

## ON THE PERIOD MAP FOR PRIME FANO THREEFOLDS OF DEGREE 10

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### Abstract

We study, after Logachev, the geometry of smooth complex Fano threefolds  $X$  with Picard number 1, index 1, and degree 10, and their period map to the moduli space of 10-dimensional principally polarized abelian varieties. We prove that a general such  $X$  has no nontrivial automorphisms. By a simple deformation argument and a parameter count, we show that  $X$  is *not* birational to a quartic double solid, disproving a conjecture of Tyurin.

Through a detailed study of the variety of conics contained in  $X$ , a smooth projective irreducible surface of general type with globally generated cotangent bundle, we construct two smooth projective two-dimensional components of the fiber of the period map through a general  $X$ : one is isomorphic to the variety of conics in  $X$ , modulo an involution, another is birationally isomorphic to a moduli space of semistable rank-2 torsion-free sheaves on  $X$ , modulo an involution. The threefolds corresponding to points of these components are obtained from  $X$  via conic and line (birational) transformations. The general fiber of the period map is the disjoint union of an even number of smooth projective surfaces of this type.

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## 1. Introduction: two problems about Fano threefolds of degree 10

There are 17 families of smooth Fano threefolds with Picard number 1: the projective space  $\mathbf{P}^3$ , the smooth quadric  $Q \subset \mathbf{P}^4$ , the five families  $\mathcal{Y}_d$ ,  $d \in \{1, \dots, 5\}$  of Fano threefolds of index 2 and degree  $d$ , and the 10 families  $\mathcal{X}_{2g-2}$ ,  $g \in \{2, 3, \dots, 10, 12\}$  of Fano threefolds of index 1 and degree  $2g - 2$  (see table in [IsP], §12.2). We will denote by  $Y_d$  a Fano threefold belonging to the family  $\mathcal{Y}_d$ , and by  $X_{2g-2}$  a member of  $\mathcal{X}_{2g-2}$ .

Any threefold from the 8 families  $\mathbf{P}^3$ ,  $Q$ ,  $\mathcal{Y}_4$ ,  $\mathcal{Y}_5$ ,  $\mathcal{X}_{12}$ ,  $\mathcal{X}_{16}$ ,  $\mathcal{X}_{18}$ , and  $\mathcal{X}_{22}$  is rational.

For threefolds  $X$  from the remaining 9 families  $\mathcal{Y}_1$ ,  $\mathcal{Y}_2$  (quartic double solids),  $\mathcal{Y}_3$  (cubics), and  $\mathcal{X}_{2g-2}$ ,  $g \in \{2, \dots, 6, 8\}$ , there has been two approaches to proving or disproving rationality:

- studying the group  $\text{Bir}(X)$  of birational automorphisms of  $X$ . This group is known for  $X_2$ ,  $X_4$ ,  $X_6$ , and  $Y_1$  ([Is2], [G]);
- studying the intermediate Jacobian  $J(X)$  of  $X$ . This principally polarized abelian variety has been well-studied for  $Y_2$ ,  $Y_3$ , and  $X_8$ . In particular, the Torelli theorem holds ([CG], [V], [D]).

One important outcome is that any  $X$  from the above 7 families is not rational: in the first case because  $\text{Bir}(X)$  differs from the Cremona group  $\text{Bir}(\mathbf{P}^3)$ , and in the second case because  $J(X)$  is not a product of Jacobians of curves.

The two remaining families are  $\mathcal{X}_{14}$  and  $\mathcal{X}_{10}$ .

It was known to Fano that any  $X_{14}$  is birational to a smooth cubic threefold  $Y_3$ . In particular, no  $X_{14}$  is rational ([B], Theorem 5.6 (i)). Moreover, this implies that the 5-dimensional intermediate Jacobians  $J(X_{14})$  and  $J(Y_3)$  are isomorphic. Together with the Torelli theorem for cubic threefolds, this implies that the set of all  $X_{14}$  that are birational to a given cubic  $Y_3$  is an irreducible fivefold birational to  $J(Y_3)$  and that this set is the fiber through  $[X_{14}]$  of the period map  $\mathcal{X}_{14} \rightarrow \mathcal{A}_5$  ([IMr]).

Many geometrical properties of  $X_{10}$  were discovered in 1982 by Logachev (see [L], posted 22 years later), but neither is the group  $\text{Bir}(X_{10})$  known, nor has the intermediate Jacobian  $J(X_{10})$  been studied.

As noticed by Tyurin, the Fano threefold  $X_{10}$  shares certain properties with the quartic double solid  $Y_2$ . In particular, their intermediate Jacobians are both 10-dimensional.<sup>1</sup> Thinking that the situation would be analogous to the correspondences between  $X_{14}$  and  $Y_3$  described above, Tyurin stated the following ([T], p. 739).

**Conjecture** (Tyurin, 1979). *The general Fano threefold  $X_{10}$  is birational to a quartic double solid.*

We give a negative answer to this conjecture (Corollary 5.4).

Very little is known about the group  $\text{Bir}(X_{10})$  ([Is3], Problem 4 (b)). As far as we know, the only approach was initiated in the 80's by Khashin: in the short note [K], he describes birational involutions of  $X_{10}$  (associated with points, twisted cubics, and elliptic quartics contained in  $X_{10}$ ). Given a line  $\ell \subset X_{10}$  (resp. a conic  $c \subset X_{10}$ ), he also constructs a birational isomorphism  $\psi_\ell : X_{10} \dashrightarrow X_\ell$  (resp.  $\psi_c : X_{10} \dashrightarrow X_c$ ), where  $X_\ell$  (resp.  $X_c$ ) is also in  $\mathcal{X}_{10}$  (see §7.2 and §7.1).<sup>2</sup>

Since  $X_\ell$  and  $X_c$  have the same intermediate Jacobian as  $X_{10}$ , the matter of deciding whether they are isomorphic is of great interest for the Torelli problem. However, this problem remained unsolved for years ([IsP], Problem 11.4.2 (ii)).

**Problem** (Khashin, 1986). *Are  $X_\ell$  and  $X_c$  isomorphic to  $X_{10}$ ?*

<sup>1</sup>Unknown to Tyurin at the time, there are also numerical coincidences between the invariants of the so-called Fano surface  $F(Y_2)$  of lines on  $Y_2$  (computed by Welters in [W]), and those of (the minimal model of) the Fano surface  $F(X_{10})$  of conics on  $X_{10}$  (computed by Logachev in [L]): they both satisfy  $p_g = 101$ ,  $q = 10$ ,  $c_1^2 = 720$ , and  $c_2 = 384$ . This coincidences stem from the fact that a general  $Y_2$  is isomorphic to a *singular*  $X_{10}$  (see Proposition 5.2).

<sup>2</sup>Khashin also stated without proof that the existence of a birational isomorphism  $f$  between  $X_{10}$  and any other Fano threefold  $Y$  of index one, should imply that  $Y$  is also of type  $\mathcal{X}_{10}$ , and that  $f$  is a composition of birational isomorphisms of the five types described above.

We also give a negative answer to this problem, by proving that in general, neither  $X_\ell$ , nor  $X_c$ , is isomorphic to  $X_{10}$ .

Both answers are consequences of our study of the period map  $\wp : \mathcal{N}_{10} \rightarrow \mathcal{A}_{10}$ , which maps a point  $[X]$  in  $\mathcal{N}_{10}$  to its 10-dimensional intermediate Jacobian  $[J(X)] \in \mathcal{A}_{10}$ . We show (Theorem 7.4) that the set of all conic transforms  $X_c$ , as  $c$  varies in the Fano surface  $F(X_{10})$  of conics contained in  $X_{10}$ , forms a connected component  $\mathcal{F}_X$  of the fiber of  $\wp$  through  $[X_{10}]$  which is birationally isomorphic to the quotient of  $F(X_{10})$  by a geometrically meaningful involution  $\iota$ .

On the other hand, consider points of  $\mathcal{X}_{10}$  corresponding to general line transforms  $X_\ell$ . Their conic transforms form another 2-dimensional component  $\mathcal{F}_X^*$  of the fiber of  $\wp$  over  $[J(X_{10})]$  which is birationally isomorphic to the quotient by a geometrically meaningful involution of the moduli space  $\mathcal{M}_X(2; 1, 5)$  of stable rank-2 torsion-free sheaves on  $X_{10}$  with Chern numbers  $c_1 = 1$ ,  $c_2 = 5$ , and  $c_3 = 0$ , itself a smooth projective irreducible surface (Proposition 8.1 and Theorem 8.2).

We conjecture that a general fiber of the period map  $\wp$  is just the disjoint union of these two surfaces  $\mathcal{F}_X$  and  $\mathcal{F}_X^*$ . Although the study of certain nodal degenerations of elements of  $\mathcal{X}_{10}$  provide strong evidence for this conjecture ([DIM]), our lack of knowledge about the properness of  $\wp$  makes it very difficult to obtain more information about it at the moment. What we obtain here (with the help of a result from [DIM]) is that *the general fiber of  $\wp$  is the disjoint union of an even number of smooth projective irreducible surfaces* (Remark 7.5).

## 2. Notation

- As a general rule,  $V_m$  denotes an  $m$ -dimensional vector space, and  $\Gamma_d^g$  a degree- $d$  curve with geometric genus  $g$ . Fiber bundles are denoted by script letters; for example,  $\mathcal{S}_{r,V}$  is the rank- $r$  tautological subbundle on the Grassmannian  $G(r, V)$ .

- $V_5$  is a 5-dimensional complex vector space,  $V_{10} = \wedge^2 V_5$ , and the Plücker map embeds  $G(2, V_5)$  into  $\mathbf{P}_9 = \mathbf{P}(V_{10})$ .

- $V_8 \subset \wedge^2 V_5$  is a codimension-2 linear subspace whose orthogonal  $L \subset \wedge^2 V_5^\vee$  consists of skew forms on  $V_5$  that are all of maximal rank 4, and  $\mathbf{P}_7 = \mathbf{P}(V_8)$ .

- $W$  is the *smooth* 4-fold  $G(2, V_5) \cap \mathbf{P}_7$ , of degree 5 in  $\mathbf{P}_9$  (§3.2).

- $U_3 \subset V_5$  is the unique 3-dimensional subspace totally isotropic for all forms in  $V_8^\perp$ , with dual line  $L_U \subset \mathbf{P}(V_5^\vee)$ .

- $\Pi$  is the 2-plane  $G(2, U_3) \subset W$ .
- $\Omega$  is a quadric in  $\mathbf{P}_9$  such that  $X = W \cap \Omega = G(2, V_5) \cap \mathbf{P}_7 \cap \Omega \subset \mathbf{P}_9$  is a (smooth) Fano threefold (§4).
- $c_X = \Pi \cap \Omega$  is the unique  $\rho$ -conic on  $X$  (§3.1).
- $F_g(X)$  is the smooth connected surface that parametrizes conics on  $X$  (the letter  $g$  stands for “geometric”) (§6.1).
- $L_\sigma \subset F_g(X)$  is the curve of  $\sigma$ -conics in  $X$  (§6.1). It is exceptional, and if  $F_g(X) \rightarrow F_m(X)$  is its contraction,  $F_m(X)$  is a smooth minimal surface of general type.
- $F(X) = \{(c, V_4) \in F_g(X) \times \mathbf{P}(V_5^\vee) \mid c \subset G(2, V_4)\}$  is a smooth surface and  $p_1 : F(X) \rightarrow F_g(X)$  is the blow-up of the point corresponding to  $c_X$  (§6.1, §6.3).
- $\iota$  is the fixed-point-free involution on  $F(X)$  defined in §6.2. We set  $F_\iota(X) = F(X)/\iota$  and  $F_{m,\iota}(X) = F_m(X)/\iota$ . There is a diagram

$$\begin{array}{ccc}
 F(X) & \xrightarrow{\text{quotient by } \iota} & F_\iota(X) \\
 \downarrow \text{blow-up of } [c_X] \text{ } p_1 & & \downarrow \text{blow-up of } \pi([c_X]) \\
 F_g(X) & & \\
 \downarrow \text{blow-up of } \iota([c_X]) & & \\
 F_m(X) & \xrightarrow[\pi]{\text{quotient by } \iota} & F_{m,\iota}(X)
 \end{array}$$

- For any hyperplane  $V_4 \subset V_5$ , we let  $M_{V_4}$  be the 4-dimensional vector space  $\wedge^2 V_4 \cap V_8$ .
- $Q_{W,V_4}$  is the quadric surface  $G(2, V_4) \cap \mathbf{P}(M_{V_4})$ , contained in  $W$ .
- $Q_{\Omega,V_4}$  is the quadric surface  $\Omega \cap \mathbf{P}(M_{V_4})$ .

### 3. The fourfold $W$

Except for the cohomology calculations of §3.7, all the material in this section is either classical, or is due to Logachev ([L]).

**3.1. Lines, 2-planes, and conics in  $G(2, V_5)$ .** We denote by  $V_i$  an arbitrary subspace of  $V_5$  of dimension  $i$ .

All lines in  $G(2, V_5)$  are of the type:

$$\{[V_2] \mid V_1 \subset V_2 \subset V_3\}.$$

Any 2-plane in  $G(2, V_5)$  is of one of the following types:

- an  $\alpha$ -plane:  $\{[V_2] \mid V_1 \subset V_2 \subset V_4\}$ ;
- a  $\beta$ -plane:  $\{[V_2] \mid V_2 \subset V_3\}$ .

Any (possibly nonreduced or reducible) conic  $c$  in  $G(2, V_5)$  is of one of the following types:

- a  $\tau$ -conic: the 2-plane  $\langle c \rangle$  is not contained in  $G(2, V_5)$ , there is a unique hyperplane  $V_4 \subset V_5$  such that  $c \subset G(2, V_4)$ , the conic  $c$  is reduced and, if it is smooth, the union of the corresponding lines in  $\mathbf{P}(V_5)$  is a smooth quadric surface in  $\mathbf{P}(V_4)$ ;
- a  $\sigma$ -conic: the 2-plane  $\langle c \rangle$  is an  $\alpha$ -plane, there is a unique hyperplane  $V_4 \subset V_5$  such that  $c \subset G(2, V_4)$ , and the union of the corresponding lines in  $\mathbf{P}(V_5)$  is a quadric cone in  $\mathbf{P}(V_4)$ ;
- a  $\rho$ -conic: the 2-plane  $\langle c \rangle$  is a  $\beta$ -plane and the union of the corresponding lines in  $\mathbf{P}(V_5)$  is this 2-plane.

**3.2. The fourfold  $W$ .** Choose a linear space  $\mathbf{P}_7 = \mathbf{P}(V_8)$  of codimension 2 in  $\mathbf{P}_9 = \mathbf{P}(\wedge^2 V_5)$  whose dual pencil  $\mathbf{P}(V_8^\perp) \subset \mathbf{P}(\wedge^2 V_5^\vee)$  consists of skew forms on  $V_5$ , all of maximal rank 4. This is equivalent to saying that the intersection

$$W = G(2, V_5) \cap \mathbf{P}_7 \subset \mathbf{P}(\wedge^2 V_5)$$

is a smooth fourfold (of degree 5).

The map  $\mathbf{P}(V_8^\perp) \rightarrow \mathbf{P}(V_5)$  that sends a skew form to its kernel has image a smooth conic  $c_U$  that spans a 2-plane  $\mathbf{P}(U_3)$  ([PV], Proposition 6.3), where  $U_3 \subset V_5$  is the unique common maximal isotropic subspace for all forms in  $V_8^\perp$ . A normal form for matrices in the pencil  $\mathbf{P}(V_8^\perp)$  is given in [PV], Proposition 6.4; it shows that all  $W$  are isomorphic under the action of  $\mathrm{PGL}(V_5)$ .

More precisely, if we choose  $U_2$  such that  $V_5 = U_2 \oplus U_3$ , we have  $U_3 \simeq \mathrm{Sym}^2 U_2^\vee$ . As  $\mathfrak{sl}_2$ -modules,

$$\wedge^2 V_5 \simeq U_1 \oplus U_2 \oplus U_3 \oplus U_4,$$

where  $U_k$  is an irreducible  $\mathfrak{sl}_2$ -module of dimension  $k$ . In particular,  $\wedge^2 V_5$  contains a unique codimension-two submodule  $V_8 = U_1 \oplus U_3 \oplus U_4$ , which can also be defined as the kernel of the natural map

$$\wedge^2 V_5 \rightarrow U_2 \otimes U_3 \rightarrow U_2^\vee,$$

where the rightmost map is the contraction  $x \otimes q \mapsto q(x, \cdot)$ , for  $x$  in  $U_2$  and  $q$  a symmetric bilinear form on  $U_2$ . One can check that  $\mathbf{P}(V_8)$  meets the Grassmannian  $G(2, V_5)$  transversely, so that the intersection is our smooth fourfold  $W$ .

**3.3. 2-planes in  $W$ .** The 2-plane  $\Pi = G(2, U_3)$  is the unique  $\beta$ -plane of  $G(2, V_5)$  contained in  $W$ .

An  $\alpha$ -plane contained in  $W$  corresponds to a pair  $V_1 \subset V_4 \subset V_5$  such that  $\omega(v, w) = 0$  for all  $v \in V_1$ , all  $w \in V_4$ , and all  $\omega$  in the pencil  $\mathbf{P}(V_8^\perp)$ . It follows that  $[V_1]$  must be in  $c_U$  and  $V_4$  is the common orthogonal of  $v$  for all  $\omega$

in  $\mathbf{P}(V_8^\perp)$ . There is a line  $L_U \subset \mathbf{P}(V_5^\vee)$  of such 2-planes  $\Pi_{V_4}$ , corresponding to hyperplanes  $V_4 \subset V_5$  that contain  $U_3$ .

The intersection  $\Pi \cap \Pi_{V_4}$  is a line which is tangent to the conic  $c_U^\vee \subset \Pi$  dual to  $c_U$ , whereas two distinct  $\Pi_{V_4}$  and  $\Pi_{V_4'}$  meet at a point, which is on  $\Pi$ .

**3.4. The vector bundle  $\mathcal{M}$ .** For any hyperplane  $V_4 \subset V_5$ , the vector space  $M_{V_4} = \wedge^2 V_4 \cap V_8$  has dimension 4,<sup>3</sup> so this defines a rank-4 vector bundle

$$\mathcal{M} \rightarrow \mathbf{P}(V_5^\vee)$$

with fiber  $M_{V_4}$  at  $[V_4]$ . The 3-plane  $\mathbf{P}(M_{V_4})$  contains the quadric surface<sup>4</sup>

$$(3.1) \quad Q_{W, V_4} = G(2, V_4) \cap \mathbf{P}(M_{V_4}) \subset W$$

which is reducible if and only if it is of the form  $\Pi \cup \Pi_{V_4}$ ; this happens if and only if  $U_3 \subset V_4$ , i.e.,  $[V_4] \in L_U$ .

**3.5. Automorphisms of  $W$ .** Any automorphism of  $W$  is induced by an automorphism of  $\mathbf{P}(V_5)$  that maps the conic  $c_U$  onto itself ([PV], Theorem 1.2). The automorphism group of  $W$  has dimension 8 and fits into an exact sequence ([PV], Theorem 6.6)

$$(3.2) \quad 1 \rightarrow \mathbf{C}^4 \rtimes \mathbf{C}^* \rightarrow \text{Aut}(W) \rightarrow \text{Aut}(c_U) \simeq \text{PGL}(2, \mathbf{C}) \rightarrow 1.$$

For later use, we will describe this sequence more explicitly with the help of a decomposition  $V_5 = U_2 \oplus U_3$  as in §3.2. Let  $\text{Aut}^+(W) \subset \text{GL}(V_5)$  be the pull-back of  $\text{Aut}(W) \subset \text{PGL}(V_5)$ . As we saw in §3.2, any  $\varphi \in \text{Aut}^+(W)$  must preserve  $U_3$ , so it decomposes as

$$\varphi = \begin{pmatrix} \varphi_2 & 0 \\ \varphi_0 & \varphi_3 \end{pmatrix},$$

and the projection  $\varphi \mapsto \varphi_2$  induces the map  $\text{Aut}(W) \rightarrow \text{Aut}(c_U)$  in (3.2). It also needs to preserve  $V_8 \subset \wedge^2 V_5$ , and this implies that  $\varphi_3$  must be a nonzero multiple of  $\text{Sym}^2 \varphi_2$ , the map induced by  $\varphi_2$  on  $U_3 = \text{Sym}^2 U_2^\vee$ ; this nonzero multiple gives the  $\mathbf{C}^*$ -factor in the sequence above.

The  $\mathbf{C}^4$ -factor corresponds to the case where  $\varphi_2$  and  $\varphi_3$  are the identity, in which case  $\varphi_0 \in \text{Hom}(U_2, U_3) = \text{Hom}(U_2, \text{Sym}^2 U_2^\vee)$  must be such that

$$\forall u, v \in U_2 \quad \varphi_0(u)(v, \cdot) = \varphi_0(v)(u, \cdot).$$

This is equivalent to the condition that  $\varphi_0$  be completely symmetric, that is, belong to the image of the natural map  $\text{Sym}^3 U_2^\vee \rightarrow \text{Hom}(U_2, \text{Sym}^2 U_2^\vee)$ . In

<sup>3</sup>Otherwise, some form in the pencil  $V_8^\perp$  would vanish on  $V_4$  hence would have rank  $\leq 2$  ([L], Lemma 3.1).

<sup>4</sup>This is indeed a surface since  $\text{Pic}(W)$  is generated by  $\mathcal{O}_W(1)$ , hence all threefolds in  $W$  have degree divisible by 5.

particular, the  $\mathbf{C}^4$ -factor in the sequence above is a copy of  $\mathrm{Sym}^3 U_2^\vee$  as an  $\mathrm{SL}(2, \mathbf{C})$ -module.

Finally, the group  $\mathrm{Aut}(W)$  acts on  $W$  with four orbits ([PV], Proposition 6.8):

- $O_1 = c_U^\vee \subset \Pi$ ,
- $O_2 = \Pi - c_U^\vee$ ,
- $O_3 = \bigcup_{V_4 \in L_U} \Pi_{V_4} - \Pi$ ,
- $O_4 = W - O_3$ ,

where  $\dim(O_j) = j$ .

**3.6. Rationality of  $W$ .** Once we have chosen, as in §3.2, a decomposition  $V_5 = U_2 \oplus U_3$ , with  $U_3 = \mathrm{Sym}^2 U_2^\vee$ , we get an induced action of  $\mathrm{GL}(2, \mathbf{C})$  on  $W$ .

**Proposition 3.1.** *The smooth fourfold  $W$  is a compactification of the affine homogeneous space  $\mathrm{GL}(2, \mathbf{C})/\mathfrak{S}_3$ .*

*Proof.* If  $x$  is a general point in  $W \subset G(2, V_5)$ , the corresponding subspace in  $V_5$  maps isomorphically to  $U_2$ , and is therefore the graph of a map  $g_x : U_2 \rightarrow U_3$ . The map  $y \mapsto g_x(y)(y)$  is a general cubic form on  $U_2$ , and the stabilizer of  $x$  in  $\mathrm{GL}(2, \mathbf{C})$  must preserve the zeroes of this cubic form, hence be equal to the symmetric group  $\mathfrak{S}_3$ .  $\square$

As a consequence,  $W$  is rational. For the proof of Logachev's reconstruction theorem 9.1, we will need a precise description of an explicit birational isomorphism with  $\mathbf{P}^4$ . Recall that  $W$  sits in  $\mathbf{P}(V_8) = \mathbf{P}(U_1 \oplus U_3 \oplus U_4)$ .

**Proposition 3.2.** *The projection from  $\mathbf{P}(U_3)$  defines a birational isomorphism  $\kappa : W \dashrightarrow \mathbf{P}(U_1 \oplus U_4)$ . The inverse  $\kappa^{-1}$  is defined by the linear system of quadrics containing a rational normal cubic curve  $\Gamma_3^0 \subset \mathbf{P}(U_4) \subset \mathbf{P}(U_1 \oplus U_4)$ .*

*Proof.* A point in  $G(2, V_5)$  corresponds to a tensor of the form

$$(u + P) \wedge (v + Q) = u \wedge v + u \otimes Q - v \otimes P + P \wedge Q,$$

with  $u$  and  $v$  in  $U_2$ , and  $P$  and  $Q$  in  $U_3 \simeq \mathrm{Sym}^2 U_2^\vee$ . It belongs to  $W$  if and only if  $Q(u, \cdot) = P(v, \cdot)$  as linear forms on  $U_2$ . We must prove that the projection sending this tensor to  $u \wedge v + u \otimes Q - v \otimes P$  is generically injective. This follows from the fact there there is a natural map

$$\mathrm{Sym}^2 U_4 \hookrightarrow \mathrm{Sym}^2(U_2 \otimes U_3) \rightarrow \wedge^2 U_2 \otimes \wedge^2 U_3 \simeq U_3,$$

sending  $(u \otimes Q)(v \otimes P)$  to  $(u \wedge v)(P \wedge Q)$ , from which we can get the component  $P \wedge Q$  back.

Moreover, this shows that the inverse map is defined by the space of quadrics orthogonal to the kernel of this morphism in  $\mathrm{Sym}^2(U_1 \oplus U_4)^\vee$ , and these are precisely the quadrics containing the rational normal cubic curve  $\Gamma_3^0$  image of  $\mathbf{P}(U_2^\vee)$  in  $\mathbf{P}(U_4) = \mathbf{P}(\mathrm{Sym}^3 U_2^\vee)$ .  $\square$



The rational map  $\kappa$  is not defined along the 2-plane  $\Pi$ , whose total image is the hyperplane  $\mathbf{P}(U_4)$ . More precisely, through any point  $x$  in  $\Pi$ , there are two (possibly equal) lines tangent to  $c_U^\vee$ , which meet this conic at (possibly equal) points  $x_1$  and  $x_2$ , and the total image of  $x$  by  $\kappa$  is the line  $\ell_x$  bisecant (or tangent) to  $\Gamma_3^0$  through  $x_1$  and  $x_2$ .

If a hyperplane  $V_4 \subset V_5$  does not contain  $U_3$ , the quadric surface  $Q_{W,V_4}$  (see (3.1)) meets  $\Pi$  at the unique point  $[U_3 \cap V_4]$ , so the restriction of  $\kappa$  to  $Q_{W,V_4}$  sends it to a 2-plane in  $\mathbf{P}(U_1 \oplus U_4)$ . If  $V_4$  does contain  $U_3$ , that is, if  $V_4 = V_1^\perp$  for some line  $V_1 \subset U_2$  defining a point in  $c_U$ , the linear span of  $Q_{W,V_4}$  is mapped to the tangent line to the corresponding point of  $\Gamma_3^0$ .

**3.7. Cohomology calculations.** We gather here various results on the cohomology groups of some sheaves of twisted differentials on  $W$ .

**Proposition 3.3.** *We have*

$$(3.3) \quad H^3(W, \Omega_W^1(-3)) = 0,$$

$$(3.4) \quad H^q(W, \Omega_W^1(-2)) \simeq \begin{cases} 0 & \text{if } q \neq 3, \\ L & \text{if } q = 3, \end{cases}$$

$$(3.5) \quad H^q(W, \Omega_W^2(-1)) \simeq \begin{cases} 0 & \text{if } q \neq 3, \\ V_5/U_3 & \text{if } q = 3. \end{cases}$$

*Proof.* Set  $G = G(2, V_5)$ . By the Bott-Borel-Weil theorem, the sheaves  $\Omega_G^1(-r)$  for  $1 \leq r \leq 4$ , and  $\Omega_G^2(-r)$  for  $1 \leq r \leq 2$ , are acyclic, whereas

$$H^q(G, \Omega_G^2(-3)) = \begin{cases} 0 & \text{if } q \neq 5, \\ V_5 & \text{if } q = 5. \end{cases}$$

Using the Koszul resolution

$$0 \rightarrow \wedge^2 L \otimes \mathcal{O}_G(-2) \rightarrow L \otimes \mathcal{O}_G(-1) \rightarrow \mathcal{O}_G \rightarrow \mathcal{O}_W \rightarrow 0,$$

we obtain that  $\Omega_G^1(-2)|_W$  is acyclic, and

$$(3.6) \quad H^q(W, \Omega_G^2(-1)|_W) \simeq H^{q+2}(G, \Omega_G^2(-3)) \otimes \wedge^2 L \simeq \begin{cases} 0 & \text{if } q \neq 3, \\ \wedge^4 V_5^\vee & \text{if } q = 3. \end{cases}$$

Using the conormal sequence

$$(3.7) \quad 0 \rightarrow L \otimes \mathcal{O}_W(-1) \rightarrow \Omega_G^1|_W \rightarrow \Omega_W^1 \rightarrow 0,$$

we obtain

$$H^q(W, \Omega_W^1(-2)) \simeq H^{q+1}(W, \mathcal{O}_W(-3)) \otimes L$$

and (3.4) follows.

The vanishing of  $H^3(W, \Omega_W^1(-3))$  is a little less straightforward. From (3.7), we deduce an exact sequence

$$0 \rightarrow H^3(W, \Omega_W^1(-3)) \rightarrow L \otimes H^4(W, \mathcal{O}_W(-4)) \rightarrow H^4(W, \Omega_G^1(-3)|_W).$$

The vector space  $H^4(W, \mathcal{O}_W(-4))$  is Serre-dual to  $H^0(W, \mathcal{O}_W(1)) = \wedge^2 V_5^\vee / L$ . The acyclicity of  $\Omega_G^1(-r)$  for  $r \in \{3, 4\}$  yields

$$H^4(W, \Omega_G^1(-3)|_W) \simeq H^6(G, \Omega_G^1(-5)),$$

which is Serre-dual to  $H^0(G, T_G) \simeq \mathfrak{sl}(V_5)$ . By duality, we are therefore reduced to verifying that the natural map

$$\mathfrak{sl}(V_5^\vee) \rightarrow \text{Hom}(L, \wedge^2 V_5^\vee / L)$$

is surjective. This is checked by an explicit computation, and proves (3.3).

The exact sequence (3.7) also induces exact sequences

$$(3.8) \quad 0 \rightarrow \mathcal{K} \rightarrow \Omega_G^2|_W \rightarrow \Omega_W^2 \rightarrow 0,$$

$$(3.9) \quad 0 \rightarrow \wedge^2 L \otimes \mathcal{O}_W(-2) \rightarrow \mathcal{K} \rightarrow L \otimes \Omega_W^1(-1) \rightarrow 0.$$

From (3.4) and (3.9), it follows that the cohomology groups of  $\mathcal{K}(-1)$  are 0 in degrees 0, 1, and 2, and there is an exact sequence

$$0 \rightarrow H^3(W, \mathcal{K}(-1)) \rightarrow L \otimes L \rightarrow \wedge^2 L \rightarrow H^4(W, \mathcal{K}(-1)) \rightarrow 0.$$

Therefore  $H^q(W, \mathcal{K}(-1)) \simeq \begin{cases} 0 & \text{if } q \neq 3 \\ \text{Sym}^2 L & \text{if } q = 3 \end{cases}$ . With (3.8) and (3.6), we

obtain that  $\Omega_W^2(-1)$  has zero cohomology in degree 0, 1, and 4, and that there is an exact sequence

$$0 \rightarrow H^2(W, \Omega_W^2(-1)) \rightarrow \text{Sym}^2 L \xrightarrow{\psi} \wedge^4 V_5^\vee \rightarrow H^3(W, \Omega_W^2(-1)) \rightarrow 0.$$

The map  $\psi$  is the obtained from the inclusion  $L \hookrightarrow \wedge^2 V_5^\vee$  and the natural map  $\text{Sym}^2(\wedge^2 V_5^\vee) \rightarrow \wedge^4 V_5^\vee$  defined by the wedge product. One checks that the noninjectivity of  $\psi$  would imply the existence of a rank-2 form in  $L$ , which is not the case. Also, if we identify  $\wedge^4 V_5^\vee$  with  $V_5$ , the image of  $\psi$  is the subspace we denoted by  $U_3$ . We have therefore proved (3.5).  $\square$

## 4. The Fano threefold $X$

**4.1. The two types of Fano threefolds of type  $\mathcal{X}_{10}$ .** Let  $X$  be a smooth projective complex threefold with Picard group  $\mathbf{Z}[K_X]$  and  $(-K_X)^3 = 10$ . The linear system  $|-K_X|$  is very ample and embeds  $X$  in  $\mathbf{P}^7$  as a smooth subvariety of degree 10. Gushel and Mukai proved that it is of one of the

following types described below ([Gu], §4; [IsP], Corollary 4.1.13 and Theorem 5.1.1).

Let  $V_5$  be a 5-dimensional vector space. Let  $G(2, V_5) \subset \mathbf{P}(\wedge^2 V_5)$  be the Grassmannian in its Plücker embedding and denote by  $\mathbf{P}_m$  a linear subspace of  $\mathbf{P}(\wedge^2 V_5)$  of dimension  $m$ . Then:

- either

$$(4.1) \quad X = G(2, V_5) \cap \mathbf{P}_7 \cap \Omega = W \cap \Omega \subset \mathbf{P}_7,$$

where  $\Omega$  is a quadric;

- or  $X$  is the intersection in  $\mathbf{P}_7$  of a cone over  $G(2, V_5) \cap \mathbf{P}_6$  with a quadric.

The second case (“Gushel threefolds”) is a degeneration of the first case.<sup>5</sup> Gushel threefolds were studied in [I].

We will only consider threefolds of the first type, which we denote by  $\mathcal{X}_{10}^0$ . They depend on 22 parameters (see §4.4), whereas Gushel threefolds depend on only 19 parameters.

**4.2. The moduli space  $\mathcal{M}_X(2; 1, 4)$ .** Let us recall how the embedding of  $X$  into  $G(2, V_5)$  was obtained by Gushel and Mukai. There exists on  $X$  a smooth elliptic quartic curve  $\Gamma_4^1$  whose linear span is 3-dimensional ([IsP], Lemma 5.1.2). Serre’s construction yields a rank-2 vector bundle  $\mathcal{E}$  on  $X$  with a section vanishing exactly on  $\Gamma_4^1$  fitting into an extension

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \mathcal{I}_{\Gamma_4^1}(1) \rightarrow 0.$$

We have  $c_1(\mathcal{E}) = [\mathcal{O}_X(1)]$  and  $c_2(\mathcal{E}) = [\Gamma_4^1]$ . Moreover,  $h^0(X, \mathcal{E}) = 5$ ,  $H^i(X, \mathcal{E}) = 0$  for  $i > 0$ , and  $H^i(X, \mathcal{E}^\vee) = 0$  for  $i \geq 0$ . In particular  $H^0(X, \mathcal{E}^\vee) = H^0(X, \mathcal{E}(-1)) = 0$ , so  $\mathcal{E}$  is stable since  $\text{Pic}(X) = \mathbf{Z}[\mathcal{O}_X(1)]$ . By [IsP], Lemma 5.1.3,  $\mathcal{E}$  is globally generated and defines a morphism of  $X$  into  $G(2, H^0(X, \mathcal{E})^\vee) = G(2, V_5)$  such that  $\mathcal{E}$  is the restriction to  $X$  of the dual tautological rank-2 bundle  $\mathcal{S}_{2, V_5}^\vee$  on  $G(2, V_5)$ .<sup>6</sup>

**Proposition 4.1.** *Let  $X$  be any smooth Fano threefold of type  $\mathcal{X}_{10}$ . The vector bundle  $\mathcal{E}$  is the unique stable rank-2 vector bundle on  $X$  with Chern numbers  $c_1(\mathcal{E}) = 1$  and  $c_2(\mathcal{E}) = 4$ .*

*Proof.* We follow [M]. Let  $\mathcal{F}$  be a vector bundle on  $X$  with the same properties as  $\mathcal{E}$ . Let  $\mathcal{H} = \mathcal{H}om(\mathcal{E}, \mathcal{F})$ . A general hyperplane section  $S$  of  $X$  is a K3 surface, and again  $\text{Pic}(S) = \mathbf{Z}[\mathcal{O}_S(1)]$ .

<sup>5</sup>Intersect, in  $\mathbf{P}^{10}$ , a cone over  $G(2, V_5)$  with a quadric and a  $\mathbf{P}_7$ ; if the  $\mathbf{P}_7$  does not pass through the vertex of the cone, we are in the first case; if it does, we are in the second case. Gushel threefolds are missing in Iskovskikh’s 1977 classification of Fano threefolds with Picard number 1 (see tables in [Is1], p. 505).

<sup>6</sup>This morphism is an embedding, except for Gushel threefolds (§4.1), for which it induces a double cover of a del Pezzo threefold  $Y_5 \subset \mathbf{P}^6$  ([IsP], §5.1).

The restriction  $\mathcal{H}|_S$  has Chern numbers  $c_1(\mathcal{H}|_S) = 0$  and  $c_2(\mathcal{H}|_S) = 4c_2(\mathcal{E}|_S) - c_1(\mathcal{E}|_S)^2 = 6$ . So by the Riemann-Roch theorem,

$$\chi(S, \mathcal{H}|_S) = 4\chi(\mathcal{O}_S) + \frac{1}{2}(c_1(\mathcal{H}|_S)^2 - 2c_2(\mathcal{H}|_S)) = 2.$$

In particular  $h^0(S, \mathcal{H}om(\mathcal{E}|_S, \mathcal{F}|_S))$  or  $h^0(S, \mathcal{H}om(\mathcal{F}|_S, \mathcal{E}|_S))$  must be non-zero. Since  $\mathcal{E}|_S$  and  $\mathcal{F}|_S$  are stable bundles with the same slope, this implies that they are isomorphic.

We want to prove that any isomorphism between  $\mathcal{E}|_S$  and  $\mathcal{F}|_S$  extends to  $X$ , by showing  $H^1(X, \mathcal{H}(-1)) = 0$ . We prove by descending induction on  $k$  that  $H^1(X, \mathcal{H}(-k))$  vanishes for all  $k > 0$ . Because of the exact sequence  $0 \rightarrow \mathcal{O}_X(-1) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_S \rightarrow 0$ , we just need to check that  $H^1(S, \mathcal{H}|_S(-k))$  vanishes for all  $k > 0$ .

Now, since  $\mathcal{E}$  and  $\mathcal{F}$  are both isomorphic on  $S$  to the restriction of  $\mathcal{S}_{2, V_5}^\vee$ , we have  $\mathcal{H}|_S = \mathcal{E}nd(\mathcal{S}_{2, V_5}^\vee)|_S$ . Since  $S$  is a codimension-4 subscheme of  $G(2, V_5)$  defined by the vanishing of a section of the vector bundle  $\mathcal{N} = \mathcal{O}_{G(2, V_5)}(1)^3 \oplus \mathcal{O}_{G(2, V_5)}(2)$ , we just need to check that  $H^{i+1}(G(2, V_5), \mathcal{E}nd(\mathcal{S}_{2, V_5}^\vee) \otimes \wedge^i \mathcal{N}^\vee)$  vanishes for  $0 \leq i \leq 4$ . This is an easy consequence of the Bott-Borel-Weil theorem.  $\square$

**Corollary 4.2.** *Any isomorphism between two Fano threefolds of type  $\mathcal{X}_{10}^0$  is induced by an automorphism of  $\mathbf{P}(V_5)$  preserving  $W$ .*

**4.3. Automorphisms.** We prove that a general Fano threefold of type  $\mathcal{X}_{10}$  has no nontrivial automorphisms.

**Theorem 4.3.** *For any smooth Fano threefold  $X$  of type  $\mathcal{X}_{10}^0$ , we have*

$$H^i(X, T_X) = 0 \quad \text{for } i \neq 1.$$

*In particular, the group of automorphisms of  $X$  is finite.*

*Proof.* For  $i \geq 2$ , this follows from the Kodaira-Akizuki-Nakano (KAN) vanishing theorem since  $T_X \simeq \Omega_X^2(1)$ . The conormal exact sequence

$$(4.2) \quad 0 \rightarrow \mathcal{O}_X(-2) \rightarrow \Omega_W^1|_X \rightarrow \Omega_X^1 \rightarrow 0$$

induces a resolution of  $\Omega_X^2$ , hence of  $T_X \simeq \Omega_X^2(1)$ :

$$(4.3) \quad 0 \rightarrow \Omega_X^1(-1) \rightarrow \Omega_W^2(1)|_X \rightarrow T_X \rightarrow 0.$$

By KAN vanishing again, we deduce  $H^0(X, \Omega_W^2(1)|_X) \simeq H^0(X, T_X)$ . Since  $X$  is a quadratic section of  $W$ , there is an exact sequence

$$0 \rightarrow \Omega_W^2(-1) \rightarrow \Omega_W^2(1) \rightarrow \Omega_W^2(1)|_X \rightarrow 0.$$

Using KAN vanishing again (on  $W$ ), we obtain an isomorphism  $H^0(W, \Omega_W^2(1)) \simeq H^0(X, \Omega_W^2(1)|_X)$ . Since  $\Omega_W^2(1)$  is Serre-dual to  $\Omega_W^2(-1)$ , the theorem follows from (3.5).  $\square$

**Theorem 4.4.** *A general Fano threefold of type  $\mathcal{X}_{10}$  has no nontrivial automorphisms.*

*Proof.* Assume  $\varphi \in \mathrm{GL}(V_5)$  induces a nontrivial automorphism of  $W$ . We want to prove that the space of polynomials  $Q \in H^0(\mathbf{P}_7, \mathcal{O}_{\mathbf{P}_7}(2)) = \mathrm{Sym}^2 V_8^\vee$  such that  $\varphi$  preserves  $X = \mathbf{P}_7 \cap \{Q = 0\}$  has codimension bigger than  $8 = \dim(\mathrm{Aut}(W))$ .

This condition is equivalent to the fact that there exists a nonzero scalar  $z$  such that  $\varphi^*Q = zQ + P$  for some  $P \in H^0(\mathbf{P}_7, \mathcal{S}_W(2))$ . So the dimension we are interested in is controlled by the dimension of the eigenspaces of  $\varphi^*$ . To estimate the dimensions of these eigenspaces, we may by semicontinuity assume that the image of  $\varphi$  in  $\mathrm{PGL}(2, \mathbf{C})$  (see (3.2)) is the identity. The eigenvalues of  $\varphi$  on  $V_5$  are then 1, with multiplicity 2, and some  $\zeta^{-1}$  with multiplicity 3. The eigenvalues of  $\varphi^*$  on  $\mathrm{Sym}^2 V_8^\vee$  are thus 1,  $\zeta$ ,  $\zeta^2$ ,  $\zeta^3$ ,  $\zeta^4$ , with respective multiplicities 1, 4, 13, 12, and 6. These eigenvalues can coincide for certain values of  $\zeta$ , but it is easy to check that for  $\zeta \neq 1$ , no eigenspace can have dimension  $> 20$ . So the maximal number of parameters for  $Q$  is

$$20 + h^0(\mathbf{P}_7, \mathcal{S}_W(2)) = 25 < 36 - 8.$$

Suppose now  $\zeta = 1$ . As we saw in §3.2, this means that in a decomposition  $V_5 = U_2 \oplus U_3$ , we have

$$\varphi = \begin{pmatrix} I_2 & 0 \\ \varphi_0 & I_3 \end{pmatrix},$$

with  $\varphi_0$  in  $\mathrm{Sym}^3 U_2^\vee \subset \mathrm{Hom}(U_2, \mathrm{Sym}^2 U_2^\vee)$  completely symmetric (but non-zero). By semicontinuity again, we may suppose that  $\varphi_0$  is of the form  $\ell^3$  for some linear form  $\ell$  on  $U_2$ , so that

$$\forall u, x, y \in U_2 \quad \varphi_0(u)(x, y) = \ell(u)\ell(x)\ell(y).$$

Thus  $\varphi_0$  has rank 3, and a straightforward computation shows that the endomorphism  $\Phi$  of  $\mathrm{Sym}^2 V_8^\vee$  induced by  $\varphi$  is such that  $\mathrm{Id} - \Phi$  has rank 18. Therefore we get at least  $18 - 5 > 8$  conditions on  $Q$ . This concludes the proof.  $\square$

**Remark 4.5.** The proof shows that Fano threefolds of type  $\mathcal{X}_{10}^0$  with nontrivial automorphisms form a family of codimension  $\geq 3$ .

**4.4. Moduli.** The most natural way to define an moduli “space” for Fano varieties of type  $\mathcal{X}_{10}^0$  would be as the quotient of an open subset of  $|\mathcal{O}_W(2)|$  by the action of  $\mathrm{Aut}(W)$ . However, the latter group being nonreductive, the question of whether this quotient is a scheme is difficult to settle. On the other hand, it is not difficult to construct this same “space” as the quotient of a quasi-projective variety by the action of the (reductive) group  $\mathrm{SL}(V_5)$ , but again, it is not clear whether this subset corresponds to (semi)stable points

for some polarization, or even that the action is proper. We hope to come back to this interesting question in the future, but for the time being, we will just consider the algebraic *stack*  $\mathcal{N}_{10}$  of Fano threefolds of type  $\mathcal{X}_{10}^0$ . It is an algebraic, smooth, irreducible stack of dimension

$$\dim |\mathcal{O}_W(2)| - \dim(\text{Aut}(W)) = 30 - 8 = 22.$$

This will be sufficient for our purposes. Locally around a point corresponding to a threefold  $X$ , this stack is given by the local Kuranishi space of  $X$ , a smooth 22-dimensional variety, acted on by the finite group  $\text{Aut}(X)$ .

### 5. The period map

The period map  $\varphi : \mathcal{N}_{10} \rightarrow \mathcal{A}_{10}$  sends a point  $[X]$  in  $\mathcal{N}_{10}$  to its 10-dimensional principally polarized intermediate Jacobian  $[J(X)] \in \mathcal{A}_{10}$ .

**5.1. The differential of the period map.** Let  $X$  be a Fano threefold of type  $\mathcal{X}_{10}^0$ . The differential of  $\varphi$  at the point defined by  $X$  is the map

$$d\varphi : H^1(X, T_X) \rightarrow \text{Hom}(H^{1,2}(X), H^{2,1}(X))$$

defined by the natural pairing  $H^1(X, T_X) \otimes H^1(X, \Omega_X^2) \rightarrow H^2(X, \Omega_X^1)$ . We want to describe the kernel of  $d\varphi$ .

**Theorem 5.1.** *The kernel of  $d\varphi$  is naturally identified with the quotient  $V_5/U_3$ . In particular  $\varphi$  is smooth and its image has dimension 20.*

*Proof.* Following [F], we use the long exact sequences induced by the exact sequences (4.2) and (4.3) to get a commutative diagram

$$(5.1) \quad \begin{array}{ccccc} H^1(X, \Omega_W^2(1)|_X) & \otimes & H^1(X, \Omega_X^2) & \rightarrow & H^2(X, \Omega_W^1|_X) \\ \downarrow & & \parallel & & \downarrow \\ H^1(X, T_X) & \otimes & H^1(X, \Omega_X^2) & \rightarrow & H^2(X, \Omega_X^1) \\ \downarrow & & \parallel & & \downarrow \\ H^2(X, \Omega_X^1(-1)) & \otimes & H^1(X, \Omega_X^2) & \rightarrow & H^3(X, \mathcal{O}_X(-2)). \end{array}$$

If we set  $G = G(2, V_5)$ , the kernel of the restriction map

$$\wedge^2 V_5^\vee = H^0(G, \mathcal{O}_G(1)) \rightarrow H^0(W, \mathcal{O}_W(1))$$

is  $V_8^\perp$ . From the normal sequences of  $X$  in  $W$  and  $W$  in  $G$ , we deduce isomorphisms

$$H^2(X, \Omega_W^1|_X) \simeq H^3(W, \Omega_W^1(-2)) \simeq H^4(W, V_8^\perp \otimes \mathcal{O}_W(-3)) \simeq V_8^\perp$$

(because  $\omega_W = \mathcal{O}_W(-3)$ ). On the other hand,

$$H^3(X, \mathcal{O}_X(-2)) \simeq H^0(X, \mathcal{O}_X(1))^\vee \simeq H^0(W, \mathcal{O}_W(1))^\vee \simeq V_8.$$

The rightmost column of the diagram (5.1) is therefore exact since the top arrow is injective (because  $H^2(X, \mathcal{O}_X(-2)) = 0$  by Kodaira vanishing) and  $H^2(X, \Omega_X^1)$  is 10-dimensional.

We check that the pairing of the bottom row of (5.1) is nondegenerate with respect to the first factor. By [F], Lemma 2.9, this follows from the fact that the following diagram commutes up to sign:

$$\begin{array}{ccccc} H^2(X, \Omega_X^1(-1)) & \otimes & H^1(X, \Omega_X^2) & \rightarrow & H^3(X, \mathcal{O}_X(-2)) \\ \downarrow & & \uparrow & & \parallel \\ H^3(X, \mathcal{O}_X(-3)) & \otimes & H^0(X, \mathcal{O}_X(1)) & \rightarrow & H^3(X, \mathcal{O}_X(-2)). \end{array}$$

Indeed, the last row of this diagram is Serre-dual to the multiplication map  $H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \mathcal{O}_X(1)) \rightarrow H^0(X, \mathcal{O}_X(2))$ ; in particular, it is nondegenerate with respect to the first factor, and the same conclusion follows for the first row because the map

$$H^2(X, \Omega_X^1(-1)) \rightarrow H^3(X, \mathcal{O}_X(-3))$$

is injective. To prove that, we need to check that  $H^2(X, \Omega_W^1(-1)|_X)$  vanishes. But this follows from the normal sequence of  $X$  in  $W$ , (3.5), and (3.3).

We can now conclude that the kernel of  $d\varphi$  at  $[X]$  is contained in the image of  $H^1(X, \Omega_W^2(1)|_X)$  in  $H^1(X, T_X)$ . We compute the dimension of this space. From the normal sequence of  $X$  in  $W$ , we get an exact sequence

$$\begin{aligned} H^1(W, \Omega_W^2(-1)) &\rightarrow H^1(W, \Omega_W^2(1)) \\ &\rightarrow H^1(X, \Omega_W^2(1)|_X) \rightarrow H^2(W, \Omega_W^2(-1)) \end{aligned}$$

whose extreme terms vanish by (3.5). Moreover,  $H^1(W, \Omega_W^2(1))$  is dual to  $H^3(W, \Omega_W^2(-1))$ , which is 2-dimensional by (3.5) again.

Since we already know that the kernel of  $d\varphi$  has dimension at least 2, we can conclude that this kernel is exactly  $H^1(X, \Omega_W^2(1)|_X)$ . We have therefore proved the theorem.  $\square$

The theorem implies that fibers of  $\varphi$  are smooth surfaces. We will give a more precise description of these fibers in Theorem 7.4.

**5.2. Fano threefolds of degree 10 and quartic double solids.** Quartic double solids are double covers of  $\mathbf{P}^3$  branched along a smooth quartic surface. They have Picard number 1, index 2, and form the family  $\mathcal{B}_2$  described in the introduction. They were extensively studied by Clemens in [C], who computed, among many other things, that their intermediate Jacobian has dimension 10.

**Proposition 5.2.** *A general quartic double solid is birational to a 2-dimensional family of nodal Fano threefolds which are degenerations of smooth threefolds of type  $\mathcal{X}_{10}$ .*

*Proof.* This is a construction that can be found in [BCZ] §4.4.1 and [CSP], Example 1.11, to which we refer for the proofs. Let  $\pi : Y \rightarrow \mathbf{P}^3$  be a quartic double solid, with involution  $\iota$ , and let  $v : Y \rightarrow \mathbf{P}^{10}$  be the morphism defined by the anticanonical linear system  $| -K_Y | = |2\pi^*H|$ . Choose any line  $\ell \subset Y$  (there is a 2-dimensional family of such lines). The projection of  $v(Y) \subset \mathbf{P}^{10} \dashrightarrow \mathbf{P}^7$  from the 2-plane spanned by the (conic) image of  $\ell$  is a singular Fano threefold  $\bar{X} \subset \mathbf{P}^7$ , obtained as the image of the blow-up  $\varepsilon : \tilde{Y} \rightarrow Y$  of  $\ell$ , with exceptional divisor  $E$ , by the morphism  $\varphi$  associated with the anticanonical linear system  $| -K_{\tilde{Y}} | = |2\varepsilon^*\pi^*H - E|$  (compare with diagram (7.1)). This morphism only contracts the curve  $\iota(\ell)$  and  $\bar{X}$  is a singular Fano threefold of degree 10 with one node (from the family  $(T_7)$  of [CSP], Theorem 1.6).

By [N],  $\bar{X}$  can be deformed into a smooth Fano threefold  $X$  of degree 10, whose Picard number is 1 by [JR]; it is therefore of type  $\mathcal{X}_{10}$ .  $\square$

Keeping the notation of the proof, lines on  $Y$  map to conics on  $\bar{X}$ . This gives a birational map  $F(Y) \dashrightarrow F(\bar{X})$  which explains the coincidences between the numerical invariants of the minimal surface  $F(Y)$  and those of the minimal model of  $F(X)$  (see footnote 1).

**Corollary 5.3.** *The endomorphism ring of the intermediate Jacobian of a very general Fano threefold of type  $\mathcal{X}_{10}$  is isomorphic to  $\mathbf{Z}$ .*

*Proof.* By Proposition 5.2, the intermediate Jacobian of a quartic double solid is a degeneration of intermediate Jacobians of Fano threefolds of type  $\mathcal{X}_{10}$ . By [C], Theorem (5.67), intermediate Jacobians of quartic double solids degenerate in turn to semi-abelian varieties which are extensions by  $\mathbf{C}^*$  of Jacobians of complete intersection curves  $C \subset \mathbf{P}^3$  of bidegree  $(2, 4)$ , where the extension class  $\varepsilon$  is the difference between the two  $g_4^1$  on  $C$  ([C], end of §7). In general, the endomorphism ring of  $J(C)$  is trivial ([CvG]) and  $\varepsilon$  has infinite order. This proves the corollary.  $\square$

**Corollary 5.4.** *A general Fano threefold of type  $\mathcal{X}_{10}$  is not birationally isomorphic to a smooth quartic double solid.*

*Proof.* Let  $X$  be a general Fano threefold of type  $\mathcal{X}_{10}$ . It was observed by Clemens and Griffiths ([CG]; see also [IsP], §8.1) that the Griffiths component  $J_G(X)$  (the product of the principally polarized factors of  $J(X)$  that are *not* Jacobians of curves) is a birational invariant of  $X$ .

It follows from the proof of Corollary 5.3 that  $J(X)$  can degenerate to the intermediate Jacobian of a general quartic double solid, whose theta divisor has a singular locus of codimension 5 ([D], th. (8.1)). It follows that the singular locus of the theta divisor of  $J(X)$  has codimension  $\geq 5$ , and in particular,  $J_G(X) = J(X)$ .



Assume now that  $X$  is birationally isomorphic to a smooth quartic double solid  $Y$ . We then have  $J(Y) = J_G(Y) \simeq J_G(X) = J(X)$ . But this is impossible, since the  $J(X)$  form a 20-dimensional family (Theorem 5.1), whereas quartic double solids form a 19-dimensional family.  $\square$

## 6. The varieties of conics contained in $X$

Most of the material in this section is due to Logachev ([L]).

**6.1. The surfaces  $F_g(X)$  and  $F(X)$ .** Let  $X$  be a Fano threefold of type  $\mathcal{X}_{10}^0$ . Let  $F_g(X)$  be the variety of (possibly nonreduced or reducible) conics contained in  $X$ . It follows from deformation theory and [IsP], Proposition 4.2.5.(iii), that  $F_g(X)$  has dimension 2 everywhere.

As seen in §3.1, any conic  $c$  in  $G(2, V_5)$  is contained in some  $G(2, V_4)$ , where the hyperplane  $V_4 \subset V_5$  is uniquely determined by  $c$  unless  $\langle c \rangle$  is a  $\beta$ -plane. It is therefore natural to introduce the incidence variety

$$F(X) = \{(c, [V_4]) \in F_g(X) \times \mathbf{P}(V_5^\vee) \mid c \subset G(2, V_4)\}.$$

The first projection  $p_1 : F(X) \rightarrow F_g(X)$  is an isomorphism except over the point  $[c_X]$  that corresponds to the  $\rho$ -conic  $\Pi \cap \Omega$ , where the fiber  $L_\rho$  is isomorphic to the line  $L_U$  via the second projection  $p_2 : F(X) \rightarrow \mathbf{P}(V_5^\vee)$ . The conic  $c_X$  is the only  $\rho$ -conic on  $X$ .

For any  $[V_4]$  in  $L_U$ , the intersection  $\Pi_{V_4} \cap \Omega$  is a  $\sigma$ -conic in  $X$ . These conics describe a curve  $L_\sigma \subset F(X)$  isomorphic to  $L_U$  via  $p_2$ . These are the only  $\sigma$ -conics on  $X$ .

If  $c$  is a conic contained in  $X$ , its span  $\langle c \rangle$  is a 2-plane such that  $\langle c \rangle \cap X = c$ .<sup>7</sup>

If  $(c, V_4)$  is a  $\sigma$ -conic, the intersection  $\Pi \cap \Pi_{V_4}$  is a line in  $\Pi$  tangent to the conic  $c_U^\vee$ . In general, it meets  $c_X$  in the two points of  $c_X \cap c$ . Through any of these two points passes one other tangent to  $c_U^\vee$ , hence  $c$  meets exactly two other  $\sigma$ -conics, at points of  $c_X$ . In particular, two general  $\sigma$ -conics are disjoint. Also, there are two  $\sigma$ -conics through a general point of  $c_X$ .

Any point of a  $\sigma$ -conic  $(c, V_4)$  corresponds to a line in  $\mathbf{P}(V_4)$ , which must meet  $\mathbf{P}(U_3)$ . Therefore, the union of all  $\sigma$ -conics in  $X$  is contained in its section with the hyperplane  $\{V_2 \subset V_5 \mid V_2 \cap U_3 \neq \{0\}\}$ . This section being irreducible, they are equal. A general conic contained in  $X$  meets this hyperplane at two points, hence meets exactly two  $\sigma$ -conics.

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<sup>7</sup>This is because  $X$  contains no 2-planes (footnote 8) and is an intersection of quadrics.

**6.2. The involution  $\iota$  on  $F(X)$ .** Let  $V_4 \subset V_5$  be a hyperplane. In the 3-plane  $\mathbf{P}(M_{V_4}) = \mathbf{P}(\wedge^2 V_4) \cap \mathbf{P}_7$  introduced in §3.4, let us define the quadric

$$Q_{\Omega, V_4} = \Omega \cap \mathbf{P}(M_{V_4})$$

and set (see (3.1))

$$(6.1) \quad \Gamma_{4, V_4}^1 = X \cap \mathbf{P}(M_{V_4}) = Q_{W, V_4} \cap Q_{\Omega, V_4},$$

which is a genus-1 degree-4 1-cycle.<sup>8</sup> We have

$$(c, V_4) \in F(X) \iff c \subset \Gamma_{4, V_4}^1.$$

Assume now that  $X$  is general. Easy parameters counts such as the ones used in the proof of [L], Lemma 3.7 (see [L], (3.6)), then show that the residual curve is another conic  $\iota(c) \subset X$  that meets  $c$  in two points. This defines a fixed-point-free involution  $\iota$  on  $F(X)$  and the quotient  $F_\iota(X) = F(X)/\iota$  maps injectively to  $\mathbf{P}(V_5^\vee)$  by the projection  $p_2$ .

For any  $[V_4]$  in  $L_U$ , the quadric  $Q_{W, V_4}$  is already reducible (see §3.4), hence  $L_U \subset p_2(F(X))$ . Moreover,  $p_2^{-1}([V_4])$  has exactly two points, which correspond to the conics  $\Pi \cap \Omega$  and  $\Pi_{V_4} \cap \Omega$ . We have  $p_2^{-1}(L_U) = L_\rho \sqcup L_\sigma$  and  $\iota(L_\rho) = L_\sigma$ .

Here is a characterization of the involution  $\iota$ .

**Lemma 6.1.** *Let  $c$  and  $c'$  be two conics on a general  $X$  of type  $\mathcal{X}_{10}$ , with no common component. If  $\dim(\langle c, c' \rangle) = 3$  and  $\text{length}(c \cap c') = 2$ , we have  $c' = \iota(c)$ .*

*Proof.* If  $\langle c \rangle$  is contained in  $W$ , the line  $\langle c \rangle \cap \langle c' \rangle$  and the conic  $c'$  are in  $W \cap \langle c' \rangle$ . The line is not contained in  $c'$  (because  $X \cap \langle c \rangle = c$ ) hence  $W$ , which is an intersection of quadrics, contains  $\langle c' \rangle$ . One of the conics is then  $c_X$  and the other is a  $\sigma$ -conic so we are done.

Since  $W$  is an intersection of quadrics, if it contains neither planes  $\langle c \rangle$  and  $\langle c' \rangle$ , it does not contain the line  $\langle c \rangle \cap \langle c' \rangle$ . Any point  $z$  on the line  $\langle c \rangle \cap \langle c' \rangle$  but not on  $c \cup c'$  is not on  $G(2, V_5)$ . We may write  $x = v_1 \wedge v_2$ ,  $y = v_3 \wedge v_4$ , and  $z = v_1 \wedge v_2 + v_3 \wedge v_4$ . The vectors  $v_1, \dots, v_4$  span a hyperplane  $V_4 \subset V_5$  and every bisecant line to  $G(2, V_5)$  passing through  $z$  is contained in  $\mathbf{P}(\wedge^2 V_4)$ . It follows that  $\mathbf{P}(M_{V_4})$  contains  $c$  and  $c'$ , hence  $c' = \iota(c)$ .  $\square$

**Remark 6.2.** One checks, by a case-by-case analysis, that the involution  $\iota$  can still be defined on  $F(X)$  for *any* smooth  $X$ . It can however have fixed points. More precisely, for a smooth conic  $c \subset X$ , the following conditions are equivalent:

- (i)  $\iota(c) = c$ ;

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<sup>8</sup>Since  $\text{Pic}(X)$  is generated by  $\mathcal{O}_X(1)$ , all surfaces in  $X$  have degree divisible by 10, hence  $\Gamma_{4, V_4}^1$  is a curve.

- (ii)  $c$  is a  $\tau$ -conic with normal bundle  $N_{c/X} \simeq \mathcal{O}_c(2) \oplus \mathcal{O}_c(-2)$ ;
- (iii) there is a  $\mathbf{P}^3 \subset \mathbf{P}_7$  such that  $X \cap \mathbf{P}^3 = 2c$  (this generalizes Lemma 6.1).

Here is a quick proof. If (i) holds, we have  $X \cap \mathbf{P}(M_{V_4}) = 2c$ , hence (iii) holds. If (iii) holds, the conic  $c \subset \mathbf{P}^3$  is the intersection of a double plane and a quadric  $Q$ . With the notation of §7.1, the normal direction to  $c$  in  $Q$  at each point of  $c$  corresponds to a curve in the exceptional divisor  $E$  which is contracted by the projection from  $c$ , hence  $N_{c/X}$  must be  $\mathcal{O}_c(2) \oplus \mathcal{O}_c(-2)$ , and (ii) holds. Finally, if  $(c, V_4)$  is a  $\tau$ -conic,  $N_{c/W} \simeq \mathcal{O}_c(2) \oplus \mathcal{O}_c(1) \oplus \mathcal{O}_c(1)$ . In particular, the  $\mathcal{O}_c(2)$ -factor is uniquely determined, and it is the image of  $N_{c/Q_{W, V_4}}$ . If (ii) holds, the  $\mathcal{O}_c(2)$ -factor of  $N_{c/X}$  must map to  $N_{c/Q_{W, V_4}}$ , hence the induced morphism  $N_{c/Q_{W, V_4}} \rightarrow N_{X/W}$  must be zero. This means that the quadric  $\Omega$  vanishes to order 2 on  $c$ , hence (i) holds.

These conics correspond to singular points of the surface  $F(X)$ .

**6.3. Logachev's tangent bundle theorem.** *Assume again that  $X$  is general.* Logachev shows that  $F(X)$  and  $F_g(X)$  are smooth connected surfaces of general type ([L], Theorem 0.13 and Corollary 4.2). The map  $p_1 : F(X) \rightarrow F_g(X)$  is the contraction of the exceptional curve  $L_\rho$  to  $[c_X]$ . The curve  $L_\sigma = \iota(L_\rho)$  is therefore also exceptional. Let  $F_g(X) \rightarrow F_m(X)$  be its contraction and let  $r : F(X) \rightarrow F_m(X)$  be the composition.

Let  $\beta : F(X) \rightarrow G(2, V_8)$  be the map that sends a conic  $c$  to the projective line  $\langle c \cap \iota(c) \rangle$ , and let  $\mathcal{S}_{2, V_8}$  be the tautological bundle on  $G(2, V_8)$ . On the open set  $F(X)^0 = F(X) - (L_\rho \sqcup L_\sigma)$ , there is an isomorphism ([L], Theorem 4.14)

$$T_{F(X)^0} \xrightarrow{\sim} (\beta^* \mathcal{S}_{2, V_8})|_{F(X)^0}.$$

This ‘‘tangent bundle theorem’’ has the following geometric interpretation ([L], Theorem 7.2): in the diagram

$$\begin{array}{ccc} \mathbf{P}(T_{F_m(X)}) - \xrightarrow{\psi} \mathbf{P}(H^0(F_m(X), \Omega_{F_m(X)})^\vee) \simeq \mathbf{P}^9 & & \\ \downarrow \pi & & \downarrow \pi' \\ F_m(X) & & \mathbf{P}_7, \end{array}$$

where  $\psi$  is the *cotangent map* and  $\pi'$  the projection from the line  $\psi(\pi^{-1}([c_X]))$ , the composition  $\pi' \circ \psi$  sends a general fiber  $\pi^{-1}([c]) = \mathbf{P}(T_{F_m(X), [c]})$  to the line  $\langle c \cap \iota(c) \rangle$  in  $\mathbf{P}_7$ .

Since  $\mathcal{S}_{2, V_8}^\vee$  is generated by global sections on  $G(2, V_8)$ , so is  $\Omega_{F_m(X)}$ , except possibly at the points  $[c_X]$  and  $\iota([c_X])$ . We will show later (Corollary 7.3) that it is in fact globally generated everywhere.

**Remark 6.3.** For *any*  $X$  of type  $\mathcal{X}_{10}^0$ , the scheme  $F(X)$  is still an irreducible surface (Corollary 8.3) which may have singular points (Remark 6.2).

Note that the Fano surface of a Gushel threefold (see §4.1) has two components ([I], Proposition (2.1.2)).

## 7. Elementary transformations

**7.1. Elementary transformation along a conic.** Let  $c$  be a *smooth* conic contained in a Fano threefold  $X$  of type  $\mathcal{X}_{10}^0$  and let  $\pi_c : \mathbf{P}_7 \dashrightarrow \mathbf{P}^4$  be the projection from the 2-plane  $\langle c \rangle$ . If  $\varepsilon : \tilde{X} \rightarrow X$  is the blow-up of  $c$ , with exceptional divisor  $E$ , the composition  $\pi_c \circ \varepsilon : \tilde{X} \rightarrow \mathbf{P}^4$  is the *morphism* defined by the linear system  $|-K_{\tilde{X}}| = |-\varepsilon^*K_X - E|$ . The only curves contracted by this morphism are ([IsP], Proposition 4.4.1.(ii)):

- the strict transforms of the lines in  $X$  that meet  $c$ ;
- the strict transforms of conics  $c' \neq c$  such that  $\dim(\langle c, c' \rangle) = 3$  and  $\deg(c \cdot c') = 2$ ;
- when the normal bundle to  $c$  in  $X$  is  $\mathcal{O}(2) \oplus \mathcal{O}(-2)$ , the exceptional section of the ruled surface  $E \simeq \mathbf{F}_4$ .

It is easy to check that only finitely many lines meet  $c$  when *either*  $c$  is *general* in  $F(X)$  ([IsP], Lemma 4.2.6), or  $X$  *itself* is *general* and  $c$  is *any smooth conic*.

Assume from now on that  $X$  is *general* and  $c \neq c_X$ . It follows from Lemma 6.1 that the only conic contracted by  $\pi_c$  is  $\iota(c)$ , hence the morphism  $\varphi_{|-K_{\tilde{X}}|}$  contracts only finitely many curves. Let

$$\varphi_{|-K_{\tilde{X}}|} : \tilde{X} \xrightarrow{\varphi} \bar{X} \longrightarrow \mathbf{P}^4$$

be its Stein factorization. The variety  $\bar{X}$  has terminal hypersurface singularities,  $-K_{\bar{X}}$  is ample, and  $\varphi^*K_{\bar{X}} = K_{\tilde{X}}$ . The divisor  $-E$  is  $\varphi$ -antiample. In this situation, there exists a  $(-E)$ -flop ([IsP], Theorem 1.4.15)

$$\chi : \tilde{X} \xrightarrow{\varphi} \bar{X} \xleftarrow{\varphi'} \tilde{X}'$$

which is an isomorphism in codimension 1. The projective threefold  $\tilde{X}'$  is smooth and, if  $\bar{H}$  is a hyperplane section of  $\bar{X}$ , we have  $-K_{\tilde{X}'} = \varphi'^*\bar{H}$  and  $\chi(-E)$  is  $\varphi'$ -ample.

We have  $\rho(\tilde{X}') = 2$ . Since the extremal ray generated by the class of curves contracted by  $\varphi'$  has  $K_{\tilde{X}'}$ -degree 0 and  $K_{\tilde{X}'}$  is not nef, the other extremal ray is  $K_{\tilde{X}'}$ -negative and defines a contraction  $\varepsilon' : \tilde{X}' \rightarrow X'$ . We have ([IsP], Proposition 4.4.11.(ii)):

- $X'$  is again a smooth Fano threefold of degree 10 in  $\mathbf{P}^7$ ;

- $\varepsilon'$  is the blow-up of a smooth conic  $c'$  in  $X'$ , with exceptional divisor  $E' \equiv -2K_{\tilde{X}'} - \chi(E)$ .

There is a commutative diagram

$$(7.1) \quad \begin{array}{ccccc} \tilde{X} & \overset{\chi}{\dashrightarrow} & & \dashrightarrow & \tilde{X}' \\ & \searrow \varphi & & \swarrow \varphi' & \\ & & \tilde{X} & & \\ & \swarrow \pi_c & & \searrow \pi_{c'} & \\ X & \overset{\psi_c}{\dashrightarrow} & & \dashrightarrow & X' \\ & \swarrow \varepsilon & & \searrow \varepsilon' & \end{array}$$

If  $H'$  is a hyperplane section of  $X'$ , we have

$$\chi^* \varepsilon'^* H' \equiv \chi^* (-K_{\tilde{X}'} + E') \equiv \chi^* (-3K_{\tilde{X}'} - \chi(E)) \equiv -3\varepsilon^* K_X - 4E$$

hence  $\psi_c$  is associated with a linear subsystem of  $|\mathcal{S}_c^4(3)|$ .<sup>9</sup>

Note that  $\varepsilon'^* H' - E' \equiv -K_{\tilde{X}'} \equiv \varphi'^* \bar{H}$ , so the picture is symmetric: the elementary transformation of  $X'$  along the conic  $c'$  is  $\psi_c^{-1} : X' \dashrightarrow X$  (by construction,  $\varphi'$  automatically contracts only finitely many curves).

Finally, we remark that the intermediate Jacobians of  $X$  and of  $X'$  are isomorphic.

**Proposition 7.1.** *Let  $X$  be a general Fano threefold of type  $\mathcal{X}_{10}$  and let  $c$  be a smooth  $\tau$ -conic on  $X$ . There is a birational isomorphism*

$$\varphi_c : F_g(X) \dashrightarrow F_g(X')$$

which commutes with the (rational) involutions  $\iota$  on  $F_g(X)$  and  $\iota'$  on  $F_g(X')$ . Its inverse is  $\varphi_{c'}$ .

*Proof.* Let  $\bar{c}$  be a conic in  $X$  disjoint from  $c$ . The span  $\langle c, \bar{c} \rangle$  is a 5-plane that intersects  $X \subset \mathbf{P}_7$  along a canonically embedded genus-6 curve  $c + \bar{c} + \Gamma_{c, \bar{c}}$ , where  $\Gamma_{c, \bar{c}}$  is a sextic. Since  $X \cap \langle c \rangle = c$ , this implies that  $\Gamma_{c, \bar{c}}$  meets  $c$  in four points, and similarly for  $\bar{c}$ .

We now show that on  $X$  general, for general conics  $c$  and  $\bar{c}$ , the sextic  $\Gamma_{c, \bar{c}}$  is smooth and irreducible. Note to that effect that  $S = G(2, V_5) \cap \langle c, \bar{c} \rangle$  is a general smooth del Pezzo surface of degree 5, which can therefore be expressed as the blow-up of the plane in four general points, with exceptional divisors  $E_1, \dots, E_4$ . If  $h$  is the class of a general line in the plane, the embedding

<sup>9</sup>If  $\ell$  is the strict transform in  $\tilde{X}$  of a line in  $X$  that meets  $c$ , we have  $(3\varepsilon^* H - 4E) \cdot \ell = -1$  hence  $\ell$  is in the base locus of  $|\chi^* \varepsilon'^* H'|$ . Similarly, the strict transform in  $\tilde{X}$  of the conic  $\iota(c)$  has intersection  $-2$  with  $3\varepsilon^* H - 4E$ , hence is also in the base locus of  $|\chi^* \varepsilon'^* H'|$ . Calculations on the blow-up of  $\iota(c)$  in  $\tilde{X}$  show that the base ideal is in fact contained in  $\mathcal{S}_{\iota(c)}^2$ . So, for  $c$  general, the rational map  $\psi_c$  is associated with a linear subsystem of  $|\mathcal{S}_{\ell_1} \otimes \dots \otimes \mathcal{S}_{\ell_{20}} \otimes \mathcal{S}_{\iota(c)}^2 \otimes \mathcal{S}_c^4(3)|$ , where  $\ell_1, \dots, \ell_{20}$  are the 20 lines in  $X$  that meet  $c$ .

$S \rightarrow \langle c, \bar{c} \rangle$  is given by the linear system  $|3h - E_1 - \dots - E_4|$ . The family of conics in  $S$  is the union of the five pencils  $|h - E_i|$ , for  $i \in \{1, \dots, 4\}$ , and  $|2h - E_1 - \dots - E_4|$ . Since  $c$  and  $\bar{c}$  are disjoint, they must belong to the same pencil, say  $|h - E_1|$  or  $|2h - E_1 - \dots - E_4|$ . The curve  $c + \bar{c} + \Gamma_{c, \bar{c}}$  is the intersection of  $S$  with a general quadric containing  $c$  and  $\bar{c}$ . It follows that  $\Gamma_{c, \bar{c}}$  is a general member of the linear systems  $|4h - 2E_2 - \dots - 2E_4|$  or  $|2h|$ , which are both base-point-free. In particular,  $\Gamma_{c, \bar{c}}$  is a smooth rational curve of degree 6.

The rational map  $\psi_c$  is defined on  $\Gamma_{c, \bar{c}}$  by a linear subsystem of  $|\mathcal{S}_c^4(3)|$ , of degree  $\leq 3 \deg(\Gamma_{c, \bar{c}}) - 4c \cdot \Gamma_{c, \bar{c}} = 2$ . For  $\bar{c}$  general, its image on  $X'$  is therefore a conic. This defines, for  $X$  and  $c$  general, a rational map

$$\varphi_c : F_g(X) \dashrightarrow F_g(X').$$

Alternatively, a case-by-case analysis using footnote 9 shows that for any smooth  $\tau$ -conic  $c$ , the curve  $\psi_c(\Gamma_{c, \bar{c}})$  is a conic as soon as  $\bar{c}$  and  $\iota(\bar{c})$  are  $\tau$ -conics that meet none of the lines that meet  $c$  or  $\iota(c)$ . This implies that the map  $\varphi_c$  is still defined in this case.

The composition  $\pi_{\langle c \rangle} = \varphi \circ \varepsilon^{-1} : X \dashrightarrow \mathbf{P}^4$  is the projection from the 2-plane  $\langle c \rangle$ , hence  $\pi_{\langle c' \rangle} = \varphi' \circ \varepsilon'^{-1} : X' \dashrightarrow \mathbf{P}^4$  is the projection from the 2-plane  $\langle c' \rangle$ . It follows that the curve  $\psi_c(c + \bar{c} + \Gamma_{c, \bar{c}})$  in  $X'$  lies in the 5-plane  $\pi_{\langle c' \rangle}^{-1}(\pi_{\langle c \rangle}(\langle c, \bar{c} \rangle))$ . It contains the conic  $\psi_c(\Gamma_{c, \bar{c}})$ , the sextic  $\psi_c(c')$ , and the conic  $c'$ . Thus, the rational map  $\varphi_{c'} : F_g(X') \dashrightarrow F_g(X)$  is the inverse of  $\varphi_c$ , which is therefore birational.

The section of the quartic  $\pi_c(X) \subset \mathbf{P}^4$  by the 2-plane  $\pi_c(\langle c, \bar{c} \rangle)$  is the union of the conics  $\pi_c(\bar{c})$  and  $\pi_c(\Gamma_{c, \bar{c}})$ . Similarly, its section by the 2-plane  $\pi_c(\langle c, \iota(\bar{c}) \rangle)$  is the union of the conics  $\pi_c(\iota(\bar{c}))$  and  $\pi_c(\Gamma_{c, \iota(\bar{c})})$ . Since  $\bar{c}$  and  $\iota(\bar{c})$  together span a 3-plane, these two 2-planes meet along a line that meets  $\pi_c(X)$  in four points, including the two points of  $\pi_c(\bar{c}) \cap \pi_c(\iota(\bar{c}))$ . It follows that the intersection of the conics  $\pi_c(\Gamma_{c, \bar{c}})$  and  $\pi_c(\Gamma_{c, \iota(\bar{c})})$  with this line must be the same. In particular, these conics meet in two points, hence so do their images  $\varphi_c([\bar{c}])$  and  $\varphi_c(\iota([\bar{c}]))$  in  $X'$ . This proves  $\varphi_c(\iota([\bar{c}])) = \iota'(\varphi_c([\bar{c}]))$ .  $\square$

**Proposition 7.2.** *Let  $X$  be a general Fano threefold of type  $\mathcal{X}_{10}$  and let  $c$  be a smooth  $\tau$ -conic on  $X$ . The map  $\varphi_c : F_g(X) \dashrightarrow F_g(X')$  sends*

- the curve of  $\sigma$ -conics on  $X$  to the point  $[c']$ ;
- the point  $[c_X]$  to  $\iota'([c'])$ ;
- the point  $\iota([c])$  to  $[c_{X'}]$ ;

and factors as

$$F_g(X) \longrightarrow F_m(X) \xrightarrow{\sim} F_m(X') \longleftarrow F_g(X').$$

In other words, the surface  $F_g(X')$  is isomorphic to the surface  $F_m(X)$  blown up at the point  $[c]$ .

*Proof.* We can assume that  $c$  is general. Let  $\bar{c}$  be a general  $\sigma$ -conic on  $X$ . We have

$$X \cap \langle c, \bar{c} \rangle = c + \bar{c} + \Gamma_{c, \bar{c}},$$

where  $\Gamma_{c, \bar{c}}$  is a smooth irreducible sextic. To show  $\psi_c(\Gamma_{c, \bar{c}}) = c'$ , since

$$\chi^* E' \cdot \Gamma_{c, \bar{c}} = (-2\varepsilon^* K_X - 3E) \cdot \Gamma_{c, \bar{c}} = 0,$$

it is enough to show that (the strict transform of)  $\Gamma_{c, \bar{c}}$  meets the divisor  $\chi^* E'$ , because it will then have to be contained in it.

Recall from §6.1 that  $c$  meets exactly two  $\sigma$ -conics, say  $c_1$  and  $c_2$ . Since  $c_1$  meets  $c$  in one point,  $\iota(c)$  in one point, is not contained in the indeterminacy locus of  $\psi_c$ , and has degree  $-2 - 4 + 3 \times 2 = 0$  on the linear system  $|\mathcal{S}_{\iota(c)}^2 \otimes \mathcal{S}_c^4(3)|$  that defines  $\psi_c$  (see footnote 9), it maps to a point of  $c'$  on  $X'$ , hence to a fiber of the map  $\varepsilon|_{E'} : E' \rightarrow c'$  on  $\tilde{X}'$ . It is therefore enough to show that  $\Gamma_{c, \bar{c}}$  meets  $c_1$  outside of  $c$ .

Note that  $c_1$  and  $c_2$  both meet  $c$ , and they both meet  $c_X$  in two points. They are therefore contained in  $\langle c, c_X \rangle$ , and the sextic  $\Gamma_{c, c_X}$  is the union of  $c_1$ ,  $c_2$ , and a conic that meets  $c_1$  and  $c_2$  each in one point, and  $c$  in two points, hence must be  $\iota(c)$ .

The 1-cycle

$$\begin{aligned} \Gamma &= (c + \bar{c} + \Gamma_{c, \bar{c}}) + (c + c_X + \Gamma_{c, c_X}) \\ &= 2c + \bar{c} + \Gamma_{c, \bar{c}} + c_X + c_1 + c_2 + \iota(c) \end{aligned}$$

is the complete intersection in  $\mathbf{P}_7$  of  $X$  with the hyperplane  $\langle c, \bar{c}, c_X \rangle$  and the reducible quadric  $\langle c, \bar{c} \rangle \cup \langle c, c_X \rangle$ , hence  $\omega_\Gamma = \mathcal{O}_\Gamma(2)$ . In particular,  $c_1$  must meet the other components in six points. Since it meets  $c_X$  twice,  $c$  and  $\iota(c)$  simply, and neither  $\bar{c}$  nor  $c_2$ , it must meet  $\Gamma_{c, \bar{c}}$ .

We have therefore shown that  $\varphi_c$  is defined at general points of the curve  $L_\sigma \subset F(X)$  of  $\sigma$ -conics in  $X$ , and that its value at these points is the point  $[c'] \in F(X')$ . Since  $\varphi_c$  commutes with  $\iota$ , it is also defined at  $[c_X]$ , where it takes the value  $\iota'([c'])$ .

The rest of the statements follows from the symmetry  $\varphi_c^{-1} = \varphi_{c'}$  and the facts that  $\varphi_c$  commutes with the involutions and the surfaces  $F_m(X)$  and  $F_m(X')$  are of general type (§6.3).  $\square$

**Corollary 7.3.** *If  $X$  is a general Fano threefold of type  $\mathcal{X}_{10}$ , the sheaf  $\Omega_{F_m(X)}$  is generated by its global sections. In particular, the surface  $F_m(X)$  is minimal and the only rational curves in  $F(X)$  are  $L_\rho$  and  $L_\sigma$ .*

*Proof.* We saw in §6.3 that  $\Omega_{F_m(X)}$  is generated by global sections, except possibly at  $[c_X]$  and  $\iota([c_X])$ . Because of Proposition 7.2, this holds everywhere.  $\square$

We will now write  $X_c$  instead of  $X'$  to highlight the dependence on  $c$ . For  $X$  general, as  $[c]$  varies in the open subset of  $F(X)$  consisting of smooth  $\tau$ -conics, the assignment  $[c] \mapsto [X_c]$  defines a rational map  $F_g(X) \dashrightarrow \mathcal{N}_{10}$  whose image is contained in a fiber of the period map.

**Theorem 7.4.** *Let  $X$  be a general Fano threefold of type  $\mathcal{X}_{10}$ . For any conic  $c \subset X$ , one can define a smooth birational model  $X_c$  of  $X$  such that  $F_g(X_c)$  is isomorphic to the surface  $F_m(X)$  blown up at the point  $[c]$ .*

*The assignment  $[c] \mapsto [X_c]$  induces an isomorphism between  $F_{m,\iota}(X)$  and the connected component through  $[X]$  of the fiber of the period map  $\wp : \mathcal{N}_{10} \rightarrow \mathcal{A}_{10}$ .*

In particular, the general fiber of the period map  $\wp : \mathcal{N}_{10} \rightarrow \mathcal{A}_{10}$  is the disjoint union of finitely many smooth projective irreducible surfaces.

*Proof.* The idea is to prove that for a general conic  $d \subset X$ , the conic

$$\varphi_d(c) = \psi_d(\Gamma_{d,c}) \subset X_d$$

is a smooth  $\tau$ -conic. We can then set

$$X_c = (X_d)_{\varphi_d(c)}.$$

Note that we define the variety  $X_c$ , but not a particular birational map  $X \dashrightarrow X_c$ . According to Proposition 7.2, the surface  $F(X_c)$  is then isomorphic to the surface  $F_m(X)$  blown-up at the point  $[c]$ . By the Reconstruction Theorem 9.1, the isomorphism class of the variety  $X_c$  is independent of the choice of  $d$ . This procedure therefore defines, for each general  $X$ , a morphism  $F_g(X) \rightarrow \mathcal{N}_{10}$  which, since all  $\sigma$ -conics have the same images in  $F_m(X)$ , factors through  $F_m(X)$ .

If  $c$  is a  $\sigma$ -conic (resp. the  $\rho$ -conic), we established during the proof of Proposition 7.2 that  $\psi_d(\Gamma_{d,c})$  is the smooth  $\tau$ -conic  $d'$  (resp.  $\iota'(d')$ ).

So we may assume that  $c = \ell \cup \ell'$  is a reducible  $\tau$ -conic. We proceed by contradiction, assuming that the conic  $\psi_d(\Gamma_{d,c}) \subset X_d$  is reducible. The degree-6 1-cycle  $\Gamma_{d,c}$  must then split as the sum of two degree-3 1-cycles  $\Gamma$  and  $\Gamma'$ , each of which meets  $d$  in two points.

We assume that  $d$  is a general  $\tau$ -conic; more precisely, that *neither  $d$  nor  $\iota(d)$  meets  $c$ ,  $\iota(c)$ , or any of the (finitely many) lines contained in  $X$  that meet  $c$* . Furthermore, if we write (see §3.1)

$$(7.2) \quad \ell = \{[V_2] \mid \langle e_1 \rangle \subset V_2 \subset V_3\} \quad \text{and} \quad \ell' = \{[V_2] \mid \langle e'_1 \rangle \subset V_2 \subset V'_3\},$$

we assume that *the unique hyperplane  $V_4^d \subset V_5$  such that  $d \subset G(2, V_4^d)$*  (see §6.1) *contains neither  $e_1$  nor  $e'_1$* . Note that since  $c$  is a  $\tau$ -conic, we have



$\ell \cap \ell' = \{e_1 \wedge e'_1\}$  and  $V_4^c = V_3 + V_3'$ . We assume that *the 2-planes  $\mathbf{P}(V_3)$  and  $\mathbf{P}(V_3')$  each meet the smooth quadric surface  $Q_d \subset \mathbf{P}(V_5)$  swept out by the lines corresponding to points of  $d$  (see §3.1) transversely in two points, and no two of these four points are on a line contained in  $Q_d$ .*

*Step 1. The curve  $\Gamma$  is irreducible and meets each of the lines  $\ell$  and  $\ell'$  in one point.*

If  $\Gamma$  is reducible, it contains a line  $m$ . If  $m$  meets  $\ell$  (or  $\ell'$ ), we have by assumption  $m \cap d = \emptyset$ , hence the residual conic must meet  $m$ , and  $d$  in two points: it is therefore  $\iota(d)$ , which is absurd since we assumed  $m \cap \iota(d) = \emptyset$ . If  $\Gamma$  is reducible, it must therefore contain a smooth conic meeting  $\ell$  and  $\ell'$ , and this conic is  $\iota(c)$ . The residual line must then meet  $d$  in two points, which again contradicts our assumptions.

It follows that  $\Gamma$  is irreducible. Since a cubic curve contained in  $X$  cannot be bisecant to a line contained in  $X$  (because the corresponding line transform would contract the cubic and this would contradict [IsP], Proposition 4.3.1; see §7.2),  $\Gamma$  meets each of the lines  $\ell$  and  $\ell'$  in one point.

*Step 2. There exists a line  $\mathbf{P}(W_2) \subset \mathbf{P}(V_5)$  that meets all lines parametrized by  $\Gamma$ . It is contained in  $\mathbf{P}(V_4^d)$  but not in the quadric  $Q_d$ .*

The restriction to  $\Gamma$  of the tautological subbundle  $\mathcal{S}_{2,V_5}$  is isomorphic to  $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$ , or to  $\mathcal{O} \oplus \mathcal{O}(-3)$ . In the latter case, all lines parametrized by  $\Gamma$  pass through a fixed point. But this cannot happen since,  $d$  being a  $\tau$ -conic, the lines in  $\mathbf{P}(V_5)$  corresponding to the two points of  $\Gamma \cap d$  are disjoint. We are therefore in the first case, hence  $\Gamma$  can be parametrized by  $\gamma : t \mapsto [w(t) \wedge v(t)]$ , where  $w$  is linear in  $t$ , and  $v$  is quadratic in  $t$ . Take for  $W_2 \subset V_5$  the 2-dimensional vector space spanned by the  $w(t)$ . More precisely, we may assume  $\Gamma \cap d = \{\gamma(0), \gamma(\infty)\}$  and

$$\gamma(t) = [(w_0 + tw_\infty) \wedge (v_0 + tv_1 + t^2v_\infty)],$$

with  $W_2 = \langle w_0, w_\infty \rangle$  and  $V_4^d = \langle w_0, w_\infty, v_0, v_\infty \rangle$ . We may further assume  $\Gamma \cap \ell = \{\gamma(1)\}$  and  $\Gamma \cap \ell' = \{\gamma(t_0)\}$  for some  $t_0 \notin \{0, 1, \infty\}$ . Set

$$(7.3) \quad w = w_0 + w_\infty \quad \text{and} \quad w' = w_0 + t_0w_\infty,$$

so that  $W_2 = \langle w, w' \rangle$ . Since we assumed  $e_1 \notin V_4^d$ , we have  $e_1 \notin W_2$ , hence  $\gamma(1) = [e_1 \wedge w]$ . Similarly,  $\gamma(t_0) = [e'_1 \wedge w']$ . If  $\mathbf{P}(W_2)$  is contained in  $Q_d$ , the point  $[w] \in \mathbf{P}(V_5)$  must be one of the two points of  $\mathbf{P}(V_3) \cap Q_d$ , and  $[w']$  must be one of the two points of  $\mathbf{P}(V_3') \cap Q_d$ . Since we assumed that none of the four lines joining these points are contained in  $Q_d$ , this is absurd.

Finally, we have

$$(7.4) \quad v_0 + v_1 + v_\infty \in \langle e_1, w \rangle \quad \text{and} \quad v_0 + t_0v_1 + t_0^2v_\infty \in \langle e'_1, w' \rangle,$$

hence  $v_0 + t_0 v_\infty$  is in the hyperplane  $\langle e_1, e'_1, w, w' \rangle$ .

Conversely, let us start with  $[w] \in \mathbf{P}(V_3 \cap V_4^d)$  and  $[w'] \in \mathbf{P}(V_3' \cap V_4^d)$ . Assume  $w \notin V_3'$  and  $w' \notin V_3$ , and that the line  $\langle [w], [w'] \rangle$  meets  $Q_d$  in two points,  $[w_0]$  and  $[w_\infty]$ . These points determine uniquely the points  $[w_0 \wedge v_0]$  and  $[w_\infty \wedge v_\infty]$  of  $d$ , together with the point  $[e_1 \wedge w]$  of  $\ell$ , and the point  $[e'_1 \wedge w']$  of  $\ell'$ . Note that  $w_0$  and  $w_\infty$  are defined only up to multiplication by nonzero scalars. But our choice of parametrization of  $\Gamma$  imposes  $w_0 + w_\infty \in \mathbf{C}w$  (see (7.3)), so these scalars must be the same, and  $w_0 + t_0 w_\infty \in \mathbf{C}w'$ , which determines  $t_0$  uniquely.

Again,  $v_0$  and  $v_\infty$  are defined only up to multiplication by nonzero scalars and addition of multiples of  $w_0$  and  $w_\infty$  respectively. But since  $v_0 + t_0 v_\infty$  must be in the hyperplane  $\langle e_1, e'_1, w, w' \rangle$ , these scalars must again be the same. These changes can be achieved by noting that the span of  $w_0 + t w_\infty$  and  $v_0 + t v_1 + t^2 v_\infty$  is also the span of  $w_0 + t w_\infty$  and  $\lambda v_0 + \mu w_0 + t(\lambda v_1 + \mu w_\infty + \nu w_0) + t^2(\lambda v_\infty + \nu w_\infty)$ .

Finally,  $v_1$  is uniquely determined by (7.4), hence the curve  $\Gamma$  is uniquely determined by the choice of  $[w]$  and  $[w']$ . Since both vary in a projective line,  $\Gamma$  belongs to a two-dimensional (irreducible) family.

Let us now look at quadrics  $\Omega$  containing  $c$  and  $d$ . Containing a given twisted cubic  $\Gamma$  as above imposes three further conditions. Since  $\Gamma$  varies in a two-dimensional family, a general quadric containing  $d$  and  $c$  contains no such cubic  $\Gamma$ . This implies what we need.

Finally, we will show in Theorem 9.1 that the conic transforms  $X_c$  and  $X_d$  are isomorphic if and only if there exists an automorphism  $\sigma$  of  $F_m(X)$  such that  $\sigma([c]) = [d]$ . By Corollary 9.3,  $\sigma$  is either trivial or  $\iota$ , so this proves that the assignment  $[c] \mapsto [X_c]$  defines an injective morphism from the projective surface  $F_{m,\iota}(X)$  to the moduli stack  $\mathcal{N}_{10}$ .

Since the intermediate Jacobians of  $X$  and of  $X_c$  are isomorphic, the image of this morphism is contained in the fiber  $\mathcal{F}$  of the period map passing through the point  $[X]$ . By Theorem 5.1, the image is a smooth surface and a connected component of  $\mathcal{F}$ . Since  $F_{m,\iota}(X)$  is a minimal surface, this component is actually isomorphic to  $F_{m,\iota}(X)$ .  $\square$

**7.2. Elementary transformation along a line.** Let  $\ell$  be a line contained in a threefold  $X$  of type  $\mathcal{X}_{10}^0$ . We can define an elementary transformation along  $\ell$  as in §7.1. If  $\varepsilon: \tilde{X} \rightarrow X$  is the blow-up of  $\ell$ , with exceptional divisor  $E$ , projection from  $\ell$  induces a birational morphism  $\varphi_{|-K_{\tilde{X}}|}: \tilde{X} \rightarrow \mathbf{P}^5$ , whose image  $\tilde{X}$  has degree 10 and has terminal hypersurface singularities. The only curves contracted by  $\varphi_{|-K_{\tilde{X}}|}$  are the strict transforms of the lines

that meet  $\ell$  and, when the normal bundle to  $\ell$  in  $X$  is  $\mathcal{O}(1) \oplus \mathcal{O}(-2)$ , the exceptional section of the ruled surface  $E \simeq \mathbf{F}_3$ ; in any event, there are finitely many such curves ([IsP], Proposition 4.3.1 and Corollary 4.3.2).

Performing a flop, we end up as in §7.1 with a diagram

$$(7.5) \quad \begin{array}{ccc} \tilde{X} & \overset{\chi}{\dashrightarrow} & \tilde{X}' \\ \varepsilon \downarrow & \begin{array}{c} \searrow \varphi \\ \swarrow \varphi' \end{array} & \downarrow \varepsilon' \\ X & \overset{\psi_\ell}{\dashrightarrow} & X_\ell \end{array}$$

where ([IsP], Proposition 4.3.3.(iii)):

- $X_\ell$  is again a smooth Fano threefold of degree 10 in  $\mathbf{P}^7$ ;
- $\varepsilon'$  is the blow-up of a line  $\ell'$  in  $X_\ell$ , with exceptional divisor  $E' \equiv -K_{\tilde{X}'} - \chi(E)$ .

If  $H'$  is a hyperplane section of  $X_\ell$ , we have

$$\chi^* \varepsilon'^* H' \equiv \chi^* (-K_{\tilde{X}'} + E') \equiv \chi^* (-2K_{\tilde{X}'} - \chi(E)) \equiv -2\varepsilon^* K_X - 3E,$$

hence  $\psi_\ell$  is associated with a linear subsystem of  $|\mathcal{S}_\ell^3(2)|$ .

As in §7.1, the elementary transformation of  $X_\ell$  along the line  $\ell'$  is  $\psi_\ell^{-1} : X_\ell \dashrightarrow X$ .

As for conic transforms, the intermediate Jacobians of  $X$  and of  $X_\ell$  are isomorphic.

**Remark 7.5.** Let  $X$  be general. Using iterated conic and line transformations, we can construct many threefolds with the same intermediate Jacobian as  $X$ . We will explain where these transforms land in the fiber of the period map.

We just saw that the connected component  $\mathcal{F}_X$  of the fiber  $\mathcal{F}$  of the period map passing through the point corresponding to  $X$  is a projective surface isomorphic to  $F_{m,i}(X)$ . If  $\ell$  is a line contained in  $X$ , we get another component  $F_{m,i}(X_\ell)$  of  $\mathcal{F}$ . By continuity, this component  $\mathcal{F}_X^*$  is independent of the choice of  $\ell$  (in §8, we will prove directly that these surfaces are all abstractly isomorphic to a quotient of the moduli space  $\mathcal{M}_X(2; 1, 5)$  by an involution). Now take a conic  $c \subset X$ ; by the same reasoning, conic transforms of  $X_c$  also land in the component  $\mathcal{F}_X$  of  $\mathcal{F}$ , whereas line transforms of  $X_c$  land in the component  $\mathcal{F}_X^*$ . In other words, conic transformations leave each of the two components  $\mathcal{F}_X$  and  $\mathcal{F}_X^*$  of  $\mathcal{F}$  invariant, whereas line transformations switch them.

In [DIM], §7.3, we prove by a degeneration argument that the components  $\mathcal{F}_X$  and  $\mathcal{F}_X^*$  are different. Therefore, a general fiber of the period map is the disjoint union of an *even number* of smooth projective irreducible surfaces.

### 8. The moduli space $\mathcal{M}_X(2; 1, 5)$

Let  $X$  be *any* threefold of type  $\mathcal{X}_{10}^0$ . We study the moduli space  $\mathcal{M}_X(2; 1, 5)$  of semistable rank-2 torsion-free sheaves on  $X$ , with Chern numbers  $c_1 = 1$ ,  $c_2 = 5$ , and  $c_3 = 0$ , along the lines of [IM]. A noteworthy difference is that a vector bundle in this moduli space is not necessarily generated by its global sections. Our treatment of the non globally generated bundles is directly inspired by [BF]. The main result of this section is Theorem 8.2.

*First remarks.* Let  $\mathcal{E}$  be a semistable rank-2 vector bundle on  $X$ , with Chern numbers  $c_1 = 1$  and  $c_2 = 5$ . By the Riemann-Roch theorem, we have

$$\chi(X, \mathcal{E}nd(\mathcal{E})) = 4\chi(\mathcal{O}_X) + \frac{1}{2}c_1(X)(c_1(\mathcal{E})^2 - 4c_2(\mathcal{E})) = -1.$$

We claim that  $\mathcal{E}$  has nonzero sections. Indeed  $\chi(X, \mathcal{E}) = 4$  and  $H^3(X, \mathcal{E}) = 0$  by Serre duality. So we just need to prove that  $H^2(X, \mathcal{E})$  vanishes, which can be proved as in [IM], Lemma 5.1.

Restrict  $\mathcal{E}$  to a general hyperplane section  $S$  of  $X$ , a K3 surface of degree 10 with  $\text{Pic}(S) = \mathbf{Z}[\mathcal{O}_S(1)]$ . We have  $\chi(S, \mathcal{E}|_S) = 4$  and  $H^2(S, \mathcal{E}|_S) = H^1(S, \mathcal{E}|_S^\vee) = 0$ , hence  $h^0(S, \mathcal{E}|_S) \geq 4$ . Choosing a nonzero section  $s$  of  $\mathcal{E}|_S$ , we get an extension

$$0 \rightarrow \mathcal{O}_S \xrightarrow{s} \mathcal{E}|_S \rightarrow \mathcal{I}_Z(1) \rightarrow 0,$$

where  $Z$  is a zero-dimensional scheme of length 4. Since  $\mathcal{E}|_S$  is locally free, we have  $h^1(S, \mathcal{I}_Z(1)) \geq 1$ , and there is in fact equality since otherwise  $Z$  would generate a line contained in  $S$ , which is impossible. Looking at the associated long exact sequence, we get  $H^1(S, \mathcal{E}|_S) = 0$ , thus  $h^0(S, \mathcal{E}|_S) = 4$ . Back to  $X$ , we obtain  $H^1(X, \mathcal{E}) = 0$  and  $h^0(X, \mathcal{E}) = 4$ . Moreover,  $\mathcal{E}$  is globally generated in codimension two.

*Non locally free sheaves.* Let  $\mathcal{F}$  be a semistable rank-2 torsion-free sheaf on  $X$  with Chern numbers  $c_1 = 1$ ,  $c_2 = 5$ , and  $c_3 = 0$ , which is *not* locally free. We claim that there exists a unique line  $\ell$  in  $X$  such that  $\mathcal{F}$  fits in an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow T_X^\vee \rightarrow \mathcal{O}_\ell \rightarrow 0.$$

This is the same statement as Proposition 5.11 in [IM] and the proof is the same: the bidual  $\mathcal{G} = \mathcal{F}^{\vee\vee}$  is locally free outside a finite set, so its restriction

to a general K3 section  $S$  is locally free, the quotient  $(\mathcal{G}/\mathcal{F})|_S$  has finite support, and  $\text{length}((\mathcal{G}/\mathcal{F})|_S) = 5 - c_2(\mathcal{G}|_S)$ . Moreover  $\mathcal{G}|_S$  is semistable and the moduli space of simple sheaves on  $S$  is smooth at  $\mathcal{G}|_S$ , of dimension  $4 - 4\text{length}((\mathcal{G}/\mathcal{F})|_S)$ . The same arguments as in [IM] yields that the length of  $(\mathcal{G}/\mathcal{F})|_S$  is nonzero, hence must in fact be 1. This means that  $\mathcal{G}/\mathcal{F}$  is supported on a line  $\ell$  plus possibly a finite set of points.

Moreover  $\mathcal{G}|_S$ , being semistable with Chern numbers  $c_1 = 1$  and  $c_2 = 4$ , must be the restriction of  $T_X^\vee$  to  $S$ . Finally, exactly as in [IM],  $\mathcal{G} \simeq T_X^\vee$  and  $\mathcal{G}/\mathcal{F} \simeq \mathcal{O}_\ell$ .

*Non globally generated bundles.* Now, let  $\mathcal{E}$  be a semistable rank-2 vector bundle on  $X$ , with Chern numbers  $c_1 = 1$  and  $c_2 = 5$ , which is *not* generated by global sections. We follow the arguments of [BF], Lemma 6.12. Denote by  $\mathcal{I} \subset \mathcal{E}$  the image of the evaluation map  $H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E}$  and by  $\mathcal{K}$  its kernel. Set also  $\mathcal{T} = \mathcal{E}/\mathcal{I}$ , a nonzero sheaf supported in codimension two. The torsion-free sheaf  $\mathcal{I}$  is stable of rank two, with  $c_1(\mathcal{I}) = 1$  and  $c_2(\mathcal{I}) \geq 5$ . The reflexive sheaf  $\mathcal{K}$  is also stable of rank two, with  $c_1(\mathcal{K}) = -1$ . By restricting to a general K3 section and applying Mukai's formula for the dimension of the moduli space, we get  $c_2(\mathcal{K}) \geq 4$ , hence  $c_2(\mathcal{I}) = 10 - c_2(\mathcal{K}) \leq 6$ .

Suppose  $c_2(\mathcal{K}) = 5$ ; then  $c(\mathcal{T}) = 1 + c_3(\mathcal{K})$  and, by Riemann-Roch,  $\ell(\mathcal{T}) = c_3(\mathcal{K})/2 \leq 0$ . So  $\mathcal{T} = 0$ , a contradiction.

So  $c_2(\mathcal{K}) = 4$ . But the dual sheaf  $\mathcal{K}^\vee$  must then be the dual  $T_X^\vee$  of the tautological sheaf. Finally, a computation yields  $\chi(\mathcal{T}(k)) = k$ . Since  $H^0(X, \mathcal{T}) = 0$ , this implies  $\mathcal{T} = \mathcal{O}_\ell(-1)$  for some line  $\ell$  in  $X$ . We get an exact sequence

$$0 \rightarrow T_X \rightarrow H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \mathcal{O}_\ell(-1) \rightarrow 0.$$

Note that if we apply the functor  $\mathcal{H}om(\cdot, \mathcal{O}_X)$ , we get the dual sequence

$$0 \rightarrow \mathcal{E}^\vee \rightarrow H^0(X, \mathcal{E})^\vee \otimes \mathcal{O}_X \rightarrow T_X^\vee \rightarrow \mathcal{O}_\ell \rightarrow 0.$$

This means that  $\mathcal{E}^\vee$  can be identified with the kernel of the evaluation map  $H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \rightarrow \mathcal{F}$  of the non locally free sheaf  $[\mathcal{F}] \in \mathcal{M}_X(2; 1, 5)$  defined by  $\ell$ . In particular,  $\mathcal{E}$  is uniquely defined by  $\ell$ , and  $H^0(X, \mathcal{E}) \simeq H^0(X, \mathcal{F})^\vee$ .

*An involution on the moduli space.* Let  $[\mathcal{E}] \in \mathcal{M}_X(2; 1, 5)$  be a globally generated vector bundle. Observe that the kernel  $\mathcal{K}$  of the evaluation map  $H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E}$  is a rank-2 vector bundle with  $c_1 = -1$  and  $c_2 = 5$ , with no nonzero global sections. Its dual  $\iota\mathcal{E}$  is therefore stable, with Chern numbers  $c_1(\iota\mathcal{E}) = 1$  and  $c_2(\iota\mathcal{E}) = 5$ . Moreover,  $\iota\mathcal{E}$  is globally generated and we have a natural identification  $H^0(X, \iota\mathcal{E}) \simeq H^0(X, \mathcal{E})^\vee$ .

More precisely, taking global sections in the sequence above, we get an exact sequence of vector spaces

$$0 \rightarrow H^0(X, \mathcal{E})^\vee \rightarrow H^0(X, T_X^\vee) \simeq V_5^\vee \rightarrow V_1^\vee \rightarrow 0,$$

where  $V_1 \subset V_5$  is the ‘‘vertex’’ of the line  $\ell \subset X$  (§3.1). We can thus identify  $H^0(X, \mathcal{E})$  with  $V_5/V_1$ , hence  $\mathbf{P}(H^0(X, \mathcal{E}))$  with the set of  $V_2 \subset V_5$  containing  $V_1$ . Geometrically, the codimension-2 Schubert cycle  $\sigma_{20}(V_2) \subset G(2, V_5)$  meets  $X$  along a 1-cycle of the form  $\ell + \Gamma_5^1$ , where  $\Gamma_5^1$  is an elliptic quintic bisecant to  $\ell$ . This cycle is the zero-locus of the section of  $\mathcal{E}$  defined (up to scalars) by  $V_2$ .

**Proposition 8.1.** *Let  $X$  be a Fano threefold of type  $\mathcal{X}_{10}^0$  and let  $\ell \subset X$  be a line. There is a birational isomorphism*

$$\varphi_\ell : \mathcal{M}_X(2; 1, 5) \dashrightarrow F(X_\ell)$$

which is compatible with the involutions on  $\mathcal{M}_X(2; 1, 5)$  and  $F(X_\ell)$ .

*Proof.* Let  $[\mathcal{E}] \in \mathcal{M}_X(2; 1, 5)$  be a globally generated vector bundle and consider the exact sequence

$$0 \rightarrow \mathcal{I}_\ell \otimes \mathcal{E} \rightarrow \mathcal{E} \rightarrow \mathcal{E}|_\ell \simeq \mathcal{O}_\ell \oplus \mathcal{O}_\ell(1) \rightarrow 0.$$

Since every proper subspace of  $H^0(\ell, \mathcal{O}_\ell(1))$  has a base-point, and  $\mathcal{E}$  is globally generated, the restriction map  $H^0(X, \mathcal{E}) \rightarrow H^0(\ell, \mathcal{E}|_\ell)$  is surjective. Hence  $H^0(X, \mathcal{I}_\ell \otimes \mathcal{E})$  is one-dimensional, generated by  $s$ . The zero-locus of  $s$  is  $\Gamma_4^0 \cup \ell$ , where  $\Gamma_4^0$  is a rational quartic curve bisecant to  $\ell$ . The line transform  $\psi_\ell$  maps  $\Gamma_4^0$  to a conic on  $X_\ell$ . This defines a rational map  $\varphi_\ell$  as required.

Conversely, the inverse image by  $\psi_\ell$  of a general conic on  $X_\ell$  is a rational quartic  $\Gamma_4^0$  bisecant to  $\ell$ . Applying the Serre construction to the degenerate elliptic quintic  $\Gamma_4^0 \cup \ell$ , we obtain a stable vector bundle of rank 2 on  $X$  with Chern numbers 1 and 5 and a section that vanishes on  $\Gamma_4^0 \cup \ell$ . The map  $\varphi_\ell$  is therefore birational. We now prove that it is compatible with the involutions.

Recall that  $\iota\mathcal{E}$  is dual to the kernel of the evaluation map  $H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E}$ . Let  $s$  (resp.  $s'$ ) be a section of  $\mathcal{E}$  (resp.  $\iota\mathcal{E}$ ) generating  $H^0(X, \mathcal{I}_\ell \otimes \mathcal{E})$  (resp.  $H^0(X, \mathcal{I}_\ell \otimes \iota\mathcal{E})$ ), and denote its zero-locus by  $\Gamma_4^0 \cup \ell$  (resp.  $\Gamma_4^0 \cup \ell$ ). By the definition of  $\iota\mathcal{E}$ , the section  $s$  defines on  $\Gamma_4^0$  a section  $\sigma$  of  $(\iota\mathcal{E})^\vee$ , and this section does not vanish (since it defines a nonzero section of  $\mathcal{E}|_{\Gamma_4^0}$ ). We get an exact sequence

$$0 \rightarrow \mathcal{O}_{\Gamma_4^0} \rightarrow (\iota\mathcal{E})^\vee|_{\Gamma_4^0} \rightarrow \mathcal{L}^\vee \rightarrow 0,$$

where  $\mathcal{L}$  is a line bundle on  $\Gamma_4^0$  of degree 4. On the other hand, the pairing  $(\sigma, s'|_{\Gamma_4^0}) \in \mathcal{O}_{\Gamma_4^0}$  is identically zero, since  $s'$  vanishes on  $\ell$ , which meets  $\Gamma_4^0$ . This means that on  $\Gamma_4^0$ , the section  $s'$  of  $\iota\mathcal{E}$  is in fact a section of  $\mathcal{L}$ . Therefore  $s'$  must vanish at four points of  $\Gamma_4^0$ : the two intersection points with  $\ell$ , and

two other points which must be on  $\Gamma_4^0$ . Thus  $\Gamma_4^0$  and  $\Gamma_4^{\prime 0}$  meet in two points, hence the corresponding conics  $c = \psi_\ell(\Gamma_4^0)$  and  $c' = \psi_\ell(\Gamma_4^{\prime 0})$  also meet in two points. By Lemma 6.1, these two conics are in involution.  $\square$

**Theorem 8.2.** *Let  $X$  be any Fano threefold of type  $\mathcal{X}_{10}^0$ . The moduli space  $\mathcal{M}_X(2; 1, 5)$  of semistable rank-2 torsion-free sheaves on  $X$  with Chern numbers  $c_1 = 1$ ,  $c_2 = 5$ , and  $c_3 = 0$ , is a smooth projective irreducible surface.*

*The non locally free sheaves and the non globally generated vector bundles in  $\mathcal{M}_X(2; 1, 5)$  are respectively parametrized by two copies  $\Gamma_{\text{nf}}$  and  $\Gamma_{\text{ng}}$  of the curve  $\Gamma(X)$  of lines in  $X$ . These two curves are exchanged by the involution  $\iota$  on the moduli space.*

When  $X$  is general,  $\Gamma(X)$  is a smooth projective irreducible curve of genus 71 ([IsP], Theorem 4.2.7).

*Proof.* First suppose that  $[\mathcal{E}] \in \mathcal{M}_X(2; 1, 5)$  is a globally generated vector bundle. A general section of  $\mathcal{E}$  vanishes along a smooth elliptic quintic  $\Gamma_5^1$  and yields an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \mathcal{I}_{\Gamma_5^1}(1) \rightarrow 0.$$

Twisting by  $\mathcal{E}^\vee$  and taking cohomology, we obtain

$$h^2(X, \mathcal{E}nd(\mathcal{E})) = h^2(X, \mathcal{I}_{\Gamma_5^1} \otimes \mathcal{E}) = h^1(\Gamma_5^1, \mathcal{E}|_{\Gamma_5^1}).$$

By the Atiyah classification,  $\mathcal{E}|_{\Gamma_5^1}$  is the direct sum of two line bundles, both globally generated.

Suppose that  $[\mathcal{E}]$  is *not* a smooth point of  $\mathcal{M}_X(2; 1, 5)$ . The vector space  $H^1(\Gamma_5^1, \mathcal{E}|_{\Gamma_5^1})$  is nonzero, and one of these line bundles is trivial:  $\mathcal{E}|_{\Gamma_5^1} = \mathcal{O}_{\Gamma_5^1} \oplus \mathcal{O}_{\Gamma_5^1}(1)$ . But  $\mathcal{E}|_{\Gamma_5^1}$  is the normal bundle to  $\Gamma_5^1$  in  $X$ . In the normal sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & N_{\Gamma_5^1/X} & \rightarrow & N_{\Gamma_5^1/G(2, V_5)} & \rightarrow & (N_{X/G(2, V_5)})|_{\Gamma_5^1} \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \rightarrow & \mathcal{O}_{\Gamma_5^1} \oplus \mathcal{O}_{\Gamma_5^1}(1) & \rightarrow & \mathcal{O}_{\Gamma_5^1}(1)^5 & \rightarrow & \mathcal{O}_{\Gamma_5^1}(1)^2 \oplus \mathcal{O}_{\Gamma_5^1}(2) \rightarrow 0 \end{array}$$

the image in  $N_{\Gamma_5^1/G(2, V_5)}$  of the factor  $\mathcal{O}_{\Gamma_5^1}(1)$  of  $N_{\Gamma_5^1/X}$  maps to zero in  $(N_{X/G(2, V_5)})|_{\Gamma_5^1}$ , which means that the corresponding linear form defines a tangent hyperplane to  $X$ . This contradicts the smoothness of  $\Gamma_5^1$ .

Suppose now that  $[\mathcal{F}] \in \mathcal{M}_X(2; 1, 5)$  is not locally free. We saw that  $\mathcal{F}$  fits into an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow T_X^\vee \rightarrow \mathcal{O}_\ell \rightarrow 0$$

for a unique line  $\ell$  in  $X$ . We are thus in the same situation as in [IM], Proposition 5.12, and the same proof yields  $\text{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$ .

Finally, suppose that  $\mathcal{E}$  is locally free but not globally generated. We saw that  $\mathcal{E}$  fits into an exact sequence

$$0 \rightarrow \mathcal{E}^\vee \rightarrow H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow 0,$$

where the non locally free sheaf  $[\mathcal{F}] \in \mathcal{M}_X(2; 1, 5)$  is as above. From this sequence we infer

$$H^2(X, \mathcal{E}nd(\mathcal{E})) \simeq H^1(X, \mathcal{E} \otimes \mathcal{F})$$

and

$$H^1(X, \mathcal{E} \otimes \mathcal{F}) \simeq \text{Ext}^2(\mathcal{F}, \mathcal{F})$$

(apply the functor  $\text{Hom}(\cdot, \mathcal{F})$  and use  $H^1(X, \mathcal{F}) = H^2(X, \mathcal{F}) = 0$ ). So by the previous case we obtain the vanishing of  $H^2(X, \mathcal{E}nd(\mathcal{E}))$ .

We can now conclude that  $\mathcal{M}_X(2; 1, 5)$  is a smooth surface. Since it is birational to  $F(X_\ell)$  (Proposition 8.1), it is also irreducible for  $X$  *general* (see §6.3). But the vanishing of  $\text{Ext}^2(\mathcal{F}, \mathcal{F})$ , for any point  $[\mathcal{F}]$  of  $\mathcal{M}_X(2; 1, 5)$ , also proves that for any family  $\mathcal{X} \rightarrow S$  of Fano threefolds of type  $\mathcal{X}_{10}^0$ , the relative moduli scheme  $\mathcal{M}_{\mathcal{X}}(2; 1, 5) \rightarrow S$  is smooth over  $S$ . In particular, since its general fibers are connected, *any* fiber is connected. This proves that  $\mathcal{M}_X(2; 1, 5)$  is always connected.

The involution  $\iota$  was defined by mapping a globally generated vector bundle  $\mathcal{E}$  to the globally generated vector bundle  $\iota\mathcal{E}$  defined by the exact sequence

$$0 \rightarrow (\iota\mathcal{E})^\vee \rightarrow H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow 0$$

If we replace  $\mathcal{E}$  by the non locally free sheaf  $\mathcal{F} \in \mathcal{M}_X(2; 1, 5)$  defined by the line  $\ell$ , this exact sequence gives for  $\iota\mathcal{F}$  the non globally generated vector bundle associated to the same line. This shows that  $\iota$  extends to the curve  $\Gamma_{\text{nf}}$  of non locally free sheaves in  $\mathcal{M}_X(2; 1, 5)$ , which is mapped bijectively to the curve  $\Gamma_{\text{ng}}$  of non globally generated vector bundles. Being an involution,  $\iota$  also extends to  $\Gamma_{\text{ng}}$ . We can finally conclude that  $\iota$  defines a regular involution of the smooth surface  $\mathcal{M}_X(2; 1, 5)$  which permutes the curves  $\Gamma_{\text{nf}}$  and  $\Gamma_{\text{ng}}$ .  $\square$

**Corollary 8.3.** *For any Fano threefold  $X$  of type  $\mathcal{X}_{10}^0$ , the surface  $F(X)$  is irreducible.*

*Proof.* This follows directly from Theorem 8.2, Proposition 8.1, and the fact that  $X$  is itself a line transform (§7.2).  $\square$

## 9. The reconstruction theorem

Following Logachev ([L]), we show that a general  $X_{10}$  can be recovered from its Fano surface.



**Theorem 9.1** (Logachev). *Let  $X$  and  $X'$  be general Fano threefolds of type  $\mathcal{X}_{10}$  and let  $f : F_g(X) \xrightarrow{\sim} F_g(X')$  be an isomorphism. There exists an isomorphism  $\varphi : X \xrightarrow{\sim} X'$  inducing  $f$ .*

**Corollary 9.2.** *Assume that  $X$  is general. For any smooth  $\tau$ -conic  $c \subset X$ , the threefolds  $X_c$  and  $X_{\iota(c)}$  are isomorphic.*

*Proof.* This follows from Theorem 9.1 and Proposition 7.2.  $\square$

**Corollary 9.3.** *For a general  $X$ , the only nontrivial automorphism of the minimal surface  $F_m(X)$  is the involution  $\iota$ .*

*Proof.* The cotangent bundle of  $F_m(X)$  is generated by global sections (Corollary 7.3) and defines a morphism

$$\ell_{10} : F_m(X) \rightarrow G(2, V_{10}),$$

where  $V_{10} = H^0(F_m(X), \Omega_{F_m(X)}^\vee)$ . Since global sections of  $\Omega_{F_m(X)}^1$  are anti-invariant by the involution  $\iota$  ([L], Proposition 0.5),  $\ell_{10}$  factors through the action of  $\iota$ . Let  $V_8$  be the quotient of  $V_{10}$  by the 2-dimensional vector space  $\ell_{10}([c_X])$ . By the tangent bundle theorem (§6.3), there is a diagram

$$(9.1) \quad \begin{array}{ccc} F(X) & \xrightarrow{\ell_8} & G(2, V_8) \\ \downarrow & \searrow & \uparrow \\ & F_\iota(X) & \\ & \downarrow & \\ F_m(X) & \xrightarrow{\ell_{10}} & G(2, V_{10}) \\ & \searrow & \uparrow \\ & F_{m,\iota}(X) & \end{array}$$

where the rational map  $G(2, V_{10}) \dashrightarrow G(2, V_8)$  is induced by the projection  $V_{10} \rightarrow V_8$ . The map  $\ell_{10,\iota}$  is generically injective, because  $\ell_{8,\iota}$  is:  $\ell_8$  sends  $\langle c, V_4 \rangle$  to  $\langle c \cap \iota(c) \rangle$ , the lines in  $\mathbf{P}(V_5)$  corresponding to the points of the line  $\langle c \cap \iota(c) \rangle$  span the hyperplane  $\mathbf{P}(V_4)$ , and we recover  $c \cup \iota(c)$  by intersecting  $G(2, V_4)$  with  $X$ .

Assume now that  $X$  is very general, so that the endomorphism ring of  $\text{Alb}(F_m(X)) \simeq J(X)$  is trivial (Corollary 5.3). Any nontrivial automorphism  $\sigma$  of  $F_m(X)$  then acts on the tangent space  $T_{J(X),0} = V_{10}$  as  $\pm \text{Id}$ , hence  $\ell_{10}$  factors through the action of  $\sigma$ . This implies  $\sigma = \iota$ .  $\square$

*Proof of Theorem 9.1.* We will reconstruct  $X$  from the abstract surface  $F_g(X)$ . The successive steps are the following.

1) From the abstract surface  $F_g(X)$ , we first recover the minimal surface  $F_m(X)$  with its involution  $\iota$ . By Corollary 7.3, we also recover the map

$F_\iota(X) \rightarrow G(2, V_8)$  which, when we will have constructed  $X$ , will map a conic  $c$  to the line  $\langle c \cap \iota(c) \rangle$ .

2) In  $\mathbf{P}(V_8)$ , we recover the plane  $\Pi$  and the conics  $c_X$  and  $c_U^\vee$ .

3) We consider the projection to  $\mathbf{P}(V_8) \dashrightarrow \mathbf{P}(V_8/U^\vee) = \mathbf{P}^4$  which, when we will have constructed  $X$ , will induce a birational isomorphism  $\kappa : W \dashrightarrow \mathbf{P}^4$  (Proposition 3.2). The problem at that point is that we do not know where to locate  $W$  in  $\mathbf{P}(V_8)$ . Nevertheless, we can reconstruct the rational normal cubic  $\Gamma_3^0$  in  $\mathbf{P}^4$  which defines  $\kappa^{-1}$  (Proposition 3.2).

4) The map  $\kappa^{-1}$  sends  $\mathbf{P}^4$  to a copy of  $W$  in some  $\mathbf{P}(V'_8)$ . From  $F_\iota(X)$ , we are able to find an identification between  $\mathbf{P}(V_8)$  and  $\mathbf{P}(V'_8)$ .

5) So we get  $W$  inside  $\mathbf{P}(V_8)$ , and intersecting it with the lines parametrized by  $F_\iota(X)$ , we get a surface  $S$  whose “quadratic span” is the Fano threefold  $X$  we are looking for.

*Step 1.* We know that  $L_\sigma$  and  $L'_\sigma$  are the unique rational curves in  $F_g(X)$  and  $F_g(X')$ , so  $f(L_\sigma) = L'_\sigma$ . Moreover these curves are  $(-1)$ -curves, and contracting them, we get the minimal surfaces  $F_m(X)$  and  $F_m(X')$ , and an isomorphism  $f : F_m(X) \xrightarrow{\sim} F_m(X')$  which sends  $\iota([c_X])$  to  $\iota'([c_{X'}])$ .

We keep the notation and results of the proof of Corollary 9.3. We have a commutative diagram

$$\begin{array}{ccc} F_m(X) & \xrightarrow{\ell_{10}} & G(2, V_{10}) \\ \downarrow f & & \downarrow f \\ F_m(X') & \xrightarrow{\ell'_{10}} & G(2, V'_{10}), \end{array}$$

where the maps  $\ell_{10}$  and  $\ell'_{10}$  have degree 2 onto their images, and factor through the involutions  $\iota$  and  $\iota'$  respectively. Therefore we recover  $\iota$  and  $\iota'$ , and also the points  $[c_X] \in F_g(X)$  and  $[c_{X'}] \in F_g(X')$ .

The map  $f$  induces a map from the diagram (9.1) to the corresponding diagram for  $X'$ .

One must think of the map  $\ell_8$  as sending a conic  $c$  to the line  $\langle c \cap \iota(c) \rangle$  – except that we do not know yet how to identify a point in the abstract surface  $F_g(X)$  with a conic on  $X$ . What we have is the abstract 2-dimensional family of lines  $\text{Im}(\ell_8)$  in  $\mathbf{P}(V_8)$ . If we knew how to construct  $W$  in  $\mathbf{P}(V_8)$ , intersecting it with these lines, we would get a surface  $S$ , and the base-locus of the quadrics containing  $S$  would be  $X$ .

The problem at this point is that we do not know how to locate  $W$  in our abstract  $\mathbf{P}(V_8)$ . Logachev’s idea is to use the birational isomorphism  $\kappa^{-1} : \mathbf{P}^4 \dashrightarrow W$  defined in §3.6 to reconstruct  $W$ . Since its inverse  $\kappa$  is just

the projection from the 2-plane  $\Pi = \langle c_X \rangle$ , we first need to reconstruct this plane.

*Step 2.* Let  $L_i$  be the image of  $L_\sigma$  in  $F_i(X)$ . For  $[c] \in L_i$ , the line  $\ell_8([c])$  is tangent to the conic  $c_U^\vee \subset \Pi$ . So we recover the plane  $\Pi$ , and the isomorphism  $L_\sigma \simeq c_U^\vee \simeq c_U$ . Moreover  $f$  induces an isomorphism between  $\Pi$  and the corresponding plane  $\Pi' \subset \mathbf{P}(V_8')$ , restricting to an isomorphism between the conics  $c_U^\vee$  and  $(c_U')^\vee$ .

On the other hand, there exists a finite set of points  $[c]$  in  $F_i(X) - L_i$  such that  $\ell_8([c])$  meets  $\Pi$ . Moreover the intersection points with  $\Pi$  belong to a unique conic in  $\Pi$ ,<sup>10</sup> namely  $c_X \subset \Pi$ . We can therefore recover this conic from our data, in such a way that  $f$  induces an isomorphism between  $c_X \subset \Pi$  and  $c_X' \subset \Pi'$ .

*Step 3.* Now consider the projection from  $V_8$  to  $V_8/U_3^\vee$ . The next claim is that we can reconstruct the rational cubic  $\Gamma_3^0 \subset \mathbf{P}(V_8/U_3^\vee)$ .

For this we observe that from the two conics  $c_X$  and  $c_U^\vee$  in  $\Pi$ , we can define the elliptic curve

$$\Gamma_X^1 = \{(t, x) \in c_U^\vee \times c_X \mid x \in T_{c_U^\vee, t}\},$$

a double cover of both  $c_X$  and  $c_U^\vee$ . Let  $\mathcal{S}_{2, V_8}$  denote the rank-2 tautological bundle on  $G(2, V_8)$ . There is an obvious map from  $\mathbf{P}(\mathcal{S}_{2, V_8})$  to  $\mathbf{P}(V_8)$ . On the other hand, we have a chain of inclusions

$$\Gamma_X^1 \hookrightarrow \mathbf{P}(\ell_8^* \mathcal{S}_{2, V_8})|_{L_i} \hookrightarrow \mathbf{P}(\ell_8^* \mathcal{S}_{2, V_8}).$$

The rational map defined as the composition

$$\mathbf{P}(\ell_8^* \mathcal{S}_{2, V_8}) \rightarrow \mathbf{P}(\mathcal{S}_{2, V_8}) \rightarrow \mathbf{P}(V_8) \dashrightarrow \mathbf{P}(V_8/U_3^\vee)$$

is not defined on  $\Gamma_X^1$ . However, since  $\mathbf{P}(\ell_8^* \mathcal{S}_{2, V_8})|_{L_i}$  is a divisor in  $\mathbf{P}(\ell_8^* \mathcal{S}_{2, V_8})$ , the restriction to this divisor is a well-defined rational map. And since  $\Gamma_X^1$  is itself a divisor in  $\mathbf{P}(\ell_8^* \mathcal{S}_{2, V_8})|_{L_i}$ , the restriction is again well-defined. We end up with a well-defined map  $\Gamma_X^1 \rightarrow \mathbf{P}(V_8/U_3^\vee)$ .

Of course we can do the same for  $X'$ , and we get a compatible isomorphism between  $\Gamma_X^1$  and  $\Gamma_{X'}^1$ .

Now recall (§3.6) that the total transform of a point  $x$  of  $\Pi$  by the birational isomorphism  $\kappa : W \dashrightarrow \mathbf{P}(V_8/U_3)$  is a line  $\ell_x$  that meets  $\Gamma_3^0$  at the points corresponding to those two points in  $c_U^\vee$  whose tangents pass through  $x$ . This implies that a point  $(t, x)$  in  $\Gamma_X^1$  must be mapped in  $\mathbf{P}(V_8/U_3)$  to some point  $p(t, x)$  of the bisecant  $\ell_t$ .

<sup>10</sup>One must check that we get at least five such intersection points; see [L].

But there is another point  $(t', x)$  on  $\Gamma_X^1$ , mapping to a point  $p(t', x)$  on the same bisecant line  $\ell_x$ . Since, in general,  $p(t, x) \neq p(t', x)$ ,<sup>11</sup> we can recover the line  $\ell_x$  as the line joining these two points. Finally, if we consider the points  $(t, x)$  and  $(t, x')$  of  $\Gamma_X^1$ , we see that the lines  $\ell_x$  and  $\ell_{x'}$  must meet on the point of  $\Gamma_3^0$  corresponding to  $t$ . So this process allows to reconstruct the curve  $\Gamma_3^0$ , and an isomorphism with  $c_U^\vee$ .

*Step 4.* From  $\Gamma_3^0 \subset \mathbf{P}(V_8/U_3)$ , we can reconstruct a copy of  $W$ , say  $W^\dagger \subset V_8^\dagger$ , as the image of the birational map defined by the linear system of quadrics through  $\Gamma_3^0$ . We get the diagram

$$\begin{array}{ccccc} \mathbf{P}(V_8) & \dashrightarrow & \mathbf{P}(V_8/U_3) & \xrightarrow{|\mathcal{S}_{\Gamma_3^0(2)}|} & W^\dagger \\ \cup & & \cup & & \cap \\ W & & \Gamma_3^0 & & \mathbf{P}(V_8^\dagger) \end{array}$$

We do not know yet where to locate  $W$  in  $\mathbf{P}(V_8)$ , but we know that the identification of  $W$  with  $W^\dagger$  induces a unique linear isomorphism between  $\mathbf{P}(V_8)$  and  $\mathbf{P}(V_8^\dagger)$ . What we need to do is to recover this isomorphism from our data.

First observe that to a general point  $[c]$  of  $F(X)$  should correspond a conic  $(c, V_4)$ , and the point  $\theta([c]) = [U_3 \cap V_4]$  in  $\Pi = G(2, U_3)$ .

From our present knowledge, this point can be recovered as follows: the line  $\ell_5([c])$ , image of  $\ell_8([c])$  in  $G(2, V_8/U_3)$ , meets the linear span of  $\Gamma_3^0$  at one point. This point belongs to a unique bisecant line to  $\Gamma_3^0$ , meeting  $\Gamma_3^0$  at two points which we can identify with two points of  $c_U^\vee$ . The intersection of the corresponding tangents is  $\theta([c])$ . Moreover, we know that the projection  $\mathbf{P}(V_8) \dashrightarrow \mathbf{P}(V_8/U_3)$ , once restricted to  $\mathbf{P}(M_{V_4})$ , can be identified with the projection through that point  $\theta([c])$ ; in particular this applies to  $\ell_8([c])$ , and we can conclude that  $\ell_5([c])$  is the image of  $\ell_8([c])$  by projection from  $\theta([c])$ .

Now Logachev considers those pairs  $(c, c')$  such that  $\ell_5([c])$  and  $\ell_5([c'])$  meet at one point, say  $m(c, c')$ ; there is a two-dimensional family of such pairs. We know the image of  $m(c, c')$  in  $\mathbf{P}(V_8^\dagger)$ , and we want to reconstruct its preimage  $n(c, c')$  in  $\mathbf{P}(V_8)$ . But this is easy:  $\ell_5([c])$  and  $\theta([c])$  generate a plane  $\Pi_c$ , which meets the corresponding plane  $\Pi_{c'}$  at a unique point, which is  $n(c, c')$ .<sup>12</sup>

We are thus able to reconstruct the isomorphism  $\mathbf{P}(V_8) \rightarrow \mathbf{P}(V_8^\dagger)$  on the special points  $n(c, c')$ . Since they are not contained in any hyperplane, they completely determine the isomorphism we were looking for.

<sup>11</sup>This can be checked by a lengthy direct calculation.

<sup>12</sup>To see this, observe that  $\Pi_c$  is contained in the 3-plane  $\Theta_c = \mathbf{P}(V_8) \cap \mathbf{P}(\wedge^2 V_4)$ , which meets  $\Theta_{c'}$  along  $\mathbf{P}(V_8) \cap \mathbf{P}(\wedge^2(V_4 \cap V_4'))$ ; but in general this is just a point. So it must be  $n(c, c')$ .

*Step 5.* In the preceding steps, we have reconstructed  $W$  in  $\mathbf{P}(V_8)$ . Moreover, we have done that from purely projective constructions in terms of the family of lines  $\ell_8([c])$ , when  $[c]$  describes the abstract surface  $F_m(X)$ . This implies that the isomorphism  $f : F_g(X) \xrightarrow{\sim} F_g(X')$  induces an isomorphism  $\varphi$  between  $\mathbf{P}(V_8)$  and  $\mathbf{P}(V'_8)$ , mapping  $W$  to an isomorphic copy  $W'$ , and compatible with  $\ell_8$  and  $\ell'_8$ . This means that we can recover a surface  $S$  in  $X$  by intersecting the lines  $\ell_8(c)$  with  $W$ , and that  $f$  maps  $S$  to the corresponding surface  $S'$  in  $X'$ .

But  $S$  determines  $X$ , as the intersection of the quadrics in  $\mathbf{P}(V_8)$  containing  $S$ <sup>13</sup> and this implies that  $\varphi$  maps  $X$  isomorphically onto  $X'$ .

Finally,  $u = f^{-1} \circ \varphi^*$  is an automorphism of  $F_g(X)$  which descends to an automorphism  $u_m$  of  $F_m(X)$  such that  $\ell_8 \circ u_m = \ell_8$ . Since  $\ell_8$  is generically injective,  $u_m$  is the identity, hence so is  $u$ . The theorem is proved.  $\square$

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<sup>13</sup>Indeed, Logachev shows that the surface  $S$  is cut out on  $X$  by a hypersurface of degree 21.

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