VARIETIES WITH VANISHING HOLOMORPHIC EULER CHARACTERISTIC

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Abstract. We study smooth complex projective varieties X of maximal Albanese dimension and of general type satisfying $\chi(X, \mathscr{O}_X) = 0$. We prove that the Albanese variety of X has at least three simple factors. Examples were constructed by Ein and Lazarsfeld, and we prove that in dimension 3, these examples are (up to abelian étale covers) the only ones. By results of Ueno, another source of examples is provided by varieties X of maximal Albanese dimension and of general type satisfying $h^0(X, \omega_X) = 1$. Examples were constructed by Chen and Hacon, and again, we prove that in dimension 3, these examples are (up to abelian étale covers) the only ones. We also formulate a conjecture on the general structure of these varieties in all dimensions.

1. Introduction

A smooth complex projective variety X is said to have *maximal Albanese dimension* if its Albanese mapping $X \to \mathrm{Alb}(X)$ is generically finite (onto its image).

Green and Lazarsfeld showed in [GL] that such a variety satisfies $\chi(X, \omega_X) \geq 0$. Ein and Lazarsfeld later constructed in [EL] a smooth projective threefold X of maximal Albanese dimension and of general type with $\chi(X, \omega_X) = 0$ (see Examples 4.1 and 4.2).

We are interested here in describing the structure of varieties X of maximal Albanese dimension (and of general type) with $\chi(X, \omega_X) = 0$. This class of varieties is stable by modifications, étale covers, and products with any other variety of maximal Albanese dimension (and of general type). More generally, if X is a smooth projective variety of maximal Albanese dimension with a fibration whose general fiber F satisfies $\chi(F, \omega_F) = 0$, then $\chi(X, \omega_X) = 0$ ([HP], Proposition 2.5).

So we study smooth projective varieties X of general type with $\chi(X, \omega_X) = 0$ and a generically finite morphism $X \to A$ to an abelian variety. In §3, we prove a general structure theorem (Theorem 3.1) which implies among other things that *A has at least three simple factors.* Examples where A is the product of any three given non-zero factors can be constructed following Ein and Lazarsfeld, and we speculate that their construction should (more or less) describe all cases where A has three simple factors but, although we prove several results in §4 in this direction (Propositions 4.3, 4.5, and 4.7) and arrive at the rather rigid picture (6), we are only able to get a complete description when X has dimension

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3: we prove that *a smooth projective threefold X of maximal Albanese dimension and of general type satisfies $\chi(X,\omega_X)=0$ if and only if it has an abelian étale cover which is an Ein-Lazarsfeld threefold* (Theorem 5.1).

Another source of examples is provided by varieties X of maximal Albanese dimension and $h^0(X,\omega_X)=1$: it follows from work of Ueno ([U]) that they satisfy $\chi(X,\omega_X)=0$. Chen and Hacon constructed examples of general type (see Example 4.2). We gather some properties of these varieties in §6. However, this class of examples is not stable under étale covers and does not lend itself well to our methods of study, *except in dimension 3,* where the precise Theorem 5.1 allows us to give *a complete description of all smooth projective threefolds X of maximal Albanese dimension and of general type, such that $h^0(X,\omega_X)=1$: they all have abelian étale covers that are Chen-Hacon threefolds* (Theorem 6.3).

In §7, we propose a conjecture on the possible general structure of smooth projective varieties X of maximal Albanese dimension and of general type satisfying $\chi(X,\omega_X)=0$. It seems difficult to give a complete classification, but based on the examples that we know, we conjecture that, after taking modifications and étale covers, there should exists a non-trivial fibration $X \twoheadrightarrow Y$ which is either isotrivial, or whose general fiber F satisfies $\chi(F,\omega_F)=0$. For the converse, one does have $\chi(X,\omega_X)=0$ in the second case by [HP], Proposition 2.5, but not necessarily in the first case, of course. Both cases do happen (Example 7.2).

We work over the field of complex numbers.

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2. Notation and preliminaries

For any smooth projective variety X, we set $\widehat{X}=\mathrm{Pic}^0(X)$. For $\xi\in\widehat{X}$, we will denote by P_ξ an algebraically trivial line bundle on X that represents ξ.

Following standard terminology, we will say that a morphism $f:X\to A$ to an abelian variety A is *minimal* if the induced group morphism $\widehat{f}:\widehat{A}\to\widehat{X}$ is injective. Equivalently, $f(X)$ generates A as an algebraic group and f factors through no non-trivial abelian étale covers of A. The Albanese mapping a_X has this property. Any $f:X\to A$ factors as $f:X\xrightarrow{f'}A'\to A$, where A' is an abelian variety and f' is minimal.

An *algebraic fibration* (or simply a fibration) is a surjective morphism between normal projective varieties, with connected fibers.

In the rest of this section, X will be a smooth projective variety, of dimension n, with a *generically finite* morphism $f:X\to A$ to an abelian variety A. In particular, X has maximal Albanese dimension.

2.1. **Cohomological loci.** For each integer i, we define the cohomological loci

$$\begin{aligned} V_i(\omega_X, f) &= \{\xi \in \widehat{A} \mid H^i(X, \omega_X \otimes f^*P_\xi) \neq 0\} \\ &= \{\xi \in \widehat{A} \mid H^{n-i}(X, f^*P_{-\xi}) \neq 0\}. \end{aligned}$$

If Y is a smooth projective variety and $\varepsilon : Y \twoheadrightarrow X$ is birational, we have $R^j\varepsilon_*\omega_Y = 0$ for $j > 0$ and $\varepsilon_*\omega_Y \simeq \omega_X$ ([K2], Theorem 2.1 and Proposition 7.6), hence $\chi(X, \omega_X) = \chi(Y, \omega_Y)$ and $V_i(\omega_X, f) = V_i(\omega_Y, f \circ \varepsilon)$ for all i. In particular, these loci do not change when X is replaced with Y.

2.1.1. Since $R^j f_*\omega_X = 0$ for $j > 0$ ([K2], Theorem 2.1), we have for all i

$$V_i(\omega_X, f) = V_i(f_*\omega_X) := \{\xi \in \widehat{A} \mid H^i(A, f_*\omega_X \otimes P_\xi) \neq 0\}.$$

2.1.2. Each irreducible component of $V_i(\omega_X, f)$ is an abelian subvariety of $\widehat{A}$ of codimension $\geq i$ ([EL], Remark 1.6 and Theorem 1.2) translated by a torsion point ([Si]).

2.1.3. There is a chain of inclusions ([EL], Lemma 1.8)

$$\text{Ker}(\widehat{f}) = V_n(\omega_X, f) \subseteq V_{n-1}(\omega_X, f) \subseteq \cdots \subseteq V_0(\omega_X, f) \subseteq \widehat{A} \tag{1}$$

and $\text{codim}\big(V_n(\omega_X, f)\big) \geq n$.

2.1.4. If $V_0(\omega_X, f)$ has a component V of codimension i, this component is contained in (hence is an irreducible component of) $V_i(\omega_X, f)$ ([EL], (1.10)), so that we have $i \leq n$ and $f(X)$ is stable by translation by the i-dimensional abelian variety $\text{Ker}(A \twoheadrightarrow \widehat{V})^0$ ([EL], Theorem 3).

2.1.5. For $\xi \in \widehat{A}$ general, $\chi(X, \omega_X) = h^0(X, \omega_X \otimes f^*P_\xi) \geq 0$ (use §2.1.2) and

$$\begin{aligned} \chi(X, \omega_X) = 0 &\iff V_0(\omega_X, f) \neq \widehat{A} \\ &\iff V_0(\omega_X, f) \text{ has a component of codimension } i \\ &\qquad\qquad \text{for some } i \in \{1, \ldots, n\} \\ &\implies V_i(\omega_X, f) \text{ has a component of codimension } i \\ &\qquad\qquad \text{for some } i \in \{1, \ldots, n\}, \end{aligned}$$

where the last implication is not always an equivalence.

2.1.6. The variety X is of general type if and only if $V_0(\omega_X, f)$ generates $\widehat{A}$ ([CH1], Theorem 2.3).

2.1.7. If $V_0(\omega_X, f)$ is finite, Ein and Lazarsfeld proved that X is birational to an abelian variety ([CH1], Theorem 1.3.2). In particular, if f is moreover minimal, it is a birational isomorphism.

2.2. **Composing f with a generically finite morphism.** If Y is smooth projective and $h : Y \twoheadrightarrow X$ is surjective and generically finite, the trace map $h_*\omega_Y \twoheadrightarrow \omega_X$ splits the natural inclusion $\omega_X \to h_*\omega_Y$, hence $H^i(X, \omega_X \otimes f^*P_\xi)$ injects into $H^i(X, h_*\omega_Y \otimes f^*P_\xi)$ for all $\xi \in \widehat{A}$. Thus, using also §2.1.1, we obtain

$$V_i(\omega_X, f) \subseteq V_i(h_*\omega_Y, f) = V_i(\omega_Y, f \circ h).$$

Moreover,

$$\chi(Y, \omega_Y) \geq \chi(X, \omega_X). \tag{2}$$

When h is étale, we have

$$\chi(Y, \omega_Y) = \deg(h)\chi(X, \omega_X).$$

Finally, when h is obtained from an isogeny $\eta : B \twoheadrightarrow A$ as in the cartesian diagram

$$\begin{array}{ccc} Y = X \times_A B & \xrightarrow{\;g\;} & B \\ {\scriptstyle h}\big\downarrow & \square & \big\downarrow{\scriptstyle \eta} \\ X & \xrightarrow[\;f\;]{} & A \end{array}$$

we have $V_i(\omega_Y, g) = \widehat{\eta}(V_i(\omega_X, f))$. Combining this with §2.1.2, we see that after making a suitable étale base change, we can always make all the components of $V_i(\omega_X, f)$ pass through 0.

3. Components of V_i of codimension i

Let X be a smooth projective variety with a generically finite morphism $f : X \to A$ to an abelian variety. If $\chi(X, \omega_X) = 0$, it follows from §2.1.5 that $V_i(\omega_X, f)$ has a component of codimension i for some $i \in \{1, \dots, n\}$. We prove a structure theorem under this weaker assumption.

Theorem 3.1. *Let X be a smooth projective variety of dimension n, let A be an abelian variety, and let $f : X \to A$ be a minimal generically finite morphism. Assume that for some $i \in \{0, \dots, n\}$, the locus $V_i(\omega_X, f)$ has a component V of codimension i in $\widehat{A}$. Let B be the abelian variety $\widehat{V}$. For a suitable modification X' of an abelian étale cover of X, the Stein factorization of the morphism $X' \to A \twoheadrightarrow B$ induces a surjective morphism $X' \twoheadrightarrow Y$ where Y is smooth of dimension $n - i$, of general type, with $\chi(Y, \omega_Y) > 0$.*

Proof of Theorem 3.1. We follow the proof of [EL], Theorem 3. Let K be the i-dimensional abelian variety $\mathrm{Ker}(A \twoheadrightarrow B)^0$. By §2.1.2 and §2.2, we may assume, after isogeny, $A = B \times K$ and $V = \widehat{B}$. Let $p : A \twoheadrightarrow B$ be the projection. Considering the Stein factorization of $\pi = p \circ f : X \to B$, and replacing X by a suitable modification, we may assume that π factors as

$$\pi : X \stackrel{g}{\twoheadrightarrow} Y \stackrel{h}{\to} B,$$

where Y is smooth, h is generically finite, and g is surjective with connected fibers. It is shown in [EL] that Y has dimension $n - i$ (so that general fibers of g have dimension i) and that $f(X)$ is stable by translation by K (§2.1.4). This implies $R^i g_*\omega_X \simeq \omega_Y$ ([K2], Proposition 7.6). Moreover, the sheaves $R^k g_*\omega_X$ on Y satisfy the generic vanishing theorem ([HP], Theorem 2.2), hence

$$V_j(R^k g_*\omega_X, h) \neq \widehat{B} \quad \text{for all } j > 0 \text{ and all } k.$$

For all

$$\xi \in \widehat{B} - \bigcup_{j>0,\ k} V_j(R^k g_*\omega_X, h),$$

we have

$$H^j(Y, R^k g_*\omega_X \otimes h^*P_\xi) = 0 \quad \text{for all } j > 0 \text{ and all } k.$$

Hence, by the Leray spectral sequence, we obtain

$$h^i(X, \omega_X \otimes f^*P_\xi) = h^i(X, \omega_X \otimes \pi^*P_\xi) = h^0(Y, R^i g_*\omega_X \otimes h^*P_\xi) = h^0(Y, \omega_Y \otimes h^*P_\xi)$$

and these numbers are non-zero because $\widehat{B} = V \subseteq V_i(\omega_X, f)$. In particular, $V_0(\omega_Y, h) = \widehat{B}$. By §2.1.6 and §2.1.5, Y is of general type and $\chi(Y, \omega_Y) > 0$. This completes the proof. □

We prove a partial converse to Theorem 3.1: assume that there is a generically finite morphism $f : X \to A$ and a quotient abelian variety $A \twoheadrightarrow B$ such that $f(X) + K = f(X)$, where $K := \mathrm{Ker}(A \twoheadrightarrow B)^0$, and denote by $X \dashrightarrow Y \to B$ a modification of the Stein factorization of $X \to A \twoheadrightarrow B$, where Y is smooth of dimension $n - i$ (we set $i := \dim(K)$).

Proposition 3.2. *In this situation, if Y is not birational to an abelian variety, $V_j(\omega_X, f)$ has a component of codimension j for some $j \in \{i, \dots, n-1\}$.*

Proof. Replacing X with a modification of an étale cover (which is allowed by §2.1 and §2.2), we may assume that we have a factorization

$$f : X \xrightarrow{(g,k)} Y \times K \xrightarrow{h \times \mathrm{Id}_K} B \times K,$$

where (g, k) is surjective and $h : Y \to B$ is generically finite. We obtain, as in the proof of Theorem 3.1, for ξ general in $\widehat{B}$,

$$h^i(X, \omega_X \otimes f^*P_\xi) = h^0(Y, R^i g_*\omega_X \otimes h^*P_\xi) = h^0(Y, \omega_Y \otimes h^*P_\xi). \tag{3}$$

If $\chi(Y, \omega_Y) > 0$, we have $V_0(\omega_Y, h) = \widehat{B}$, the number on the right-hand-side of (3) is non-zero for all ξ, hence $V_i(\omega_X, f)$ contains the i-codimensional abelian subvariety $\widehat{B}$ of $\widehat{A}$.

If $\chi(Y, \omega_Y) = 0$, since Y is not birational to an abelian variety, $V_0(\omega_Y, h)$ has (by §2.1.7 and §2.1.5) a component of codimension $l \in \{1, \dots, n-i-1\}$ in $\widehat{B}$. Thus we can apply Theorem 3.1 to $h : Y \to B$: after taking an étale cover and a modification, h factors through a morphism $Y \to Z \times C$, where C is an abelian variety of dimension l, $\chi(Z, \omega_Z) > 0$, and $\dim(Z) = n - i - l$. We are therefore reduced to the first case and we conclude again that $V_{i+l}(\omega_X, f)$ contains an $(i + l)$-codimensional component. □

Remark 3.3. Under the hypotheses of Theorem 3.1 and the assumption $A = B \times K$ made in its proof, we obtain a surjective morphism $k : X \xrightarrow{f} B \times K \overset{p_2}{\twoheadrightarrow} K$ and, from its Stein factorization, morphisms

$$k : X \overset{l}{\twoheadrightarrow} Z \overset{m}{\twoheadrightarrow} K,$$

where Z is smooth of dimension i, m is generically finite, and l has connected (generically $(n-i)$-dimensional) fibers. We have again $R^{n-i}l_*\omega_X \simeq \omega_Z$ and, for ξ general in $\widehat{K}$,

$$h^{n-i}(X, \omega_X \otimes f^*P_\xi) = h^0(Z, \omega_Z \otimes m^*P_\xi).$$

Then,

a) either $V_0(\omega_Z, m) \subsetneq \widehat{K}$ and $\chi(Z, \omega_Z) = 0$;
b) or $V_0(\omega_Z, m) = \widehat{K}$ and $\chi(Z, \omega_Z) > 0$. There is a surjective generically finite map $X \twoheadrightarrow Y \times Z$, hence $\chi(X, \omega_X) \geq \chi(Y, \omega_Y)\chi(Z, \omega_Z) > 0$ by (2) (this also follows from Corollary 3.4.a) below).

Finally, if F is a general fiber of $l : X \twoheadrightarrow Z$, there is a surjective generically finite map $F \twoheadrightarrow Y$, hence $\chi(F, \omega_F) > 0$ (see (2)).

We now deduce some consequences of Theorem 3.1 on the possible components of $V_0(\omega_X, f)$ and the number of simple factors of the abelian variety A.

Corollary 3.4. *Let X be a smooth projective variety with $\chi(X, \omega_X) = 0$ and a generically finite morphism $f : X \to A$ to an abelian variety.*

a) *The locus $V_0(\omega_X, f)$ does not have complementary components.*[1]
b) *If X is in addition of general type, A has at least three simple factors.*

Proof. If $V_0(\omega_X, f)$ has complementary components $V_1, \ldots, V_r$, with duals $B_1, \ldots, B_r$, the image $f(X)$ is stable by translation by $\prod_{j \neq i} B_j$ for each i (§2.1.4), hence f is surjective if $r \geq 2$. We obtain from Theorem 3.1, after passing to an étale cover and a modification of X, a generically finite surjective map $X \twoheadrightarrow Y_1 \times \cdots \times Y_r$, with $\chi(Y_i, \omega_{Y_i}) > 0$ for all i. Since $\chi(X, \omega_X) \geq \prod_i \chi(Y_i, \omega_{Y_i})$ (by (2)), this is absurd. This proves a). Item b) then follows from §2.1.6. □

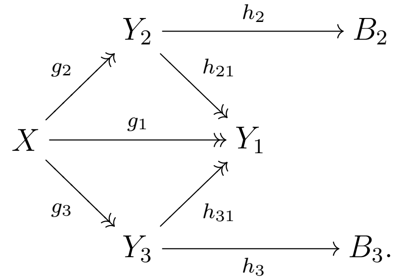
Proposition 3.5. *Let X be a smooth projective variety of general type with $\chi(X, \omega_X) = 0$ and a generically finite morphism $f : X \to A$ to an abelian variety. Then $V_0(\omega_X, f)$ has no 1-dimensional components.*

Proof. Assume $V_0(\omega_X, f)$ has a 1-dimensional component and write it as $\tau_1 + \widehat{B}_1$ for some torsion point $\tau_1 \in \widehat{A}$ and some quotient elliptic curve $A \twoheadrightarrow B_1$. By [J], Proposition 1.7, and since $V_0(\omega_X, f)$ generates $\widehat{A}$ (§2.1.6), $\widehat{B}_1$ cannot be maximal for inclusion: more precisely, there must exist two maximal components (in the sense of [J], Definition 1.6) $\tau_2 + \widehat{B}_2$ and $\tau_3 + \widehat{B}_3$ of $V_0(\omega_X, f)$ with $\widehat{B}_1 \subsetneq \widehat{B}_j \subsetneq \widehat{A}$ for each $j \in \{2, 3\}$ and $\widehat{B}_2 \neq \widehat{B}_3$. There are corresponding factorizations $A \twoheadrightarrow B_j \twoheadrightarrow B_1$.

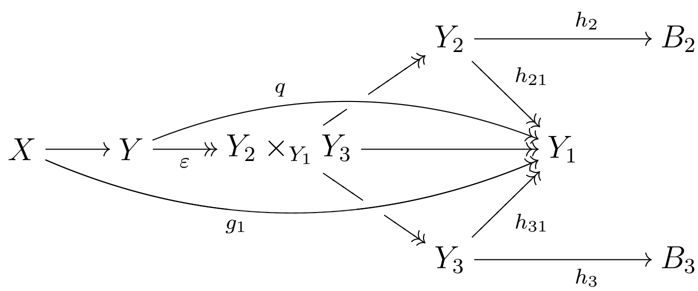
As in the proof of Theorem 3.1, after passing to an étale cover and a modification of X, we may assume $\tau_1 = \tau_2 = \tau_3 = 0$ and that we have

[1]By that, we mean components such that the sum morphism induces an isogeny from their product onto $\widehat{A}$.

- for each $i \in \{1, 2, 3\}$, Stein factorizations
$$X \xrightarrow{g_i} Y_i \xrightarrow{h_i} B_i,$$
where h_i is generically finite, Y_i is smooth, Y_1 is a curve of genus ≥ 2, and $\chi(Y_2, \omega_{Y_2})$ and $\chi(Y_3, \omega_{Y_3})$ are both positive (Theorem 3.1),
- a commutative diagram



We may further assume that the induced morphism $X \to Y_2 \times_{Y_1} Y_3$ factors as:



where ε is a resolution of singularities.

Now take $\xi_2 \in \widehat{B}_2$ and $\xi_3 \in \widehat{B}_3$. By [M], Lemma 4.10.(ii),[2] there is an inclusion
$$q_*(\omega_{Y/Y_1} \otimes \varepsilon^*(h_2^* P_{\xi_2} \otimes h_3^* P_{\xi_3})) \subseteq h_{21*}(\omega_{Y_2/Y_1} \otimes h_2^* P_{\xi_2}) \otimes h_{31*}(\omega_{Y_3/Y_1} \otimes h_3^* P_{\xi_3})$$
of locally free sheaves of the same rank on the curve Y_1. Moreover, we saw during the proof of Theorem 3.1 that for $j \in \{2, 3\}$, we have
$$0 \neq h^0(Y_j, \omega_{Y_j} \otimes h_j^* P_{\xi_j}) = h^0(Y_1, h_{j1*}(\omega_{Y_j} \otimes h_j^* P_{\xi_j})).$$
It follows that the sheaf $h_{j1*}(\omega_{Y_j} \otimes h_j^* P_{\xi_j})$ is non-zero, hence so is the sheaf $h_{j1*}(\omega_{Y_j/Y_1} \otimes h_j^* P_{\xi_j})$. All in all, we have obtained that the locally free sheaf $q_*(\omega_{Y/Y_1} \otimes \varepsilon^*(h_2^* P_{\xi_2} \otimes h_3^* P_{\xi_3}))$ is non-zero.

Assume now that ξ_2 and ξ_3 are torsion. By [V], Corollary 3.6,[3] this vector bundle is nef, hence has non-negative degree. Since Y_1 is a curve of genus ≥ 2, the Riemann-Roch theorem then implies
$$0 \neq h^0(Y_1, q_*(\omega_Y \otimes \varepsilon^*(h_2^* P_{\xi_2} \otimes h_3^* P_{\xi_3}))) = h^0(Y, \omega_Y \otimes \varepsilon^*(h_2^* P_{\xi_2} \otimes h_3^* P_{\xi_3})).$$
Finally, note that X and Y both have maximal Albanese dimensions. This implies that $\omega_{X/Y}$ is effective, hence $h^0(X, \omega_X \otimes f^*(P_{\xi_2} \otimes P_{\xi_3}))$ is also non-zero. It follows that $\xi_2 + \xi_3$

[2]This is stated in [M] for $\xi_2 = \xi_3 = 0$, but the same proof works in general.
[3]This is stated there for $\xi_2 = \xi_3 = 0$, but the general case follows by the étale covering trick.

is in $V_0(\omega_X, f)$, which therefore contains $\widehat{B}_2 + \widehat{B}_3$. This contradicts the fact that $\widehat{B}_2$ is maximal. □

4. Case when A has three simple factors

Ein and Lazarsfeld constructed an example of a smooth projective threefold X of maximal Albanese dimension and of general type with $\chi(X, \omega_X) = 0$, whose Albanese variety is the product of three elliptic curves. After presenting their construction (and a variant due to Chen and Hacon), we prove some general results when $\mathrm{Alb}(X)$ has three simple factors. In the next section, we will show that the Ein-Lazarsfeld example is essentially the only one in dimension 3 (Theorem 5.1).

Example 4.1 ([EL], Example 1.13)**.** Let E_1, E_2, and E_3 be elliptic curves and let $\rho_j : C_j \twoheadrightarrow E_j$ be double coverings, where C_j is a smooth curve of genus ≥ 2 and $\rho_{j*}\omega_{C_j} \simeq \mathscr{O}_{E_j} \oplus \delta_j$. Denote by ι_j the corresponding involution of C_j. Let $A = E_1 \times E_2 \times E_3$ and consider the quotient Z of $C_1 \times C_2 \times C_3$ by the involution $\iota_1 \times \iota_2 \times \iota_3$ and the tower of Galois covers:

$$C_1 \times C_2 \times C_3 \xrightarrow{g} Z \xrightarrow{f} A$$

of degrees 2 and 4 respectively. Observe that Z has rational singularities and is minimal of general type. Let $\varepsilon : X \twoheadrightarrow Z$ be any desingularization. The Albanese map of X is $a_X = f \circ \varepsilon$ and

$$a_{X*}\omega_X \simeq \mathscr{O}_A \oplus (L_1 \otimes L_2) \oplus (L_3 \otimes L_1) \oplus (L_2 \otimes L_3),$$

where L_j is the inverse image of δ_j by the projection $A \twoheadrightarrow E_j$, hence

$$V_0(\omega_X, a_X) = V_1(\omega_X, a_X) = (\widehat{E}_1 \times \widehat{E}_2 \times \{0\}) \cup (\widehat{E}_1 \times \{0\} \times \widehat{E}_3) \cup (\{0\} \times \widehat{E}_2 \times \widehat{E}_3), \tag{4}$$

whereas $V_2(\omega_X, a_X) = V_3(\omega_X, a_X) = \{0\}$.

This provides 3-dimensional examples which we call Ein-Lazarsfeld threefolds. Obviously, the same construction works starting from double coverings $\rho_j : X_j \twoheadrightarrow A_j$ of abelian varieties with smooth ample branch loci and provides examples in all dimensions ≥ 3. One can also extend it to any *odd* number $2r + 1$ of factors and get examples where the Albanese mapping is birationally a $(\mathbf{Z}/2\mathbf{Z})^{2r}$-covering.

Example 4.2 ([CH2], §4, Example)**.** A variant of the construction above was given by Chen and Hacon. Keeping the same notation, choose points $\xi_j \in \widehat{E}_j$ of order 2 and consider the induced double étale covers $C'_j \twoheadrightarrow C_j$, with associated involution σ_j, and $E'_j \twoheadrightarrow E_j$. The involution ι_j on C_j pulls back to an involution ι'_j on C'_j (with quotient E'_j). Let Z' be the quotient of $C'_1 \times C'_2 \times C'_3$ by the group of automorphisms generated by $\mathrm{id}_1 \times \sigma_2 \times \iota'_3$, $\iota'_1 \times \mathrm{id}_2 \times \sigma_3$, $\sigma_1 \times \iota'_2 \times \mathrm{id}_3$, and $\sigma_1 \times \sigma_2 \times \sigma_3$, and let $\varepsilon' : X' \twoheadrightarrow Z'$ be any desingularization. There is a morphism $f' : X' \twoheadrightarrow A$ of degree 4, the Albanese map of X' is $a_{X'} = f' \circ \varepsilon'$, and

$$a_{X'*}\omega_{\widetilde{X}'} \simeq \mathscr{O}_A \oplus (L_1 \otimes L_2^\xi \otimes P_{\xi_3}) \oplus (L_1^\xi \otimes P_{\xi_2} \otimes L_3) \oplus (P_{\xi_1} \otimes L_2 \otimes L_3^\xi), \tag{5}$$

where $L_j^\xi = L_j \otimes P_{\xi_j}$. In particular, $h^0(X', \omega_{X'}) = 1$ and

$$V_0(\omega_{X'}, a_{X'}) = V_1(\omega_{X'}, a_{X'}) = \{0\} \cup (\widehat{E}_1 \times \widehat{E}_2 \times \{\xi_3\}) \cup (\widehat{E}_1 \times \{\xi_2\} \times \widehat{E}_3) \cup (\{\xi_1\} \times \widehat{E}_2 \times \widehat{E}_3).$$

We will call the varieties X' obtained in this fashion Chen-Hacon threefolds. Of course, the étale cover $E_1' \times E_2' \times E_3' \twoheadrightarrow E_1 \times E_2 \times E_3$ pulls back to an étale cover $X'' \to X'$, where X'' is an Ein-Lazarsfeld threefold.

Again, this construction still works starting from double coverings of abelian varieties with smooth ample branch loci, providing examples in all dimensions ≥ 3, and for any *odd* number $2r+1$ of factors, providing examples where the Albanese mapping is birationally a $(\mathbf{Z}/2\mathbf{Z})^{2r}$-covering.

Proposition 4.3. *Let X be a smooth projective variety of general type with $\chi(X,\omega_X)=0$ and a generically finite morphism $f : X \to A$ to an abelian variety A with exactly three simple factors A_1, A_2, A_3.*

a) *The map f is surjective.*
b) *After passing to an abelian étale cover, we may assume $A = A_1 \times A_2 \times A_3$ and that $\widehat{A}_1 \times \widehat{A}_2 \times \{0\}$, $\widehat{A}_1 \times \{0\} \times \widehat{A}_3$, and $\{0\} \times \widehat{A}_2 \times \widehat{A}_3$ are irreducible components of $V_0(\omega_X, f)$.*

Proof. We begin with the proof of item b). As in the proof of Theorem 3.1, we may assume, after passing to an abelian étale cover, that A is $A_1 \times A_2 \times A_3$ and, by Corollary 3.4.a) and §2.1.6, that $\widehat{A}_1 \times \widehat{A}_2 \times \{0\}$ and $\widehat{A}_1 \times \{0\} \times \widehat{A}_3$ are irreducible components of $V_0(\omega_X, f)$.

Assume that the projection $V_0(\omega_X, f) \to \widehat{A}_2 \times \widehat{A}_3$ is not surjective. For ξ_2 and ξ_3 general torsion points in $\widehat{A}_2$ and $\widehat{A}_3$ respectively, we then have

$$(\xi_2 + \xi_3 + \widehat{A}_1) \cap V_0(\omega_X, f) = \varnothing.$$

Consider the morphism $f_1 = p_1 \circ f : X \to A_1$ and the sheaf $\mathscr{E} = f_{1*}(\omega_X \otimes f_2^* P_{\xi_2} \otimes f_3^* P_{\xi_3})$ on A_1. By [HP], Theorem 2.2, the cohomological loci of $\mathscr{E}$ satisfy (1). On the other hand, for all $\xi \in \widehat{A}_1$, we have $H^0(A_1, \mathscr{E} \otimes P_\xi) = 0$, hence $V_0(\mathscr{E}) = \varnothing$. It follows that for all $\xi \in \widehat{A}_1$ and all $i \geq 0$, we have $H^i(A_1, \mathscr{E} \otimes P_\xi) = 0$. This implies $\mathscr{E} = 0$ by Fourier-Mukai duality ([Mu]). But the rank of $\mathscr{E}$ is at least $\chi(F_1, \omega_{F_1})$, where F_1 is a component of a general fiber of f_1 ([HP], Corollary 2.3) and this is impossible: F_1 is of general type and generically finite over $A_2 \times A_3$, hence $\chi(F_1, \omega_{F_1}) > 0$ (Corollary 3.4.b)).

The projection $V_0(\omega_X, f) \twoheadrightarrow \widehat{A}_2 \times \widehat{A}_3$ is therefore surjective. Since A_1 is simple, this implies that, after passing to a (split) étale cover of A, there are morphisms $u_2 : \widehat{A}_2 \to \widehat{A}_1$ and $u_3 : \widehat{A}_3 \to \widehat{A}_1$ such that

$$\{u_2(\xi_2) + u_3(\xi_3) + \xi_2 + \xi_3 \mid \xi_2 \in \widehat{A}_2,\ \xi_3 \in \widehat{A}_3\}$$

is a component of $V_0(\omega_X, f)$. Composing this cover with the automorphism $(a_1, a_2, a_3) \mapsto (a_1, a_2 - \widehat{u}_2(a_1), a_3 - \widehat{u}_3(a_1))$ of A, we obtain b).

Item a) then follows from the fact that $f(X)$ is stable by translation by each A_i (§2.1.4) hence is equal to A. □

Remark 4.4. As shown by considering the product with a curve of genus ≥ 2 of any variety X of general type and maximal Albanese dimension with $\chi(X, \omega_X) = 0$, the conclusion of Proposition 4.3.a) does not hold in general as soon as A has at least four simple factors.

Proposition 4.5. *Let X be a smooth projective variety of general type of dimension n with $\chi(X,\omega_X)=0$ and a generically finite morphism $X\to A$ to an abelian variety A with exactly three simple factors. We have:*

a) *$q(X)=n$;*
b) *the general fiber F of any non-constant fibration $X\twoheadrightarrow Y$ satisfies $\chi(F,\omega_F)>0$;*
c) *any morphism from X to a curve of genus ≥ 2 is constant;*
d) *$V_{n-1}(\omega_X,a_X)=\{0\}$.*

Proof. Let us prove b) first. The fiber F is of general type and is generically finite but not surjective over A, hence $\chi(F,\omega_F)>0$ by Proposition 4.3.a). The other items then follow from the lemma below. □

Lemma 4.6. *Let X be a smooth projective variety of dimension n, of maximal Albanese dimension, of general type, with $\chi(X,\omega_X)=0$. Assume that the general fiber F of any non-constant fibration $X\twoheadrightarrow Y$ satisfies $\chi(F,\omega_F)>0$. Then,*

a) *the Albanese mapping a_X is surjective ($q(X)=n$);*
b) *any morphism from X to a curve of genus ≥ 2 is constant;*
c) *$V_{n-1}(\omega_X,a_X)=\{0\}$.*

Proof. Item a) follows from [CH2], Theorem 4.2, and item b) from [HP], Theorem 2.4. Let us prove c).

If $V_{n-1}(\omega_X,a_X)\smallsetminus\{0\}$ is non-empty, by §2.1.2, it contains a torsion point which defines a connected étale cover $\pi:\widetilde{X}\twoheadrightarrow X$ such that $q(\widetilde{X})>q(X)=n$. By [CH2], Theorem 4.2, again, there exists a non-constant fibration $\widetilde{X}\twoheadrightarrow Y$ with general fiber F of maximal Albanese dimension, of general type, with $\chi(F,\omega_F)=0$, such that $a_{\widetilde{X}}(F)$ is a translate of a fixed abelian subvariety $\widetilde{K}$ of $\mathrm{Alb}(\widetilde{X})$ and $\dim(\widetilde{K})=\dim(F)<\dim(X)$. We consider the image K of $\widetilde{K}$ by the induced map $\mathrm{Alb}(\pi):\mathrm{Alb}(\widetilde{X})\twoheadrightarrow\mathrm{Alb}(X)$ and the commutative diagram:

$$\begin{array}{ccccc} F & \xrightarrow{\pi|_F} & \pi(F) & \xrightarrow{a_X|_{\pi(F)}} & K+x_F \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{X} & \xrightarrow{\pi} & X & \xrightarrow{a_X} & \mathrm{Alb}(X) \\ {\scriptstyle f}\downarrow & & & \searrow^{h} & \downarrow \\ Y & & & & \mathrm{Alb}(X)/K. \end{array}$$

The map $a_X|_{\pi(F)}$ is generically finite, hence $\dim(K)=\dim\pi(F)=\dim(F)$, and

$$\dim(\mathrm{Alb}(X)/K)=n-\dim(K)=n-\dim(F).$$

It follows that $\pi(F)$ is a general fiber of the Stein factorization of h. Since $\chi(\pi(F),\omega_{\pi(F)})=0$, this contradicts our hypothesis on X. □

Assume now that we are in the situation of Proposition 4.3.b) and consider, as in Remark 3.3, the maps $f_i := p_i \circ f : X \twoheadrightarrow A_i$. Since $\chi(X, \omega_X) = 0$, we are in case a) of that remark, hence, with the notation therefrom, $V_0(\omega_Z, m)$ must be finite, because A_i is simple. If f is minimal, so is f_i and it follows from 2.1.7 that Z is birational to A_i, hence f_i is a fibration. A general fiber F_i satisfies $\chi(F_i, \omega_{F_i}) > 0$ by Proposition 4.5.b).

Proposition 4.7. *Let X be a smooth projective variety of general type with $\chi(X, \omega_X) = 0$ and a minimal generically finite morphism $f : X \to A$ to an abelian variety A product of three simple factors A_1, A_2, and A_3.*

Let $\{i, j, k\} = \{1, 2, 3\}$. For ξ_j and ξ_k general torsion points in $\widehat{A}_j$ and $\widehat{A}_k$ respectively, the sheaf $f_{i*}(\omega_X \otimes f_j^* P_{\xi_j} \otimes f_k^* P_{\xi_k})$ on A_i is locally free, homogeneous, of positive rank $\chi(F_i, \omega_{F_i})$.

Proof. We follow the proof of [CH2], Corollary 2.3. As in the proof of Proposition 4.3, since ξ_j and ξ_k are torsion, the cohomological loci of $\mathscr{E} := f_{i*}(\omega_X \otimes f_j^* P_{\xi_j} \otimes f_k^* P_{\xi_k})$ satisfy (1). On the other hand, since ξ_j and ξ_k are general and A_i is simple, the intersection

$$(\xi_j + \xi_k + \widehat{A}_i) \cap V_0(\omega_X, f)$$

is finite (§2.1.2), hence so is $V_0(\mathscr{E})$. It follows that all $V_l(\mathscr{E})$ are finite, hence $\mathscr{E}$ is locally free and homogeneous ([Mu], Example 3.2). Its rank is $h^0(F_i, \omega_{F_i} \otimes f_j^* P_{\xi_j} \otimes f_k^* P_{\xi_k}) = \chi(F_i, \omega_{F_i})$. □

Remark 4.8. Recall that a homogeneous vector bundle is a direct sum of twists of unipotent vector bundles (successive extensions of trivial line bundles) by algebraically trivial line bundles which, in our case, are torsion by Simpson's theorem (or rather its extension [HP], Theorem 2.2.b)).

When A_i is an elliptic curve, the sheaf of Proposition 4.7 is actually a direct sum of torsion line bundles (this is explained at the bottom of page 362 of [K3] when $\xi_j = \xi_k = 0$ and holds in general by the étale covering trick).

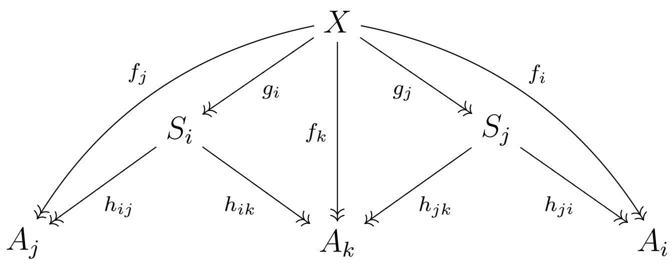
Finally, from the proof of Theorem 3.1, we have (after replacing X with a suitable modification), for each $\{i, j, k\} = \{1, 2, 3\}$, Stein factorizations

$$p_{jk} \circ f : X \xrightarrow{g_i} S_i \xrightarrow{(h_{ij}, h_{ik})} A_j \times A_k,$$

where S_i is smooth of general type with $\chi(S_i, \omega_{S_i}) > 0$ (this follows also from Corollary 3.4.b)). Since f_j has connected fibers, so does $h_{ij} : S_i \twoheadrightarrow A_j$.

All in all, we have for each $\{i, j, k\} = \{1, 2, 3\}$ a commutative diagram:

(6)



where all the morphisms are fibrations.

Question 4.9. Are the h_{ij} isotrivial? Are the f_i isotrivial? Is X rationally dominated by a product $X_1 \times X_2 \times X_3$, where X_i dominates and is generically finite over A_i? We are inclined to think that the answers to all these questions should be affirmative, but we were only able to go further in the case where the A_i are all elliptic curves.

5. The 3-dimensional case

We now come to our main result, which completely describes all smooth projective threefolds X of maximal Albanese dimension and of general type, with $\chi(X, \omega_X) = 0$.

Theorem 5.1. *Let X be a smooth projective threefold of maximal Albanese dimension and of general type with $\chi(X, \omega_X) = 0$.*

There exist elliptic curves E_1, E_2, and E_3, double coverings $C_j \twoheadrightarrow E_j$ with associated involutions ι_j, and a commutative diagram

$$\begin{array}{ccccc} & & (C_1 \times C_2 \times C_3)/\iota_1 \times \iota_2 \times \iota_3 & & \\ & \overset{\varepsilon}{\nearrow} & & \searrow & \\ \widetilde{X} & & \xrightarrow{\quad a_{\widetilde{X}} \quad} & & E_1 \times E_2 \times E_3 \\ \big\downarrow & & \square & & \big\downarrow \eta \\ X & & \xrightarrow[\quad a_X \quad]{} & & \mathrm{Alb}(X), \end{array}$$

where η is an isogeny and ε a desingularization.

In other words, up to abelian étale covers, the Ein-Lazarsfeld examples (Example 4.1) are the only ones (in dimension 3)! Note also that a_X is not finite, but that it is finite on the canonical model of X.

Corollary 5.2. *Under the hypotheses of the theorem, the Albanese mapping of X is birationally a $(\mathbf{Z}/2\mathbf{Z})^2$-covering.*

Proof. With the notation of Example 4.1, we set $A := E_1 \times E_2 \times E_3$ and $\widetilde{X}_i := \mathrm{Spec}(\mathscr{O}_A \oplus L_i^\vee)$, so that f factors through the double coverings $\widetilde{f}_i := \widetilde{X}_i \twoheadrightarrow A$. Since the action of $\mathrm{Ker}(\eta)$ on A by translations lifts to $\widetilde{X}$, it leaves

$$a_{\widetilde{X}*}\mathscr{O}_{\widetilde{X}} \simeq \mathscr{O}_A \oplus (L_1^\vee \otimes L_2^\vee) \oplus (L_3^\vee \otimes L_1^\vee) \oplus (L_2^\vee \otimes L_3^\vee)$$

invariant, hence also each $L_i^\vee$. It follows that this action lifts to each $\widetilde{X}_i$, hence $\widetilde{f}_i$ descends to a double covering $X_i \twoheadrightarrow \mathrm{Alb}(X)$ through which a_X factors. □

Proof of the theorem. By Proposition 4.5.a), a_X is surjective and $q(X) = 3$ (see also [CH2], Corollary 4.3).

Moreover, by Corollary 3.4.b), $\mathrm{Alb}(X)$ is isogeneous to the product of three elliptic curves and, after passing to étale covers, we may assume that a_X can be written as

$$a_X : X \xrightarrow{(f_1, f_2, f_3)} E_1 \times E_2 \times E_3,$$

where each $f_i : X \twoheadrightarrow E_i$ is a fibration, and that $\widehat{E}_1 \times \widehat{E}_2 \times \{0\}$, $\widehat{E}_1 \times \{0\} \times \widehat{E}_3$, and $\{0\} \times \widehat{E}_2 \times \widehat{E}_3$ are irreducible components of $V_0(\omega_X, a_X)$ (Proposition 4.3.b)).

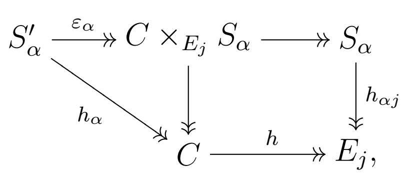
Let $\{i, j, k\} = \{1, 2, 3\}$. As in §4, we have a commutative diagram (6), where each S_i is a smooth minimal surface of general type and $A_i = E_i$. The proof of the theorem is very long, so we will divide it in several steps. The general scheme of proof goes as follows:

- In the diagram (6), the fibrations $h_{ij} : S_i \twoheadrightarrow E_j$ are all isotrivial (Step 1); we let C_{ij} be a (constant) general fiber.
- For every $i \in \{1, 2, 3\}$, there exist a finite group G_i, and for every $j \neq i$, a curve C_{ij} acted on by G_i, such that $C_{ij}/G_i \simeq E_j$, the surface S_i is birational to $(C_{ij} \times C_{ik})/G_i$, and h_{ij} and h_{ik} are the two projections (Step 2).
- At this point, it is quite easy to show that X dominates a threefold Y which is dominated by a product of three curves (Step 3).
- Taking an étale cover of X, we may assume that all the irreducible components of $V_0(\omega_X, a_X)$ pass through 0. We then show (Step 4) that $V_0(\omega_X, a_X)$ has the same form as the corresponding locus of an Ein-Lazarsfeld threefold (see (4)), from which we deduce that X is birationally isomorphic to Y (Step 5) hence is also dominated by a product of 3 curves.
- Using the fact that $V_0(\omega_X, a_X)$ has no "extra" components, we finish the proof by showing that the groups G_i all have order 2 (Step 6).

Step 1. *The fibrations $h_{ij} : S_i \twoheadrightarrow E_j$ are all isotrivial.*

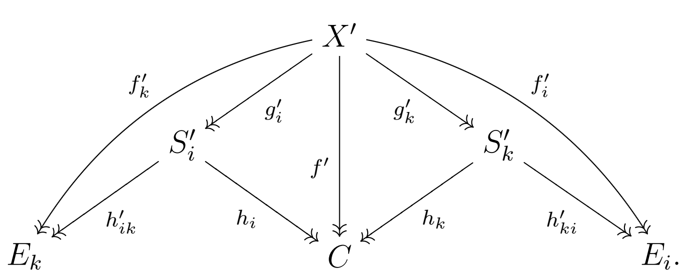
We will denote by C_{ij} a general (constant) fiber of h_{ij}.

Proof. Fix $j \in \{1, 2, 3\}$ and let as above $\{i, j, k\} = \{1, 2, 3\}$. By the semi-stable reduction theorem ([KKMS], Chapter II), there exist a finite cover $h : C \twoheadrightarrow E_j$, where C is a smooth curve, and commutative diagrams (for each $\alpha \in \{i, k\}$)



where ε_α is a modification and the fibers of h_α are all reduced connected curves, with non-singular components crossing transversally.

We also make a modification $\tau : X' \to C \times_{E_j} X$ such that there exists a commutative diagram of morphisms between smooth varieties:



Let $\xi_\alpha \in \widehat{E}_\alpha$. By [V], Lemma 3.1,[4] we have an inclusion

$$f'_*(\omega_{X'/C} \otimes f_i'^* P_{\xi_i} \otimes f_k'^* P_{\xi_k}) \subseteq h^* f_{j*}(\omega_X \otimes f_i^* P_{\xi_i} \otimes f_k^* P_{\xi_k}) \tag{7}$$

of locally free sheaves on C. Since h_α is flat with irreducible general fibers, $S'_i \times_C S'_k$ is irreducible, and we have a surjective morphism

$$g'_{ik} : X' \xrightarrow{(g'_i, g'_k)} S'_i \times_C S'_k.$$

Moreover, h_i and h_k are semistable, hence by [AK], Proposition 6.4, $S'_i \times_C S'_k$ has only rational Gorenstein singularities. After further modification of X', we may assume that g'_{ik} factors through a desingularization of $S'_i \times_C S'_k$:

$$g'_{ik} : X' \twoheadrightarrow Y'_{ik} \xrightarrow{\varepsilon} S'_i \times_C S'_k.$$

By definition of rational Gorenstein singularities, we have $\varepsilon_* \omega_{Y'_{ik}} = \omega_{S'_i \times_C S'_k}$. Since $\omega_{X'/Y'_{ik}}$ is effective, we obtain an inclusion

$$p_i^*(\omega_{S'_i/C} \otimes {h'_{ik}}^* P_{\xi_k}) \otimes p_k^*(\omega_{S'_k/C} \otimes {h'_{ki}}^* P_{\xi_i}) \subseteq g'_{ik*}(\omega_{X'/C} \otimes f_i'^* P_{\xi_i} \otimes f_k'^* P_{\xi_k})$$

of sheaves on $S'_i \times_C S'_k$. Pushing forward these sheaves to C, we obtain

$$\begin{aligned} h_{i*}(\omega_{S'_i/C} \otimes {h'_{ik}}^* P_{\xi_k}) \otimes h_{k*}(\omega_{S'_k/C} \otimes {h'_{ki}}^* P_{\xi_i}) &\subseteq f'_*(\omega_{X'/C} \otimes f_i'^* P_{\xi_i} \otimes f_k'^* P_{\xi_k}) \\ &\subseteq h^* f_{j*}(\omega_X \otimes f_i^* P_{\xi_i} \otimes f_k^* P_{\xi_k}), \end{aligned} \tag{8}$$

where the second inclusion comes from (7). Let $\{\alpha, \beta\} = \{i, k\}$. The sheaf $h_{\alpha*}(\omega_{S'_\alpha/C} \otimes {h'_{\alpha\beta}}^* P_{\xi_\beta})$ is nef ([V], Corollary 3.6). On the other hand, for ξ_i and ξ_k general and torsion, the sheaf in (8) has degree 0 by Proposition 4.7, hence

$$\deg(h_{\alpha*}(\omega_{S'_\alpha/C} \otimes {h'_{\alpha\beta}}^* P_{\xi_\beta})) = 0.$$

By [K1], Corollary 10.15, the sheaf $R^1 h_{\alpha*}(\omega_{S'_\alpha/C} \otimes {h'_{\alpha\beta}}^* P_{\xi_\beta})$ is torsion-free and generically 0, hence 0.[5] On the other hand, we have $R^1 h_{\alpha*}\omega_{S'_\alpha/C} = \mathscr{O}_C$ ([K2], Proposition 7.6) hence, by the Grothendieck-Riemann-Roch theorem,

$$[\operatorname{ch}(h_{\alpha*}(\omega_{S'_\alpha/C})) - \operatorname{ch}(\mathscr{O}_C)]\operatorname{Td}(C) = \operatorname{ch}(h_{\alpha*}(\omega_{S'_\alpha/C} \otimes {h'_{\alpha\beta}}^* P_{\xi_\beta}))\operatorname{Td}(C)$$

in the ring of cycles modulo numerical equivalence on C. Comparing the terms in $H^2(C, \mathbf{C})$ in this equality, we get $\deg(h_{\alpha*}(\omega_{S'_\alpha/C})) = \deg(h_{\alpha*}(\omega_{S'_\alpha/C} \otimes {h'_{\alpha\beta}}^* P_{\xi_\beta})) = 0$. This implies that h_α is locally trivial (see e.g., [BPV], Theorem III.17.3). Hence $h_{\alpha j}$ is isotrivial (for each $\alpha \in \{i, k\}$). □

Step 2. *For every $i \in \{1,2,3\}$, there exists a finite group G_i, and for every $j \in \{1,2,3\} - \{i\}$, there exists a curve C_{ij} acted on by G_i such that $C_{ij}/G_i \simeq E_j$, the surface S_i is birational to the quotient $(C_{ij} \times C_{ik})/G_i$ (where $\{i,j,k\} = \{1,2,3\}$) for the diagonal action of G_i, and h_{ij} and h_{ik} are identified with the two projections.*

This is a consequence of the following (probably classical) result.

[4] This is proved there for $\xi_i = \xi_k = 0$, but the same proof works in general.

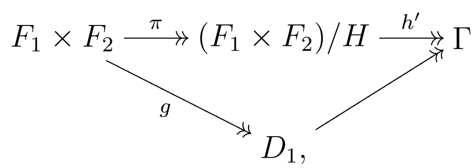
[5] Note that up to this point, the proof works in the more general situation where a_X is surjective and $\operatorname{Alb}(X)$ has a 1-dimensional simple factor E_1.

Lemma 5.3. *Let S be a smooth projective surface with an isotrivial fibration $h_1 : S \twoheadrightarrow \Gamma_1$ onto an irrational curve with (constant) irrational general fiber F_1.*

a) *There exist a smooth curve F_2 and a finite group H acting faithfully on F_1 and F_2 such that Γ_1 is isomorphic to F_2/H, the surface S is birationally isomorphic to the diagonal quotient $(F_1 \times F_2)/H$, and h_1 is the composition $S \overset{\sim}{\dashrightarrow} (F_1 \times F_2)/H \twoheadrightarrow F_2/H \simeq \Gamma_1$. Let h_2 be the composition $S \overset{\sim}{\dashrightarrow} (F_1 \times F_2)/H \twoheadrightarrow F_1/H$.*

b) *Assume S is of general type. Any isotrivial fibration $h : S \twoheadrightarrow \Gamma$ onto an irrational curve Γ is either h_1 or h_2 followed by an isomorphism between F_1/H or F_2/H with Γ.*

Proof. Item a) is well-known and can be found in [S]. Let us prove b). Since Γ is irrational and $(F_1 \times F_2)/H$ has rational singularities, h induces an isotrivial fibration $h' : (F_1 \times F_2)/H \twoheadrightarrow \Gamma$. Let D_2 be a general (constant irrational) fiber of h'. The quotient map $\pi : F_1 \times F_2 \twoheadrightarrow (F_1 \times F_2)/H$ is étale outside a finite set. Hence the Stein factorization g of $h' \circ \pi$ in the diagram

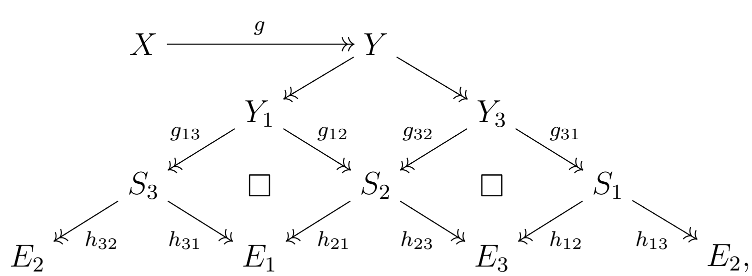


is also isotrivial, with general fiber D_2' a (fixed) étale cover of D_2. By a), there is a base change $D_1' \twoheadrightarrow D_1$ and a surjective morphism $t = (t_1, t_2) : D_1' \times D_2' \twoheadrightarrow F_1 \times F_2$. Since S is of general type, F_1 and F_2 are each of genus ≥ 2. A classical theorem of de Franchis says that there are no continuous non-constant systems of surjective morphisms $D_j' \to F_i$. It follows that each t_i must factor through one of the projections $p_j : D_1' \times D_2' \twoheadrightarrow D_j'$.

If h factors through neither h_1 nor h_2, the curve D_2' dominates both F_1 and F_2, hence t_1 and t_2 cannot factor through p_1. Thus they must factor through p_2, which contradicts the fact that t is surjective. □

Let now Y_j be a resolution of singularities of the irreducible threefold $S_i \times_{E_j} S_k$ and let Y be a resolution of singularities of a component of $Y_1 \times_{E_2 \times E_3} S_1$ that dominates both Y_1 and Y_3. After modification of X, we obtain a diagram

(9)



where the squares are birationally cartesian and the isotrivial morphisms $h_{ij} : S_i \twoheadrightarrow E_j$ fit into diagrams

$$\begin{array}{ccccc} & & C_{ik} \times C_{ij} & \xrightarrow{p_2} & C_{ij} \\ & & \Big\downarrow{\scriptstyle /G_i} & & {\scriptstyle \rho_{ij}}\Big\downarrow{\scriptstyle /G_i} \\ C_{ik} & \xhookrightarrow{\text{fiber}} & S_i & \xrightarrow[h_{ij}]{} & E_j. \end{array}$$

Step 3. *The threefold Y is dominated by a product of three curves.*

The dominant maps $C_{31} \times C_{32} \dashrightarrow S_3$ and $C_{21} \times C_{23} \dashrightarrow S_2$ induce a factorization

$$((\rho_{31}, \rho_{21}), \rho_{32}, \rho_{23}) : (C_{31} \times_{E_1} C_{21}) \times C_{32} \times C_{23} \dashrightarrow S_3 \times_{E_1} S_2 \twoheadrightarrow E_1 \times E_2 \times E_3.$$

The (Stein factorization of the) morphism $Y_1 \twoheadrightarrow E_2 \times E_3$ is therefore isotrivial (its fibers are dominated by the curve $C_{31} \times_{E_1} C_{21}$). Thus, Y is dominated by the product

$$(C_{31} \times_{E_1} C_{21}) \times (C_{12} \times_{E_2} C_{32}) \times (C_{23} \times_{E_3} C_{13}) \tag{10}$$

of three (possibly reducible) curves.

Going back to the proof of Theorem 5.1, after passing to an étale cover, we may and will assume, from now on, that the following holds (§2.2):

(11) *All the irreducible components of $V_0(\omega_X, a_X)$ pass through* 0.

One checks, using §2.1, §2.2, and Proposition 4.5.d), that if we want X to be birationally covered by a product $\Gamma_1 \times \Gamma_2 \times \Gamma_3$, with morphisms $\Gamma_i \twoheadrightarrow E_i$, as in the conclusion of the theorem, we must have the following.

Step 4. *We have*

$$V_0(\omega_X, a_X) = (\widehat{E}_1 \times \widehat{E}_2 \times \{0\}) \cup (\widehat{E}_1 \times \{0\} \times \widehat{E}_3) \cup (\{0\} \times \widehat{E}_2 \times \widehat{E}_3).$$

We already know that $V_0(\omega_X, a_X)$ contains the right-hand-side (Proposition 4.3.b)) and we must prove that it has no other components.

Proof. Assume $V_0(\omega_X, a_X)$ has another component $\widehat{T}$. It has dimension 2 (Corollary 3.4.a)) and, after possibly permuting the indices, we may assume the neutral component $\widehat{E}'_1$ of $\widehat{T} \cap (\widehat{E}_1 \times \widehat{E}_2 \times \{0\})$ is neither $\widehat{E}_1 \times \{0\} \times \{0\}$ nor $\{0\} \times \widehat{E}_2 \times \{0\}$. This yields an elliptic curve E'_1 which is a quotient of $E_1 \times E_2$ which does not factor through either projection. As we saw right before Proposition 4.7, the induced map $f_4 : X \twoheadrightarrow E'_1$ is a fibration. It factors as

$$f_4 : X \longrightarrow S_3 \overset{h_{34}}{\twoheadrightarrow} E'_1$$

where, by Step 1, h_{34} is isotrivial. By Lemma 5.3.b), h_{34} must factor through one of the projections $h_{31} : S_3 \twoheadrightarrow E_1$ or $h_{32} : S_3 \twoheadrightarrow E_2$ so we reach a contradiction. □

Step 5. *The morphism $g : X \twoheadrightarrow Y$ is birational.*

Proof. Consider, in the diagram (9), the generically finite morphism $v_1 : X \twoheadrightarrow Y_1$ and the three fibrations $f'_\alpha : Y_1 \twoheadrightarrow E_\alpha$, for $\alpha \in \{1, 2, 3\}$. Since X, Y_1, and S_3 are all of maximal Albanese dimensions, ω_{X/Y_1} and ω_{Y_1/S_3} are effective, hence

$$h^0(X, \omega_X \otimes a_X^* P_\xi) \geq h^0(Y_1, \omega_{Y_1} \otimes (f'_1, f'_2)^* P_\xi) \geq h^0(S_3, \omega_{S_3} \otimes (h_{31}, h_{32})^* P_\xi)$$

for all $\xi \in \widehat{E}_1 \times \widehat{E}_2$. Moreover, for ξ non-zero, we have by Proposition 4.5.d)

$$h^2(X, \omega_X \otimes a_X^* P_\xi) = h^3(X, \omega_X \otimes a_X^* P_\xi) = 0$$

hence, since $\chi(X, \mathscr{O}_X) = 0$,

$$h^0(X, \omega_X \otimes a_X^* P_\xi) = h^1(X, \omega_X \otimes a_X^* P_\xi).$$

Finally, for ξ general in $\widehat{E}_1 \times \widehat{E}_2$, we have, as in the proof of Theorem 3.1, since $g_3 : X \twoheadrightarrow S_3$ has connected fibers,

$$h^1(X, \omega_X \otimes a_X^* P_\xi) = h^0(S_3, \omega_{S_3} \otimes (h_{31}, h_{32})^* P_\xi).$$

Therefore, for $\xi \in \widehat{E}_1 \times \widehat{E}_2$ general, we obtain

$$h^0(X, \omega_X \otimes a_X^* P_\xi) = h^0(Y_1, \omega_{Y_1} \otimes (f'_1, f'_2)^* P_\xi).$$

The induced morphism $Y \twoheadrightarrow E_1 \times E_2 \times E_3$ is the Albanese mapping of Y. Since in any event, we always have

$$h^0(X, \omega_X \otimes a_X^* P_\xi) \geq h^0(Y, \omega_Y \otimes a_Y^* P_\xi) \geq h^0(Y_1, \omega_{Y_1} \otimes (f'_1, f'_2)^* P_\xi)$$

for all $\xi \in \widehat{E}_1 \times \widehat{E}_2$, we obtain

$$h^0(X, \omega_X \otimes a_X^* P_\xi) = h^0(Y, \omega_Y \otimes a_Y^* P_\xi) \tag{12}$$

for ξ general in $\widehat{E}_1 \times \widehat{E}_2$, hence also, by Step 3, for ξ general in $V_0(\omega_X, a_X)$. But for $\xi \notin V_0(\omega_X, a_X)$, both sides of (12) vanish. By Lemma 5.4 below, we conclude that g is a birational morphism. □

The following lemma (used in the proof above) is in the spirit of [HP], Theorem 3.1.

Lemma 5.4. *Let $X \overset{g}{\twoheadrightarrow} Y \overset{f}{\twoheadrightarrow} A$ be generically finite morphisms between smooth projective threefolds, where A is an abelian threefold, such that f and $f \circ g$ are both minimal. Assume that X is of general type with $\chi(X, \omega_X) = 0$ and that there exists an open subset $U \subseteq \widehat{A}$ with* $\operatorname{codim}_{\widehat{A}}(\widehat{A} - U) \geq 2$ *such that*

$$h^0(X, \omega_X \otimes g^* f^* P_\xi) = h^0(Y, \omega_Y \otimes f^* P_\xi)$$

for all $\xi \in U$. Then g is birational.

Proof. By §2.2, we can write $g_* \omega_X \simeq \omega_Y \oplus \mathscr{E}$, and we need to show that the sheaf $\mathscr{E}$ is zero. Since $\mathscr{E}$ is torsion-free and f is generically finite, it is sufficient to prove $f_* \mathscr{E} = 0$.

As we saw at the beginning of §5, we have $q(X) = 3$, hence $f \circ g$ is the Albanese mapping of X. By Proposition 4.5.d), for each $i \in \{2, 3\}$, we then have $\{0\} = V_i(\omega_X, f \circ g) = V_i(g_* \omega_X, f)$, hence $V_i(f_* \mathscr{E}) \subseteq \{0\}$.

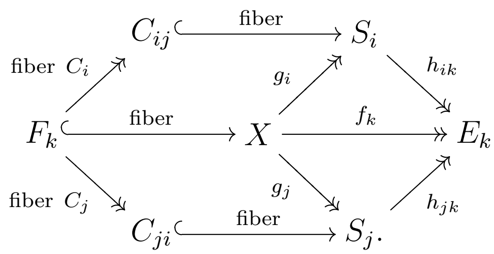
Since $q(X) = 3$, we also have $q(Y) = 3$, hence $h^i(Y, g_* \omega_X) = h^i(Y, \omega_Y)$ for $i \in \{2, 3\}$. It follows that $V_i(f_* \mathscr{E})$ is empty.

The assumption $\chi(X,\omega_X)=0$ implies $\chi(Y,\omega_Y)=0$ by (2). Thus,

$$V_0(f_*\mathscr{E}) = V_1(f_*\mathscr{E}) \subseteq \widehat{A} \smallsetminus U.$$

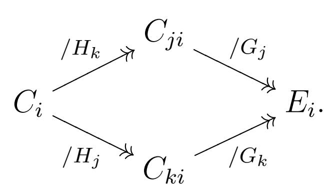
Since $\operatorname{codim}_{\widehat{A}}(\widehat{A}\smallsetminus U)>1$, the sheaf $f_*\mathscr{E}$ is therefore M-regular in the sense of [PP], Definition 2.1 (see also Remark 2.3), hence continuously globally generated ([PP], Definition 2.10 and Proposition 2.13). Since $H^0(A, f_*\mathscr{E}\otimes P_\xi)=0$ for all $\xi\in U$, we obtain $f_*\mathscr{E}=0$. □

Let us summarize what we know. Let $\{i,j,k\}=\{1,2,3\}$. The curve C_{ij} is the (constant) general fiber of the isotrivial fibration $S_i \twoheadrightarrow E_k$; it is acted on by a group G_i and $C_{ij}/G_i \simeq E_j$ (Step 2). The fibration $g_i : X \twoheadrightarrow S_i$ is also isotrivial; as we saw in Step 3, its general fiber C_i is dominated by the curve $C_{ji}\times_{E_i} C_{ki}$ but also maps onto C_{ji} and C_{ki}. The situation is summarized in the following diagram:



Note that a general fiber F_k of the isotrivial fibration $f_k : X \twoheadrightarrow E_k$ is an isotrivial fibration over C_{ij} with (constant) general fiber C_i. By Lemma 5.3, there exists a finite group H_k acting faithfully on C_i and C_j such that $C_{ij}\simeq C_j/H_k$, $C_{ji}\simeq C_i/H_k$, and F_k is isomorphic to the diagonal quotient $(C_i\times C_j)/H_k$. Moreover, the maps to C_{ij} and C_{ji} are the natural projections. So we have diagrams

(13)



Let D_1 be the Galois closure of C_1 over E_1 and set $G=\operatorname{Gal}(D_1/E_1)$. Let $\{j,k\}=\{2,3\}$. There is a normal subgroup $N_j \lhd G$ such that $G_j = G/N_j$ and G acts on C_{jk} via this quotient. By Step 2, the surface S_j is birationally isomorphic to $(C_{j1}\times C_{jk})/G_j$, hence to $(D_1\times C_{jk})/G$. Therefore, the modification Y_1 of $S_2\times_{E_1} S_3$ (see (9)) is birationally isomorphic to $(D_1\times C_{23}\times C_{32})/G$.

Step 6. *The group G is isomorphic to $\mathbf{Z}/2\mathbf{Z}$.*

We begin with a lemma which is probably well-known. We denote by $\operatorname{Irr}(G)$ the set of isomorphism classes of irreducible representations of G.

Lemma 5.5. *Let E be an elliptic curve and let $\pi : D \twoheadrightarrow E$ be a Galois cover with group G. We can write*

$$\pi_*\mathscr{O}_D = \bigoplus_{\chi\in\operatorname{Irr}(G)} \bigoplus_i \mathscr{V}_{\chi,i},$$

where each vector bundle $\mathscr{V}_{\chi,i}$ is semistable and G-invariant, and the representation of G on the general fiber of each $\mathscr{V}_{\chi,i}$ is a direct sum of χ. Moreover, for each $\chi \neq 1$, the dual vector bundle $\mathscr{V}_{\chi,i}^{\vee}$ is either ample or a direct sum of non-zero torsion line bundles.

Proof. The groups G acts on $\pi_*\mathscr{O}_D$. Identifying each representation χ with its character, we consider the endomorphism $\sum_{g\in G}\chi(g)g$ of $\pi_*\mathscr{O}_D$ and we denote by $\mathscr{V}_\chi$ its image. We then have

$$\pi_*\mathscr{O}_D = \bigoplus_{\chi}\mathscr{V}_\chi, \tag{14}$$

where the general fiber of $\mathscr{V}_\chi$ is, as a G-module, a (non-zero) direct sum of copies of χ.

The Harder-Narasimhan filtration

$$0 = \mathscr{V}_\chi^\ell \subseteq \mathscr{V}_\chi^{\ell-1} \subseteq \cdots \subseteq \mathscr{V}_\chi^0 = \mathscr{V}_\chi$$

is preserved by the G-action. The G-invariant semistable bundles $V_{\chi,i} := \mathscr{V}_\chi^{i-1}/\mathscr{V}_\chi^i$, for $1 \le i \le \ell$, have increasing slopes hence, since E is an elliptic curve, $\mathscr{V}_\chi$ is isomorphic to the direct sum $\bigoplus_{i=1}^\ell V_{\chi,i}$ (see for instance [T], Appendix A).

As a direct summand of $\pi_*\omega_D = (\pi_*\mathscr{O}_D)^\vee$, each vector bundle $\mathscr{V}_{\chi,i}^\vee$ is nef ([V], Corollary 3.6). Moreover, it is ample if it has positive degree. Consider the maximal degree-0 subsheaf $\mathscr{F}$ of $\pi_*\mathscr{O}_D$, i.e., the direct sum of all $\mathscr{V}_{\chi,1}$ that have degree 0. By [KP], Lemma 3.2 and 3.4, $\mathscr{F}$ is a G-invariant subalgebra and induces an étale cover of E, hence is a direct sum of torsion line bundles. □

Let us continue with the Galois cover $\pi : D \twoheadrightarrow E$ with group G as in the lemma and assume moreover that for each $j \in \{2,3\}$, we have a Galois cover $\pi_j : D_j \twoheadrightarrow E_j$ with Galois group $G_j = G/N_j$, where $g(D_j) \ge 2$ and E_j is an elliptic curve.

Then G acts on $D_1 \times D_2 \times D_3$ diagonally. Let Z be the quotient, let $\varepsilon : Y \twoheadrightarrow Z$ be a resolution, and consider

$$t : Y \xrightarrow{\varepsilon} Z \twoheadrightarrow E_1 \times E_2 \times E_3.$$

Lemma 5.6. *Assume*

$$V_0(\omega_Y, t) \subseteq (\widehat{E}_1 \times \widehat{E}_2 \times \{0\}) \cup (\widehat{E}_1 \times \{0\} \times \widehat{E}_3) \cup (\{0\} \times \widehat{E}_2 \times \widehat{E}_3)$$

and $V_2(\omega_Y, t) \cup V_3(\omega_Y, t) \subseteq \{0\}$. Then $N_2 = N_3$ and $G_2 \simeq G_3 \simeq \mathbf{Z}/2\mathbf{Z}$.

Proof. We decompose $\pi_*\mathscr{O}_D$ as in (14) and write similarly

$$\pi_{j*}\mathscr{O}_{D_j} = \bigoplus_{\mu\in \mathrm{Irr}(G_j)}\bigoplus_i \mathscr{V}_{\mu,i}^j.$$

Since quotient singularities are rational, we have as in Example 4.1

$$t_*\omega_Y \simeq (q_*\mathscr{O}_Z)^\vee \simeq \big((\pi_*\mathscr{O}_D)^\vee \boxtimes (\pi_{2*}\mathscr{O}_{D_2})^\vee \boxtimes (\pi_{3*}\mathscr{O}_{D_3})^\vee\big)^G.$$

Let μ be a non-trivial element of $\mathrm{Irr}(G_2)$. Since G_2 is a quotient of G, the representation μ and its complex conjugate $\overline{\mu}$ are also in $\mathrm{Irr}(G)$. Then, the vector bundle

$$\mathscr{G} := (\mathscr{V}_{\overline{\mu},1}^\vee \boxtimes \mathscr{V}_{\mu,1}^{2\vee} \boxtimes \mathscr{O}_{E_3})^G$$

on $E_1\times E_2\times E_3$ is a non-zero direct summand of both $\mathscr{V}^{\vee}_{\overline{\mu},1}\boxtimes\mathscr{V}^{2\vee}_{\mu,1}\boxtimes\mathscr{O}_{E_3}$ and $t_*\omega_Y$.

Assume that $\mathscr{V}^2_{\mu,1}$ has degree 0, hence is a direct sum of non-trivial torsion line bundles.

- If $\deg(\mathscr{V}^{\vee}_{\overline{\mu},1})=0$, the sheaf $\mathscr{G}$ is a direct sum of non-trivial torsion line bundles, which is impossible since $V_3(\mathscr{G})\subseteq V_3(\omega_Y,t)=\{0\}$.
- If $\mathscr{V}^{\vee}_{\overline{\mu},1}$ is ample, we can write
$$\mathscr{G}=\bigoplus_k(\mathscr{G}_k\boxtimes P_{\xi_k}\boxtimes\mathscr{O}_{E_3}),$$
where $\mathscr{G}_k$ is a direct summand of $\mathscr{V}^{\vee}_{\overline{\mu},1}$, hence ample, and the ξ_k are non-zero torsion points in $\widehat{E}_2$. This is again impossible, because $V_2(\mathscr{G})\subseteq V_2(\omega_Y,t)=\{0\}$.

Therefore, $\mathscr{V}^{j\vee}_{\mu,1}$ is ample for all μ non-trivial in $\operatorname{Irr}(G_j)$.

If $\operatorname{Card}(G_2)>2$, or if $N_2\neq N_3$, we may take non-trivial $\chi\in\operatorname{Irr}(G)$, $\mu\in\operatorname{Irr}(G_2)$, and $\nu\in\operatorname{Irr}(G_3)$ such that χ is a subrepresentation of $\mu\otimes\nu$. The vector bundle
$$\mathscr{H}:=(\mathscr{V}^{\vee}_{\overline{\chi},1}\boxtimes\mathscr{V}^{2\vee}_{\mu,1}\boxtimes\mathscr{V}^{3\vee}_{\nu,1})^G$$
is then non-zero and a direct summand of $t_*\omega_Y$ (and $\mathscr{V}^{2\vee}_{\mu,1}$ and $\mathscr{V}^{3\vee}_{\nu,1}$ are ample).

If $\mathscr{V}^{\vee}_{\overline{\chi},1}$ is ample, since $\mathscr{H}$ is a direct summand of $\mathscr{V}^{\vee}_{\overline{\chi},1}\boxtimes\mathscr{V}^{2\vee}_{\mu,1}\boxtimes\mathscr{V}^{3\vee}_{\nu,1}$, we have $V_m(\mathscr{H})=\varnothing$ for all $m\in\{1,2,3\}$. Hence $h^0(E_1\times E_2\times E_3,\mathscr{H}\otimes P_\xi)$ is a non-zero constant for all $\xi\in\widehat{E}_1\times\widehat{E}_2\times\widehat{E}_3$ and $V_0(\mathscr{H})=\widehat{E}_1\times\widehat{E}_2\times\widehat{E}_3$, which contradicts our assumptions.

If $\mathscr{V}^{\vee}_{\overline{\chi},1}$ is a direct sum of non-trivial torsion line bundles, we may write
$$\mathscr{H}=\bigoplus_k(P_{\xi_k}\boxtimes\mathscr{H}_k),$$
where the ξ_k are non-zero torsion points in $\widehat{E}_1$ and $\mathscr{H}_k$ is a direct summand of $\mathscr{V}^{2\vee}_{\mu,1}\boxtimes\mathscr{V}^{3\vee}_{\nu,1}$. Then $V_0(\mathscr{H})$, hence also $V_0(\omega_Y,t)$, contains $\{-\xi_1\}\times\widehat{E}_2\times\widehat{E}_3$, which contradicts our assumptions. □

We now apply this second lemma to the Galois covers $\pi:D_1\twoheadrightarrow E_1$, $\pi_2:C_{23}\twoheadrightarrow E_2$, and $\pi_3:C_{32}\twoheadrightarrow E_3$. The variety Y of the lemma is the variety Y_1 of the proof, and since $V_0(\omega_{Y_1},t)\subseteq V_0(\omega_X,a_X)$ (see §2.2), the hypotheses of the lemma are satisfied (Step 4).

We obtain $N_2=N_3$, hence the coverings $C_{ji}\twoheadrightarrow E_i$ and $C_{ki}\twoheadrightarrow E_i$ are the same (see (13)), and also $G/N_j\simeq\mathbf{Z}/2\mathbf{Z}$, so that they are double covers. Denote them by $C'_i\twoheadrightarrow E_i$. By the proof of Step 3 (see (10)), X is birational to $(C'_1\times C'_2\times C'_3)/(\mathbf{Z}/2\mathbf{Z})$. Since the latter variety contains no rational curves, there is a birational *morphism* from X to it. This finishes the proof of Theorem 5.1. □

6. Varieties with $P_1=1$

It follows from [U] and §2.1.5 that varieties X of maximal Albanese dimension and $P_1(X):=h^0(X,\omega_X)=1$ satisfy $\chi(X,\omega_X)=0$. We presented in Example 4.2 a construction of Chen and Hacon of such a variety which is in addition of general type. We gather here some properties of these varieties (most of them taken from [U]).

Proposition 6.1. *Let X be a smooth projective variety of maximal Albanese dimension n, with $P_1(X) = 1$.*

a) *We have an isomorphism*

$$a_X^* : \bigwedge^{\bullet} H^0(A, \Omega_A) \simeq H^0(X, \Omega_X^{\bullet}).$$

In particular, $h^j(X, \mathscr{O}_X) = \binom{n}{j}$ for all j, hence $\chi(X, \omega_X) = 0$, and the Albanese mapping $a_X : X \twoheadrightarrow \mathrm{Alb}(X)$ is surjective.

b) *The point 0 is isolated in $V_0(\omega_X, a_X)$.*

Proof. Replacing X with a modification, we may assume that there is a factorization $a_X : X \twoheadrightarrow Z \to \mathrm{Alb}(X)$, where Z is a desingularization of $a_X(X)$, so that $P_1(Z) \le P_1(X) = 1$. It follows from [U] (or [M], Corollary (3.5)) that $a_X(X)$ is a translate of an abelian subvariety of $\mathrm{Alb}(X)$, hence a_X is surjective. Item a) then follows from another result of Ueno ([U], or [M], Corollary (3.4)).

By §2.2, we can write $a_{X*}\omega_X \simeq \omega_A \oplus \mathscr{E} \simeq \mathscr{O}_A \oplus \mathscr{E}$. The sheaf $\mathscr{E}$ then satisfies $V_i(\mathscr{E}) - \{0\} = V_i(\omega_X, a_X) - \{0\}$ for all i. Since $1 = P_1(X) = 1 + h^0(A, \mathscr{E})$, the point 0 is not in the closed set $V_0(\mathscr{E})$, hence is isolated in $V_0(\omega_X, a_X)$. This proves b). □

Remark 6.2. Regarding item b), to be more precise, a smooth projective variety X of maximal Albanese dimension satisfies $P_1(X) = 1$ *if and only if* 0 is isolated in $V_0(\omega_X, a_X)$.

Theorem 6.3. *Let X be a smooth projective threefold of maximal Albanese dimension and of general type. If $P_1(X) = 1$, the variety X has an abelian étale cover which is a Chen-Hacon threefold.*

Proof. Replacing a_X with its Stein factorization, we will assume that X is normal and a_X is finite. By Theorem 5.1, applied to a desingularization of X, there exist elliptic curves E_1, E_2, and E_3, and a Cartesian diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{a_{\widetilde{X}}} & E_1 \times E_2 \times E_3 \\ \big\downarrow & \square & \big\downarrow \eta \\ X & \xrightarrow[a_X]{} & \mathrm{Alb}(X), \end{array}$$

where η is an isogeny (the variety $\widetilde{X}$ is the variety Z of Example 4.1) and both a_X and $a_{\widetilde{X}}$ are $(\mathbf{Z}/2\mathbf{Z})^2$-Galois coverings. In particular, X has rational singularities. Moreover,

$$a_{\widetilde{X}*}\omega_{\widetilde{X}} \simeq \mathscr{O}_{E_1\times E_2\times E_3} \oplus (p_1^*\delta_1 \otimes p_2^*\delta_2) \oplus (p_3^*\delta_3 \otimes p_1^*\delta_1) \oplus (p_2^*\delta_2 \otimes p_3^*\delta_3),$$

where the line bundle δ_i on E_i defines the double covering $\rho_i : C_i \twoheadrightarrow E_i$.

Let $\pi_i : \mathrm{Alb}(X) \twoheadrightarrow A_i$ be the quotient by the image F_i of $E_i \to \mathrm{Alb}(X)$. The natural morphism $h_i : X \twoheadrightarrow A_i$ is an isotrivial fibration and we denote by D_i its general (constant) fiber. We have a commutative diagram

$$\begin{array}{ccccc}
\widetilde{X} & \xrightarrow{a_{\widetilde{X}}} & E_1 \times E_2 \times E_3 & & \\
\downarrow & \square & \downarrow{\scriptstyle \lambda=(\lambda_1,\lambda_2,\lambda_3)} & & \\
Y & \xrightarrow{a_Y} & F_1 \times F_2 \times F_3 & & \\
\downarrow & \square & \downarrow & & \\
X & \xrightarrow[a_X]{} & \mathrm{Alb}(X) & \xrightarrow{\pi_i} & A_i
\end{array}$$

and, by §2.2, $a_{Y*}\omega_Y$ is the direct sum of $\mathscr{O}_{F_1\times F_2\times F_3}$ and translates of $p_1^*\mu_1 \otimes p_2^*\mu_2$, $p_3^*\mu_3 \otimes p_1^*\mu_1$, and $p_2^*\mu_2 \otimes p_3^*\mu_3$ by torsion points, for some line bundles μ_i on F_i.

Assume $P_1(Y) > 1$. Then, say, $\{0\} \times \widehat{F}_2 \times \widehat{F}_3$ is a component of $V_0(\omega_Y, a_Y)$. Consider the Stein factorization $Y \stackrel{\alpha}{\twoheadrightarrow} S_1 \stackrel{\beta}{\twoheadrightarrow} F_2 \times F_3$. We are in the situation of the proof of Theorem 3.1, and it shows that the surface S_1 is of general type, so that β has degree > 1. Let C_1 be a general fiber of α. On the one hand, the natural map $C_1 \twoheadrightarrow F_1$ has degree $4/\deg(\beta) < 4$; on the other hand, it factors through $D_1 \twoheadrightarrow F_1$, which has degree 4. This is a contradiction, hence $P_1(Y) = 1$.

We now claim that Y is a Chen-Hacon threefold. We write as above (see [P])

$$a_{Y*}\omega_Y = \bigoplus_{\chi \in (\mathbf{Z}/2\mathbf{Z})^{2*}} L_\chi = \mathscr{O}_{F_1\times F_2\times F_3} \oplus L_{\chi_1} \oplus L_{\chi_2} \oplus L_{\chi_3}, \tag{15}$$

where L_{χ_1}, L_{χ_2}, and L_{χ_3} are line bundles on $F_1 \times F_2 \times F_3$. By [P], Theorem 2.1, we have the following "building data": there are effective divisors D_1, D_2, and D_3 on $F_1 \times F_2 \times F_3$ satisfying

$$L_{\chi_i} + L_{\chi_j} \sim_{\mathrm{lin}} L_{\chi_k} + D_k \quad \text{and} \quad L_{\chi_i}^2 \sim_{\mathrm{lin}} D_j + D_k$$

for any $\{i, j, k\} = \{1, 2, 3\}$. These data pull back to the analogous building data on $\widetilde{X}$, hence $\lambda^* D_i$ is the pull-back on $E_1 \times E_2 \times E_3$ of the branch divisor $\Delta_i \sim_{\mathrm{lin}} 2\delta_i$ of ρ_i. It follows that there exists an ample line bundle δ'_i on F_i which pulls back to δ_i on E_i and such that D_i is also the pull-back on $F_1 \times F_2 \times F_3$ of a divisor $\Delta'_i \sim_{\mathrm{lin}} 2\delta'_i$ on F_i. Let L'_i be the pull-back on $F_1 \times F_2 \times F_3$ of δ'_i. Because of the relations $L_{\chi_i}^2 \sim_{\mathrm{lin}} D_j + D_k$, we can write

$$L_{\chi_i} \simeq P_{\xi_i} \boxtimes (L'_j \otimes P_{\xi_{i,j}}) \boxtimes (L'_k \otimes P_{\xi_{i,k}}),$$

where $\xi_i \in \widehat{E}_i$, $\xi_{i,j} \in \widehat{E}_j$, and $\xi_{i,k} \in \widehat{E}_k$ are 2-torsion points. From (15) and the equality $P_1(Y) = 1$, we deduce $H^0(F_1 \times F_2 \times F_3, L_{\chi_i}) = 0$, hence each ξ_i has order 2 and is in the kernel of $\widehat{\lambda}_i$. From the relations $L_{\chi_i} + L_{\chi_j} \sim_{\mathrm{lin}} L_{\chi_k} + D_k$, we deduce

$$\xi_{i,k} + \xi_{j,k} = \xi_k \quad \text{and} \quad \xi_{i,j} + \xi_j = \xi_{k,j}.$$

Since $\lambda_1^*\xi_1 = 0$, we may always change L'_1 to $L'_1 \otimes P_{\xi_1}$, so we may assume $\xi_{3,1} = 0$ and similarly, $\xi_{1,2} = 0$ and $\xi_{2,3} = 0$. The $\mathscr{O}_{F_1\times F_2\times F_3}$ algebra $a_{Y*}\mathscr{O}_Y$ is then the algebra associated with a Chen-Hacon threefold (see (5)). We conclude that Y is a Chen-Hacon threefold. $\square$

7. A conjecture

As mentioned in the introduction, we end this article with a conjecture on the possible general structure of smooth projective varieties X of maximal Albanese dimension, of general type, with $\chi(X, \omega_X) = 0$.

Conjecture. *Let X be a smooth projective variety of maximal Albanese dimension, of general type, with $\chi(X, \omega_X) = 0$. Then there exist a smooth projective variety X', a morphism $X' \twoheadrightarrow X$ which is a composition of modifications and abelian étale covers, and a fibration $g : X' \twoheadrightarrow Y$ with general fiber F, such that $0 < \dim(Y) < \dim(X)$ and*

a) *either g is isotrivial;*
b) *or $\chi(F, \omega_F) = 0$.*

Remarks 7.1. 1) Conversely, in the situation b) above, $\chi(X, \omega_X) = 0$ ([HP], Proposition 2.5). Moreover, $\mathrm{Alb}(X)$ has at least 4 simple factors by Corollary 3.4.b) and Proposition 4.5.b). Of course, in case a), without further constraints, one might have $\chi(X, \omega_X) > 0$, but we were unable to find necessary and sufficient conditions on the isotrivial fibration g (assuming X does not fall into case b)) to ensure $\chi(X, \omega_X) = 0$.

3) If we are *not* in case b), it follows from Lemma 4.6 that if $X' \twoheadrightarrow X$ is any composition of modifications and abelian étale covers, we have $q(X') = \dim(X)$ and any morphism from X' to a curve of genus ≥ 2 is constant.

The Ein-Lazarsfeld example (Example 4.1) falls into case a) of the conjecture, and not into case b) by Remark 7.1.1) above. We present an example that falls into case b), but not into case a). It is basically a non-isotrivial fibration whose general fibers are Ein-Lazarsfeld threefolds.

Example 7.2. Consider a smooth projective curve C of genus ≥ 2 and elliptic curves E_1, E_2, and E_3. For each $j \in \{1, 2, 3\}$, let L_j be an ample line bundle on $C \times E_j$ and let $D_j \in |2L_j|$ be a smooth divisor. For $\{i, j, k\} = \{1, 2, 3\}$, let $\mathscr{L}_i$ be the line bundle $p_j^* L_j \otimes p_k^* L_k$ on the variety $C \times E_1 \times E_2 \times E_3$, where p_i is the natural projection onto $C \times E_i$. We may assume that $p_1^* D_1 + p_2^* D_2 + p_3^* D_3$ is a simple normal crossing divisor. Considering the building data

$$\mathscr{L}_i + \mathscr{L}_j \sim_{\mathrm{lin}} \mathscr{L}_k + p_k^* D_k,$$

we get an $(\mathbf{Z}/2\mathbf{Z})^2$-Galois covering $Z \to C \times E_1 \times E_2 \times E_3$. A local computation shows that Z has rational singularities. Let $X \twoheadrightarrow Z$ be a desingularization. The variety X is of general type and has maximal Albanese dimension because $C \times E_1 \times E_2 \times E_3$ does. A general fiber of the fibration $X \twoheadrightarrow C$ is one of the examples constructed in Example 4.1, hence $\chi(X, \omega_X) = 0$ by [HP], Proposition 2.5, and X falls into case b) of the conjecture. One can prove that it does not fall into case a).

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