(\textstyle\bigwedge^2 V_5) \subset \mathbf{P}(\textstyle\bigwedge^2 V_5) \quad \text{and} \quad Q_0(X) := Q(X) \cap \mathbf{P}(\textstyle\bigwedge^2 V_5') \subset \mathbf{P}(\textstyle\bigwedge^2 V_5').$$

Proposition 3.26. *In the above setup, assume that the line $V_5'^\perp$ is not contained in the hyperplane $V_5 \subset V_6$. Then the duality between V_6 and $V_6^\vee$ induces a duality between the spaces V_5' and V_5 under which the quadrics $Q_0(X)$ and $Q_0(X')$ are projectively dual.*

Proof. The first statement is obvious. For the second, choose generators $v \in V_5'^\perp \subset V_6$ and $v' \in V_5^\perp \subset V_6^\vee$. By assumption, $v \notin V_5$ and $v' \notin V_5'$. Consider the direct sum decomposition

$$\mathbb{V} = \left(\textstyle\bigwedge^3 V_5 \oplus L \right) \oplus \left((v \wedge \textstyle\bigwedge^2 V_5) \oplus \mathbf{k} \right).$$

The quadric $Q(X') = Q(v)$ is the quadric corresponding to the Lagrangian subspace $\hat{A} \subset \mathbb{V}$ via Proposition C.1, denoted there by $Q_2^{\hat{A}}$. By Lemma C.3, the quadric $Q_0(X')$ is the result of the isotropic reduction with respect to the isotropic subspace $L \subset \mathbb{V}$. Therefore, $Q_0(X')$ is the quadric corresponding to the Lagrangian subspace $A \subset \mathbb{V}_0 = \bigwedge^3 V_6$ with respect to the direct sum decomposition

$$\textstyle\bigwedge^3 V_6 = \textstyle\bigwedge^3 V_5 \oplus (v \wedge \textstyle\bigwedge^2 V_5),$$

i.e., $Q_0(X') = Q_2^A$. The same argument shows $Q_0(X) = Q_2^{A^\perp}$ for the direct sum decomposition

$$\textstyle\bigwedge^3 V_6^\vee = \textstyle\bigwedge^3 V_5' \oplus (v' \wedge \textstyle\bigwedge^2 V_5').$$

Finally, note that the isomorphism $\bigwedge^3 V_6 \xrightarrow{\sim} \bigwedge^3 V_6^\vee$ given by the symplectic form interchanges the summands of these decompositions. To prove this, since the summands are Lagrangian, it is enough to show that $v' \wedge \bigwedge^2 V_5'$ annihilates $\bigwedge^3 V_5$ and that $v \wedge \bigwedge^2 V_5$ annihilates $\bigwedge^3 V_5'$. This follows from the fact that v' annihilates V_5 (by its definition) and that v annihilates V_5'.

Since X and X' are dual, this isomorphism also takes A to $A^\perp$, hence identifies $Q_2^{A^\perp}$ with $Q_1^A \subset \mathbf{P}(\bigwedge^3 V_5)$. The proposition follows, since the quadrics Q_1^A and Q_2^A are projectively dual by Proposition C.1.   □

The variety X is determined by the quadric $Q_0(X')$: if X is ordinary, we have $X = \mathsf{Gr}(2, V_5) \cap Q_0(X')$, and if X is special, $X = (\mathsf{Gr}(2, V_5) \cap Q_0(X'))^{\mathrm{spe}}$. Analogously, X' is determined by the quadric $Q_0(X)$. Proposition 3.26 thus shows that projective duality of quadrics governs duality of GM varieties.

Remark 3.27. One can also show that if X and X' are both special GM varieties, the duality between V_5 and V_5' can be extended to a duality between $\bigwedge^2 V_5 \oplus \mathbf{k}$ and $\bigwedge^2 V_5' \oplus \mathbf{k}$ in such a way that the quadric $Q(X') \subset \mathbf{P}(\bigwedge^2 V_5 \oplus \mathbf{k})$ is projectively dual to the quadric $Q(X) \subset \mathbf{P}(\bigwedge^2 V_5' \oplus \mathbf{k})$.

4. RATIONALITY AND BIRATIONALITIES

The goal of this section is to prove that smooth complex GM varieties X_1 and X_2 of dimension ≥ 3 which are either period partners in the sense of Definition 3.20 (so that the Lagrangian $A(X_2)$ is isomorphic to $A(X_1)$) or dual in the sense of Definition 3.24 (so that $A(X_2)$ is isomorphic to $A(X_1)^\perp$) are birationally isomorphic.

The proof is based on the construction of two quadric fibrations. Although all dimensions could be treated with our methods, we prefer, for the sake of simplicity, to prove first that all smooth complex GM varieties of dimensions 5 or 6 are rational, and then concentrate on the cases of dimensions 3 or 4, for which birationality is really meaningful.

4.1. Rationality of smooth complex GM varieties of dimensions 5 or 6. The rationality of a general complex GM fivefold is explained in [R, Section 5]. We give a different argument, showing that all smooth GM fivefolds and sixfolds are rational.

Lemma 4.1. *Any smooth complex GM 5-old or sixfold X contains a smooth quintic del Pezzo surface.*

Proof. Let (V_6, V_5, A) be a Lagrangian data corresponding to X and let $Y_A \subset \mathbf{P}(V_6)$ be the corresponding EPW sextic. By Theorem 3.14, A contains no decomposable vectors. By Lemma B.6, there is a point $v \in Y_A^2 \setminus \mathbf{P}(V_5)$ and by (21), the corresponding quadric $Q(v) \subset \mathbf{P}(W)$ has corank 2 and $X = \mathsf{CGr}(2, V_5) \cap Q(v)$. If $\dim(X) = 6$, the quadric $Q(v)$ has rank 9 in $\mathbf{P}^{10}$; if $\dim(X) = 5$, it has rank 8 in $\mathbf{P}^9$. In both cases, it has an isotropic subspace $I \subset W$ of dimension 6. Then $\mathbf{P}(I) \cap X = \mathbf{P}(I) \cap \mathsf{CGr}(2, V_5)$ is a quintic del Pezzo surface if the intersection is dimensionally transverse and smooth. It remains to show that this is true for a general choice of I.

Consider the space of all $Q(v)$-isotropic 6-subspaces in W. If $\dim(X) = 6$, this is $\mathsf{OGr}(4, 9)$; if $\dim(X) = 5$, this is $\mathsf{OGr}(4, 8)$. In both cases, it is a smooth variety which we denote by B. Let $\mathscr{I} \subset W \otimes \mathscr{O}_B$ be the tautological bundle of $Q(v)$-isotropic subspaces over B

and let $\mathbf{P}_B(\mathscr{I})$ be its projectivization. It comes with a natural map $\mathbf{P}_B(\mathscr{I}) \to \mathbf{P}(\bigwedge^2 V_5 \oplus \mathbf{k})$ induced by the embedding $\mathscr{I} \subset W \otimes \mathscr{O}_B \subset (\bigwedge^2 V_5 \oplus \mathbf{k}) \otimes \mathscr{O}_B$. The scheme

$$\mathscr{X} = \mathbf{P}_B(\mathscr{I}) \times_{\mathbf{P}(\bigwedge^2 V_5 \oplus \mathbf{k})} \mathsf{CGr}(2, V_5)$$

maps to B and the fibers are the intersections $\mathbf{P}(I) \cap X$.

To show that the general fiber is dimensionally transverse and smooth, it is enough to show that $\mathscr{X}$ is itself smooth of the expected dimension. For this, consider the projection $\mathscr{X} \to \mathsf{CGr}(2, V_5)$. Since the fibers of $\mathbf{P}_B(\mathscr{I})$ are contained in $Q(v)$, the image of $\mathscr{X}$ is contained in $\mathsf{CGr}(2, V_5) \cap Q(v) = X$. Moreover, the fiber of this projection over a point $w \in X \subset \mathbf{P}(W)$ is the space of all $Q(v)$-isotropic 6-subspaces $I \subset W$ such that $w \in I$.

Since w cannot be in the kernel of $Q(v)$ (otherwise, X would be singular), the space of such I is parameterized by $\mathsf{OGr}(3, 7)$ if $\dim(X) = 6$, and by $\mathsf{OGr}(3, 6)$ if $\dim(X) = 5$. In both cases, all fibers of the projection $\mathscr{X} \to X$ are smooth of the same dimension, hence $\mathscr{X}$ is smooth. As observed earlier, the general fiber of the projection $\mathscr{X} \to B$ is therefore a smooth quintic del Pezzo surface. □

We can now prove rationality of smooth GM 5-olds and sixfolds.

Proposition 4.2. *Any smooth complex GM variety of dimension 5 or 6 is rational.*

Proof. We use the notation of the previous proof. Let $Q = Q(v)$ be a quadric of corank 2 in $\mathbf{P}(W)$ such that $X = \mathsf{CGr}(2, V_5) \cap Q$. Let $I_0 \subset W$ be a general Q-isotropic 6-subspace, so that $S_0 := \mathbf{P}(I_0) \cap X$ is a smooth quintic del Pezzo surface. Consider the linear projection

$$\pi_{S_0} \colon X \dashrightarrow \mathbf{P}^{n-2}$$

from the linear span $\mathbf{P}(I_0)$ of S_0, where $n = \dim(X)$. A general fiber is also a quintic del Pezzo surface. Indeed, the intersection of a general $\mathbf{P}^6$ containing $\mathbf{P}(I_0) = \mathbf{P}^5$ with X is a 2-dimensional GM intersection containing the quintic del Pezzo surface S_0. It is the union of S_0 and a residual quintic del Pezzo surface S (more precisely, the intersection of Q with this $\mathbf{P}^6$ is the union of $\mathbf{P}(I_0)$ and a residual space $\mathbf{P}(I)$, where I is another Q-isotropic 6-subspace, and $S = \mathbf{P}(I) \cap X$).

The argument of the proof of Lemma 4.1 shows that for a general choice of S_0, the general fiber of π_{S_0} is smooth. Hence the field of rational functions on X is the field of rational functions on a smooth quintic del Pezzo surface defined over the field of rational functions on $\mathbf{P}^{n-2}$. But a smooth quintic del Pezzo surface is rational over any field by a theorem of Enriques (see [S-B2]). □

Remark 4.3. Let X be a smooth complex GM fourfold such that $Y_{A(X)}^3 \smallsetminus \mathbf{P}(V_5(X)) \neq \varnothing$. The same argument shows that X is rational. More generally, using Theorem 4.15, one can show that X is rational as soon as $Y_{A(X)}^3 \neq \varnothing$.

4.2. **The first quadric fibration.** Let X be a smooth GM variety of dimension $n \geq 3$, with Lagrangian data (V_6, V_5, A) and GM data $(W, V_6, V_5, L, \mu, \mathbf{q}, \varepsilon)$. We consider its Gushel bundle $\mathscr{U}$ and projectivization $\mathbf{P}_X(\mathscr{U})$. By Theorem 3.14, A contains no decomposable vectors. The canonical embedding $\mathscr{U} \hookrightarrow V_5 \otimes \mathscr{O}_X$ induces a regular map

$$\rho_1 \colon \mathbf{P}_X(\mathscr{U}) \to \mathbf{P}(V_5).$$

In Proposition 4.5, we prove that ρ_1 is a quadric fibration and identify its discriminant locus.

Let $W_0^\perp = A \cap \bigwedge^3 V_5$ be the even part of the space of linear equations of X. We consider $W_0^\perp$ as a subspace of the space $\bigwedge^2 V_5^\vee$ of skew forms on V_5. Let

$$\Sigma_1(X) \subset \mathbf{P}(V_5)$$

be the union of the kernels of all nonzero skew forms in the space $W_0^\perp$. It can be described in terms of the incidence correspondence $\widehat{Y}_A$ and its two projections $p \colon \widehat{Y}_A \to Y_A \subset \mathbf{P}(V_6)$ and $q \colon \widehat{Y}_A \to Y_{A^\perp} \subset \mathbf{P}(V_6^\vee)$ (see (38)).

Lemma 4.4. *We have $\Sigma_1(X) = p(q^{-1}([V_5]))$. In particular, $\Sigma_1(X) \subset Y_A \cap \mathbf{P}(V_5)$. Moreover, $\dim(\Sigma_1(X)) \leq 1$ if X is a GM fourfold, and $\dim(\Sigma_1(X)) \leq 2$ if X is a GM threefold.*

Proof. The lemma follows immediately from Remark B.4.   □

We now prove an analogue of Proposition 3.16.

Proposition 4.5. *Let X be a smooth GM variety of dimension $n \geq 3$.*

Over $\mathbf{P}(V_5) \smallsetminus \Sigma_1(X)$, the map $\rho_1 \colon \mathbf{P}_X(\mathscr{U}) \to \mathbf{P}(V_5)$ is a relative quadratic hypersurface in a $\mathbf{P}^{n-2}$-fibration, whose fiber over a point v has corank k if and only if $v \in Y_A^k \cap \mathbf{P}(V_5)$. In particular, in $\mathbf{P}(V_5) \smallsetminus \Sigma_1(X)$, its discriminant set is $Y_A \cap (\mathbf{P}(V_5) \smallsetminus \Sigma_1(X))$.

Over $\Sigma_1(X)$, the map ρ_1 is a relative quadratic hypersurface in a $\mathbf{P}^{n-1}$-fibration, whose fiber over a point v has corank k if and only if $v \in Y_A^{k+1}$.

Proof. Choose an arbitrary $v_0 \in V_6 \smallsetminus V_5$ and let $v \in V_5$. The fiber $\rho_1^{-1}(v)$ is the set of $V_2 \subset V_5$ containing v which correspond to points of X. This is the intersection $Q'(v)$ of the quadric $Q(v_0) \subset \mathbf{P}(W)$ defining X with the subspace $\mathbf{P}\big(((v \wedge V_5) \oplus \mathbf{k}) \cap W\big) \subset \mathbf{P}(W)$.

On the other hand, by the argument of Theorem 3.6, the quadric $Q(v_0)$ is the quadric corresponding to the Lagrangian decomposition $\mathbb{V} = \mathbb{L}_1 \oplus \mathbb{L}_2$ of (18), with $\mathbb{L}_1 = \bigwedge^3 V_5 \oplus L$ and $\mathbb{L}_2 = (v_0 \wedge \bigwedge^2 V_5) \oplus \mathbf{k}$, under the Lagrangian-quadratic correspondence of Appendix C (it is the quadric $Q_2^{\hat{A}}$ of Proposition C.1). Therefore, the intersection

$$Q'(v) = Q(v_0) \cap \mathbf{P}\big((v \wedge V_5) \oplus \mathbf{k}) \cap W\big)$$

can be described by the isotropic reduction of Corollary C.4.

For this, we set $\overline{\mathbb{L}}_2 := (v_0 \wedge v \wedge V_5) \oplus \mathbf{k} \subset \mathbb{L}_2$ and we consider the isotropic subspace

$$\mathbb{I}_v := \mathbb{L}_1 \cap \overline{\mathbb{L}}_2^\perp = v \wedge \bigwedge^2 V_5 \subset \mathbb{L}_1.$$

According to Corollary C.4, the codimension in $\mathbf{P}(\overline{\mathbb{L}}_2)$ of the span of the quadric $Q'(v)$ is equal to $\dim\big((\hat{A} \cap \mathbb{L}_1)/(\hat{A} \cap \mathbb{I}_v)\big)$ and its kernel is equal to $\big(\hat{A} \cap (\mathbb{I}_v \oplus \overline{\mathbb{L}}_2)\big)/(\hat{A} \cap \mathbb{I}_v)$, so it remains to describe the intersections of $\hat{A}$ with $\mathbb{I}_v$, $\mathbb{L}_1$, and $\mathbb{I}_v \oplus \overline{\mathbb{L}}_2$.

First, we have $\hat{A} \cap \mathbb{I}_v = A \cap (v \wedge \bigwedge^2 V_5) \subset A \cap \bigwedge^3 V_5 = W_0^\perp$ and the latter space is the space of linear equations of M_X'. A skew form on V_5, considered as an element of $\bigwedge^3 V_5$, belongs to $v \wedge \bigwedge^2 V_5$ if and only if v is in its kernel. This means that $\hat{A} \cap \mathbb{I}_v$ is non-zero only for $v \in \Sigma_1(X)$. Moreover, the intersection $\hat{A} \cap \mathbb{I}_v$ is then 1-dimensional, since otherwise the space $A \cap (v \wedge \bigwedge^2 V_5)$ would contain decomposable vectors, so

$$\tag{25} \dim(\hat{A} \cap \mathbb{I}_v) = \begin{cases} 1 & \text{if } v \in \Sigma_1(X), \\ 0 & \text{if } v \notin \Sigma_1(X). \end{cases}$$

Furthermore, the space $\hat{A} \cap \mathbb{L}_1 = W^\perp$ has dimension $6 - n$, hence the codimension of the span of $Q'(v)$ in $\mathbf{P}(\overline{\mathbb{L}}_2)$ is $6 - n$ for $v \notin \Sigma_1(X)$ and $5 - n$ for $v \in \Sigma_1(X)$. Since the dimension of $\mathbf{P}(\overline{\mathbb{L}}_2)$ is 4, it follows that $Q'(v)$ is a quadratic hypersurface in $\mathbf{P}^{n-2}$ or in $\mathbf{P}^{n-1}$ depending on whether v is out of $\Sigma_1(X)$ or in it. Finally, note that

$$\mathbb{I}_v \oplus \overline{\mathbb{L}}_2 = ((v \wedge {\textstyle\bigwedge}^2 V_5) \oplus (v_0 \wedge v \wedge V_5)) \oplus \mathbf{k} = v \wedge {\textstyle\bigwedge}^2 V_6 \oplus \mathbf{k}.$$

Therefore, the dimension of the intersection $\hat{A} \cap (\mathbb{I}_v \oplus \overline{\mathbb{L}}_2) = A \cap (v \wedge {\textstyle\bigwedge}^2 V_6)$ is detected by the position of $v \in \mathbf{P}(V_6)$ with respect to the EPW stratification defined by A on $\mathbf{P}(V_6)$. Taking into account (25), we obtain that for $v \in Y_A^k \smallsetminus \Sigma_1(X)$, the corank of $Q'(v)$ is k, and for $v \in Y_A^k \cap \Sigma_1(X)$, the corank of $Q'(v)$ is $k - 1$ (this gives another proof of the inclusion $\Sigma_1(X) \subset Y_A$). $\qquad \square$

4.3. **Birationality of period partners of dimension 3.** Let X_1 and X_2 be smooth complex GM threefolds. Assume that they are period partners, so that they are constructed from Lagrangian data with the same V_6 and A, but possibly different Plücker hyperplanes $V_5^1 \subset V_6$ and $V_5^2 \subset V_6$ and possibly different Lagrangian subspaces A_1 and A_1'. The aim of this section is to prove that the varieties X_1 and X_2 are birationally isomorphic.

Consider the diagrams

$$
\begin{array}{ccccccc}
& \mathbf{P}_{X_1}(\mathscr{U}_{X_1}) & & & & \mathbf{P}_{X_2}(\mathscr{U}_{X_2}) & \\
& \swarrow \quad \searrow {\scriptstyle \rho_1^1} & & {\scriptstyle \rho_1^2} \swarrow & & \quad \searrow & \\
X_1 & & \mathbf{P}(V_5^1) & & \mathbf{P}(V_5^2) & & X_2,
\end{array}
$$

where ρ_1^1 and ρ_1^2 are the first quadratic fibrations associated with the threefolds X_1 and X_2 respectively. Denote by $\Sigma_1(X_1) \subset Y_A \cap \mathbf{P}(V_5^1)$ and $\Sigma_1(X_2) \subset Y_A \cap \mathbf{P}(V_5^2)$ the associated subsets of the previous section. Assume moreover $V_5^1 \neq V_5^2$, set $V_4 := V_5^1 \cap V_5^2 \subset V_6$, and restrict everything to the common base $\mathbf{P}(V_4)$. We get a diagram

$$
\begin{array}{ccccccc}
& \widetilde{X}_1 & & & & \widetilde{X}_2 & \\
& \swarrow \quad \searrow {\scriptstyle \tilde{\rho}_1^1} & & {\scriptstyle \tilde{\rho}_1^2} \swarrow & & \quad \searrow & \\
X_1 & & \mathbf{P}(V_4) & & & & X_2,
\end{array}
$$

where $\widetilde{X}_i = \mathbf{P}_{X_i}(\mathscr{U}_{X_i}) \times_{\mathbf{P}(V_5^1)} \mathbf{P}(V_4)$ and $\tilde{\rho}_1^i$ is the restriction of ρ_1^i to $\widetilde{X}_i$.

Lemma 4.6. *For each $i \in \{1, 2\}$, the map $\widetilde{X}_i \to X_i$ is the blow up of $X_i \cap \mathsf{CGr}(2, V_4)$. In particular, $\widetilde{X}_i$ is irreducible, generically reduced, and birational to X_i.*

Proof. By definition, $\widetilde{X}_1 = \mathbf{P}_{X_1}(\mathscr{U}_{X_1}) \times_{\mathbf{P}(V_5)} \mathbf{P}(V_4)$ is a divisor in a smooth 4-dimensional variety $\mathbf{P}_{X_1}(\mathscr{U}_{X_1})$, hence each irreducible component of $\widetilde{X}_1$ is at least 3-dimensional. The fiber of $\widetilde{X}_1 \to X_1$ over a point $w \in X_1$ is a linear section of the fiber of $\mathbf{P}_{X_1}(\mathscr{U}_{X_1}) \to X_1$, so it is a point if $\mathscr{U}_{X_1,w} \not\subset V_4$, or the line $\mathbf{P}(\mathscr{U}_{X_1,w})$ if $\mathscr{U}_{X_1,w} \subset V_4$. So, the locus of non-trivial fibers is $X_1 \cap \mathsf{CGr}(2, V_4)$.

Since X_1 is cut out in $\mathsf{CGr}(2, V_5)$ by three hyperplanes and a quadric, the intersection $X_1 \cap \mathsf{CGr}(2, V_4)$ is also cut out in $\mathsf{CGr}(2, V_4)$ by three hyperplanes and a quadric, hence its expected dimension is 1; moreover, its degree is at most 4. But X_1 is a smooth threefold, so by the Lefschetz Theorem, it cannot contain a surface of degree < 10, hence we have $\dim(X_1 \cap \mathsf{CGr}(2, V_4)) \leq 1$. Therefore, the exceptional set of $\widetilde{X}_1 \to X_1$ has dimension at most 2

and thus cannot be an irreducible component. It follows that $\widetilde{X}_1$ is integral and that the map $\widetilde{X}_1 \to X_1$ is the blow up of $X_1 \cap \mathsf{CGr}(2, V_4)$. The same argument works for $\widetilde{X}_2$ as well.   □

Theorem 4.7. *Let X_1 and X_2 be smooth complex GM threefolds which are period partners. Over an open subset $U \subset \mathbf{P}(V_4)$, there is an isomorphism $(\tilde{\rho}_1^1)^{-1}(U) \simeq (\tilde{\rho}_1^2)^{-1}(U)$. In particular, $\widetilde{X}_1$ is birationally isomorphic to $\widetilde{X}_2$ over $\mathbf{P}(V_4)$ and X_1 is birationally isomorphic to X_2.*

Proof. By Proposition 4.5, the morphism $\rho_1^1 \colon \mathbf{P}_{X_1}(\mathscr{U}_{X_1}) \to \mathbf{P}(V_5)$ is, outside the locus $\Sigma_1(X_1) \subset \mathbf{P}(V_5)$ (which has dimension ≤ 2 by Lemma 4.4), a flat double cover branched along the sextic $Y_A \cap \mathbf{P}(V_5)$. The morphism $\tilde{\rho}_1^1 \colon \widetilde{X}_1 \to \mathbf{P}(V_4)$ is obtained from ρ_1^1 by a base change.

If $\widetilde{X}_1 \xrightarrow{g_1} \bar{X}_1 \xrightarrow{h_1} \mathbf{P}(V_4)$ is its Stein factorization, g_1 is birational, $\bar{X}_1$ is isomorphic to the restriction of $\mathbf{Spec}(\rho_{1*}^1 \mathscr{O}_{\mathbf{P}_{X_1}(\mathscr{U}_{X_1})})$ to $\mathbf{P}(V_4)$, hence h_1 is the double cover of $\mathbf{P}(V_4)$ branched along $Y_A \cap \mathbf{P}(V_4)$ (note that $Y_A \cap \mathbf{P}(V_4) \neq \mathbf{P}(V_4)$ since $\widetilde{X}_1$, hence also $\bar{X}_1$, is generically reduced).

The same argument shows that in the Stein factorization $\tilde{\rho}_1^2 \colon \widetilde{X}_2 \xrightarrow{g_2} \bar{X}_2 \xrightarrow{h_2} \mathbf{P}(V_4)$, the map g_2 is also birational and h_2 is again the double covering branched along the same sextic $Y_A \cap \mathbf{P}(V_4)$.

Since the branch divisors coincide, $\bar{X}_1$ and $\bar{X}_2$ are isomorphic, hence $\widetilde{X}_1$ and $\widetilde{X}_2$ are birationally isomorphic. By Lemma 4.6, X_1 and X_2 are birationally isomorphic as well.   □

Remark 4.8. One can explicitly decompose the birational transformation $X_1 \dashrightarrow X_2$ into several steps. The first half can be described as follows:

- the map $\widetilde{X}_1 \to X_1$ is the blow up of the union $c_1 \cup c_2$ of two conics (intersecting in two points);
- the map $\widetilde{X}_1 \to \bar{X}_1$ contracts all lines intersecting $c_1 \cup c_2$ and all conics intersecting $c_1 \cup c_2$ twice (all of them are so-called σ-conics corresponding to points of $\Sigma_1 \cap \mathbf{P}(V_4)$),

and similarly for the second part. This transformation is very similar to the conic transformation used in [DIM, Theorem 7.4] to prove Theorem 4.7 when X_1 is a general GM threefold.

One can use the same idea to prove birationality of period partners in any dimension $n \geq 3$. It requires, however, a long and cumbersome analysis, so we decided to omit it. Instead, we will use birationalities for dual GM varieties (proved later in this section) to establish the result in dimension 4 (Theorem 4.15). Since for $n \geq 5$, all smooth GM varieties are rational, this shows the result holds in all dimensions $n \geq 3$.

4.4. **The second quadric fibration.** Let X be a smooth GM variety of dimension $n \geq 3$. Instead of the Gushel bundle $\mathscr{U}$, we now consider the quotient $(V_5 \otimes \mathscr{O}_X)/\mathscr{U}$, which we denote simply by $V_5/\mathscr{U}$, its projectivization $\mathbf{P}_X(V_5/\mathscr{U})$, and the map $\rho_2 \colon \mathbf{P}_X(V_5/\mathscr{U}) \to \mathsf{Gr}(3, V_5)$. In Proposition 4.10, we prove that ρ_2 is a quadric fibration and identify its discriminant locus.

Let as before $W_0^\perp = A \cap \bigwedge^3 V_5$ be the even part of the space of linear equations of X. Considering its elements as skew forms on V_5, we denote by

(26) $$\Sigma_2^{\geq k}(X) \subset \mathsf{Gr}(3, V_5)$$

the set of all 3-dimensional subspaces V_3 in V_5 which are isotropic for some k-dimensional space of these skew forms. As usual, put $\Sigma_2(X) := \Sigma_2^{\geq 1}(X)$.

Finally, we will use the EPW sequence of $\mathsf{Gr}(3, V_6)$ defined in (43),

$$Z_A^{\geq k} = \{V_3 \in \mathsf{Gr}(3, V_6) \mid \dim(A \cap (V_6 \wedge \textstyle\bigwedge^2 V_3)) \geq k\}$$

and $Z_A = Z_A^{\geq 1}$. It plays for the second quadratic fibration the same role as the EPW sequence of $\mathbf{P}(V_6)$ plays for the first.

In the next lemma, we consider 1-dimensional subspaces $V_1 \subset V_5$ and identify the Grassmannian $\mathsf{Gr}(2, V_5/V_1)$ with the subscheme of $\mathsf{Gr}(3, V_5)$ parameterizing all 3-dimensional subspaces containing V_1.

Lemma 4.9. *For any smooth GM variety X of dimension $n \geq 3$, we have $\Sigma_2(X) \subset Z_A \cap \mathsf{Gr}(3, V_5)$.*

Assume moreover that either $n = 3$ and X ordinary, or $n \geq 4$. If $V_1 \notin \Sigma_1(X)$, we have $\dim(\Sigma_2(X) \cap \mathsf{Gr}(2, V_5/V_1)) \leq 2$.

Proof. Let $V_3 \in \Sigma_2(X)$, i.e., the subspace V_3 is isotropic for some non-zero $\omega \in A \cap \bigwedge^3 V_5$, considered as a skew form on V_5. Then $\omega \in \bigwedge^2 V_3 \wedge V_5$. It follows that $V_3 \in Z_A$, which proves the first assertion of the lemma.

Let $\omega \in A \cap \bigwedge^3 V_5$ be non-zero. Since X is smooth of dimension ≥ 3, the Lagrangian A contains no decomposable vectors hence ω, viewed as a skew form on V_5, has rank 4 and its kernel is a 1-dimensional subspace $K_1(\omega) \subset V_5$. Any 3-dimensional subspace V_3 of V_5 which is isotropic for ω contains $K_1(\omega)$, hence the set of such subspaces can be identified with the set of 2-dimensional subspaces in $V_5/K_1(\omega)$ which are isotropic for the non-degenerate skew form induced by ω, i.e., with $\mathsf{LGr}(2, 4)$.

It remains to describe which of these subspaces V_3 contain V_1. Since $V_1 \notin \Sigma_1(X)$, it projects to a non-zero subspace $V_1' \subset V_5/K_1(\omega)$. The subset of $\mathsf{LGr}(2, 4)$ consisting of subspaces containing V_1' is $\mathbf{P}(V_1'^\perp/V_1') = \mathbf{P}^1$. It follows that $\Sigma_2(X) \cap \mathsf{Gr}(2, V_5/V_1)$ is dominated by a $\mathbf{P}^1$-fibration over $\mathbf{P}(A \cap \bigwedge^3 V_5)$. Since $\dim(A \cap \bigwedge^3 V_5) \leq 2$ for GM fourfolds and ordinary GM threefolds (see (22)), the claim follows. $\square$

Proposition 4.10. *Let X be a smooth GM variety of dimension $n \geq 3$.*

Over $\mathsf{Gr}(3, V_5) \smallsetminus \Sigma_2(X)$, the map $\rho_2 \colon \mathbf{P}_X(V_5/\mathscr{U}) \to \mathsf{Gr}(3, V_5)$ is a relative quadratic hypersurface in a $\mathbf{P}^{n-3}$-fibration, whose fiber over a point U_3 has corank k if and only if $U_3 \in Z_A^k$. In particular, in $\mathsf{Gr}(3, V_5) \smallsetminus \Sigma_2(X)$, its discriminant set is $Z_A \cap (\mathsf{Gr}(3, V_5) \smallsetminus \Sigma_2(X))$.

Over $\Sigma_2^l(X)$, the map $\rho_2 \colon \mathbf{P}_X(V_5/\mathscr{U}) \to \mathsf{Gr}(3, V_5)$ is a relative quadratic hypersurface in a $\mathbf{P}^{n+l-3}$-fibration, whose fiber over a point U_3 has corank k if and only if $U_3 \in Z_A^{k+l}$.

Proof. We follow the proof of Proposition 4.5 and use the notation introduced therein. Choose $v_0 \in V_6 \smallsetminus V_5$ and let $U_3 \subset V_5$. The fiber $\rho_2^{-1}(U_3)$ is the set of $V_2 \subset U_3$ which correspond to points of X. This is the intersection

$$Q'(U_3) := Q(v_0) \cap \mathbf{P}\big((\textstyle\bigwedge^2 U_3 \oplus \mathbf{k}) \cap W\big),$$

so it can be described by the isotropic reduction of Corollary C.4.

For this, we let $\overline{\mathbb{L}}_2 := \bigwedge^2 U_3 \oplus \mathbf{k} \subset \mathbb{L}_2$ and we consider the isotropic subspace

$$\mathbb{I}_{U_3} := \mathbb{L}_1 \cap \overline{\mathbb{L}}_2^\perp = (\textstyle\bigwedge^3 V_5 \oplus L) \cap (\bigwedge^2 U_3 \oplus \mathbf{k})^\perp = V_5 \wedge \bigwedge^2 U_3.$$

According to Corollary C.4, the codimension in $\mathbf{P}(\overline{\mathbb{L}}_2)$ of the span of the quadric $Q'(U_3)$ is equal to $\dim\big((\hat{A} \cap \mathbb{L}_1)/(\hat{A} \cap \mathbb{I}_{U_3})\big)$ and its kernel is $(\hat{A} \cap (\mathbb{I}_{U_3} \oplus \overline{\mathbb{L}}_2))/(\hat{A} \cap \mathbb{I}_{U_3})$. It remains to describe the intersections of $\hat{A}$ with $\mathbb{I}_{U_3}$, $\mathbb{L}_1$, and $\mathbb{I}_{U_3} \oplus \overline{\mathbb{L}}_2$.

We have $\hat{A} \cap \mathbb{I}_{U_3} = A \cap (V_5 \wedge \bigwedge^2 U_3) \subset A \cap \bigwedge^3 V_5 = W_0^\perp$ and the latter is the space of linear equations of M'_X. A skew form on V_5, considered as an element of $\bigwedge^3 V_5$, is contained in the subspace $V_5 \wedge \bigwedge^2 U_3$ if and only if U_3 is isotropic for it. This means that for $U_3 \in \Sigma_2^l(X)$, we have

$$(27) \qquad \dim(\hat{A} \cap \mathbb{I}_{U_3}) = l.$$

Furthermore, the space $\hat{A} \cap \mathbb{L}_1 = W^\perp$ has dimension $6 - n$, hence the codimension of the span of $Q'(U_3)$ in $\mathbf{P}(\overline{\mathbb{L}}_2)$ is $6 - n - l$. Since the dimension of $\mathbf{P}(\overline{\mathbb{L}}_2)$ is 3, it follows that $Q'(U_3)$ is a quadratic hypersurface in $\mathbf{P}^{n+l-3}$. Finally, we have

$$\mathbb{I}_{U_3} \oplus \overline{\mathbb{L}}_2 = (V_5 \wedge \bigwedge^2 U_3) \oplus (v_0 \wedge \bigwedge^2 U_3) \oplus \mathbf{k} = V_6 \wedge \bigwedge^2 U_3 \oplus \mathbf{k}.$$

Therefore, the dimension of $\hat{A} \cap (\mathbb{I}_{U_3} \oplus \overline{\mathbb{L}}_2) = A \cap (V_6 \wedge \bigwedge^2 U_3)$ is detected by the position of $U_3 \in \mathsf{Gr}(3, V_5)$ with respect to the stratification of $\mathsf{Gr}(3, V_6)$ defined by A (see (43)). Taking (27) into account, we deduce that for $U_3 \in Z_A^k \cap \Sigma_2^l(X)$, the corank of $Q'(v)$ is $k - l$. $\qquad\square$

4.5. Birationality of dual varieties of dimension 4.

Let X and X' be smooth dual GM varieties of the same dimension $n \geq 3$. This means that they are constructed from dual Lagrangian data (V_6, V_5, A, A_1) and $(V_6^\vee, V_5', A^\perp, A_1')$. Set $V_1' := V_5'^\perp \subset V_6$ and assume additionally

$$(28) \qquad V_1' \subset V_5$$

(we will see later that the general case reduces to this one).

Remark 4.11. The condition (28) is symmetric with respect to X and X'. Indeed, it can be reformulated as the degeneracy of the restriction of the natural pairing $V_6 \otimes V_6^\vee \to \mathbf{k}$ to the subspace $V_5 \otimes V_5' \subset V_6 \otimes V_6^\vee$.

The construction of birationalities is very similar to the one used in Section 4.3. Assume $n = 3$ or $n = 4$, and consider the diagrams

$$\mathbf{P}_X(V_5/\mathscr{U}_X) \qquad\qquad\qquad \mathbf{P}_{X'}(V_5'/\mathscr{U}_{X'})$$
$$\swarrow \quad \searrow{\scriptstyle\rho_2} \qquad\qquad {\scriptstyle\rho_2'}\swarrow \quad \searrow$$
$$X \qquad\qquad \mathsf{Gr}(3, V_5) \qquad \mathsf{Gr}(3, V_5') \qquad\qquad X',$$

where ρ_2 and ρ_2' are the second quadric fibrations defined in Section 4.4, together with the subsets $\Sigma_2(X) \subset Z_A \cap \mathsf{Gr}(3, V_5)$ and $\Sigma_2(X') \subset Z_A \cap \mathsf{Gr}(3, V_5')$. Since $\mathsf{Gr}(3, V_5) \subset \mathsf{Gr}(3, V_6)$ and $\mathsf{Gr}(3, V_5') \subset \mathsf{Gr}(3, V_6^\vee) = \mathsf{Gr}(3, V_6)$, we can take their intersection which, since (28) is satisfied, is

$$\mathsf{Gr}(2, V_5/V_1') \simeq \mathsf{Gr}(2, 4)$$

(it is empty if (28) does not hold). We restrict everything to the common base $\mathsf{Gr}(2, V_5/V_1')$ and get a diagram

$$\widetilde{X} \qquad\qquad\qquad \widetilde{X}'$$
$$\swarrow \quad \searrow{\scriptstyle\tilde{\rho}_2} \qquad {\scriptstyle\tilde{\rho}_2'}\swarrow \quad \searrow$$
$$X \qquad\qquad \mathsf{Gr}(2, V_5/V_1') \qquad\qquad X',$$

where $\widetilde{X} = \mathbf{P}_X(V_5/\mathscr{U}_X) \times_{\mathsf{Gr}(3,V_5)} \mathsf{Gr}(2, V_5/V_1')$ and $\tilde{\rho}_2$ is the restriction of ρ_2 to $\widetilde{X}$—and analogously for $\widetilde{X}'$ and $\tilde{\rho}_2'$.

Lemma 4.12. *If*

$$(29) \qquad\qquad V_1' \notin \Sigma_1(X) \cup Y_{A(X)}^{\geq n-1},$$

the map $\widetilde{X} \to X$ is the blow up of $X \cap \mathsf{CGr}(1, V_5/V_1')$. In particular, $\widetilde{X}$ is irreducible, generically reduced, and birational to X. Analogous claims hold for $\widetilde{X}'$.

Proof. The proof is analogous to that of Lemma 4.6. The fiber of the map $\widetilde{X} \to X$ over a point $w \in X$ is a point if $V_1' \not\subset \mathscr{U}_w$, and $\mathbf{P}(V/\mathscr{U}_w) \simeq \mathbf{P}^2$ otherwise. The locus of non-trivial fibers is therefore $X \cap \mathsf{CGr}(1, V_5/V_1')$. Since $X \subset \mathsf{CGr}(2, V_5)$ is an intersection of $6 - n$ hyperplanes and a quadric, $X \cap \mathsf{CGr}(1, V_5/V_1')$ is the intersection of $6 - n$ hyperplanes and a quadric in $\mathsf{CGr}(1, V_5/V_1') = \mathbf{P}^4$, hence its expected dimension is $n - 3$.

The dimension can jump if

- either one of the hyperplanes contains $\mathsf{CGr}(1, V_5/V_1')$, i.e., $V_1' \in \Sigma_1(X)$;
- or the quadric contains the intersection $\mathbf{P}(W) \cap \mathsf{CGr}(1, V_5/V_1') = \mathbf{P}^{n-2}$; this is equivalent to $V_1' \in Y_{A(X)}^{\geq n-1}$ by Proposition 4.5.

Therefore, if (29) is satisfied, the dimension does not jump and $\dim\big(X \cap \mathsf{CGr}(1, V_5/V_1')\big) \leq n - 3$. The exceptional set of $\widetilde{X} \to X$ has dimension at most $n - 1$ and thus cannot be an irreducible component of $\widetilde{X}$. It follows that $\widetilde{X}$ is integral and the map $\widetilde{X} \to X$ is the blow up of $X \cap \mathsf{CGr}(1, V_5/V_1')$. The same argument works for $\widetilde{X}'$. □

Proposition 4.13. *Let X and X' be smooth dual complex GM fourfolds satisfying (28) and such that (29) hold for both X and X'. Over a dense open subset $U \subset \mathsf{Gr}(2, V_5/V_1')$, there is an isomorphism $\tilde{\rho}_2^{-1}(U) \simeq \tilde{\rho}_2'^{-1}(U)$. In particular, $\widetilde{X}$ is birationally isomorphic to $\widetilde{X}'$ over $\mathsf{Gr}(2, V_5/V_1')$ and X is birationally isomorphic to X'.*

Proof. By Proposition 4.10, the morphism $\rho_2 \colon \mathbf{P}_X(V_5/\mathscr{U}_X) \to \mathsf{Gr}(3, V_5)$ is, outside the locus $\Sigma_2(X) \subset \mathsf{Gr}(3, V_5)$, a flat double cover branched along the quartic $Z_A \cap \mathsf{Gr}(3, V_5)$. The morphism $\tilde{\rho}_2 \colon \widetilde{X} \to \mathsf{Gr}(2, V_5/V_1')$ is obtained from ρ_2 by a base change. Outside of $\Sigma_2(X) \cap \mathsf{Gr}(2, V_5/V_1')$ (which has dimension at most 2 by Lemma 4.9), $\tilde{\rho}_2$ is therefore a double covering branched along $Z_A \cap \mathsf{Gr}(2, V_5/V_1')$.

Let $\widetilde{X} \xrightarrow{g} \bar{X} \xrightarrow{h} \mathsf{Gr}(2, V_5/V_1')$ be its Stein factorization. As in the proof of Proposition 4.7, we see that g birational and h is the double cover branched along $Z_A \cap \mathsf{Gr}(2, V_5/V_1')$.

The same argument shows that in the Stein factorization $\tilde{\rho}_2' \colon \widetilde{X}' \xrightarrow{g'} \bar{X}' \xrightarrow{h'} \mathsf{Gr}(2, V_5/V_1')$, the map g' is also birational and h' is again the double covering branched along the same quartic $Z_A \cap \mathsf{Gr}(2, V_5/V_1')$ (Lemma B.10). As the branch divisors coincide, $\bar{X}$ is isomorphic to $\bar{X}'$, hence $\widetilde{X}$ and $\widetilde{X}'$ are birationally isomorphic. By Lemma 4.12, we also have a birational isomorphism between X and X'. □

Remark 4.14. Again, one can explicitly describe the resulting birational transformation. The map $\widetilde{X} \to X$ is the blow up of a σ-conic c and the map $\widetilde{X} \to \bar{X}$ contracts all lines intersecting c (they are parameterized by the curve $Z_{\bar{A}}^{\geq 2} \cap \mathsf{Gr}(2, V_5/v')$) as well as all conics intersecting c twice (all of them are ρ-conics corresponding to points of $\Sigma_2(X) \cap \mathsf{Gr}(2, V_5/V_1')$). The second half of the transformation is analogous.

4.6. **Combined birationalities.** In this section, we combine birational isomorphisms from the previous sections to show that all period partners and all dual varieties of any dimension $n \geq 3$ are birationally isomorphic. We first deal with period partners.

Theorem 4.15. *Any two smooth complex GM fourfolds which are period partners are birationally isomorphic.*

Proof. Let X_1 and X_2 be smooth GM fourfolds obtained from the same Lagrangian subspace A and possibly different hyperplanes $V_5^1 \subset V_6$ and $V_5^2 \subset V_6$. By Theorem 3.25, smooth ordinary GM fourfolds that are dual to both X and X' are parametrized by points V_1' of the locus Y_A^1 defined in (35). If we moreover impose conditions (28) and (29) for both V_5^1 and V_5^2, we need to take V_1' in $Y_A^1 \cap \mathbf{P}(V_5^1 \cap V_5^2)$, but not in $\Sigma_1(X_1) \cup \Sigma_1(X_2)$ (which has dimension at most 1 by Lemma 4.4). If $Y_A \cap \mathbf{P}(V_5^1 \cap V_5^2)$ has dimension ≥ 3, this is certainly possible; if it has dimension 2, it is a surface of degree 6, whereas $Y_A^{\geq 2}$ is an integral surface of degree 40 (Theorem B.2), so it is again possible.

We make this choice of V_1' and we let X' be the corresponding GM fourfold, dual to both X_1 and X_2. Condition (28) holds for both pairs (X_1, X') and (X_2, X'); also, we have $V_1' \notin \Sigma_1(X_1) \cup \Sigma_1(X_2) \cup Y_A^{\geq 3}$ by construction. Moreover, for each $i \in \{1, 2\}$, we have $(V_5^i)^\perp \notin \Sigma_1(X')$ (by Lemma 4.4, $(V_5^i)^\perp \in \Sigma_1(X')$ implies $(V_1', V_5^i) \in \widehat{Y}_A$, hence $V_1' \in \Sigma_1(X_i)$) and $V_5^i \notin Y_{A^\perp}^{\geq 3}$, because X_1 and X_2 are fourfolds. Thus Condition (29) also holds for both pairs. Applying Proposition 4.13, we conclude that X_1 and X' are birationally isomorphic, and so are X_2 and X'. This proves the theorem. $\qquad\square$

Corollary 4.16. *Smooth complex period partners of the same dimension $n \geq 3$ are birationally isomorphic.*

Proof. For $n = 3$, this is Theorem 4.7; for $n = 4$, this is Theorem 4.15; for $n \geq 5$, this is Proposition 4.2. $\qquad\square$

The following corollary was brought to our attention by T. de Fernex.

Corollary 4.17. *No smooth complex GM variety X of dimension $n \geq 3$ is birationally rigid: there are smooth prime Fano varieties that are birationally isomorphic to, but not biregularly isomorphic to, X.*

Proof. By Theorem 3.23, there is a bijection between the set of isomorphism classes of period partners of X and the set $\left(Y_{A^\perp}^{5-n} \sqcup Y_{A^\perp}^{6-n}\right) / \mathrm{PGL}(V_6)_A$. Since the group $\mathrm{PGL}(V_6)_A$ is finite (Theorem B.9(a)), these sets are infinite. By Corollary 4.16, all these period partners of X are birationally isomorphic to X. This proves the corollary. $\qquad\square$

For threefolds, the assumption (29) is very restrictive; for instance, it rules out all ordinary dual varieties. We replace it with the more flexible assumption

(30) $$V_1 \in (Y_A^2 \cap \mathbf{P}(V_5)) \smallsetminus \Sigma_1(X).$$

The space $\tilde{X} \subset \mathbf{P}_X(V_5/\mathscr{U}_X)$ is then reducible. We describe its irreducible components and the restriction of the map $\tilde{\rho}_2$ to these components.

Lemma 4.18. *Let X be a smooth ordinary GM threefold and let $V_1 \subset V_5$ be a subspace such that (30) holds. Then $\widetilde{X}$ has two irreducible components: the proper preimage $\widetilde{X}_1$ of X, and*

the total preimage $\widetilde{X}_2$ of a particular line $M \subset X$. The restriction of the map $\tilde{\rho}_2$ to $\widetilde{X}_2$ is a birational isomorphism onto a Schubert hyperplane section of $\mathsf{Gr}(2, V_5/V_1)$.

Proof. The proof of Lemma 4.12 shows that the locus of non-trivial fibers of the projection $\widetilde{X} \to X$ is $X \cap \mathsf{Gr}(1, V_5/V_1)$, i.e., the intersection in $\mathsf{Gr}(1, V_5/V_1) = \mathbf{P}^3$ of two hyperplanes and a quadric. Since $V_1 \notin \Sigma_1(X)$, the intersection of the hyperplanes is a line, which we denote by M. Its intersection with the remaining quadric can be identified with the fiber of the first quadric fibration over the point $V_1 \in \mathbf{P}(V_5)$. So, under Condition (30), its corank is 2, hence it coincides with M. Since the non-trivial fibers of the projection $\widetilde{X} \to X$ are $\mathbf{P}^2$, the preimage of M in $\widetilde{X}$ has dimension 3, hence is an irreducible component of $\widetilde{X}$ which we denote by $\widetilde{X}_2$. The other irreducible component is the proper preimage $\widetilde{X}_1$ of X in $\widetilde{X}$.

Since M is a line in $\mathsf{Gr}(1, V_5/V_1) = \mathbf{P}(V_5/V_1)$, it can be written, as a subvariety of $\mathsf{Gr}(2, V_5)$, as $M = \{U_2 \in \mathsf{Gr}(2, V_5) \mid V_1 \subset U_2 \subset V_3\}$, for some 3-dimensional subspace $V_3 \subset V_5$ containing V_1. From the definition of the fibers of the map $\widetilde{X} \to X$, we get

$$\widetilde{X}_2 \simeq \{(U_2, U_3) \in \mathsf{Fl}(2,3;V_5) \mid V_1 \subset U_2 \subset V_3 \ \text{and} \ U_2 \subset U_3 \subset V_5\}.$$

The map $\tilde{\rho}_2 \colon \widetilde{X} \to \mathsf{Gr}(3, V_5)$ takes a point $(U_2, U_3) \in \widetilde{X}_2$ to $U_3/V_1 \in \mathsf{Gr}(2, V_5/V_1)$, hence maps $\widetilde{X}_2$ birationally onto the Schubert hyperplane section of $\mathsf{Gr}(2, V_5/V_1)$ of all 2-dimensional subspaces intersecting V_3/V_1 nontrivially (in fact, $\widetilde{X}_2$ is a Springer-type resolution of singularities of the Schubert hyperplane). □

We now show that under assumption (30), we still have birational threefolds.

Proposition 4.19. *Let X and X' be dual smooth ordinary complex GM threefolds. Assume that V_1' satisfies (30) for X and that $V_5^\perp$ satisfies (30) for X'. Then X and X' are birationally isomorphic.*

Proof. Let $\widetilde{X} = \widetilde{X}_1 \cup \widetilde{X}_2$ and $\widetilde{X}' = \widetilde{X}_1' \cup \widetilde{X}_2'$ be the decompositions into irreducible components of Lemma 4.18. Consider the maps $\tilde{\rho}_2$ and $\tilde{\rho}_2'$ outside of $(\Sigma_2(X) \cup \Sigma_2(X')) \cap \mathsf{Gr}(2, V_5/V_1')$ (this locus is 2-dimensional (Lemma 4.9), so it does not affect the birational geometry of X and X').

Over a point of $\mathsf{Gr}(2, V_5/V_1')$ not in $\Sigma_2(X)$, the fiber of $\widetilde{X}$ is, by Proposition 4.10, a quadric in $\mathbf{P}^0$. It is non-empty if and only if the corresponding quadratic form is zero, i.e., if its kernel is non-zero. By Proposition 4.10 again, this happens precisely for points of $Z_A \cap \mathsf{Gr}(2, V_5/V_1')$. This means that away from $\Sigma_2(X) \cap \mathsf{Gr}(2, V_5/V_1')$, the map $\tilde{\rho}_2$ is an isomorphism from $\widetilde{X}$ onto $Z_A \cap \mathsf{Gr}(2, V_5/V_1')$. Analogously, away from $\Sigma_2(X') \cap \mathsf{Gr}(2, V_5/V_1')$, the map $\tilde{\rho}_2'$ is an isomorphism from $\widetilde{X}'$ onto $Z_A \cap \mathsf{Gr}(2, V_5/V_1')$.

By Lemma 4.18, the components $\widetilde{X}_2$ and $\widetilde{X}_2'$ map birationally onto the Schubert hyperplanes of $\mathsf{Gr}(2, V_5/V_1')$ given by the three-dimensional spaces $V_3 \subset V_6$ and $V_3' \subset V_6^\vee$ corresponding to the lines $M_{V_1'} \subset X$ and $M_{V_5^\perp} \subset X'$ respectively. Let us show $V_3' = V_3^\perp$, so that the Schubert hyperplanes coincide.

By definition of the line $M_{V_1'}$, the space V_3 can be constructed as follows. Let

$$A \cap (V_1' \wedge {\bigwedge}^2 V_6) = \langle v \wedge \xi_1, v \wedge \xi_2 \rangle,$$

where v is a generator of V_1' and $\xi_1, \xi_2 \in \bigwedge^2 V_6$. Let $f \in V_6^\vee$ be an equation of V_5. Then V_3 is spanned by v and the contractions $\xi_i(f, -)$. Analogously, if

$$A^\perp \cap (V_5^\perp \wedge \textstyle\bigwedge^2 V_6^\vee) = \langle f \wedge \eta_1, f \wedge \eta_2 \rangle,$$

the space V_3' is spanned by f and the contractions $\eta_i(v, -)$ of $\eta_i \in \bigwedge^2 V_6^\vee$ with $v \in V_6$. To identify V_3' with $V_3^\perp$, it is therefore enough to show

$$(\xi_i(f, -), \eta_j(v, -)) = 0$$

for all $i, j \in \{1, 2\}$. This follows from the equality

$$(v \wedge \xi_i, f \wedge \eta_j) = (\xi_i(f, -), \eta_j(v, -))$$

which in its turn follows from $f(v) = 0$.

We proved that $\widetilde{X}_2$ and $\widetilde{X}_2'$ are mapped by $\tilde\rho_2$ and $\tilde\rho_2'$ onto the same Schubert hyperplane section $H \subset \mathsf{Gr}(2, V_5/V_1')$. Since both $\widetilde{X}$ and $\widetilde{X}'$ map birationally onto the quartic hypersurface $Z_A \cap \mathsf{Gr}(2, V_5/V_1')$, it follows that this quartic has two components H and Z', where Z' is a cubic hypersurface in $\mathsf{Gr}(2, V_5/V_1')$, and that $\widetilde{X}_1$ and $\widetilde{X}_1'$ both map birationally onto Z'. Thus $\widetilde{X}_1$ is birational to $\widetilde{X}_1'$, hence X is birational to X'. $\qquad\square$

The birational isomorphism of Proposition 4.19 coincides with the transformation of X (with respect to the line M) defined in [DIM, Section 7.2].

We can now prove the main result of this section.

Theorem 4.20. *Any two dual smooth complex GM varieties of the same dimension $n \geq 3$ are birationally isomorphic.*

Proof. We keep the same notation as above. For $n \geq 5$, the result is trivial by Proposition 4.2, so we may assume $n \in \{3, 4\}$.

Assume first $n = 4$. As in the proof of Theorem 4.15, we may choose $V_1'' \subset V_5$ such that the corresponding smooth GM fourfold X'' is dual to X. Then X is birationally isomorphic to X'' by Proposition 4.13 and X' and X'' are birationally isomorphic by Theorem 4.15, hence X is birationally isomorphic to X'.

Now let X be a smooth GM threefold and set $A := A(X)$. By Lemma B.5, we can find $v \in \mathbf{P}(V_6)$ and $V_5 \subset V_6$ such that

$$(31) \qquad v \in Y_A^2, \qquad [V_5] \in Y_{A^\perp}^2, \qquad v \in V_5, \quad \text{and} \quad (v, V_5) \notin (p \times q)(\widehat{Y}_A).$$

Let X' be the ordinary GM threefold corresponding to the Lagrangian A and the Plücker hyperplane $V_5 \subset V_6$, and let X'' be the ordinary GM threefold corresponding to the Lagrangian $A^\perp$ and the Plücker hyperplane $v^\perp \subset V_6^\vee$.

Then X' is a period partner of X, hence X' is birational to X by Theorem 4.7. Furthermore, X'' is dual to X' and the conditions of Proposition 4.19 are satisfied for the pair (X', X'') (the last condition in (31) is equivalent to $v \notin \Sigma_1(X')$ by Lemma 4.4). Therefore, X'' is birational to X' by Proposition 4.19. Combining these two birationalities, we conclude that X'' is birational to X. It remains to note that any other dual GM threefold of X is a period partner of X'', hence is birational to X'' by Theorem 4.7. $\qquad\square$

Appendix A. Excess conormal sheaves

Let W be a **k**-vector space and let $X \subset \mathbf{P}(W)$ be a closed **k**-subscheme which is an intersection of quadrics, i.e., the twisted ideal sheaf $\mathscr{I}_X(2)$ on $\mathbf{P}(W)$ is globally generated. Let

$$V_X := H^0(\mathbf{P}(W), \mathscr{I}_X(2))$$

be the space of quadrics through X. The canonical map $V_X \otimes \mathscr{O}_{\mathbf{P}(W)}(-2) \to \mathscr{I}_X$ is surjective and its restriction to X induces an epimorphism $V_X \otimes \mathscr{O}_X(-2) \twoheadrightarrow \mathscr{I}_X/\mathscr{I}_X^2 = \mathscr{N}_{X/\mathbf{P}(W)}^\vee$ onto the conormal sheaf of X.

Definition A.1. The excess conormal sheaf $\mathscr{E}\mathscr{N}_{X/\mathbf{P}(W)}^\vee$ of an intersection of quadrics $X \subset \mathbf{P}(W)$ is the kernel of the canonical map $V_X \otimes \mathscr{O}_X(-2) \twoheadrightarrow \mathscr{N}_{X/\mathbf{P}(W)}^\vee$.

When the ambient space $\mathbf{P}(W)$ is clear, we will simply write $\mathscr{E}\mathscr{N}_X^\vee$. This sheaf is defined over **k** and fits in the *excess conormal sequence*

$$0 \to \mathscr{E}\mathscr{N}_{X/\mathbf{P}(W)}^\vee \to V_X \otimes \mathscr{O}_X(-2) \to \mathscr{N}_{X/\mathbf{P}(W)}^\vee \to 0.$$

We will also often consider the twist

$$(32) \qquad 0 \to \mathscr{E}\mathscr{N}_{X/\mathbf{P}(W)}^\vee(2) \to V_X \otimes \mathscr{O}_X \to \mathscr{N}_{X/\mathbf{P}(W)}^\vee(2) \to 0.$$

In this appendix, we discuss some properties of the excess conormal sheaf.

Lemma A.2. *The excess conormal sheaf is locally free on the lci locus of X.*

Proof. This follows from the local freeness of the conormal bundle on the lci locus of X. $\qquad \square$

Proposition A.3. *Consider a closed subscheme $X \subset \mathbf{P}(W)$ and a cone $\mathsf{C}_K X \subset \mathbf{P}(W \oplus K)$. Let $W' \subset W \oplus K$ be a vector subspace and let $X' \subset \mathbf{P}(W') \cap \mathsf{C}_K X$ be a closed subscheme. Set $X_0' := X' \cap \mathsf{C}_K^\circ X$. If both X and X' are intersections of quadrics, there is a canonical commutative diagram*

$$(33)$$
$$
\begin{array}{ccccccc}
\mu^* \mathscr{E}\mathscr{N}_{X/\mathbf{P}(W)}^\vee & \longrightarrow & V_X \otimes \mathscr{O}_{X_0'}(-2) & \longrightarrow & \mu^* \mathscr{N}_{X/\mathbf{P}(W)}^\vee & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow \mathscr{E}\mathscr{N}_{X_0'/\mathbf{P}(W')}^\vee & \longrightarrow & V_{X'} \otimes \mathscr{O}_{X_0'}(-2) & \longrightarrow & \mathscr{N}_{X_0'/\mathbf{P}(W')}^\vee & \longrightarrow & 0
\end{array}
$$

of sheaves on X_0', where μ denotes the natural projection $\mathbf{P}(W \oplus K) \smallsetminus \mathbf{P}(K) = \mathsf{C}_K^\circ \mathbf{P}(W) \to \mathbf{P}(W)$ as well as its restriction to X_0'.

Proof. The embedding $W' \subset W \oplus K$ induces a map $\mathsf{S}^2 W^\vee \to \mathsf{S}^2 W'^\vee$. Denote by V' the image of V_X under this map. Clearly, $\mathbf{P}(W') \cap \mathsf{C}_K X$ is cut out in $\mathbf{P}(W')$ by the quadrics in V'. Hence the space $V_{X'}$ of quadrics cutting out X' in $\mathbf{P}(W')$ contains V' and we obtain a canonical map $V_X \to V_{X'}$. Furthermore, on the open subset $\mathbf{P}(W') \smallsetminus \mathbf{P}(K) \subset \mathbf{P}(W')$, the pullback $\mu^* \mathscr{I}_X$ of the ideal of X in $\mathbf{P}(W)$ is contained in the ideal $\mathscr{I}_{X_0'}$ of X_0' in $\mathbf{P}(W') \smallsetminus \mathbf{P}(K)$, hence we have a commutative diagram

$$
\begin{array}{ccc}
V_X \otimes \mathscr{O}_{\mathbf{P}(W') \smallsetminus \mathbf{P}(K)}(-2) & \longrightarrow & \mu^* \mathscr{I}_X \\
\downarrow & & \downarrow \\
V_{X'} \otimes \mathscr{O}_{\mathbf{P}(W') \smallsetminus \mathbf{P}(K)}(-2) & \longrightarrow & \mathscr{I}_{X_0'}.
\end{array}
$$

Restricting it to X_0' and extending lines to excess conormal sequences proves the claim. $\qquad \square$

In some cases, the vertical map in (33) between excess conormal sheaves is an isomorphism.

Lemma A.4. *In the situation of Proposition* A.3, *assume moreover* $W' = W \oplus \mathbf{k}$ *and* $X' = \mathsf{C}_{\mathbf{k}}X \subset \mathbf{P}(W')$, *so that* $X'_0 = \mathsf{C}^\circ_{\mathbf{k}}X$ *is the punctured cone. If* $\mu \colon X'_0 \to X$ *is the natural projection, we have* $\mathscr{E}\mathscr{N}^\vee_{X'}|_{X'_0} = \mu^* \mathscr{E}\mathscr{N}^\vee_X$.

Proof. The intersection in $\mathbf{P}(W')$ of the quadrics in $V_X \subset \mathsf{S}^2 W^\vee \subset \mathsf{S}^2 W'^\vee$ is the cone $X' = \mathsf{C}_{\mathbf{k}}X$. Moreover, $\mathscr{N}_{X'_0/\mathbf{P}(W')} \simeq \mu^* \mathscr{N}_{X/\mathbf{P}(W)}$. Thus, the central and right vertical arrows in (33) are isomorphisms. Moreover, the map μ is flat, hence the top line is exact on the left. Therefore the left arrow is an isomorphism as well. $\square$

Lemma A.5. *Assume that* $X \subset \mathbf{P}(W)$ *is an intersection of quadrics and is linearly normal. Let* $W' \subset W$ *be a hyperplane such that* $X' := X \cap \mathbf{P}(W')$ *is a dimensionally transverse intersection. Then* $X' \subset \mathbf{P}(W')$ *is an intersection of quadrics and* $\mathscr{E}\mathscr{N}^\vee_{X'_{\mathrm{lci}}} = \mathscr{E}\mathscr{N}^\vee_{X_{\mathrm{lci}}}|_{X'_{\mathrm{lci}}}$, *where* X_{lci} *is the lci locus of* X *and* $X'_{\mathrm{lci}} = X' \cap X_{\mathrm{lci}}$.

Proof. We have an exact sequence

$$0 \to \mathscr{I}_X(-1) \to \mathscr{I}_X \to \mathscr{I}_{X'/\mathbf{P}(W')} \to 0.$$

Twisting it by $\mathscr{O}_{\mathbf{P}(W)}(2)$ and using the linear normality of X, we conclude that X' is an intersection of quadrics and $V_{X'} = V_X$. The diagram of Proposition A.3 gives a commutative diagram

$$
\begin{array}{ccccccc}
\mathscr{E}\mathscr{N}^\vee_X|_{X'} & \longrightarrow & V_X \otimes \mathscr{O}_{X'}(-2) & \longrightarrow & \mathscr{N}^\vee_{X/\mathbf{P}(W)}|_{X'} & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow & & \\
0 \longrightarrow \mathscr{E}\mathscr{N}^\vee_{X'} & \longrightarrow & V_{X'} \otimes \mathscr{O}_{X'}(-2) & \longrightarrow & \mathscr{N}^\vee_{X'/\mathbf{P}(W')} & \longrightarrow & 0.
\end{array}
$$

The right vertical arrow is an isomorphism by the dimensionally transverse intersection condition. Moreover, the top line is exact on the lci locus, since the conormal bundle is locally free. Hence the left arrow is an isomorphism. $\square$

Lemma A.6. *Assume* $H^0(X, \mathscr{O}_X) = \mathbf{k}$ *and let* $Q \subset \mathbf{P}(W)$ *be a quadratic hypersurface such that* $X' = X \cap Q$ *is a dimensionally transverse intersection. Then* $\mathscr{E}\mathscr{N}^\vee_{X'_{\mathrm{lci}}} = \mathscr{E}\mathscr{N}^\vee_{X_{\mathrm{lci}}}|_{X'_{\mathrm{lci}}}$, *where again* X_{lci} *is the lci locus of* X *and* $X'_{\mathrm{lci}} = X' \cap X_{\mathrm{lci}}$.

Proof. Tensoring the exact sequences

$$0 \to \mathscr{I}_X \to \mathscr{O}_{\mathbf{P}(W)} \to \mathscr{O}_X \to 0 \qquad \text{and} \qquad 0 \to \mathscr{O}_{\mathbf{P}(W)}(-2) \to \mathscr{O}_{\mathbf{P}(W)} \to \mathscr{O}_Q \to 0$$

and taking into account the dimensional transversality of the intersection $X' = X \cap Q$, we obtain an exact sequence

$$0 \to \mathscr{I}_X(-2) \to \mathscr{I}_X \oplus \mathscr{O}_{\mathbf{P}(W)}(-2) \to \mathscr{I}_{X'} \to 0.$$

Twisting it by $\mathscr{O}_{\mathbf{P}(W)}(2)$ and taking into account that the condition $H^0(X, \mathscr{O}_X) = \mathbf{k}$ implies $H^0(\mathbf{P}(W), \mathscr{I}_X) = H^1(\mathbf{P}(W), \mathscr{I}_X) = 0$, we obtain an isomorphism $V_{X'} = V_X \oplus \mathbf{k}$. It follows

that the diagram of Proposition A.3 extends to a commutative diagram

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
\mathscr{E}\mathscr{N}_X^\vee|_{X'} & \longrightarrow & V_X \otimes \mathscr{O}_{X'}(-2) & \longrightarrow & \mathscr{N}_{X/\mathbf{P}(W)}^\vee|_{X'} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 \longrightarrow \mathscr{E}\mathscr{N}_{X'}^\vee & \longrightarrow & V_{X'} \otimes \mathscr{O}_{X'}(-2) & \longrightarrow & \mathscr{N}_{X'/\mathbf{P}(W')}^\vee & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
& & \mathscr{O}_{X'}(-2) & = & \mathscr{O}_{X'}(-2) & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0. & &
\end{array}
$$

Its middle column is exact by the above argument and the right column is exact by the dimensionally transverse intersection condition. Moreover, the top line is exact on X_{lci}, hence the left vertical arrow is an isomorphism. $\qquad\square$

Finally, we compute the excess conormal bundle for some examples. Let $\mathscr{U}$ be the rank-2 tautological vector bundle on the Grassmannian $\mathsf{Gr}(2, V_5)$.

Proposition A.7. *Let* $X = \mathsf{Gr}(2, V_5) \subset \mathbf{P}(\bigwedge^2 V_5)$. *Then* $\mathscr{E}\mathscr{N}_X^\vee \simeq \det(V_5^\vee) \otimes \mathscr{U}(-2)$.

Proof. The standard resolution of the structure sheaf of X is

$$0 \to \det(V_5^\vee)^{\otimes 2} \otimes \mathscr{O}(-5) \to \det(V_5^\vee) \otimes V_5^\vee \otimes \mathscr{O}(-3) \to \det(V_5^\vee) \otimes V_5 \otimes \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_X \to 0,$$

where the middle map is induced by the contraction $V_5^\vee \otimes \bigwedge^2 V_5 \to V_5$. In particular, the space $\det(V_5^\vee) \otimes V_5 = \bigwedge^4 V_5^\vee$ can be identified with the space of quadrics through X. Tensoring this resolution with $\mathscr{O}_X$ and taking into account that $\mathrm{Tor}_i(\mathscr{O}_X, \mathscr{O}_X) = \bigwedge^i \mathscr{N}_X^\vee$, we deduce an exact sequence

$$0 \to \bigwedge^2 \mathscr{N}_X^\vee \to \det(V_5^\vee) \otimes V_5^\vee \otimes \mathscr{O}_X(-3) \to \det(V_5^\vee) \otimes V_5 \otimes \mathscr{O}_X(-2) \to \mathscr{N}_X^\vee \to 0.$$

The above description of the middle map shows that (up to a twist by $\det(V_5^\vee)$) it factors as a composition of the epimorphism $V_5^\vee \otimes \mathscr{O}_X(-3) \twoheadrightarrow \mathscr{U}^\vee(-3)$, an isomorphism $\mathscr{U}^\vee(-3) \xrightarrow{\sim} \mathscr{U}(-2)$, and the monomorphism $\mathscr{U}(-2) \hookrightarrow V_5 \otimes \mathscr{O}_X(-2)$, hence the claim. $\qquad\square$

Let V_6 be a **k**-vector space and let again $\mathscr{U}$ be the rank-2 tautological vector bundle on the Grassmannian $\mathsf{Gr}(2, V_6)$.

Proposition A.8. *Let* $X = \mathsf{Gr}(2, V_6) \subset \mathbf{P}(\bigwedge^2 V_6)$. *Then* $\mathscr{E}\mathscr{N}_X^\vee$ *is isomorphic to* $\det(V_6^\vee) \otimes \big((V_6 \otimes \mathscr{U}(-2))/(\mathsf{S}^2\mathscr{U})(-2)\big)$.

Proof. The standard resolution of the structure sheaf of X is

$$\cdots \to \det(V_6^\vee) \otimes \mathfrak{sl}(V_6) \otimes \mathscr{O}(-3) \to \det(V_6^\vee) \otimes \bigwedge^2 V_6 \otimes \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_X \to 0,$$

where the second map is given by the natural Lie algebra action $\mathfrak{sl}(V_6) \otimes \bigwedge^2 V_6 \to \bigwedge^2 V_6$. Tensoring this resolution with $\mathscr{O}_X$, we deduce an exact sequence

$$\det(V_6^\vee) \otimes \mathfrak{sl}(V_6) \otimes \mathscr{O}_X(-3) \to \det(V_6^\vee) \otimes \bigwedge^2 V_6 \otimes \mathscr{O}_X(-2) \to \mathscr{N}_X^\vee \to 0.$$

The above description of the middle map shows that it factors (again, up to a twist by $\det(V_6^\vee)$) as a composition

$$\mathfrak{sl}(V_6) \otimes \mathscr{O}_X(-3) \twoheadrightarrow V_6 \otimes \mathscr{U}^\vee(-3) \xrightarrow{\sim} V_6 \otimes \mathscr{U}(-2) \to \bigwedge^2 V_6 \otimes \mathscr{O}_X(-2).$$

Since the first arrow is surjective, the image of the composition (i.e., the excess conormal bundle of the Grassmannian) is isomorphic (up to a twist) to the image of $V_6 \otimes \mathscr{U}(-2)$. The kernel of the last map is $(\mathsf{S}^2\mathscr{U})(-2)$, hence the claim. $\qquad\square$

Appendix B. Eisenbud–Popescu–Walter sextics

These sextics were introduced by Eisenbud, Popescu, and Walter in [EPW, Example 9.3] and were thoroughly investigated, over $\mathbf{C}$, by O'Grady in the series of papers [O1, O2, O3, O4, O5, O6, O7]. For their remarkable properties, we refer to these papers. In this appendix, we sketch the original construction of EPW sextics from Lagrangian subspaces and discuss some of their properties. The only new results here are Lemmas B.5 and B.6. To be able to use O'Grady's result, we work over $\mathbf{C}$, although many of his results are valid over more general fields.

B.1. **Overview of EPW sextics.** Let V_6 be a 6-dimensional complex vector space and let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace for the $\det(V_6)$-valued symplectic form ω defined by $\omega(\xi, \eta) = \xi \wedge \eta$.

Definition B.1. For any integer ℓ, we set

(34) $$Y_A^{\geq \ell} := \left\{ [v] \in \mathbf{P}(V_6) \mid \dim\left(A \cap (v \wedge \bigwedge^2 V_6)\right) \geq \ell \right\}$$

and endow it with a scheme structure as in [O1, Section 2]. The locally closed subsets

(35) $$Y_A^\ell := Y_A^{\geq \ell} \smallsetminus Y_A^{\geq \ell+1}$$

form the EPW stratification of $\mathbf{P}(V_6)$ and the sequence of inclusions

$$\mathbf{P}(V_6) = Y_A^{\geq 0} \supset Y_A^{\geq 1} \supset Y_A^{\geq 2} \supset \cdots$$

is called the EPW sequence. When the scheme $Y_A := Y_A^{\geq 1}$ is not the whole space $\mathbf{P}(V_6)$, it is a sextic hypersurface ([O1, (1.8)]) called an EPW sextic. The scheme $Y_A^{\geq 2}$ is non-empty and has everywhere dimension ≥ 2 ([O1, (2.9)]).

Non-zero elements of A which can be written as $v_1 \wedge v_2 \wedge v_3$ are called decomposable vectors, so that A contains no decomposable vectors if

$$\Theta_A := \mathbf{P}(A) \cap \mathsf{Gr}(3, V_6)$$

is empty. When $\mathbf{P}(V_6) \neq \bigcup_{V_3 \in \Theta_A} \mathbf{P}(V_3)$ (e.g., when $\dim(\Theta_A) \leq 2$), the scheme Y_A is not $\mathbf{P}(V_6)$ and its singular locus can be described as ([O3, Corollary 2.5])

(36) $$\mathrm{Sing}(Y_A) = Y_A^{\geq 2} \cup \left(\bigcup_{V_3 \in \Theta_A} \mathbf{P}(V_3) \right).$$

The following theorem gathers various results of O'Grady's.

Theorem B.2 (O'Grady)**.** *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace. If A contains only finitely many decomposable vectors,*

(a) *Y_A is an integral normal sextic hypersurface in $\mathbf{P}(V_6)$;*
(b) *$Y_A^{\geq 2}$ is a surface;*
(c) *$Y_A^{\geq 3}$ is finite;*
(d) *$Y_A^{\geq 5}$ is empty.*

If moreover A contains no decomposable vectors,

   (b$'$) $Y_A^{\geq 2} = \mathrm{Sing}(Y_A)$ *is an integral normal Cohen–Macaulay surface of degree* 40;
   (c$'$) $Y_A^{\geq 3} = \mathrm{Sing}(Y_A^{\geq 2})$ *is finite and smooth, and is empty for A general;*
   (d$'$) $Y_A^{\geq 4}$ *is empty.*

Note that if A is the dual Lagrangian (see Section B.2) associated with a strongly smooth ordinary GM curve, A contains 5 decomposable vectors and $Y_A^{\geq 4}$ is non-empty (Remark 3.15).

Proof. The finite union $S := \bigcup_{V_3 \in \Theta_A} \mathbf{P}(V_3)$ is a surface if Θ_A is non-empty (finite) and it is empty otherwise. Relation (36) holds, hence (a) will follow from (b).

As we already mentioned, the scheme $Y_A^{\geq 2}$ has everywhere dimension ≥ 2 ([O1, (2.9)]). We now show that $Y_A^{\geq 2}$ has dimension ≤ 2 at any point $v \notin S$. If $v \in Y_A^2$, this is [O3, Proposition 2.9]. Assume $v \in Y_A^{\geq 3}$; the proof of [O4, Claim 3.7] still applies because $v \notin S$. It gives $v \in Y_A^3$ and v isolated in $Y_A^{\geq 3}$, which proves (c). This implies that in any positive-dimensional component T of $Y_A^{\geq 2} \smallsetminus S$, the open subset $T \cap Y_A^2$ is dense and has dimension ≤ 2. Therefore, $\dim(T) \leq 2$ and (b) is proved.

Assume finally $v \in Y_A^{\geq 5}$. Inside $\mathbf{P}(\bigwedge^2(V_6/v)) = \mathbf{P}^9$, the linear space $\mathbf{P}\big(A \cap (v \wedge \bigwedge^2(V_6/v))\big)$ (of dimension ≥ 4) and the Grassmaniann $\mathsf{Gr}(2, V_6/v)$ (of dimension 6) meet along a positive-dimensional locus contained in Θ_a, which is absurd. This proves (d).

For the proof of (b$'$), (c$'$), and (d$'$), we have $\Theta_A = \varnothing$.

Again, we have $\mathrm{Sing}(Y_A) = Y_A^{\geq 2}$ by (36); moreover, Y_A^2 is smooth of the expected dimension 2 ([O3, Proposition 2.9]), so that $\mathrm{Sing}(Y_A^{\geq 2}) \subset Y_A^{\geq 3}$. On the other hand, [O4, Claim 3.7] says that (d$'$) holds; the proof actually shows that the tangent space to $Y_A^{\geq 3}$ at any point is zero, so that is smooth. It is moreover empty for A general ([O1, Claim 2.6]).

All this implies that determinantal locus $Y_A^{\geq 2}$ has everywhere the expected dimension, 2. Therefore, it is Cohen–Macaulay and $Y_A^{\geq 3} \subset \mathrm{Sing}(Y_A^{\geq 2})$. This completes the proof of (c$'$). It also implies that $Y_A^{\geq 2}$ is normal since it is Cohen–Macaulay and its singular locus is finite. Its degree is 40 by [O1, (2.9)].

It remains to show that $Y_A^{\geq 2}$ is irreducible. When A is general, this follows from [O7, Theorem 1.1]. To deduce irreducibility for any A with no decomposable vectors, we use a standard trick. Let $(T, 0)$ be a smooth pointed connected curve that parametrizes Lagrangians $(A_t)_{t \in T}$ with no decomposable vectors such that for $t \in T$ general, $Y_{A_t}^{\geq 2}$ is smooth irreducible and $A_0 = A$.

Let $H \subset \mathbf{P}(V_6)$ be a general hyperplane, so that $H \cap Y_{A_0}^{\geq 2}$ is a smooth curve and $H \cap Y_{A_t}^{\geq 2}$ is smooth irreducible for general t. In $T \times \mathbf{P}^5$, consider the union of all surfaces $\{t\} \times Y_{A_t}^{\geq 2}$. As a determinantal locus, it has everywhere codimension ≤ 3, hence its intersection $\mathscr{Y}$ with $T \times H$ has everywhere dimension ≥ 2. Since the fiber $\mathscr{Y}_0 = H \cap Y_{A_0}^{\geq 2}$ is 1-dimensional, any component of $\mathscr{Y}_0$ deforms to a neighborhood of 0 in T. Since $\mathscr{Y}_t$ is irreducible for t general, so is $\mathscr{Y}_0$, hence so is $Y_{A_0}^{\geq 2}$. This finishes the proof of the theorem. $\qquad\square$

B.2. **Duality of EPW sextics.** If $A \subset \bigwedge^3 V_6$ is a Lagrangian subspace, its orthogonal $A^\perp \subset \bigwedge^3 V_6^\vee$ is also a Lagrangian subspace. In the dual projective space $\mathbf{P}(V_6^\vee) = \mathsf{Gr}(5, V_6)$, the

EPW sequence for $A^\perp$ can be described as

$$(37) \qquad\qquad Y_{A^\perp}^{\geq \ell} = \{V_5 \in \mathsf{Gr}(5, V_6) \mid \dim(A \cap \textstyle\bigwedge^3 V_5) \geq \ell\}.$$

The canonical identification $\mathsf{Gr}(3, V_6) \simeq \mathsf{Gr}(3, V_6^\vee)$ induces an isomorphism between the scheme Θ_A of decomposable vectors in $\mathbf{P}(A)$ and the scheme $\Theta_{A^\perp}$ of decomposable vectors in $\mathbf{P}(A^\perp)$ ([O3, (2.82)]). In particular, A contains no decomposable vectors if and only if the same is true for $A^\perp$.

One of the interesting properties of a EPW sextic is that its projective dual is often also an EPW sextic. The proof we give is essentially equivalent to O'Grady's ([O2, Corollary 3.6]) but is written in more geometrical terms.

Consider the scheme

$$(38) \qquad\qquad \widehat{Y}_A := \{(v, V_5) \in \mathsf{Fl}(1, 5; V_6) \mid A \cap (v \wedge \textstyle\bigwedge^2 V_5) \neq 0\}$$

and the projections

$$
\begin{array}{ccc}
& \widehat{Y}_A & \\
{}^{p}\swarrow & & \searrow{}^{q} \\
Y_A & & Y_{A^\perp}.
\end{array}
$$

Proposition B.3. *If A has only finitely many decomposable vectors, the hypersurfaces $Y_A \subset \mathbf{P}(V_6)$ and $Y_{A^\perp} \subset \mathbf{P}(V_6^\vee)$ are projectively dual. If A contains no decomposable vectors, $\widehat{Y}_A$ is irreducible and realizes the projective duality between these two hypersurfaces.*

Proof. We define

$$\widehat{Y}_A' := \{(a, v, V_5) \in \mathbf{P}(A) \times \mathsf{Fl}(1, 5; V_6) \mid a \in \mathbf{P}(A \cap (v \wedge \textstyle\bigwedge^2 V_5))\}$$

with the forgetful map $\widehat{Y}_A' \to \widehat{Y}_A$ and the maps $p' \colon \widehat{Y}_A' \to \widehat{Y}_A \xrightarrow{p} Y_A$ and $q' \colon \widehat{Y}_A' \to \widehat{Y}_A \xrightarrow{q} Y_{A^\perp}$.

Let us determine the fiber of p' over $v \in Y_A$. Take $a \in A \cap (v \wedge \textstyle\bigwedge^2 V_6)$ non-zero and write $a = v \wedge \eta$ with $\eta \in \textstyle\bigwedge^2(V_6/v)$.

If a is not decomposable, so is η, hence its rank is 4 and there is a unique hyperplane $V_5 \subset V_6$ containing v such that $\eta \in \textstyle\bigwedge^2(V_5/v)$. Moreover, the space V_5 corresponding to $a = v \wedge \eta$ is given by the 4-form

$$\eta \wedge \eta \in \textstyle\bigwedge^4(V_6/v) \simeq (V_6/v)^\vee = v^\perp \subset V_6^\vee.$$

If a is decomposable, i.e., in some $\textstyle\bigwedge^3 V_3 \subset \textstyle\bigwedge^3 V_6$, the hyperplanes $V_5 \subset V_6$ such that $a \in v \wedge \textstyle\bigwedge^2 V_5$ satisfy $\textstyle\bigwedge^3 V_3 \subset \textstyle\bigwedge^3 V_5$, hence $V_3 \subset V_5$. This means that the natural map

$$p'^{-1}(v) \to \mathbf{P}(A \cap (v \wedge \textstyle\bigwedge^2 V_6))$$

is a $\mathbf{P}^2$-fibration over $\mathbf{P}(A \cap (v \wedge \textstyle\bigwedge^2 V_6)) \cap \mathsf{Gr}(3, V_6)$ and is an isomorphism over its complement. If $v \notin S$, the second case does not happen, hence

$$(39) \qquad\qquad p'^{-1}(v) \simeq \mathbf{P}(A \cap (v \wedge \textstyle\bigwedge^2 V_6)).$$

Since $\widehat{Y}_A'$ can be defined as the zero-locus of a section of a rank-14 vector bundle on the smooth 18-dimensional variety $\mathbf{P}(A) \times \mathsf{Fl}(1, 5; V_6)$, any irreducible component of $\widehat{Y}_A'$ has dimension at least 4. On the other hand, the description (39) of the fibers of p' outside of the surface $S := \bigcup_{V_3 \in \Theta_A} \mathbf{P}(V_3)$ implies $\dim(p'^{-1}(Y_A^1 \smallsetminus S)) = 4$, and, using also Theorem B.2,

$$\dim(p'^{-1}(Y_A^{\geq 2} \smallsetminus S)) \leq \max\{2 + 1, 0 + 2, 0 + 3\} = 3.$$

Using the description of the fibers of p' over the surface S, we conclude that the irreducible components of $\widehat{Y}'_A$ are

- a "main" component $\widehat{Y}''_A$ which dominates Y_A;
- one component $\{a\} \times \mathbf{P}(V_3) \times \mathbf{P}(V_3^\perp)$ for each decomposable vector $a = \bigwedge^3 V_3 \in A$.

It now remains to prove that $\widehat{Y}''_A$ defines the projective duality between Y_A and $Y_{A^\perp}$. Let $v \in Y^1_A \smallsetminus S$ and set $V_5 := q(p^{-1}(v))$. Let us show that the hyperplane $\mathbf{P}(V_5) \subset \mathbf{P}(V_6)$ is tangent to Y_A at v. Since p' is birational over a neighborhood of v, the tangent space $\mathbf{T}_{Y^1_A,v}$ is identified with the tangent space $\mathbf{T}_{\widehat{Y}'_A,(a,v,V_5)}$. Let $(a + a't, v + v't, V_5 + V_5't)$ be a tangent vector to $\widehat{Y}'_A$, i.e., a $\mathbf{k}[t]/t^2$-point of this variety, with $a' \in A$. Since A is Lagrangian, we have

(40) $$a \wedge a' = 0.$$

On the other hand, by definition of $\widehat{Y}'_A$, we can write $a + a't = (v + v't) \wedge (\eta + \eta't)$, which gives $a = v \wedge \eta$ and $a' = v \wedge \eta' + v' \wedge \eta$. Substituting into (40), we get

(41) $$0 = (v \wedge \eta) \wedge (v \wedge \eta' + v' \wedge \eta) = v \wedge v' \wedge \eta \wedge \eta.$$

But v' is the image of the tangent vector in $\mathbf{T}_{Y^1_A,v}$, while $v \wedge \eta \wedge \eta$ is the equation of V_5. Equation (41) therefore implies $v' \in V_5$. Since this holds for any v', we deduce $\mathbf{T}_{Y^1_A,v} \subset \mathbf{P}(V_5)$. Since v is a smooth point of the hypersurface Y_A, this implies $\mathbf{T}_{Y_A,v} = \mathbf{P}(V_5)$, as required. $\qquad\square$

Remark B.4. Assume that A contains no decomposable vectors. First, the forgetful map $\widehat{Y}'_A \to \widehat{Y}_A$ is a bijection: if $(a, v, V_5), (a', v, V_5) \in \widehat{Y}'_A$ with $a \neq a'$, there would be decomposable vectors in A in the span of a and a'.

Second, the definition of $\widehat{Y}_A$ is symmetric with respect to V_6 and $V_6^\vee$. In particular, we obtain from (39) an identification $q^{-1}(V_5) \simeq \mathbf{P}(A \cap \bigwedge^3 V_5)$ and p takes $a \in A \cap \bigwedge^3 V_5$ to the kernel of a considered as a skew form on V_5 via the isomorphism $\bigwedge^3 V_5 \simeq \bigwedge^2 V_5^\vee$.

The following two lemmas were used in the article.

Lemma B.5. *If A contains no decomposable vectors, we have*

$$(Y_A^2 \times Y_{A^\perp}^2) \cap \mathsf{Fl}(1, 5; V_6) \not\subset (p \times q)(\widehat{Y}_A).$$

Proof. Assume by contradiction $(Y_A^{\geq 2} \times Y_{A^\perp}^{\geq 2}) \cap \mathsf{Fl}(1, 5; V_6) \subset (p \times q)(\widehat{Y}_A)$. The left side is the intersection in $\mathbf{P}(V_6) \times \mathbf{P}(V_6^\vee)$ of the fourfold $Y_A^{\geq 2} \times Y_{A^\perp}^{\geq 2}$ with the hypersurface $\mathsf{Fl}(1, 5; V_6)$, hence has dimension everywhere at least 3. On the other hand, it sits in $E := p^{-1}(Y_A^{\geq 2})$, the exceptional divisor of the birational morphism p and it follows from the proof of Proposition B.3 that E is irreducible of dimension 3. Therefore, we have

$$(Y_A^{\geq 2} \times Y_{A^\perp}^{\geq 2}) \cap \mathsf{Fl}(1, 5; V_6) = E.$$

By symmetry, we also have

$$(Y_A^{\geq 2} \times Y_{A^\perp}^{\geq 2}) \cap \mathsf{Fl}(1, 5; V_6) = q^{-1}(Y_{A^\perp}^{\geq 2}) =: E',$$

the exceptional divisor of the birational morphism q. It follows that $E = E'$.

Denote by H and H' the respective restrictions to Y_A and $Y_{A^\perp}$ of the hyperplane classes from $\mathbf{P}(V_6)$ and $\mathbf{P}(V_6^\vee)$. The birational isomorphism $q \circ p^{-1} \colon Y_A \dashrightarrow Y_{A^\perp}$ is given by the Gauss map (i.e., by the partial derivatives of the equation of Y_A). Since the equation of Y_A has degree

6 and any point of Y_A^2 has multiplicity 2 on Y_A ([O3, Corollary 2.5]), the partial derivatives have degree 5 and multiplicity 1, and we get

$$H' \underset{\text{lin}}{\equiv} 5H - E.$$

By symmetry, we also have $H \underset{\text{lin}}{\equiv} 5H' - E'$. Combining these two equations, we obtain

$$E' \underset{\text{lin}}{\equiv} 24H - 5E.$$

From $E' = E$, we obtain $24H \underset{\text{lin}}{\equiv} 6E$, which is a contradiction, since E has negative intersection with curves contracted by p, while H has zero intersection with such curves. $\square$

Lemma B.6. *If A contains no decomposable vectors, the scheme $Y_A^{\geq 2} \subset \mathbf{P}(V_6)$ is not contained in a hyperplane.*

Proof. We use the notation introduced in the proof of Lemma B.5. If $Y_A^{\geq 2}$ is contained in a hyperplane, the linear system $|H - E|$ is non-empty, hence the linear system

$$|E'| = |24H - 5E| = |19H + 5(H - E)|$$

is movable. But E' is the exceptional divisor of the birational morphism q, hence E' is rigid, so this is a contradiction. $\square$

B.3. **Double EPW sextics.** The properties of EPW sextics that are of most interest come from the existence of a (finite) canonical double cover $f_A \colon \widetilde{Y}_A \to Y_A$ ([O4, Section 1.2]) with the following properties.

Theorem B.7 (O'Grady). *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace which contains no decomposable vectors and let $Y_A \subset \mathbf{P}(V_6)$ be the associated EPW sextic.*

(a) *The double cover $f_A \colon \widetilde{Y}_A \to Y_A$ is branched over the surface $Y_A^{\geq 2}$ and induces the universal cover of Y_A^1.*
(b) *The variety $\widetilde{Y}_A$ is irreducible and normal, and its singular locus is the finite set $f_A^{-1}(Y_A^{\geq 3})$.*
(c) *When $Y_A^{\geq 3}$ is empty, $\widetilde{Y}_A$ is a smooth hyperkähler fourfold which is a deformation of the symmetric square of a K3 surface.*

Proof. Item (a) is proved in [O4, Proof of Theorem 4.15, p. 179], item (b) follows from statement (3) in the introduction of [O4], and item (c) is [O1, Theorem 1.1(2)]. $\square$

We draw a consequence of this theorem.

Proposition B.8. *Let $A \subset \bigwedge^3 V_6$ and $A' \subset \bigwedge^3 V_6'$ be Lagrangian subspaces and let $Y_A \subset \mathbf{P}(V_6)$ and $Y_{A'} \subset \mathbf{P}(V_6')$ be the schemes defined in (34).*

(a) *Any linear isomorphism $\varphi \colon V_6 \xrightarrow{\sim} V_6'$ such that $(\bigwedge^3 \varphi)(A) = A'$ induces an isomorphism $Y_A \xrightarrow{\sim} Y_{A'}$.*
(b) *Assume $Y_A \neq \mathbf{P}(V_6)$. Any isomorphism $Y_A \xrightarrow{\sim} Y_{A'}$ is induced by a linear isomorphism $\varphi \colon V_6 \xrightarrow{\sim} V_6'$ and, if A contains no decomposable vectors, A' contains no decomposable vectors and $(\bigwedge^3 \varphi)(A) = A'$.*

Proof. Item (a) follows from the definition of Y_A.

(b) If $Y_A \neq \mathbf{P}(V_6)$, it follows from the Lefschetz theorem [SGA2, Exposé XII, cor. 3.7] that $\mathrm{Pic}(Y_A)$ is generated by the hyperplane class. Any isomorphism $Y_A \xrightarrow{\sim} Y_{A'}$ is therefore induced by a linear isomorphism $\varphi\colon V_6 \xrightarrow{\sim} V_6'$.

If A contains no decomposable vectors, $\mathrm{Sing}(Y_A)$ is an integral surface of degree 40 (Theorem B.2) hence contains no planes, and the same holds for $\mathrm{Sing}(Y_{A'})$. It follows from (36) that A' contains no decomposable vectors. Applying Theorem B.2 again, we obtain $\varphi(Y_A^1) = Y_{A'}^1$. This isomorphism lifts to the universal covers to an isomorphism $f_A^{-1}(Y_A^1) \xrightarrow{\sim} f_{A'}^{-1}(Y_{A'}^1)$, and extends to the normal completions to an isomorphism $\widetilde{Y}_A \xrightarrow{\sim} \widetilde{Y}_{A'}$ (Theorem B.7). By [O5, Proof of Proposition 1.0.5], this implies $(\bigwedge^3 \varphi)(A) = A'$. $\square$

Finally, we characterize the automorphism group of an EPW sextic.

Proposition B.9 (O'Grady). *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace which contains no decomposable vectors and let $Y_A \subset \mathbf{P}(V_6)$ be the associated EPW sextic.*

(a) *The automorphism group of Y_A is finite and equal to the group*

$$(42) \qquad \mathrm{PGL}(V_6)_A := \{g \in \mathrm{PGL}(V_6) \mid (\textstyle\bigwedge^3 g)(A) = A\}.$$

(b) *When A is very general, these groups are trivial.*

Proof. (a) The equality $\mathrm{Aut}(Y_A) = \mathrm{PGL}(V_6)_A$ follows from Proposition B.8. These groups are finite because A is a stable point of $\mathsf{LGr}(\bigwedge^3 V_6)$ for the action of $\mathrm{PGL}(V_6)$: the non-stable locus has 12 components, listed in [O5, Table 1], and one checks that they are all contained in the locus of Lagrangians which contain decomposable vectors.

(b) Let $f_A\colon \widetilde{Y}_A \to Y_A$ be the canonical double cover, where $\widetilde{Y}_A$ is a smooth hyperkähler fourfold (Theorem B.7(c)). As explained in the proof of Proposition B.8(b), any automorphism g of Y_A is linear and lifts to an automorphism $\tilde{g}$ of $\widetilde{Y}_A$.

Let h be the class in $H^2(\widetilde{Y}_A, \mathbf{Z})$ of $f_A^*(\mathscr{O}_{Y_A}(1))$. The automorphism $\tilde{g}$ fixes h hence acts on its orthogonal $h^\perp \subset H^2(\widetilde{Y}_A, \mathbf{C})$ for the Beauville–Bogomolov form as a linear map which we denote by $\tilde{g}^*$. The line $H^{2,0}(\widetilde{Y}_A) \subset H^2(\widetilde{Y}_A, \mathbf{C})$ is contained in an eigenspace of $\tilde{g}^*$ and the small deformations A_t of A for which g extends must maintain $H^{2,0}(\widetilde{Y}_{A_t})$ in that same eigenspace. It follows that if g extends for all small deformations of A, the eigenspace containing $H^{2,0}(\widetilde{Y}_A)$ is all of $h^\perp$. Thus $\tilde{g}^* = \zeta \, \mathrm{Id}_{h^\perp}$, where ζ is some root of unity. Since $\tilde{g}^*$ is real, we have $\zeta = \pm 1$.

But the morphism $\mathrm{Aut}(\widetilde{Y}_A) \to \mathrm{Aut}(H^2(\widetilde{Y}_A, \mathbf{Z}))$ is injective: its kernel is deformation invariant ([HT, Theorem 2.1]) and it is trivial for symmetric squares of K3 surfaces ([B, prop. 10]); the statement therefore follows from Theorem B.7(c). This implies that $\tilde{g}$ is either the identity or the involution associated with the double covering f_A. In both cases, $g = \mathrm{Id}_{Y_A}$. $\square$

B.4. **EPW quartics.** We introduce certain quartic hypersurfaces in $\mathsf{Gr}(3, V_6)$ also associated with Lagrangian subspaces $A \subset \bigwedge^3 V_6$. For each $\ell \geq 0$, we set

$$(43) \qquad Z_A^{\geq \ell} := \{V_3 \in \mathsf{Gr}(3, V_6) \mid \dim(A \cap (V_6 \wedge \textstyle\bigwedge^2 V_3)) \geq \ell\}.$$

The subschemes $Z_A^{\geq \ell} \subset \mathsf{Gr}(3, V_6)$ were recently studied in [IKKR]. In particular, if A contains no decomposable vectors, the scheme $Z_A = Z_A^{\geq 1}$ is an integral quartic hypersurface ([IKKR, Corollary 2.10]). We will call it the **EPW quartic** associated with A. We will only need the following duality property.

Lemma B.10. *The natural isomorphism* $\mathsf{Gr}(3, V_6) \simeq \mathsf{Gr}(3, V_6^\vee)$ *induces an identification between the EPW quartics* $Z_A \subset \mathsf{Gr}(3, V_6)$ *and* $Z_{A^\perp} \subset \mathsf{Gr}(3, V_6^\vee)$.

Proof. The isomorphism of the Grassmannians takes $V_3 \subset V_6$ to $V_3^\perp \subset V_6^\vee$. It is therefore enough to show that the isomorphism $\bigwedge^3 V_6 \to \bigwedge^3 V_6^\vee$ given by the symplectic form takes the subspace $V_6 \wedge \bigwedge^2 V_3$ to the subspace $V_6^\vee \wedge \bigwedge^2 V_3^\perp$. As both subspaces are Lagrangian, it is enough to verify that the pairing between $V_6 \wedge \bigwedge^2 V_3$ and $V_6^\vee \wedge \bigwedge^2 V_3^\perp$ induced by the pairing between V_6 and $V_6^\vee$ is trivial, and this is straightforward. $\square$

Appendix C. Projective duality of quadrics and Lagrangian subspaces

Let V be a **k**-vector space. We discuss projective duality of quadrics (of all possible dimensions) in $\mathbf{P}(V)$ and its interpretation in terms of Lagrangian geometry. To simplify the statements, it is convenient to define a quadric Q in $\mathbf{P}(V)$ as a subvariety of a (possibly empty) linear subspace $\mathbf{P}(W) \subset \mathbf{P}(V)$ defined by a (possibly zero) quadratic form $q \in \mathsf{S}^2 W^\vee$ (if $q = 0$, then $Q = \mathbf{P}(W)$ is a *linear* subspace of $\mathbf{P}(V)$; it should not be confused with rank-1 quadrics, which are *double* linear subspaces). With each such form q, we can associate its kernel space $K \subset W$. The induced form on the (possibly zero) quotient space W/K is then non-degenerate.

The projective dual variety $Q^\vee \subset \mathbf{P}(V^\vee)$ is also a quadric which can be constructed as follows. The quadratic form q induces an isomorphism
$$q \colon W/K \xrightarrow{\sim} (W/K)^\vee = K^\perp/W^\perp,$$
where $W^\perp \subset K^\perp \subset V^\vee$ are the orthogonals. The inverse isomorphism
$$q^{-1} \colon K^\perp/W^\perp \xrightarrow{\sim} W/K = (K^\perp/W^\perp)^\vee$$
defines a quadratic form $q^\vee$ on $K^\perp \subset V^\vee$ with kernel $W^\perp$. The corresponding quadric in $\mathbf{P}(K^\perp) \subset \mathbf{P}(V^\vee)$ is the projective dual $Q^\vee$ of Q.

It is a classical observation that projective duality of quadrics can be described in terms of Lagrangian subspaces in a symplectic vector space equipped with a Lagrangian direct sum decomposition. For the reader's convenience, we summarize this relation and develop it a bit.

We recall how a Lagrangian subspace gives a pair of projectively dual quadrics.

Proposition C.1. *Let* $(\mathbb{V}, \omega)$ *be a symplectic* **k**-*vector space and let* $\mathbb{L}_1$ *and* $\mathbb{L}_2$ *be Lagrangian subspaces such that*
$$\mathbb{V} = \mathbb{L}_1 \oplus \mathbb{L}_2.$$
Denote by pr_1 *and* $\mathsf{pr}_2 \in \mathrm{End}(\mathbb{V})$ *the projectors to* $\mathbb{L}_1$ *and* $\mathbb{L}_2$.

(a) *For any Lagrangian subspace* $A \subset \mathbb{V}$, *the bilinear form* q^A *on* A *defined by*
$$(44) \qquad q^A(x, y) := \omega(\mathsf{pr}_1(x), \mathsf{pr}_2(y))$$
is symmetric. Its kernel is given by
$$(45) \qquad \mathrm{Ker}(q^A) = (A \cap \mathbb{L}_1) \oplus (A \cap \mathbb{L}_2).$$

(b) *The quadratic form* q^A *induces quadratic forms* q_1^A *and* q_2^A *on the subspaces*
$$W_1 := \mathsf{pr}_1(A) = (A \cap \mathbb{L}_2)^\perp \subset \mathbb{L}_1 \qquad and \qquad W_2 := \mathsf{pr}_2(A) = (A \cap \mathbb{L}_1)^\perp \subset \mathbb{L}_2$$
with respective kernels
$$K_1 := A \cap \mathbb{L}_1 \subset W_1 \qquad and \qquad K_2 := A \cap \mathbb{L}_2 \subset W_2.$$

The corresponding quadrics $Q_1^A \subset \mathbf{P}(W_1) \subset \mathbf{P}(\mathbb{L}_1)$ and $Q_2^A \subset \mathbf{P}(W_2) \subset \mathbf{P}(\mathbb{L}_2)$ are projectively dual with respect to the duality between $\mathbb{L}_1$ and $\mathbb{L}_2$ induced by the symplectic form ω.

Proof. (a) We have

$$q^A(x,y) - q^A(y,x) = \omega(\mathsf{pr}_1(x), \mathsf{pr}_2(y)) + \omega(\mathsf{pr}_2(x), \mathsf{pr}_1(y))$$
$$= \omega(\mathsf{pr}_1(x) + \mathsf{pr}_2(x), \mathsf{pr}_1(y) + \mathsf{pr}_2(y)) - \omega(\mathsf{pr}_1(x), \mathsf{pr}_1(y)) - \omega(\mathsf{pr}_2(x), \mathsf{pr}_2(y)).$$

In this last expression, the first term equals $\omega(x,y)$, hence is zero since A is Lagrangian; the second and the third terms are zero since $\mathbb{L}_1$ and $\mathbb{L}_2$ are Lagrangian. Thus the expression vanishes and q^A is symmetric.

It remains to compute the kernel of q^A. Since $x = \mathsf{pr}_1(x) + \mathsf{pr}_2(x)$ and $\mathbb{L}_2$ is Lagrangian, we have

$$q^A(x,y) = \omega(\mathsf{pr}_1(x), \mathsf{pr}_2(y)) = \omega(x - \mathsf{pr}_2(x), \mathsf{pr}_2(y)) = \omega(x, \mathsf{pr}_2(y)).$$

Since A is Lagrangian, its ω-orthogonal coincides with A, so $y \in \mathrm{Ker}(q^A)$ is equivalent to $\mathsf{pr}_2(y) \in A$. Writing $y = \mathsf{pr}_1(y) + \mathsf{pr}_2(y)$, this implies that also $\mathsf{pr}_1(y) \in A$. Thus $\mathsf{pr}_1(y) \in A \cap \mathbb{L}_1$ and $\mathsf{pr}_2(y) \in A \cap \mathbb{L}_2$. This proves one inclusion. For the other inclusion, both $A \cap \mathbb{L}_1$ and $A \cap \mathbb{L}_2$ are in the kernel of q^A: for $A \cap \mathbb{L}_1$, this is because $\mathsf{pr}_2(y) = 0$ for $y \in A \cap \mathbb{L}_1$, and for $A \cap \mathbb{L}_2$, because $\mathsf{pr}_1(x) = 0$ for $x \in A \cap \mathbb{L}_2$.

(b) By (44), the kernels $A \cap \mathbb{L}_2$ and $A \cap \mathbb{L}_1$ of the restrictions $\mathsf{pr}_1|_A$ and $\mathsf{pr}_2|_A$ are contained in the kernel of q^A, hence q^A induces quadratic forms on $W_1 = \mathsf{pr}_1(A) \subset \mathbb{L}_1$ and $W_2 = \mathsf{pr}_2(A) \subset \mathbb{L}_2$. The equalities $W_1 = (A \cap \mathbb{L}_2)^\perp$ and $W_2 = (A \cap \mathbb{L}_1)^\perp$ follow from the Lagrangian property of A. The kernels K_1 and K_2 of the induced quadratic forms q_1^A and q_2^A are the respective images of the kernel of q^A under pr_1 and pr_2, so (45) implies $K_1 = A \cap \mathbb{L}_1$ and $K_2 = A \cap \mathbb{L}_2$.

Note that $A/(K_1 \oplus K_2) \subset W_1/K_1 \oplus W_2/K_2$ is the graph of an isomorphism $W_1/K_1 \simeq W_2/K_2$. The duality between W_1/K_1 and W_2/K_2 given by the symplectic form ω identifies this isomorphism with the quadratic form q_1^A and its inverse with q_2^A. This means that the quadrics Q_1^A and Q_2^A are projectively dual. $\qquad\square$

The following construction shows that any pair of projectively dual quadrics comes from a Lagrangian subspace.

Lemma C.2. *Let $Q \subset \mathbf{P}(\mathbb{L})$ be a quadric and let $Q^\vee \subset \mathbf{P}(\mathbb{L}^\vee)$ be its projective dual. In the space $\mathbb{V} := \mathbb{L} \oplus \mathbb{L}^\vee$ endowed with the symplectic form*

$$\omega(x,y) = (x_1, y_2) - (y_1, x_2),$$

there is a unique Lagrangian subspace $A \subset \mathbb{V}$ such that $Q_1^A = Q$ and $Q_2^A = Q^\vee$.

Proof. Let $K \subset \mathbb{L}$ and $W \subset \mathbb{L}$ be the kernel and span of Q and let $q \in \mathsf{S}^2 W^\vee$ be a quadratic form defining Q. Let A be the kernel of the sum of the induced map $q \colon W \to W^\vee$ with the canonical surjection $\mathbb{L}^\vee \to W^\vee$. It fits into an exact sequence

(46) $$0 \to A \to W \oplus \mathbb{L}^\vee \to W^\vee \to 0.$$

The embedding $W \to \mathbb{L}$ induces an embedding $A \to \mathbb{L} \oplus \mathbb{L}^\vee = \mathbb{V}$. Let us show the image is Lagrangian. It can be explicitly described as

$$A = \{(x_1, x_2) \in \mathbb{L} \oplus \mathbb{L}^\vee \mid x_1 \in W \quad \text{and} \quad x_2 \equiv q(x_1) \bmod W^\perp\}.$$

Given $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in A, we have

$$\omega(x, y) = (x_1, y_2) - (y_1, x_2) = (x_1, q(y_1)) - (y_1, q(x_1)) = q(y_1, x_1) - q(x_1, y_1) = 0,$$

so A is indeed Lagrangian. Note that $\mathsf{pr}_1(A) = W$ by definition of A. Moreover, if q^A is the quadratic form on A defined in Lemma C.1, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in A, we have

$$q^A(x, y) = \omega(\mathsf{pr}_1(x), \mathsf{pr}_2(y)) = \omega(x_1, y_2) = (x_1, q(y_1)) = q(x_1, y_1) = q(\mathsf{pr}_1(x), \mathsf{pr}_1(y)).$$

This shows $q_1^A = q$, so $Q_1^A = Q$. Since Q_2^A is projectively dual to Q_1^A, we obtain $Q_2^A = Q^\vee$.

For the uniqueness, it is enough to check that the construction of this lemma is inverse to the construction of Proposition C.1, which is straightforward. $\qquad\square$

We now discuss the so-called "isotropic reduction" of Lagrangian subspaces.

Let $(\mathbb{V}, \omega)$ be a symplectic vector space. Let $\mathbb{I} \subset \mathbb{V}$ be an isotropic subspace and let $\mathbb{I}^\perp$ be its ω-orthogonal. The restriction of the symplectic form ω to $\mathbb{I}^\perp$ is degenerate and its kernel is $\mathbb{I} \subset \mathbb{I}^\perp$. Hence the form descends to a symplectic form $\bar\omega$ on $\overline{\mathbb{V}} := \mathbb{I}^\perp/\mathbb{I}$. Moreover, for each Lagrangian subspace $\mathbb{L} \subset \mathbb{V}$, the subspace

$$\overline{\mathbb{L}} := (\mathbb{L} \cap \mathbb{I}^\perp)/(\mathbb{L} \cap \mathbb{I}) \subset \overline{\mathbb{V}}$$

is Lagrangian. We call the symplectic space $\overline{\mathbb{V}}$ the **isotropic reduction** of $\mathbb{V}$, and the Lagrangian subspace $\overline{\mathbb{L}}$ the **isotropic reduction** of $\mathbb{L}$.

Let $\mathbb{V} = \mathbb{L}_1 \oplus \mathbb{L}_2$ be a Lagrangian direct sum decomposition and let $\mathbb{I} \subset \mathbb{L}_1$ (automatically isotropic). Isotropic reduction gives us a Lagrangian direct sum decomposition

$$\overline{\mathbb{V}} = \overline{\mathbb{L}}_1 \oplus \overline{\mathbb{L}}_2,$$

where $\overline{\mathbb{L}}_1 = \mathbb{L}_1/\mathbb{I}$ and $\overline{\mathbb{L}}_2 = \mathbb{L}_2 \cap \mathbb{I}^\perp$. Furthermore, if $A \subset \mathbb{V}$ is Lagrangian, its reduction $\overline{A} \subset \overline{\mathbb{V}}$ is Lagrangian as well. In particular, this produces a projectively dual pair of quadrics $(Q_1^{\overline{A}}, Q_2^{\overline{A}})$. We will say that this pair is obtained from the pair (Q_1^A, Q_2^A) by **isotropic reduction with respect to** $\mathbb{I}$. The next lemma gives a geometric relation between a pair of projectively dual quadrics and their isotropic reduction.

Lemma C.3. *In the situation described above, we have the following.*

(a) *The quadratic form $q^{\overline{A}}$ on $\overline{A}$ is induced by the restriction of the quadratic form q^A to $A \cap \mathbb{I}^\perp \subset A$. In particular,*

$$Q^A \cap \mathbf{P}(A \cap \mathbb{I}^\perp) = \mathsf{C}_{A \cap \mathbb{I}}(Q^{\overline{A}}).$$

(b) *The quadric $Q_2^{\overline{A}}$ is the linear section $Q_2^A \cap \mathbf{P}(\overline{\mathbb{L}}_2)$ of Q_2^A and the quadric $Q_1^{\overline{A}}$ is its projective dual. Moreover, the span and the kernel of $Q_2^{\overline{A}}$ are*

(47) $$\mathrm{Span}(Q_2^{\overline{A}}) = ((A \cap \mathbb{L}_1)/(A \cap \mathbb{I}))^\perp \subset \overline{\mathbb{L}}_2, \quad \mathrm{Ker}(Q_2^{\overline{A}}) = (A \cap (\mathbb{I} \oplus \overline{\mathbb{L}}_2))/(A \cap \mathbb{I}).$$

Proof. (a) First, we have $A \cap \mathbb{I} \subset A \cap \mathbb{L}_1 \subset \mathrm{Ker}(q^A)$. Thus q^A induces a quadratic form on $\overline{A}$. Moreover, the projectors $\bar{\mathsf{pr}}_1$ and $\bar{\mathsf{pr}}_2$ onto the components of the decomposition $\overline{\mathbb{V}} = \overline{\mathbb{L}}_1 \oplus \overline{\mathbb{L}}_2$ are induced by the projectors pr_1 and pr_2. Hence, for $x, y \in A \cap \mathbb{I}^\perp$, we have

$$q^{\overline{A}}(x, y) = \bar\omega(\bar{\mathsf{pr}}_1(x), \bar{\mathsf{pr}}_2(y)) = \omega(\mathsf{pr}_1(x), \mathsf{pr}_2(y)) = q^A(x, y).$$

This means that the quadratic form induced on $\overline{A}$ is $q^{\overline{A}}$ and thus proves the first equality.

(b) We prove that $Q_2^{\overline{A}}$ is this linear section of Q_2^A. Since $\mathbb{L}_1 \subset \mathbb{I}^\perp$, we have

$$\overline{W}_2 := \bar{\mathsf{pr}}_2(\overline{A}) = \mathsf{pr}_2(A \cap \mathbb{I}^\perp) = \mathsf{pr}_2(A) \cap \mathsf{pr}_2(\mathbb{I}^\perp) = W_2 \cap \overline{\mathbb{L}}_2$$

and the kernel of the projection $A \cap \mathbb{I}^\perp \to \mathsf{pr}_2(A \cap \mathbb{I}^\perp)$ is $A \cap \mathbb{L}_1$. Therefore we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A \cap \mathbb{L}_1 & \longrightarrow & A \cap \mathbb{I}^\perp & \overset{\mathsf{pr}_2}{\longrightarrow} & W_2 \cap \overline{\mathbb{L}}_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \overline{A} \cap \overline{\mathbb{L}}_1 & \longrightarrow & \overline{A} & \overset{\bar{\mathsf{pr}}_2}{\longrightarrow} & W_2 & \longrightarrow & 0.
\end{array}
$$

Since $A \cap \mathbb{L}_1$ is contained in the kernel of the quadratic form $q^A|_{A \cap \mathbb{I}^\perp}$, the quadratic form induced by it on $W_2 \cap \overline{\mathbb{L}}_2$ via the projection pr_2 coincides with the quadratic form obtained in two steps, by first inducing a quadratic form on $\overline{A}$ via the middle vertical arrow, and then on $\overline{W}_2$ via $\bar{\mathsf{pr}}_2$. The quadratic form induced via pr_2 is $q_2^A|_{W_2 \cap \overline{\mathbb{L}}_2}$, the quadratic form induced via the vertical map is $q^{\overline{A}}$, and the quadratic form induced by it via $\bar{\mathsf{pr}}_2$ is $q_2^{\overline{A}}$. So, by the commutativity of the right square, we deduce $q_2^A|_{W_2 \cap \overline{\mathbb{L}}_2} = q_2^{\overline{A}}$.

We now identify the span and the kernel of $Q_2^{\overline{A}}$. By Proposition C.1, the span is $(\overline{A} \cap \overline{\mathbb{L}}_1)^\perp$ and the kernel is $\overline{A} \cap \overline{\mathbb{L}}_2$. To describe the first, consider the commutative diagram above. Since the middle vertical map is surjective with kernel $A \cap \mathbb{I}$, the same is true for the left vertical map, hence $\overline{A} \cap \overline{\mathbb{L}}_1 = (A \cap \mathbb{L}_1)/(A \cap \mathbb{I})$.

For the second, recall that $\overline{A}$ is the image of $A \cap \mathbb{I}^\perp$ under the projection $\mathbb{I}^\perp \to \overline{\mathbb{V}}$ with kernel $\mathbb{I}$. Therefore, $\overline{A} \cap \overline{\mathbb{L}}_2$ is the image in $\overline{\mathbb{V}}$ of the intersection of $A \cap \mathbb{I}^\perp$ with the preimage of $\overline{\mathbb{L}}_2$ in $\mathbb{I}^\perp$, i.e., with $\mathbb{I} \oplus \overline{\mathbb{L}}_2$. Thus it is the image in $\overline{\mathbb{V}}$ of $A \cap (\mathbb{I} \oplus \overline{\mathbb{L}}_2)$. Since the kernel of the projection $\mathbb{I}^\perp \to \overline{\mathbb{V}}$ is $\mathbb{I}$, we see that $\overline{A} \cap \overline{\mathbb{L}}_2 = (A \cap (\mathbb{I} \oplus \overline{\mathbb{L}}_2))/(A \cap \mathbb{I})$. $\qquad \square$

Choosing an isotropic space appropriately, we can realize any subquadric in Q_2 as the result of an isotropic reduction applied to the pair (Q_1, Q_2). This gives a convenient way to control the span and the kernel of a subquadric.

Corollary C.4. *Let (Q_1, Q_2) be a projectively dual pair of quadrics corresponding to a Lagrangian decomposition $\mathbb{V} = \mathbb{L}_1 \oplus \mathbb{L}_2$ and a Lagrangian subspace $A \subset \mathbb{V}$. Let $\overline{\mathbb{L}}_2 \subset \mathbb{L}_2$ be an arbitrary subspace and set*

$$\mathbb{I} := \mathbb{L}_1 \cap \overline{\mathbb{L}}_2^\perp.$$

The isotropic reduction with respect to $\mathbb{I}$ produces a projectively dual pair of quadrics $(\overline{Q}_1, \overline{Q}_2)$, where $\overline{Q}_2 = Q_2 \cap \mathbf{P}(\overline{\mathbb{L}}_2)$. In particular, the span and the kernel of $\overline{Q}_2$ are given by (47).

Proof. Since the pairing between $\mathbb{L}_1$ and $\mathbb{L}_2$ is non-degenerate and $\mathbb{I}$ is the orthogonal of $\overline{\mathbb{L}}_2$ in $\mathbb{L}_1$, it follows that $\overline{\mathbb{L}}_2$ is the orthogonal of $\mathbb{I}$ in $\mathbb{L}_2$, i.e., $\overline{\mathbb{L}}_2 = \mathbb{L}_2 \cap \mathbb{I}^\perp$, and Lemma C.3 applies. $\qquad \square$

To conclude, we discuss a family version of the correspondence discussed in this appendix. Let S be a scheme, let $\mathscr{V}$ be a vector bundle on S equipped with a symplectic form $\omega \colon \bigwedge^2 \mathscr{V} \to \mathscr{M}$ with values in a line bundle $\mathscr{M}$, and let $\mathscr{V} = \mathscr{L}_1 \oplus \mathscr{L}_2$ be a Lagrangian direct sum decomposition. It induces an isomorphism

$$\mathscr{L}_2^\vee \simeq \mathscr{L}_1 \otimes \mathscr{M}^{-1}.$$

Let $\mathscr{A} \subset \mathscr{V}$ be another Lagrangian subbundle. Formula (44) then defines a quadratic form $q_{\mathscr{A}} \colon \mathscr{M}^{-1} \to \mathsf{S}^2 \mathscr{A}^\vee$ on $\mathscr{A}$, but if we want to consider one of the two projectively dual quadrics,

we should impose some constant rank condition to ensure that the linear span of the quadric in question is a vector bundle.

Lemma C.5. *Assume the morphism $\mathscr{A} \hookrightarrow \mathscr{V} \xrightarrow{\mathsf{pr}_2} \mathscr{L}_2$ has constant rank and let $\mathscr{W}_2 \subset \mathscr{L}_2$ be its image. Then the family of quadrics $q_{\mathscr{A}}$ induces a family $q_{\mathscr{A},2} \colon \mathscr{M}^{-1} \to \mathsf{S}^2 \mathscr{W}_2^{\vee}$ of quadrics on $\mathscr{W}_2$ and there are isomorphisms of sheaves*

$$\mathrm{Coker}(\mathscr{W}_2 \otimes \mathscr{M}^{-1} \xrightarrow{q_{\mathscr{A},2}} \mathscr{W}_2^{\vee}) \simeq \mathrm{Coker}(\mathscr{A} \otimes \mathscr{M}^{-1} \xrightarrow{\mathsf{pr}_1} \mathscr{L}_2^{\vee})$$

$$\simeq \mathrm{Coker}(\mathscr{L}_2 \otimes \mathscr{M}^{-1} \xrightarrow{\mathsf{pr}_1^{\vee}} \mathscr{A}^{\vee})$$

and similarly for the kernels.

Proof. By definition of $\mathscr{W}_2$, there are epimorphisms $\mathscr{A} \twoheadrightarrow \mathscr{W}_2$ and $\mathscr{L}_1 \otimes \mathscr{M}^{-1} \simeq \mathscr{L}_2^{\vee} \twoheadrightarrow \mathscr{W}_2^{\vee}$. They combine into a commutative diagram

$$
\begin{array}{ccccc}
\mathscr{A} \otimes \mathscr{M}^{-1} & \longrightarrow\!\!\!\!\!\rightarrow & \mathscr{W}_2 \otimes \mathscr{M}^{-1} & \lhook\joinrel\longrightarrow & \mathscr{L}_2 \otimes \mathscr{M}^{-1} \\
\downarrow{\scriptstyle \mathsf{pr}_1} & & \downarrow{\scriptstyle q_{\mathscr{A},2}} & & \downarrow{\scriptstyle \mathsf{pr}_1^{\vee}} \\
\mathscr{L}_2^{\vee} & \longrightarrow\!\!\!\!\!\rightarrow & \mathscr{W}_2^{\vee} & \lhook\joinrel\longrightarrow & \mathscr{A}^{\vee},
\end{array}
$$

where the kernels and cokernels of the horizontal compositions are identified by the maps pr_1 and $\mathsf{pr}_1^{\vee}$ (the kernels also can be characterized as the intersection of the subbundles $\mathscr{A}$ and $\mathscr{L}_1$ inside $\mathscr{V}$, twisted by $\mathscr{M}^{-1}$, and the cokernels are dual up to a twist to the kernels). It follows that the cokernels of the vertical maps pr_1, $q_{\mathscr{A},2}$, and $\mathsf{pr}_1^{\vee}$ are isomorphic, hence the lemma.   □

We also discuss a family version of isotropic reduction. Assume that in addition to the above, we are given a vector subbundle $\mathscr{I} \subset \mathscr{L}_1$. The vector bundle $\bar{\mathscr{V}}$ defined as the cohomology of the monad $\mathscr{I} \hookrightarrow \mathscr{V} \twoheadrightarrow \mathscr{I}^{\vee} \otimes \mathscr{M}$ then has a natural symplectic structure and if $\mathscr{L} \subset \mathscr{V}$ is a Lagrangian subbundle *such that the composition $\mathscr{L} \hookrightarrow \mathscr{V} \twoheadrightarrow \mathscr{I}^{\vee} \otimes \mathscr{M}$ has constant rank,* the sheaf $\bar{\mathscr{L}} := \mathrm{Ker}(\mathscr{L} \to \mathscr{I}^{\vee} \otimes \mathscr{M})/\mathrm{Ker}(\mathscr{I} \to \mathscr{L}^{\vee} \otimes \mathscr{M})$ is locally free and is a Lagrangian subbundle in $\bar{\mathscr{V}}$. In particular, when applied to $\mathscr{L} = \mathscr{L}_1$ and $\mathscr{L} = \mathscr{L}_2$, we get

$$\bar{\mathscr{L}}_1 = \mathscr{L}_1/\mathscr{I} \qquad \text{and} \qquad \bar{\mathscr{L}}_2 = \mathrm{Ker}(\mathscr{L}_2 \twoheadrightarrow \mathscr{I}^{\vee} \otimes \mathscr{M}).$$

Lemma C.6. *One has a Lagrangian direct sum decomposition $\bar{\mathscr{V}} = \bar{\mathscr{L}}_1 \oplus \bar{\mathscr{L}}_2$. Moreover, if $\mathscr{A} \subset \mathscr{V}$ is a Lagrangian subbundle such that the composition $\mathscr{A} \to \mathscr{V} \to \mathscr{I}^{\vee} \otimes \mathscr{M}$ has constant rank, the vector bundle $\bar{\mathscr{W}}_2$ is a subbundle in $\mathscr{W}_2$ and the quadratic form $q_{\bar{\mathscr{A}},2}$ on $\bar{\mathscr{W}}_2$ equals the restriction of the quadratic form $q_{\mathscr{A},2}$. Finally, the subbundle $\tilde{\mathscr{L}}_2 := \mathscr{I} \oplus \bar{\mathscr{L}}_2 \subset \mathscr{L}_1 \oplus \mathscr{L}_2 = \mathscr{V}$ is Lagrangian and there is an exact sequence*

$$0 \to \mathrm{Coker}(\bar{\mathscr{W}}_2 \otimes \mathscr{M}^{-1} \to \bar{\mathscr{W}}_2^{\vee}) \to \mathrm{Coker}(\mathscr{A} \otimes \mathscr{M}^{-1} \to \tilde{\mathscr{L}}_2^{\vee}) \to \mathrm{Coker}(\mathscr{A} \otimes \mathscr{M}^{-1} \to \mathscr{I}^{\vee}) \to 0.$$

In particular, if the map $\mathscr{A} \to \mathscr{I}^{\vee} \otimes \mathscr{M}$ is an epimorphism,

$$\mathrm{Coker}(\bar{\mathscr{W}}_2 \otimes \mathscr{M}^{-1} \to \bar{\mathscr{W}}_2^{\vee}) \simeq \mathrm{Coker}(\mathscr{A} \otimes \mathscr{M}^{-1} \to \tilde{\mathscr{L}}_2^{\vee}) \simeq \mathrm{Coker}(\tilde{\mathscr{L}}_2 \otimes \mathscr{M}^{-1} \to \mathscr{A}^{\vee}).$$

Proof. Just repeat the proof of Lemma C.3.   □

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