ON NODAL PRIME FANO THREEFOLDS OF DEGREE 10

OLIVIER DEBARRE, ATANAS ILIEV, AND LAURENT MANIVEL

Abstract. We study the geometry and the period map of nodal complex prime Fano threefolds with index 1 and degree 10. We show that these threefolds are birationally isomorphic to Verra threefolds, i.e., hypersurfaces of bidegree $(2,2)$ in $\mathbf{P}^2 \times \mathbf{P}^2$. Using Verra's results on the period map for these threefolds and on the Prym map for double étale covers of plane sextic curves, we prove that the fiber of the period map for our nodal threefolds is the union of two disjoint surfaces, for which we give several descriptions. This result is the analog in the nodal case of a result of [DIM] in the smooth case.

Dedicated to Fabrizio Catanese on his 60th birthday

Contents

O. Debarre and L. Manivel are part of the project VSHMOD-2009 ANR-09-BLAN-0104-01.

1. Introduction

Nodal prime Fano threefolds of degree 10. There are 10 irreducible families of smooth complex Fano threefolds X with Picard group $\mathbf{Z}[K_X]$, one for each degree $(-K_X)^3 = 2g-2$, where $g \in \{2, 3, \dots, 10, 12\}$. The article is a sequel to [DIM], where we studied the geometry and the period map of *smooth* complex Fano threefolds X with Picard group $\mathbf{Z}[K_X]$ and degree 10. Following again Logachev ([Lo], §5), we study here complex Fano threefolds X with Picard group $\mathbf{Z}[K_X]$ and degree 10 which are general *with one node.* They are degenerations of their smooth counterparts and their geometry is made easier to study by the fact that they are (in two ways) birationally conic bundles over $\mathbf{P}^2$.

Two conic bundle structures. More precisely, the nodal variety X is anticanonically embedded in $\mathbf{P}^7$, and it has long been known that the projection from its node O maps X birationally onto a (singular) intersection of three quadrics $X_O \subset \mathbf{P}^6$. The variety X_O, hence also X, is therefore (birationally) a conic bundle with discriminant the union of a line Γ_1 and a smooth sextic Γ_6 (see [B1], 5.6.2), and associated connected double étale cover $\pi : \widetilde{\Gamma}_6 \cup \Gamma_1^1 \cup \Gamma_1^2 \to \Gamma_6 \cup \Gamma_1$ (§4.2). On the other hand, the "double projection" of X from O (i.e., the linear projection from the 4-dimensional embedded Zariski tangent space at O) is also (birationally) a conic bundle, with discriminant curve another smooth plane sextic $\Gamma_6^\star$ and associated connected double étale cover $\pi^\star : \widetilde{\Gamma}_6^\star \to \Gamma_6^\star$ (§4.1).

Birational isomorphism with Verra threefolds. We show (Theorem 4.5) that these two conic bundle structures define a birational map from X onto a (general) hypersurface $T \subset \mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(2,2)$. These threefolds T were studied by Verra in [V]. Both projections to $\mathbf{P}^2$ are conic bundles and define the connected double étale covers π and $\pi^\star$ of the discriminant curves, which are nonisomorphic smooth sextics. In particular, the associated Prym varieties $\mathrm{Prym}(\pi)$ and $\mathrm{Prym}(\pi^\star)$ are isomorphic (to the intermediate Jacobian $J(T)$) and Verra showed that the Prym map from the space of connected double étale covers of smooth plane sextics to the moduli space of 9-dimensional principally polarized abelian varieties has degree 2.

Fibers of the period map. This information is very useful for the determination of the fiber of the period map of our nodal threefolds X, i.e., for the description of all nodal threefolds of the same type with intermediate Jacobian isomorphic to $J(X)$. This $J(X)$ is a 10-dimensional extension by $\mathbf{C}^\star$ of the intermediate Jacobian of the minimal desingularization of X, which is also the intermediate Jacobian of T, and the extension class depends only on T (§7.3). A nodal X can be uniquely reconstructed from the data of a general sextic Γ_6 and an even theta characteristic M on the union of Γ_6 and a line Γ_1: by work of Dixon, Catanese, and Beauville, the sheaf M (on $\mathbf{P}^2$) has a free resolution by a 7×7 symmetric matrix of linear forms, which defines the net of quadrics whose intersection is X_O (Theorem 6.1 and Remark 6.2). We show that given the double cover $\pi : \widetilde{\Gamma}_6 \to \Gamma_6$, the set of even theta characteristics M on $\Gamma_6 \cup \Gamma_1$, which induce π on Γ_6, is isomorphic to the quotient of the *special surface* $S^{\mathrm{odd}}(\pi)$ (as defined in [B3]) by its natural involution σ (Proposition 6.3). Together with Verra's result, this implies that *the general fiber of the period map for our nodal threefolds is birationally the union of the two surfaces* $S^{\mathrm{odd}}(\pi)/\sigma$ and $S^{\mathrm{odd}}(\pi^\star)/\sigma^\star$ (§7.3).

A result of Logachev ([Lo], Proposition 5.8) says that the surface S^{odd} is also isomorphic to the minimal model $F_m(X)$ of the normalization of the Fano surface $F_g(X)$ of conics contained in X. On the one hand, we obtain the analog in the nodal case of the reconstruction theorem of [DIM], Theorem 9.1: a general nodal X can be reconstructed from the surface $F_g(X)$. On the other hand, the present description of the fiber of the period map at a nodal X fits in with the construction in [DIM] of two (proper smooth) surfaces in the fiber of the period map at a smooth X', one of them isomorphic to $F_m(X')/\iota$ ([DIM], Theorem

6.4). In both the smooth and nodal cases, the threefolds in the fiber of the period map are obtained one from another by explicit birational transformations called line and conic transformations (see §4.3 and §5.5).

Unfortunately, because of properness issues, we cannot deduce from our present results that a general fiber of the period map for smooth prime Fano threefolds of degree 10 consists of just these two surfaces (although we prove here that these surfaces are distinct, which was a point missing from [DIM]), although we certainly think that this is the case.

Singularities of the theta divisor. The singular locus of the theta divisor of the intermediate Jacobian of a Verra threefold was described in [V]. This description fits in with a conjectural description of the singular locus of the theta divisor of the intermediate Jacobian of a smooth prime Fano threefold of degree 10 that we give in §8. This conjecture would imply the conjecture about the general fiber of the period map mentioned above.

2. Notation

• As a general rule, V_m denotes an m-dimensional (complex) vector space, $\mathbf{P}_m$ an m-dimensional projective space, and Γ_m a plane curve of degree m. We fix a 5-dimensional complex vector space V_5 and we consider the Plücker embedding $G(2,V_5)\subset\mathbf{P}_9=\mathbf{P}(\wedge^2 V_5)$ and its smooth intersection W with a general $\mathbf{P}_7$ (§3.1).

• $O\in W$ is a general point, $\Omega\subset\mathbf{P}_9$ is a general quadric with vertex O, and $X=W\cap\Omega\subset\mathbf{P}_7$ is an anticanonically embedded prime Fano threefold with one node at O; $\widetilde{X}\to X$ is the blow-up of O.

• $p_O:\mathbf{P}_7 \dashrightarrow \mathbf{P}^6_O$ is the projection from O. We write $\Omega_O=p_O(\Omega)\subset\mathbf{P}^6_O$, general quadric, $W_O=p_O(W)\subset\mathbf{P}^6_O$, base-locus of a pencil Γ_1 of quadrics or rank 6, and $X_O=p_O(X)\subset\mathbf{P}^6_O$, base-locus of the net $\Pi=\langle\Gamma_1,\Omega_O\rangle$ of quadrics.

• W_O contains the 3-plane $\mathbf{P}^3_W=p_O(\mathbf{T}_{W,O})$ and X_O contains the smooth quadric surface $Q=\mathbf{P}^3_W\cap\Omega_O$. The singular locus of W_O is a rational normal cubic curve $C_O\subset\mathbf{P}^3_W$. The singular locus of X_O is $Q\cap C_O=\{s_1,\dots,s_6\}$.

• $\widetilde{\mathbf{P}}^6_O\to\mathbf{P}^6_O$ is the blow-up along $\mathbf{P}^3_W$, $\widetilde{W}_O\subset\widetilde{\mathbf{P}}^6_O$ is the (smooth) strict transform of W_O, and $\widetilde{X}_O\subset\widetilde{W}_O$ the (smooth) strict transform of X_O.

• The projection $p_W: X\dashrightarrow\mathbf{P}^2_W$ from the 4-plane $\mathbf{T}_{W,O}$ is a birational conic bundle with associated double étale cover $\pi^\star:\widetilde{\Gamma}^\star_6\to\Gamma^\star_6$ and involution $\sigma^\star$.

• $\Gamma_7 = \Gamma_6 \cup \Gamma_1 \subset \Pi$ is the discriminant curve of the net of quadrics Π that contain X_O, with associated double étale cover $\pi : \widetilde{\Gamma}_6 \cup \Gamma_1^1 \cup \Gamma_1^2 \to \Gamma_6 \cup \Gamma_1$ and involution σ. We write $\{p_1, \dots, p_6\} = \Gamma_1 \cap \Gamma_6$ and $\{\tilde{p}_1, \dots, \tilde{p}_6\} = \Gamma_1^1 \cap \widetilde{\Gamma}_6$.

• $F_g(X)$ is the connected surface that parametrizes conics on X, with singular locus the smooth connected curve $\widetilde{\Gamma}_6^\star$ of conics on X passing through O, and $\nu : \widetilde{F}_g(X) \to F_g(X)$ is the normalization (§5.2). The smooth surface $\widetilde{F}_g(X)$ carries an involution ι; its minimal model is $\widetilde{F}_m(X)$.

• S^{even} and S^{odd} are the special surfaces associated with π, with involution σ (§5.3). There is an isomorphism $\rho : \widetilde{F}_m(X) \xrightarrow{\sim} S^{\text{odd}}$ (§5.4).

• $T \subset \mathbf{P}^2 \times \mathbf{P}^2$ is a *Verra threefold,* i.e., a smooth hypersurface of bidegree $(2, 2)$.

3. The fourfolds W and W_O

3.1. **The fourfold W.** As explained in [DIM], §3, the intersections

$$W = G(2, V_5) \cap \mathbf{P}_7 \subset \mathbf{P}(\wedge^2 V_5),$$

whenever smooth and 4-dimensional, are all isomorphic under the action of $\mathrm{PGL}(V_5)$. They correspond dually to pencils $\mathbf{P}_7^\perp$ of skew-symmetric forms on V_5 which are all of maximal rank. The map that sends a form in the pencil to its kernel has image a smooth conic $c_U \subset \mathbf{P}(V_5)$ that spans a 2-plane $\mathbf{P}(U_3)$, where $U_3 \subset V_5$ is the unique common maximal isotropic subspace to all forms in the pencil (see §9.1 for explicit equations).

3.2. **Quadrics containing W.** To any one-dimensional subspace $V_1 \subset V_5$, one associates a Plücker quadric in $|\mathscr{I}_{G(2,V_5)}(2)|$ obtained as the pull-back of $G(2, V_5/V_1)$ by the rational map $\mathbf{P}(\wedge^2 V_5) \dashrightarrow \mathbf{P}(\wedge^2 (V_5/V_1))$. This gives a linear map

$$\gamma_G : \mathbf{P}(V_5) \longrightarrow |\mathscr{I}_{G(2,V_5)}(2)| \simeq \mathbf{P}^{49}$$

whose image consists of quadrics of rank 6. Since no such quadric contains $\mathbf{P}_7$ and $|\mathscr{I}_W(2)|$ has dimension 4, we obtain an isomorphism

$$\gamma_W : \mathbf{P}(V_5) \xrightarrow{\sim} |\mathscr{I}_W(2)|. \tag{1}$$

The quadric $\gamma_W([V_1]) \subset \mathbf{P}_7$ has rank 6 except if the vertex of $\gamma_G([V_1])$, which is the 3-plane $\mathbf{P}(V_1 \wedge V_5)$ contained in $G(2, V_5)$, meets $\mathbf{P}_7$ along a 2-plane, which must be contained in W. This happens if and only if $[V_1] \in c_U$ ([DIM], §3.3).

Given a point $O \in W$, corresponding to a 2-dimensional subspace $V_O \subset V_5$, the quadrics that contain W and are singular at O correspond, via the isomorphism (1), to the projective line $\mathbf{P}(V_O) \subset \mathbf{P}(V_5)$.

3.3. **The $\mathbf{P}^2$-bundle $\mathbf{P}(\mathscr{M}_O) \to \mathbf{P}^2_O$.** As in [DIM], §3.4, we define, for any hyperplane $V_4 \subset V_5$, the 4-dimensional vector space $M_{V_4} = \wedge^2 V_4 \cap V_8$ (where $V_8 \subset \wedge^2 V_5$ is the vector space such that $\mathbf{P}_7 = \mathbf{P}(V_8)$) and the quadric surface

$$Q_{W,V_4} = G(2, V_4) \cap \mathbf{P}(M_{V_4}) \subset W.$$

Let $\mathbf{P}^2_O \subset \mathbf{P}(V_5^\vee)$ be the set of hyperplanes $V_4 \subset V_5$ that contain V_O, and consider the rank-3 vector bundle $\mathscr{M}_O \to \mathbf{P}^2_O$ whose fiber over $[V_4]$ is $M_{V_4}/\wedge^2 V_O$ and the associated $\mathbf{P}^2$-bundle $\mathbf{P}(\mathscr{M}_O) \to \mathbf{P}^2_O$.

3.4. **The fourfold W_O.** Let O be a point of W, let $V_O \subset V_5$ be the corresponding 2-dimensional subspace, and let $p_O : \mathbf{P}_7 \dashrightarrow \mathbf{P}^6_O$ be the projection from O. We set

$$W_O = p_O(W) \subset \mathbf{P}^6_O.$$

The group $\mathrm{Aut}(W)$ acts on W with four orbits $O_1, \dots, O_4$ indexed by their dimensions (§9.1), so there are four different W_O. We will restrict ourselves to the (general) case $O \in O_4$, although similar studies can be made for the other orbits.

Since $O \in O_4$, the line $\mathbf{P}(V_O) \subset \mathbf{P}(V_5)$ does not meet the conic c_U (§9.1), hence all quadrics in the pencil $\gamma_W(\mathbf{P}(V_O))$ are singular at O and have rank 6 (§3.2). This pencil projects to a pencil of rank-6 quadrics in $\mathbf{P}^6_O$ whose base-locus contains the fourfold W_O. Since W_O has degree 4, it is equal to this base-locus and contains the 3-plane $\mathbf{P}^3_W = p_O(\mathbf{T}_{W,O})$. The locus of the vertices of these quadrics is a rational normal cubic curve $C_O \subset \mathbf{P}^3_W$, which is the singular locus of W_O and parametrizes lines in W through O (see §9.2 for explicit computations).

In fact, all pencils of rank-6 quadrics in $\mathbf{P}^6$ are isomorphic ([HP], Chapter XIII, §11). In particular, all quadrics in the pencil have a common 3-plane, and the fourfold that they define in $\mathbf{P}^6$ is isomorphic to W_O.

3.5. **The $\mathbf{P}^2$-bundle $\widetilde{W}_O \to \mathbf{P}^2_W$.** Let $\mathbf{P}^2_W$ parametrize 5-planes in $\mathbf{P}_7$ that contain $\mathbf{T}_{W,O}$ (or, equivalently, 4-planes in $\mathbf{P}^6_O$ that contain $\mathbf{P}^3_W$). Let $\varepsilon : \widetilde{\mathbf{P}}^6_O \to \mathbf{P}^6_O$ be the blow-up of $\mathbf{P}^3_W$, with $\widetilde{\mathbf{P}}^6_O \subset \mathbf{P}^6_O \times \mathbf{P}^2_W$, and let $\widetilde{W}_O \subset \widetilde{\mathbf{P}}^6_O$ be the strict transform of W_O.

We will prove in §9.2 that the projection $\widetilde{W}_O \to \mathbf{P}^2_W$ is a $\mathbf{P}^2$-bundle, and that $\widetilde{W}_O$ is smooth. Furthermore, there is an isomorphism $\mathbf{P}^2_O \simeq \mathbf{P}^2_W$ such that the induced isomorphism $\mathbf{P}^2_O \times \mathbf{P}^6_O \simeq \mathbf{P}^2_W \times \mathbf{P}^6_O$ gives by restriction *an isomorphism between the* $\mathbf{P}^2$*-bundles* $\mathbf{P}(\mathscr{M}_O) \to \mathbf{P}^2_O$ *and* $\widetilde{W}_O \to \mathbf{P}^2_W$ (see §9.3).

Finally, the strict transform $\widetilde{\mathbf{P}}^3_W$ of $\mathbf{P}^3_W$ in $\widetilde{W}_O$ is the intersection of the exceptional divisor of ε with $\widetilde{W}_O$; it has therefore dimension 3 everywhere. Since it contains the inverse image of the cubic curve $C_O \subset \mathbf{P}^3_W$, which is a surface (§9.2), it must be the blow-up of $\mathbf{P}^3_W$ along C_O. The fibers of the projection $\widetilde{\mathbf{P}}^3_W \to \mathbf{P}^2_W$ are the bisecant lines to C_O.

4. The threefolds X and X_O

We consider here singular threefolds

$$X = G(2, V_5) \cap \mathbf{P}_7 \cap \Omega,$$

where Ω is a quadric in $\mathbf{P}^9$, such that X *has a unique singular point* O*, which is a node.*

Lemma 4.1 (Logachev)**.** *The intersection* $G(2, V_5) \cap \mathbf{P}_7$ *is smooth, hence isomorphic to* W*, and one may choose* Ω *to be a cone with vertex* O*.*

Proof. We follow [Lo], Lemmas 3.5 and 5.7. If the intersection $W' = G(2, V_5) \cap \mathbf{P}_7$ is singular, the corresponding pencil $\mathbf{P}_7^\perp$ of skew-symmetric forms on V_5 contains a form of rank ≤ 2, and one checks that the singular locus of W' is

$$\operatorname{Sing}(W') = \bigcup_{\omega \in \mathbf{P}_7^\perp,\ \operatorname{rank}(\omega) \leq 2} G(2, \ker(\omega)) \cap \mathbf{P}_7,$$

hence is a union of linear spaces of dimension ≥ 1. In particular, the intersection with the quadric Ω either has at least two singular points or a singular point which is not a node. It follows that $W = G(2, V_5) \cap \mathbf{P}_7$ is smooth.

Consider now the map

$$\begin{array}{ccc} |\mathscr{I}_W(2)| & \dashrightarrow & \left\{ \begin{array}{c} \text{Hyperplanes in } T_{\mathbf{P}_7, O} \\ \text{containing } T_{W,O} \end{array} \right\} \simeq \mathbf{P}^2 \\ \Omega' & \longmapsto & T_{\Omega', O}. \end{array}$$

It is not defined exactly when Ω' is singular at O, which happens along the projective line $\gamma_W(\mathbf{P}(V_O))$ (§3.2). Since it is nonconstant, it is therefore surjective.

Assume that Ω is smooth at O. Since $W \cap \Omega$ is singular at O, we have $T_{W,O} \subset T_{\Omega_O}$, hence there exists a quadric $\Omega' \supset W$ smooth at O such that $T_{\Omega', O} = T_{\Omega, O}$. Some linear combination of Ω and Ω' is then singular at O and still cuts out X on W. ∎

Conversely, we will from now on consider a *general* quadric Ω with vertex O in the orbit O_4. The intersection $X = W \cap \Omega$ is then smooth except for one node at O and $\mathrm{Pic}(X)$ *is generated by the class of* $\mathscr{O}_X(1)$ (the lines from the two rulings of the exceptional divisor of the blow-up of O in X are numerically, but not algebraically, equivalent (see §7.2; the reader will check that the proof of this fact does not use the fact that X is locally factorial!), hence the local ring $\mathscr{O}_{X,O}$ is factorial ([M], (3.31)) and the Lefschetz Theorem still applies ([G], Exp. XII, cor. 3.6; the hypotheses $H^1(X, \mathscr{O}_X(-k)) = H^2(X, \mathscr{O}_X(-k)) = 0$ for all $k > 0$ follow from Kodaira vanishing on W)).

We keep the notation of §3.4 and set $\Omega_O = p_O(\Omega)$ and $X_O = p_O(X)$. Let $\widetilde{X} \to X$ be the blow-up of O. The projection p_O from O induces a morphism

$$\widetilde{X} \stackrel{\varphi}{\twoheadrightarrow} X_O \subset W_O \subset \mathbf{P}^6_O$$

which is an isomorphism except on the union of the lines in X through O. There are six such lines, corresponding to the six points $s_1, \dots, s_6$ of $\mathrm{Sing}(W_O) \cap \Omega_O$, which are the six singular points of X_O.

The threefold $X_O \subset \mathbf{P}^6_O$ is the complete intersection of three quadrics and conversely, given the intersection of W_O with a general smooth quadric Ω_O, its inverse image under the birational map $W \dashrightarrow W_O$ is a variety of type $\mathscr{X}_{10}$ with a node at O.

4.1. **The conic bundle $p_W : X \dashrightarrow \mathbf{P}^2_W$ and the double cover $\pi^\star : \widetilde{\Gamma}^\star_6 \to \Gamma^\star_6$.** Keeping the notation of §3.5, consider the projection

$$p_W : X \dashrightarrow \mathbf{P}^2_W$$

from the 4-plane $\mathbf{T}_{W,O}$. It is also induced on $X_O \subset \mathbf{P}^6_O$ by the projection from the 3-plane $\mathbf{P}^3_W = p_O(\mathbf{T}_{W,O})$, hence is a well-defined morphism on the strict transform $\widetilde{X}_O$ of X_O in the blow-up $\widetilde{\mathbf{P}}^6_O$ of $\mathbf{P}^6_O$ along $\mathbf{P}^3_W$, where $\widetilde{\mathbf{P}}^6_O \subset \mathbf{P}^6_O \times \mathbf{P}^2_W$.

Proposition 4.2. *The variety $\widetilde{X}_O$ is smooth and the projection $\widetilde{X}_O \to \mathbf{P}^2_W$ is a conic bundle with discriminant a smooth sextic $\Gamma^\star_6 \subset \mathbf{P}^2_W$.*

We denote the associated double cover by $\pi^\star : \widetilde{\Gamma}^\star_6 \to \Gamma^\star_6$. It is étale by [B1], prop. 1.5, and connected.[1]

[1]Any nontrivial conic bundle over $\mathbf{P}^2_O$ defines a nonzero element of the Brauer group of $\mathbf{C}(\mathbf{P}^2_O)$ which, since the Brauer group of $\mathbf{P}^2_O$ is trivial, must have some nontrivial residue: the double cover of at least one component of the discriminant curve must be irreducible.

Also, fibers of $\widetilde{X}_O \to \mathbf{P}^2_W$ project to conics in X_O. Since the strict transform of $\mathbf{P}^3_W$ in $\widetilde{W}_O$ is a $\mathbf{P}^1$-bundle (§9.2), they meet the quadric Q in two points.

Proof. We saw in §3.5 that $\widetilde{W}_O \to \mathbf{P}^2_W$ is a $\mathbf{P}^2$-bundle. The smoothness of $\widetilde{X}_O$ then follows from the Bertini theorem, which also implies that the discriminant curve is smooth. The fact that it is a sextic follows either from direct calculations or from Theorem 4.5. □

With the notation of §3.3, consider now the rank-3 vector bundle $\mathscr{M}_O \to \mathbf{P}^2_O$ whose fiber over $[V_4]$ is $M_{V_4}/\wedge^2 V_O$. Inside $\mathbf{P}(\mathscr{M}_O)$, the quadric Ω_O defines a conic bundle $Q_{\Omega,O} \to \mathbf{P}^2_O$, and $Q_{\Omega,O}$ is smooth by the Bertini theorem.

Proposition 4.3. *The double cover associated with the conic bundle* $Q_{\Omega,O} \to \mathbf{P}^2_O$ *is isomorphic to* $\pi^\star : \widetilde{\Gamma}^\star_6 \to \Gamma^\star_6$.

Proof. We saw in §3.5 that the $\mathbf{P}^2$-bundles $\widetilde{W}_O \to \mathbf{P}^2_W$ and $\mathbf{P}(\mathscr{M}_O) \to \mathbf{P}^2_O$ are isomorphic and the isomorphism restricts to an isomorphism between the conic bundles $\widetilde{X}_O \to \mathbf{P}^2_W$ and $Q_{\Omega,O} \to \mathbf{P}^2_O$, because they are pull-backs of the same quadric $\Omega_O \subset \mathbf{P}^6_O$. It follows that the associated double covers are isomorphic. □

4.2. **The double cover $\widetilde{\Gamma}_6 \to \Gamma_6$.** Let $\mathbf{P}$ be the net of quadrics in $\mathbf{P}^6_O$ that contain X_O. The discriminant curve $\Gamma_7 \subset \mathbf{P}$ (which parametrizes singular quadrics in $\mathbf{P}$) is the union of the line Γ_1 of quadrics that contain W_O, and a sextic Γ_6. The pencil Γ_1 meets Γ_6 transversely at six points $p_1, \dots, p_6$ corresponding to quadrics whose vertices are the six singular points $s_1, \dots, s_6$ of X_O ([B1], §5.6.2).

All quadrics in $\mathbf{P}$ have rank at least 6 (because rank-5 quadrics in $\mathbf{P}^6_O$ have codimension 3). In particular, there is a double étale cover

$$\pi : \widetilde{\Gamma}_6 \cup \Gamma^1_1 \cup \Gamma^2_1 \to \Gamma_6 \cup \Gamma_1$$

corresponding to the choice of a (2-dimensional) family of 3-planes contained in a quadric of rank 6 in $\mathbf{P}$. The 3-plane $\mathbf{P}^3_W$ is contained in all the quadrics in the pencil Γ_1 and defines the component Γ^1_1 and points $\{\tilde{p}_1, \dots, \tilde{p}_6\} = \Gamma^1_1 \cap \widetilde{\Gamma}_6$. The curve $\widetilde{\Gamma}_6$ is smooth and connected (footnote 1).

4.3. **Line transformations.** Let ℓ be a general line contained in X. As in the smooth case ([DIM], §6.2), one can perform a *line transformation* of X along ℓ:

$$\begin{array}{ccc} \widetilde{X}_\ell & \overset{\chi}{\dashrightarrow} & \widetilde{X}'_\ell \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon'} \\ X & \overset{\psi_\ell}{\dashrightarrow} & X_\ell, \end{array}$$

where ε is the blow-up of ℓ, with exceptional divisor E, the birational map χ is a $(-E)$-flop, ε' is the blow-down onto a line $\ell' \subset X_\ell$ of a divisor $E' \equiv -K_{\widetilde{X}'_\ell} - \chi(E)$, and X_ℓ is another nodal threefold of type $\mathscr{X}_{10}$. The map ψ_ℓ is associated with a linear subsystem of $|\mathscr{I}_\ell^3(2)|$. Its inverse ψ_ℓ^{-1} is the line transformation of X_ℓ along the line ℓ'. Moreover, ψ_ℓ is defined at the node O of X and ψ_ℓ^{-1} is defined at the node O' of X_ℓ.

As explained in [IP] §4.1–4.3, this is a general process. One can also perform it with the image ℓ_O of ℓ in X_O and obtain a diagram ([IP], Theorem 4.3.3.(ii))

$$\begin{array}{ccc} \widetilde{X}_{O,\ell_O} & \dashrightarrow & \widetilde{X}'_{O,\ell_O} \\ \downarrow & & \downarrow{\scriptstyle p'_\ell} \\ X_O & \overset{p_\ell}{\dashrightarrow} & \mathbf{P}^2, \end{array}$$

where p'_ℓ is a conic bundle and p_ℓ is again associated with a linear subsystem of $|\mathscr{I}_{\ell_O}^3(2)|$. The birational conic bundle p_ℓ can be described geometrically as follows ([B1], §1.4.4 or §6.4.2): a general point $x \in X_O$ is mapped to the unique quadric in $\mathbf{P} = \mathbf{P}^2$ containing the 2-plane $\langle \ell_O, x\rangle$. Its discriminant curve is $\Gamma_7 = \Gamma_1 \cup \Gamma_6$.

Lemma 4.4. *Let $X_\ell \dashrightarrow X_{\ell,O'} \subset \mathbf{P}^6_{O'}$ be the projection from the node O' of X_ℓ and let $\ell'_{O'}$ be the image of the line $\ell' \subset X_\ell$. There is a commutative diagram*

$$\begin{array}{ccc} X_O & \overset{\psi_{\ell,O}}{\dashrightarrow} & X_{\ell,O'} \\ \downarrow{\scriptstyle p_W} & & \downarrow{\scriptstyle p_{\ell'}} \\ \mathbf{P}^2_W & = & \mathbf{P}'. \end{array}$$

Proof. A general fiber of p_W is a conic which meets Q in two points (§4.1). Its strict transform F on X is a quartic curve that passes twice through the node O and does not meet ℓ, hence its image $\psi_\ell(F) \subset X_\ell$ has degree 8 and passes twice through the node O' of X_ℓ. Moreover,

$$\begin{aligned} E' \cdot \chi(\varepsilon^{-1}(F)) &= (-K_{\widetilde{X}'_\ell} - \chi(E)) \cdot \chi(\varepsilon^{-1}(F)) \\ &= (-K_{\widetilde{X}_\ell} - E) \cdot \varepsilon^{-1}(F) \\ &= \varepsilon^* \mathscr{O}_X(1)(-2E) \cdot \varepsilon^{-1}(F) \\ &= \deg(F) = 4. \end{aligned}$$

The image of F in X_ℓ is therefore an octic that passes twice through the node O' and meets ℓ' in 4 points. Its image in $X'_{\ell,O'}$ is a rational sextic that meets $\ell'_{O'}$ in 4 points. Since the map $p_{\ell'}$ is associated with a linear subsystem of $|\mathscr{I}^3_{\ell'_{O'}}(2)|$, it contracts this sextic. In other words, images of general fibers of p_W by $\psi_{\ell,O}$ are contracted by $p_{\ell'}$, which proves the lemma. □

4.4. **The Verra threefold associated with X.** A *Verra threefold* ([V]) is a smooth hypersurface of bidegree $(2,2)$ in $\mathbf{P}^2 \times \mathbf{P}^2$.

Theorem 4.5. *A general nodal Fano threefold of type $\mathscr{X}_{10}$ is birational to a general Verra threefold.*

More precisely, let X be a general nodal Fano threefold of type $\mathscr{X}_{10}$. We show that for a suitable choice of $\ell \subset X_O$, the two birational conic bundle structures:

- $p_W : X \dashrightarrow \mathbf{P}^2_W$, with discriminant curve $\Gamma^\star_6$ (§4.1), and
- $p_\ell : X \dashrightarrow \mathbf{P}$, with discriminant curve $\Gamma_6 \cup \Gamma_1$ (§4.3),

induce a birational isomorphism

$$\psi_\ell := (p_W, p_\ell) : X \dashrightarrow T,$$

where $T \subset \mathbf{P}^2_W \times \mathbf{P}$ is a general Verra threefold. In particular, the sextics Γ_6 and $\Gamma^\star_6$ are general.

Proof of the theorem. Recall that $X_O = W_O \cap \Omega_O$ contains the smooth quadric surface $Q = \mathbf{P}^3_W \cap \Omega_O$. Instead of choosing a general line as in §4.3, we choose a line ℓ contained in Q, not passing through any of the six singular points of X_O.

In suitable coordinates, $W_O \subset \mathbf{P}^6_O$ is the intersection of the quadrics

$$\begin{aligned} \Omega_1(x) &= x_0x_1 + x_2x_3 + x_4x_5 \\ \Omega_2(x) &= x_1x_2 + x_3x_4 + x_5x_6, \end{aligned}$$

and $\mathbf{P}^3_W = \mathbf{P}(\langle e_0, e_2, e_4, e_6\rangle)$ (§9.2). We may assume $\ell = \mathbf{P}(\langle e_2, e_4\rangle)$.

The quadric Ω_O contains ℓ, hence its equation is of the form

$$\Omega_O(x) = x_2\lambda_2(x') + x_4\lambda_4(x') + q(x'),$$

where λ_2 and λ_4 are linear and q quadratic in $x' = (x_0, x_1, x_3, x_5, x_6)$.

The projection $p_W : X_O \dashrightarrow \mathbf{P}^2_W$ sends x to (x_1, x_3, x_5). The map $p_\ell : X_O \dashrightarrow \mathbf{P} = \langle \Omega_1, \Omega_2, \Omega_O \rangle$ sends a general point $x \in X_O$ to the unique quadric in $\mathbf{P}$ containing the 2-plane $\langle \ell, x \rangle$ (§4.3). We obtain

$$p_\ell(x) = (x_1\lambda_4(x') - x_3\lambda_2(x'), x_3\lambda_4(x') - x_5\lambda_2(x'), x_3^2 - x_1x_5).$$

A general (conic) fiber of p_W is mapped by p_ℓ onto a conic in $\mathbf{P}$ because p_ℓ becomes linear once restricted to a fiber of p_W.

Similarly, let us consider the fiber of p_ℓ at the general point $[\Omega_O] \in \mathbf{P}$. It is the set of points $x \in X_O$ such that the 2-plane $\langle \ell, x \rangle$ is contained in Ω_O. This means $\lambda_2(x') = \lambda_4(x') = 0$, and consequently $q(x') = 0$, so that x' describes a plane conic in its parameter space $\mathbf{P}(\langle e_0, e_1, e_3, e_5, e_6 \rangle)$. Since the projection to $\mathbf{P}^2_W$ factors through this 4-plane, a general fiber of p_ℓ is mapped by p_W birationally onto a conic in $\mathbf{P}^2_W$.

Since the restriction of p_W (resp. p_ℓ) to a general fiber of p_ℓ (resp. p_W) is birational onto its image, the product map

$$\psi_\ell = (p_W, p_\ell) : X_O \dashrightarrow \mathbf{P}^2_W \times \mathbf{P}$$

is birational onto its image $T \subset \mathbf{P}^2_W \times \mathbf{P}$. Let (d, e) be the bidegree of this hypersurface: d (resp. e) is the degree of the image by p_W (resp. p_ℓ) of a general fiber of p_ℓ (resp. p_W). Since these plane curves are conics, we have $d = e = 2$, and T is a Verra threefold.

Finally, the fact that T is general follows from a dimension count. □

Remark 4.6. Let $\rho_1 : T \to \mathbf{P}^2_W$ and $\rho_2 : T \to \mathbf{P}$ be the two projections. *The surface $\psi_\ell(Q) \subset T$ is isomorphic to the blow-up of Q at the six singular points $s_1, \ldots, s_6$ of X_O. It is also equal to $\rho_2^{-1}(\Gamma_1)$.*

For x in $Q - \ell$, the only quadrics in $\mathbf{P}$ that contain the 2-plane $\langle \ell, x \rangle$ are in the pencil Γ_1. This implies $p_\ell(Q) \subset \Gamma_1$ and $\psi_\ell(Q) \subset \rho_2^{-1}(\Gamma_1)$. A dimension count shows that given the Verra threefold $T \subset \mathbf{P}^2_W \times \mathbf{P}$, the line $\Gamma_1 \subset \mathbf{P}$ is general. It follows that $\rho_2^{-1}(\Gamma_1)$ is a smooth del Pezzo surface of degree 2 which is equal to $\psi_\ell(Q)$; its anticanonical (finite) map is the double cover $\rho_{1Q} : \psi_\ell(Q) \to \mathbf{P}^2_W$.

With the notation of §4.1, it follows from the comments at the end of §3.5 that the inverse image $\widetilde{Q}$ of Q in $\widetilde{X}_O$ is isomorphic to the blow-up of Q at the six singular points of X_O. We have a commutative diagram

$$\begin{array}{ccc} \widetilde{Q} & \overset{\psi_Q}{\dashrightarrow} & \psi_\ell(Q) \\ & {\scriptstyle p_W} \searrow \quad \swarrow {\scriptstyle \rho_{1Q}} & \\ & \mathbf{P}^2_W. & \end{array}$$

Since ρ_{1Q} is finite, ψ_Q must be a morphism; since $\widetilde{Q}$ and $\psi_\ell(Q)$ are both del Pezzo surfaces of degree 2, it is an isomorphism.

Remark 4.7. Let us analyze more closely the conic bundle structure $p_\ell : X \dashrightarrow \mathbf{P}$ for our choice of $\ell \subset Q$. Following the classical construction of [B1], 1.4.4, we set

$$X_\ell = \{(P, \Omega_p) \in G(2, \mathbf{P}^6_O) \times \mathbf{P} \mid \ell \subset P \subset \Omega_p\}$$

and consider the birational map

$$\begin{array}{rcl} \varphi_\ell : X & \dashrightarrow & X_\ell \\ x & \longmapsto & (\langle \ell, x\rangle, p_\ell(x)). \end{array}$$

Away from Γ_1, the second projection $q_\ell : X_\ell \to \mathbf{P}$ is a conic bundle with discriminant curve Γ_6, whereas $q_\ell^{-1}(\Gamma_1)$ is the union of two components:

$$Q_1 = \{(P, \Omega_p) \in G(2, \mathbf{P}^6_O) \times \Gamma_1 \mid \ell \subset P \subset \mathbf{P}^3_W\} \simeq \mathbf{P}^1 \times \Gamma_1,$$

and the closure of

$$\{(P, \Omega_p) \in G(2, \mathbf{P}^6_O) \times \Gamma_1 \mid P \cap \mathbf{P}^3_W = \ell,\ P \subset \Omega_p\}.$$

They correspond respectively to the two components Γ_1^1 and Γ_1^2 of $\pi^{-1}(\Gamma_1)$ (see §4.2). We have $\varphi_\ell(Q) = Q_1$ and lines $\ell^- \subset Q$ that meet ℓ map to sections of $Q_1 \to \Gamma_1$. Consider the diagram

$$\begin{array}{ccccc} Q_1 & \xleftarrow{\ \sim\ } & Q & \xleftarrow{\ \varepsilon\ } & \psi_\ell(Q) \\ \cap & & \cap & & \cap \\ X_\ell & \overset{\varphi_\ell}{\dashleftarrow} & X & \overset{\psi_\ell}{\dashrightarrow} & T \\ & {\scriptstyle q_\ell}\searrow & \downarrow{\scriptstyle p_\ell} & \swarrow{\scriptstyle \rho_2} & \\ & & \mathbf{P}. & & \end{array}$$

The map $\rho_{2Q} : \varphi_\ell(Q) \to \Gamma_1$ has six reducible fibers, above the points $p_1, \ldots, p_6$ of $\Gamma_1 \cap \Gamma_6$, and each contains one exceptional divisor $E_1, \ldots, E_6$ of ε. Since lines $\ell^+ \subset Q$ from the same ruling as ℓ map to fibers of ρ_{2Q}, the other components must be the strict transforms $\ell_1^+, \ldots, \ell_6^+$ of these lines passing through $s_1, \ldots, s_6$.

As mentioned above, a general line $\ell^- \subset Q$ from the other ruling maps to a section of ρ_{2Q} that does not meet $E_1, \ldots, E_6$, hence must meet $\ell_1^+, \ldots, \ell_6^+$; moreover, these components of reducible fibers of the conic bundle $\rho_2 : T \to \mathbf{P}$ correspond to the points of $\Gamma_1^1 \cap \widetilde{\Gamma}_6$, which we denoted by $\tilde{p}_i$ in §4.2. We also denote by $\ell_1^-, \ldots, \ell_6^-$ the strict transforms of those lines passing through $s_1, \ldots, s_6$. The line ℓ_i^- meets ℓ_j^+ if and only if $i \neq j$, and meets E_j if and only if $i = j$. It maps by both projections ρ_1 and ρ_2 to a line. Finally, $\ell_i^\pm + E_i$ is rationally equivalent to $\ell^\pm$.

5. The variety of conics contained in X

5.1. **The surfaces $F_g(X)$ and $F(X)$.** We follow [Lo], §5, but with the notation of [DIM], §5. In particular, $F_g(X)$ is the variety of conics contained in X. As remarked in [DIM], §3.1, any nonreduced conic contained in W is contained in a 2-plane contained in W. Since the family of these 2-planes has dimension 1 and none of them contain O, and nonreduced conics have codimension 3, all conics contained in a general X are reduced. Let

$$F(X) = \{(c, [V_4]) \in F_g(X) \times \mathbf{P}(V_5^\vee) \mid c \subset G(2, V_4)\}.$$

The projection $F(X) \to F_g(X)$ is an isomorphism except over the one point corresponding to the only ρ-conic c_X contained in X ([DIM], §5.1).

We define as in [DIM], §5.2 an involution ι on $F(X)$ as follows. For any hyperplane $V_4 \subset V_5$, define quadric surfaces

$$Q_{W,V_4} = G(2, V_4) \cap \mathbf{P}(M_{V_4}) \quad \text{and} \quad Q_{\Omega,V_4} = \Omega \cap \mathbf{P}(M_{V_4}).$$

If $(c, [V_4]) \in F(X)$, the intersection

$$X \cap \mathbf{P}(M_{V_4}) = Q_{W,V_4} \cap Q_{\Omega,V_4},$$

has dimension 1 (as we saw in §4, X is locally factorial and $\operatorname{Pic}(X)$ is generated by $\mathscr{O}_X(1)$, hence the degree of any surface contained in X is divisible by 10). Since c is reduced, one checks by direct calculation that as a 1-cycle, it is the sum of c and another (reduced) conic contained in X, which we denote by $\iota(c)$. One checks as in [Lo], Lemma 3.7, that since X is general, $\iota(c) \neq c$ for all c, and some quadric in the pencil $\langle Q_{W,V_4}, Q_{\Omega,V_4}\rangle$ is a pair of distinct planes. This defines a fixed-point-free involution ι on $F(X)$.

5.2. **Conics in X passing through O.** Since $O \in O_4$, any such conic c is a τ-conic, hence is contained in a unique $G(2, V_4)$, and $[V_4] \in \mathbf{P}^2_O$. The quadric Q_{Ω,V_4} is then a cone with vertex O.

If Q_{Ω,V_4} is reducible, i.e., if $[V_4] \in \Gamma_6^\star$, the conics c and $\iota(c)$ meet at O and another point. The two points of $\pi^{\star-1}([V_4])$ correspond to the two 2-planes contained in Q_{Ω,V_4}, hence to $[c]$ and $[\iota(c)]$. By Proposition 4.3, these conics c are parametrized by the curve $\widetilde{\Gamma}_6^\star$.

If Q_{Ω,V_4} is irreducible, $c \cup \iota(c)$ is the intersection of the cone Q_{Ω,V_4} with a pair of planes, and c is the union of two (among the six) lines in X through O.

Theorem 5.1 ([Lo], §5)**.** *The variety $F_g(X)$ is an irreducible surface. Its singular locus is the smooth connected curve $\widetilde{\Gamma}_6^\star$ of conics on X passing through O described above, and its normalization $\widetilde{F}_g(X)$ is smooth.*

Moreover, the curve of σ-conics (which is disjoint from $\widetilde{\Gamma}_6^\star$) is exceptional on $F_g(X)$, and its inverse image on $\widetilde{F}_g(X)$ can be contracted to a smooth surface $\widetilde{F}_m(X)$ (as in the smooth case; see [DIM], §5.3). The involution ι induces an involution on $\widetilde{F}_m(X)$.

Logachev also proves that the inverse image of $\widetilde{\Gamma}_6^\star$ in $\widetilde{F}_g(X)$ has two connected components $\widetilde{\Gamma}_{6,+}^\star$ and $\widetilde{\Gamma}_{6,-}^\star$, which map isomorphically to $\widetilde{\Gamma}_6^\star$ by the normalization $\nu : \widetilde{F}_g(X) \to F_g(X)$. They can be described as follows.

Let $c \subset X$ be a conic passing through O corresponding to a point of $\widetilde{\Gamma}_6^\star$. The curve $p_O(c)$ is a line $\ell \subset X_O$ that meets, but is not contained in, the smooth quadric surface $Q = \mathbf{P}^3_W \cap \Omega_O$. The two points of $\nu^{-1}([c])$ correspond to the conics $[\ell \cup \ell^+] \in \widetilde{\Gamma}_{6,+}^\star$ and $[\ell \cup \ell^-] \in \widetilde{\Gamma}_{6,-}^\star$ contained in X_O, where ℓ^+ and ℓ^- are the two lines in Q passing through its point of intersection with ℓ.

In particular, $\widetilde{\Gamma}_{6,\pm}^\star$ carries an involution $\sigma_\pm^\star$ induced by the involution $\sigma^\star$ of $\widetilde{\Gamma}_6^\star$. On the other hand, the involution ι of $\widetilde{F}_g(X)$ maps $\widetilde{\Gamma}_{6,+}^\star$ isomorphically onto $\widetilde{\Gamma}_{6,-}^\star$. The identification $\widetilde{\Gamma}_{6,+}^\star \xrightarrow{\sim} \widetilde{\Gamma}_{6,-}^\star$ induced by the normalization ν is $\iota \circ \sigma_+^\star = \sigma_-^\star \circ \iota$.

5.3. **The special surfaces S^{even} and S^{odd}.** There is an embedding $\mathbf{P}^\vee \hookrightarrow \Gamma_6^{(6)}$ that sends a line in $\mathbf{P}$ to its intersection with Γ_6. Its inverse image in $\widetilde{\Gamma}_6^{(6)}$ is a surface S with two connected components S^{even} and S^{odd}, each endowed with an involution σ. They are defined by

$$S^{\mathrm{even}} = \{[\widetilde{D}] \in S \mid h^0(\widetilde{\Gamma}_6, \pi^*\mathscr{O}_{\mathbf{P}}(1)(\widetilde{D})) \text{ even}\}$$

and similarly for S^{odd} ([B3], §2, cor.). By [B3], prop. 3, they are smooth because Γ_6, being general, has no tritangent lines.

In particular, the point $\tilde{p}_1 + \cdots + \tilde{p}_6$ of $\widetilde{\Gamma}_6^{(6)}$ defined at the end of §4.2 is in S. *We will show in Proposition 6.3 that it is in S^{odd}.*

5.4. **The isomorphism $\widetilde{F}_m(X) \xrightarrow{\sim} S^{\mathrm{odd}}$.** Let c be a conic on X such that $O \notin \langle c \rangle$. The projection $p_O(c)$ is a conic in X_O, and the set of quadrics in $\mathbf{P}$ that contain $\langle p_O(c) \rangle$ is a line $L_c \subset \mathbf{P}$. For each point p of $L_c \cap \Gamma_6$, *if the vertex v_p of Ω_p is not in the 2-plane $\langle p_O(c) \rangle$*, the 3-plane $\langle p_O(c), v_p \rangle$ defines a point $\tilde{p} \in \widetilde{\Gamma}_6$ above p. This defines a point $\rho_g([c])$ in S.

One checks by direct calculation (§9.2) that the 2-plane $\Pi = \langle c_X \rangle$ is disjoint from $\mathbf{T}_{W,O}$. It follows that c_X satisfies the conditions above, hence $\rho_g([c_X])$ is well-defined. Moreover, the line L_{c_X} is Γ_1, and for each $p_i \in \Gamma_1 \cap \Gamma_6$, the 3-planes $\mathbf{P}^3_W$ and $\langle p_O(c_X), v_{p_i} \rangle$ meet only at v_{p_i}, hence belong to different families; it follows that we have $\rho_g([c_X]) = \sigma(\tilde p_1 + \cdots + \tilde p_6)$, *which is in the surface* $S^{\rm odd}$. We have therefore defined a rational map

$$\rho_g : F_g(X) \dashrightarrow S^{\rm odd}.$$

Logachev then proves ([Lo], §5) that ρ_g induces an isomorphism

$$\rho : \widetilde F_m(X) \xrightarrow{\sim} S^{\rm odd}. \tag{2}$$

This isomorphism commutes with the involutions ι and σ: since the 3-planes $\langle p_O(c), v_p\rangle$ and $\langle p_O(\iota(c)), v_p\rangle$ meet in codimension 1, they belong to different families, hence $\rho_g \circ \iota = \sigma \circ \rho_g$.

Let us explain how ρ is defined on the normalization $\widetilde F_g(X)$. If $c \subset X$ is a conic passing through O corresponding to a point of $\widetilde\Gamma_6^\star$ and $\ell = p_O(c)$, the two points of $\nu^{-1}([c])$ correspond to the conics $[\ell \cup \ell^+]$ and $[\ell \cup \ell^-]$, where ℓ^+ and ℓ^- are the two lines in Q passing through its point of intersection with ℓ.

The images by ρ of these two points are defined as usual: the set of quadrics in $\mathbf{P}$ that contain $\langle \ell \cup \ell^\pm\rangle$ is a line $L^\pm \subset \mathbf{P}$; for each point p of $L^\pm \cap \Gamma_6$, if the vertex v_p of Ω_p is not in the 2-plane $\langle \ell\cup\ell^\pm\rangle$, the 3-plane $\langle \ell\cup\ell^\pm, v_p\rangle$ defines a point $\tilde p \in \widetilde\Gamma_6$ above p.

5.5. **Conic transformations.** Let c be a general conic contained in X. As in the smooth case ([DIM], §6.2), one can perform a *conic transformation* of X along c:

$$\begin{array}{ccc} \widetilde X_c & \dashrightarrow & \widetilde X'_c \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon'} \\ X & \overset{\psi_c}{\dashrightarrow} & X_c, \end{array}$$

where ε is the blow-up of c, the birational map χ is a flop, ε' is the blow-down onto a conic $c' \subset X_c$ of a divisor, and X_c is another nodal threefold of type $\mathscr{X}_{10}$. Moreover, ψ_c is defined at the node O of X and ψ_c^{-1} is defined at the node of X_c.

As in the smooth case, there is an isomorphism

$$\varphi_c : F_m(X) \xrightarrow{\sim} F_m(X_c)$$

which commutes with the involutions ι and $\varphi_c(\iota([c])) = [c_{X_c}]$ ([DIM], Proposition 6.2).

6. Reconstructing the threefold X

We keep the notation of §4.2: in the net $\mathbf{P}$ of quadrics in $\mathbf{P}^6_O$ which contain X_O, the discriminant curve $\Gamma_7 = \Gamma_6 \cup \Gamma_1$ parametrizes singular quadrics, and the double étale cover $\pi : \widetilde{\Gamma}_7 \to \Gamma_7$ corresponds to the choice of a family of 3-planes contained in a quadric of rank 6 in $\mathbf{P}^6_O$.

Theorem 6.1. *We have the following properties.*

a) *The morphism $v : \Gamma_7 \to \mathbf{P}^6_O$ that sends $p \in \Gamma_7$ to the unique singular point of the corresponding singular quadric $\Omega_p \subset \mathbf{P}^6_O$ is an embedding and $v^* : H^0(\mathbf{P}^6_O, \mathscr{O}_{\mathbf{P}^6_O}(1)) \to H^0(\Gamma_7, v^*\mathscr{O}_{\mathbf{P}^6_O}(1))$ is an isomorphism. Furthermore, the invertible sheaf M_X on Γ_7 defined by $M_X(1) = v^*\mathscr{O}_{\mathbf{P}^6_O}(1)$ is a theta characteristic and $H^0(\Gamma_7, M_X) = 0$.*

b) *The double étale cover $\pi : \widetilde{\Gamma}_7 \to \Gamma_7$ is defined by the point $\eta = M_X(-2)$, of order 2 in $J(\Gamma_7)$.*

c) *The variety $X_O \subset \mathbf{P}^6_O$ is determined up to projective isomorphism by the pair (Γ_7, M_X).*

In c), we prove more precisely that given an isomorphism $f : \Gamma_7 \xrightarrow{\sim} \Gamma'_7$ such that $f^*M_{X'} = M_X$, there exists a projective isomorphism $X_O \xrightarrow{\sim} X'_O$ which induces f.

Proof. Item a) is proved in the same way as [B1], lemme 6.8 and lemme 6.12.(ii); item b) as [B1], lemme 6.14; and item c) as [B1], prop. 6.19. The isomorphism $v^*\mathscr{O}_{\mathbf{P}^6_O}(2) \simeq \mathscr{O}_{\Gamma_7}(6)$ can also be seen directly by noting that a quadric in the net $\mathbf{P}$ is given by a 7×7 symmetric matrix A of linear forms on $\mathbf{P}$, and that when this matrix has rank 6, the comatrix of A (whose entries are sextics) is of the type $(v_iv_j)_{1 \le i,j \le 7}$, where $v_1, \dots, v_7$ are homogeneous coordinates of the vertex. $\square$

Remark 6.2. Conversely, given a reduced septic $\Gamma_7 \subset \mathbf{P}$ and an invertible sheaf M on Γ_7 that satisfies $H^0(\Gamma_7, M) = 0$ and $M^2 \simeq \mathscr{O}_{\Gamma_7}(4)$, there is a resolution

$$0 \longrightarrow \mathscr{O}_{\mathbf{P}}(-2)^{\oplus 7} \xrightarrow{A} \mathscr{O}_{\mathbf{P}}(-1)^{\oplus 7} \longrightarrow M \longrightarrow 0 \tag{3}$$

of M viewed as a sheaf on $\mathbf{P}$, where A is a 7×7 symmetric matrix of linear forms, everywhere of rank ≥ 6, with determinant an equation of Γ_7. Indeed, this follows from work of Catanese ([C2], Remark 2.29 and [C3], Theorem 2.3) and Beauville ([B2], Corollary 2.4; the hypothesis that Γ_7 be integral is not needed in the proof because M is invertible) who generalized an old result of Dixon's ([Di]) for smooth plane curves.

Given any smooth sextic $\Gamma_6 \subset \mathbf{P}$ that meets a line Γ_1 transversely, there exists on the septic $\Gamma_7 = \Gamma_6 \cup \Gamma_1$ an invertible theta characteristic

M such that $H^0(\Gamma_7, M) = 0$ ([C1], Theorem 7 or Proposition 13). So we obtain by (3) a net of quadrics in $\mathbf{P}^6_O$ whose base-locus is the intersection of W_O with a smooth quadric. By taking its inverse image under the birational map $W \dashrightarrow W_O$, we obtain a variety of type $\mathscr{X}_{10}$ with, in general, a single node at O.

The following proposition describes how the theta characteristic M_X is related to our previous constructions.

Proposition 6.3. *Let Γ_6 be a general plane sextic, let $\pi : \widetilde{\Gamma}_6 \to \Gamma_6$ be a connected double étale cover, with associated involution σ and line bundle η of order 2 on Γ_6. There is a commutative diagram*

$$\begin{array}{ccc} \left\{\begin{array}{c}\textit{Invertible theta characteristics} \\ M \textit{ on the union of } \Gamma_6 \textit{ and a} \\ \textit{transverse line such that } M|_{\Gamma_6} \simeq \eta(2)\end{array}\right\} & \xrightarrow{\ \theta\ } & S/\sigma \\ & \searrow & \swarrow \\ & & \mathbf{P}^\vee, \end{array}$$

where θ is an open embedding and maps even *(resp.* odd*) theta characteristics to S^{odd}/σ (resp. S^{even}/σ).*

Furthermore, if M_X is the (even) theta characteristic associated with a general nodal X, we have

$$\theta(M_X) = \rho([c_X]).$$

(The map ρ was defined in §5.4.)

Proof. For any invertible theta characteristic M on $\Gamma_7 = \Gamma_6 \cup \Gamma_1 \subset \mathbf{P}$ such that $M|_{\Gamma_6} \simeq \eta(2)$, the invertible sheaf $M(-2)$ has order 2 in $J(\Gamma_7)$, hence defines a double étale cover $\pi : \widetilde{\Gamma}_7 \to \Gamma_7$ which induces the given cover over Γ_6. The inverse image of Γ_1 splits as the disjoint union of two rational curves Γ_1^1 and Γ_1^2, and the intersections $\Gamma_1^i \cap \widetilde{\Gamma}_6$ define divisors $\widetilde{D} = \tilde{p}_1 + \cdots + \tilde{p}_6$ and $\sigma^*\widetilde{D}$, hence a well-defined point in S/σ.

This defines a morphism θ which is compatible with the morphisms to $\mathbf{P}^\vee$. These morphisms are both finite of degree 2^5 ([H], Theorem 2.14), hence θ is an open embedding. Since S/σ is smooth with two components, the set of M as in the statement of the theorem, with fixed parity, is smooth and irreducible.

Finally, for any $s \in H^0(\widetilde{\Gamma}_6, \pi^*\mathscr{O}(1)(\widetilde{D}))$, set

$$s^{\pm} = \sigma^* s \cdot s_{\widetilde{D}} \pm s \cdot s_{\sigma^*\widetilde{D}}.$$

We have

$$s^- \in H^0(\widetilde{\Gamma}_6, \pi^*\mathscr{O}(2))^- \simeq H^0(\Gamma_6, M|_{\Gamma_6})$$

and

$$s^+ \in H^0(\widetilde{\Gamma}_6, \pi^*\mathscr{O}(2))^+ \simeq H^0(\Gamma_6, \mathscr{O}(2)) \simeq H^0(\mathbf{P}, \mathscr{O}(2)).$$

Since the points $p_i = \pi(\tilde{p}_i)$, $i \in \{1, \dots, 6\}$, are distinct, we have an exact sequence

$$(4) \qquad 0 \to H^0(\Gamma_7, M) \to H^0(\Gamma_6, M|_{\Gamma_6}) \oplus H^0(\Gamma_1, M|_{\Gamma_1}) \to \bigoplus_{i=1}^{6} \mathbf{C}_{p_i},$$

and since $s^+(p_i) = s^-(p_i)$, the pair $(s^-, s^+|_{\Gamma_1})$ defines an element of $H^0(\Gamma_7, M)$. Since $s^- = 0$ if and only if $s \in s_{\widetilde{D}} \cdot H^0(\widetilde{\Gamma}_6, \pi^*\mathscr{O}(1))$, we obtain an exact sequence

$$(5) \qquad 0 \to H^0(\Gamma_6, \mathscr{O}(1)) \to H^0(\widetilde{\Gamma}_6, \pi^*\mathscr{O}(1)(\widetilde{D})) \to H^0(\Gamma_7, M).$$

Since Γ_6 is general, there is, by [C1], Theorem 7 or Proposition 13, a transverse line Γ_1 and an even theta characteristic M on $\Gamma_7 = \Gamma_6 \cup \Gamma_1$ such that $H^0(\Gamma_7, M) = 0$. Because of (5), $\theta(M)$ is in S^{odd}/σ; we proved above that the set of all even theta characteristics is irreducible, hence it must map to S^{odd}/σ, and odd theta characteristics must map to S^{even}/σ. □

Corollary 6.4. *A general nodal X of type $\mathscr{X}_{10}$ can be reconstructed, up to projective isomorphism, from the double étale cover $\pi : \widetilde{\Gamma}_6 \to \Gamma_6$ and the point $\rho([c_X])$ of S^{odd}/σ.*

Remark 6.5. The map θ is never an isomorphism: even if we allow lines $[\Gamma_1] \in \mathbf{P}^\vee$ which are (simply) tangent to Γ_6 at a point p_1 (this happens when the quadric Ω_O is tangent to the cubic curve C_O), we only get elements of S/σ of the type $2\tilde{p}_1 + \tilde{p}_3 + \cdots + \tilde{p}_6$.

To fill up the remaining 16 points $\tilde{p}_1 + \sigma(\tilde{p}_1) + \tilde{p}_3 + \cdots + \tilde{p}_6$ of S/σ above $[\Gamma_1]$, we need to let our nodal threefold "degenerate" to an X with a node O in the orbit O_3 (the pencil of quadrics that defines W_O then contains a unique quadric of rank 5 and Γ_6 becomes tangent to Γ_1 at the corresponding point). Similarly, bitangent lines Γ_1 to Γ_6 correspond to the case where the node O is in the orbit O_2 (there are then two quadrics of rank 5 in the pencil Γ_1). Although what should be done for flex lines is not clear, it seems likely that for Γ_6 general plane sextic, there should exist a family of nodal X parametrized by a suitable *proper* family of (even, possibly noninvertible) theta characteristics on the union of Γ_6 and *any* line, isomorphic over $\mathbf{P}^\vee$ to S^{odd}/σ. This would fit with the results of [DIM], where a proper surface contained in a general fiber of the period map is constructed: as explained in §7.3, this surface degenerates in the nodal case to S^{odd}/σ.

6.1. **More on Verra threefolds.** Let Π_1 and Π_2 be two copies of $\mathbf{P}^2$. Recall from §4.4 that a Verra threefold is a general (smooth) hypersurface $T \subset \Pi_1 \times \Pi_2$ of bidegree $(2,2)$. Each projection $\rho_i : T \to \Pi_i$ makes it into a conic bundle with discriminant curve a smooth plane sextic $\Gamma_{6,i} \subset \Pi_i$ and associated connected double étale covering $\pi_i : \widetilde{\Gamma}_{6,i} \to \Gamma_{6,i}$. We have

$$J(T) \simeq \mathrm{Prym}(\widetilde{\Gamma}_{6,1}/\Gamma_{6,1}) \simeq \mathrm{Prym}(\widetilde{\Gamma}_{6,2}/\Gamma_{6,2}).$$

The special subvariety S_i associated with the linear system $|\mathscr{O}_{\Pi_i}(1)|$ is the union of two smooth connected surfaces S_i^{odd} and S_i^{even} (§5.3).

Let $\mathscr{C}$ the subscheme of the Hilbert scheme of T that parametrizes reduced connected purely 1-dimensional subschemes of T of degree 1 with respect to both $\rho_1^*\mathscr{O}_{\Pi_1}(1)$ and $\rho_2^*\mathscr{O}_{\Pi_2}(1)$. For T general, the scheme $\mathscr{C}$ is a smooth surface and a general element of each irreducible component corresponds to a smooth irreducible curve in T ([V], (6.11)).

Proposition 6.6. *For T general, the surfaces $\mathscr{C}$, S_1^{even}, and S_2^{even} are smooth, irreducible, and isomorphic.*

Proof. Let $[c]$ be a general element of $\mathscr{C}$. Each projection $\rho_i|_c : c \to \Pi_i$ induces an isomorphism onto a line that meets $\Gamma_{6,i}$ in six distinct points. For each of these points p, the curve c meets exactly one of the components of $\rho_i^{-1}(p)$, hence defines an element of S_i. This defines a rational map

$$\mathscr{C} \dashrightarrow S_i$$

over $\Pi_i^\vee$.

Conversely, let $L_i \subset \Pi_i$ be a general line. By Bertini, the surface $F_1 = \rho_1^{-1}(L_1)$ is smooth and connected. It is ruled over L_1 with exactly six reducible fibers. The projection $\rho_2|_{F_1} : F_1 \to \Pi_2$ is a double cover ramified along a smooth quartic $\Gamma_4 \subset \Pi_2$, and $K_{F_1} \equiv -\rho_2^* L_2$. Let L be a bitangent line to Γ_4. The curve $\rho_2^{-1}(L)$ has two irreducible components and total degree 2 over Π_1. Either one component is contracted by ρ_1 and it is then one of the 12 components of the reducible fibers of $\rho_1|_{F_1} : F_1 \to L_1$, or both components are sections and belong to $\mathscr{C}$. Since there are 28 bitangents, the degree of $\mathscr{C} \to \Pi_1^\vee$ is $2 \times (28-12) = 32$.

More generally, for *any* line $L_1 \subset \Pi_1$, the surface $F_1 = \rho_1^{-1}(L_1)$ is irreducible, maps 2-to-1 to Π_2 with ramification a quartic, and only finitely many curves are contracted. Any smooth $[c] \in \mathscr{C}$ with $c \subset F_1$ must map to a line everywhere tangent to the ramification, and there is only a finite (nonzero) number of such lines.

It follows that every component of $\mathscr{C}$ dominates $\Pi_1^\vee$. Moreover, the morphisms $\mathscr{C} \to \Pi_i^\vee$, $S_i^{\rm odd} \to \Pi_i^\vee$, and $S_i^{\rm even} \to \Pi_i^\vee$ are all finite of degree 32. It follows that the smooth surface $\mathscr{C}$ is irreducible and maps isomorphically to a component of S_i. It remains to prove that this component is $S_i^{\rm even}$.

According to Theorem 4.5, we may assume that T is obtained from a nodal X of type $\mathscr{X}_{10}$, as explained in §4.4. In Remark 4.7, we constructed a curve ℓ_1^- of bidegree $(1,1)$ whose image in S is $\sigma(\tilde p_1) + \tilde p_2 + \cdots + \tilde p_6$. Since $\rho([c_X]) = \sigma(\tilde p_1 + \cdots + \tilde p_6)$ (§5.4) and $\rho([c_X]) \in S^{\rm odd}$ (Proposition 6.3), this finishes the proof. □

7. The extended period map

7.1. **Intermediate Jacobians.** The intermediate Jacobian $J(X)$ of our nodal X appears as an extension

$$1 \to \mathbf{C}^\star \to J(X) \to J(\widetilde X) \to 0, \tag{6}$$

with extension class $e_X \in J(\widetilde X)/\pm 1$. Since $\tilde X$ is birationally isomorphic to a general Verra threefold T (Theorem 4.5), $J(\widetilde X)$ and $J(T)$, having no factors that are Jacobians of curves, are isomorphic (see the classical argument used for example in the proof of [DIM], Corollary 7.6). In particular, by [V], we have

$$J(\widetilde X) \simeq \mathrm{Prym}(\widetilde\Gamma_6/\Gamma_6) \simeq \mathrm{Prym}(\widetilde\Gamma_6^\star/\Gamma_6^\star). \tag{7}$$

7.2. **The extension class e_X.** Choose a general line $\ell \subset X_O$. A point of $\widetilde\Gamma_6$ corresponds to a family of 3-planes contained in a singular Ω_p. In this family, there is a unique 3-plane that contains ℓ, and its intersection with X_O is the union of ℓ and a rational normal cubic meeting ℓ at two points. So we get a family of curves on $\tilde X$ parametrized by $\widetilde\Gamma_6$ hence an Abel-Jacobi map

$$J(\widetilde\Gamma_6) \to J(\widetilde X)$$

which vanishes on $\pi^* J(\Gamma_6)$ and induces (the inverse of) the isomorphism (7) (here, $\mathrm{Prym}(\widetilde\Gamma_6/\Gamma_6)$ is seen as the quotient $J(\widetilde\Gamma_6)/\pi^*J(\Gamma_6)$). Also, there is a natural map

$$\beta : S \to \mathrm{Prym}(\widetilde\Gamma_6/\Gamma_6) \subset J(\widetilde\Gamma_6)$$

defined up to translation (here, $\mathrm{Prym}(\widetilde\Gamma_6/\Gamma_6)$ is seen as the kernel of the norm morphism $\mathrm{Nm} : J(\widetilde\Gamma_6) \to J(\Gamma_6)$) which can be checked to be a closed embedding (as in [B3], C, p. 374). Finally, we have an Abel-Jacobi map

$$\alpha : \widetilde F_m(X) \to J(\widetilde X)$$

(also defined up to translation). Logachev proves that the diagram

$$\begin{array}{ccc} \widetilde{F}_m(X) & \xrightarrow[\rho]{\sim} & S^{\mathrm{odd}} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ J(\widetilde{X}) & \xleftarrow{\sim} & \mathrm{Prym}(\widetilde{\Gamma}_6/\Gamma_6) \end{array}$$

commutes up to a translation ([Lo], Proposition 5.16, although the leftmost map in the diagram there should be Φ, not 2Φ). In particular, α is a closed embedding.

The extension class e_X of (6) is the image in $J(\widetilde{X})/\pm 1$ by the Abel-Jacobi map of the difference between the (homologous) lines ℓ^+ and ℓ^- from the two rulings of the smooth quadric surface $Q = \mathbf{P}^3_W \cap \Omega_O$ ([Co], Theorem (0.4)). In particular, if $\nu : \widetilde{F}_m(X) \to F_m(X)$ is the normalization, it follows from §5.2 that we have for all $[c] \in \widetilde{\Gamma}^\star_6$

$$e_X = \alpha([c^+]) - \alpha([c^-]),$$

where $\nu^{-1}([c]) = \{[c^+], [c^-]\}$. Since α is injective, *e_X is nonzero.*

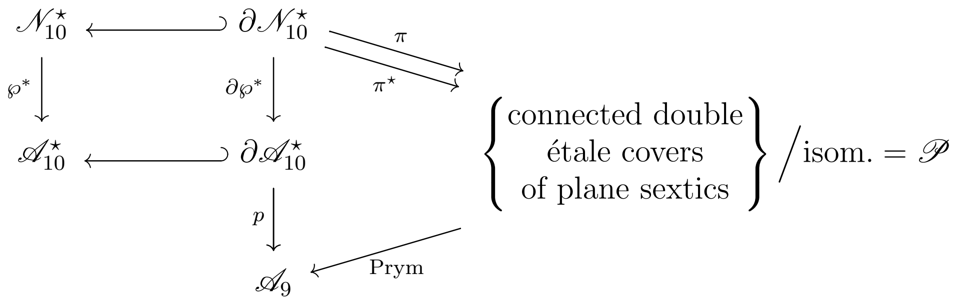
Note that $\alpha \circ \iota + \alpha$ is constant on $\widetilde{F}_m(X)$, say equal to C. We have

$$\alpha(\widetilde{\Gamma}^\star_{6,-}) = \alpha(\widetilde{\Gamma}^\star_{6,+}) - e_X, \tag{8}$$

the involution $\sigma^\star_\pm$ on $\alpha(\widetilde{\Gamma}^\star_{6,\pm})$ is given by $x \mapsto C \pm e_X - x$, and these curves are Prym-canonically embedded in $J(\widetilde{X}) \simeq \mathrm{Prym}(\widetilde{\Gamma}^\star_6/\Gamma^\star_6)$.

Remark 7.1. The semi-abelian variety $J(X)$ can be seen as the complement in the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathscr{O}_{J(\widetilde{X})} \oplus P_{e_X}) \to J(\widetilde{X})$ (where P_{e_X} is the algebraically trivial line bundle on $J(\widetilde{X})$ associated with e_X) of the two canonical sections $J(\widetilde{X})_0$ and $J(\widetilde{X})_\infty$. A (nonnormal) compactification $\overline{J(X)}$ is obtained by glueing these two sections by the translation $x \mapsto x + e_X$ and it is the proper limit of intermediate Jacobians of smooth threefolds of type $\mathscr{X}_{10}$. The Abel-Jacobi map $F_g(X) \dashrightarrow J(X)$ then defines an embedding $F_m(X) \hookrightarrow \overline{J(X)}$ (use Logachev's description of the nonnormal surface $F_m(X)$ given in §5.2 and (8)).

7.3. **Extended period map.** Let $\mathscr{N}^\star_{10}$ (resp. $\partial\mathscr{N}^\star_{10}$) be the stack of smooth or nodal threefolds of type $\mathscr{X}_{10}$ (resp. of nodal threefolds of type $\mathscr{X}_{10}$), and let $\mathscr{A}^\star_{10}$ (resp. $\partial\mathscr{A}^\star_{10}$) be the stack of principally polarized abelian varieties of dimension 10 and their rank-1 degenerations (resp. of rank-1 degenerations) ([Mu]). There is an extended period map $\wp^\star : \mathscr{N}_{10}^\star \to \mathscr{A}_{10}^\star$ and a commutative diagram



where the map $p\circ\partial\wp^* = \mathrm{Prym}\circ\pi = \mathrm{Prym}\circ\pi^\star$ sends a nodal threefold of type $\mathscr{X}_{10}$ to the intermediate Jacobian of its minimal desingularization.

A dimension count shows that $\partial\mathscr{N}_{10}^\star$ is irreducible of dimension 21. By [V], Corollary 4.10, the map Prym is 2-to-1 onto its 19-dimensional image. The irreducible variety $\mathscr{P}$ therefore carries a birational involution σ and $\pi^\star = \sigma\circ\pi$. By Corollary 6.4, the general fiber of π is birationally the surface S^{odd}/σ so, with the notation of §6.1, if T is a general Verra threefold, the fiber $(p\circ\partial\wp^*)^{-1}(J(T))$ is birationally the union of the two special surfaces $S_1^{\mathrm{odd}}/\sigma_1$ and $S_2^{\mathrm{odd}}/\sigma_2$.

We proved in [DIM], Theorem 7.1, that $\wp^*(\mathscr{N}_{10}^\star)$ has dimension 20. It follows that $\partial\wp^*(\partial\mathscr{N}_{10}^\star)$ has dimension 19, hence the fibers of π must be contracted by $\partial\wp^*$. This can also be seen by checking that the various nodal threefolds corresponding to general points in this fiber differ by conic transformations (§5.5) hence have same intermediate Jacobians (there is a birational isomorphism between them which is defined at the nodes).

From Lemma 4.4, we deduce that a line transform of a nodal X produces another nodal X_ℓ such that the covers $\pi(X_\ell)$ and $\pi^\star(X)$ are isomorphic. Since their intermediate Jacobians are isomorphic (there is a birational isomorphism between X and X_ℓ which is defined at the nodes), the surfaces $S_1^{\mathrm{odd}}/\sigma_1$ and $S_2^{\mathrm{odd}}/\sigma_2$ are in the same fiber of $\wp^\star$.

It follows that p is birational on $\wp^*(\partial\mathscr{N}_{10}^\star)$: the extension class e is canonically attached to a general Verra threefold. Moreover, since a general nonempty fiber of $\partial\wp^*$ is the union of two distinct irreducible surfaces (obtained by conic and line transformations), *the two smooth irreducible surfaces contained in a general nonempty fiber of* $\wp^\star$ constructed in [DIM] *are disjoint.*

Unfortunately, we cannot deduce from our description of a general nonempty fiber of $\partial\wp^*$ that a general nonempty fiber of $\wp^\star$ consists only of these two surfaces, since there are difficult properness issues here.

8. Singularities of the theta divisor

We keep the notation of §6.1. In [V], Theorem 4.11, Verra proves that the singular locus of a theta divisor Ξ of the intermediate Jacobian of a general Verra threefold T has (at least) three components, all 3-dimensional:[2]

$$\mathrm{Sing}^{\mathrm{ex}}_{\pi_i}(\Xi) = \pi_i^* H_i + \widetilde{\Gamma}_{6,i} + S_i^{\mathrm{odd}}$$

for $i \in \{1, 2\}$, and a union of components ([V], Remark 4.12)

$$\Xi_T \subset \mathrm{Sing}^{\mathrm{st}}_{\pi_1}(\Xi) \cap \mathrm{Sing}^{\mathrm{st}}_{\pi_2}(\Xi)$$

which Verra calls *distinguished.*

What follow are some speculations on the singular locus of the intermediate Jacobian of a general Fano threefold X' of type $\mathscr{X}_{10}$. We proved in [DIM] that a general fiber of the period map has (at least) two connected components which are quotient of smooth proper surfaces $F(X')$ and $F^\star(X')$ by involutions.

Conjecture 8.1. The singular locus of the theta divisor of the intermediate Jacobian of a general threefold X' of type $\mathscr{X}_{10}$ has dimension 4 and contains a unique component of that dimension; this component is a translate of $F(X') + F^\star(X')$.

When X' degenerates to a nodal X, with associated Verra threefold T, the surface $F(X')$ degenerates to the special surface S_1^{odd}, and the surface $F^\star(X')$ to S_2^{odd}. If $(J(T), \Xi)$ is the intermediate Jacobian of T, the singular locus of the theta divisor degenerates to a subvariety of $\overline{J(X)}$ (see Remark 7.1) which projects onto $\mathrm{Sing}(\Xi \cdot \Xi_e)$ (see [Mu], (2.4), pp. 363–364). So the degenerate version of the conjecture is the following.

Conjecture 8.2. Let $(J(T), \Xi)$ be the intermediate Jacobian of a general Verra threefold T, with canonical extension class e. The singular locus of $\Xi \cdot \Xi_e$ has dimension 4 and has a unique component of that dimension. This component is a translate of $S_1^{\mathrm{odd}} + S_2^{\mathrm{odd}}$.

According to §7.2 and (8), there is an embedding $\widetilde{\Gamma}_{6,2} \subset S_1^{\mathrm{odd}} \subset \widetilde{\Gamma}_{6,1}^{(6)}$ such that for all $\widetilde{D} \in \widetilde{\Gamma}_{6,1}^{(6)}$ in this curve, $\widetilde{D} + e$ is linearly equivalent to an effective divisor $\widetilde{D}' \in S_1^{\mathrm{odd}}$. This implies

$$\pi_1^* H_1 + e \equiv \sigma_1^* \widetilde{D} + \widetilde{D} + e \equiv \sigma_1^* \widetilde{D} + \widetilde{D}',$$

[2] Recall that the singularities of a theta divisor Ξ of the Prym variety of a double étale covering π are of two kinds: the *stable singularities* $\mathrm{Sing}^{\mathrm{st}}_\pi(\Xi)$, and the *exceptional singularities* $\mathrm{Sing}^{\mathrm{ex}}_\pi(\Xi)$ (see [D], §2, for the definitions).

hence $h^0(\widetilde{\Gamma}_{6,1}, \pi_1^*H_1 + e) \geq 2$. It follows that $\mathrm{Sing}^{\mathrm{ex}}_{\pi_1}(\Xi)$, hence also $\mathrm{Sing}^{\mathrm{ex}}_{\pi_2}(\Xi)$, is contained in Ξ_e. In particular, $\mathrm{Sing}^{\mathrm{ex}}_{\pi_i}(\Xi)$ is contained in the singular locus of $\Xi \cdot \Xi_e$, and is also contained in a translate of $S_1^{\mathrm{odd}} + S_2^{\mathrm{odd}}$ by (8).

9. Appendix: explicit computations

9.1. **The fourfold W.** It follows from [PV], Proposition 6.4, that there exists a basis $(e_1, \dots, e_5)$ for V_5 such that the pencil of skew-symmetric forms on V_5 that defines W in $G(2, V_5)$ (see §3.1) is spanned by $e_1^* \wedge e_4^* + e_2^* \wedge e_5^*$ and $e_1^* \wedge e_5^* + e_3^* \wedge e_4^*$. In other words, in coordinates for the basis

$$\mathscr{B} = (e_{12}, e_{13}, e_{14}, e_{15}, e_{23}, e_{24}, e_{25}, e_{34}, e_{35}, e_{45})$$

for $\wedge^2 V_5$, where $e_{ij} = e_i \wedge e_j$, the linear space V_8 that cuts out W has equations

$$x_{14} + x_{25} = x_{15} + x_{34} = 0.$$

The unique common maximal isotropic subspace for all forms in the pencil is

$$U_3 = \langle e_1, e_2, e_3 \rangle.$$

It contains the smooth conic

$$c_U = (x_1^2 + x_2 x_3 = x_4 = x_5 = 0),$$

which parametrizes the kernels of the forms in the pencil. The unique β-plane (see [DIM], §3.3) contained in W is

$$\Pi = G(2, U_3) = \langle e_{12}, e_{13}, e_{23} \rangle$$

and the orbits for the action of $\mathrm{Aut}(W)$ are (see [DIM], §3.5)

- $O_1 = c_U^\vee = (x_{23}^2 + 4x_{12}x_{13} = 0) \subset \Pi$,
- $O_2 = \Pi - c_U^\vee$,
- $O_3 = (W \cap (x_{45} = 0)) - \Pi$,
- $O_4 = W - O_3$.

9.2. **The fourfold W_O.** The point $O = [e_{45}]$ is in the dense orbit O_4, which can be parametrized by

$$O_4 = \{(-ux - v^2, u^2 - vy, -v, u, uv + xy, x, v, -u, y, 1) \mid (u, v, x, y) \in \mathbf{C}^4\}$$

in the basis $\mathscr{B}$, and $\mathbf{T}_{W,O} \subset \mathbf{P}_7$ has equations

$$x_{12} = x_{13} = x_{23} = x_{14} + x_{25} = x_{15} + x_{34} = 0.$$

In particular, $\mathbf{T}_{W,O} \cap \Pi = \varnothing$.

The quadrics that contain W and are singular at O correspond, via the isomorphism (1), to one-dimensional subspaces $V_1 \subset V_O = \langle e_4, e_5 \rangle$ (see §3.2). It follows that the projection $W_O \subset \mathbf{P}^6_O$ is defined by

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = x_{12}x_{35} + x_{13}x_{14} - x_{34}x_{23} = 0, \tag{9}$$

where $(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, x_{35})$ are coordinates for $V_8/\langle e_{45} \rangle$. It contains the 3-plane

$$\mathbf{P}^3_W = p_O(\mathbf{T}_{W,O}) = (x_{12} = x_{13} = x_{23} = 0)$$

and its singular locus is the twisted cubic $C_O \subset \mathbf{P}^3_W$ defined by

$$\operatorname{rank} \begin{pmatrix} x_{14} & x_{34} & -x_{35} \\ x_{24} & x_{14} & x_{34} \end{pmatrix} \leq 1. \tag{10}$$

Let $\widetilde{\mathbf{P}}^6_O \to \mathbf{P}^6_O$ be the blow-up of $\mathbf{P}^3_W$. It is defined in $\mathbf{P}^6_O \times \mathbf{P}^2_W$ by the condition

$$\operatorname{rank} \begin{pmatrix} a_{12} & a_{13} & a_{23} \\ x_{12} & x_{13} & x_{23} \end{pmatrix} \leq 1, \tag{11}$$

where a_{12}, a_{13}, a_{23} are homogeneous coordinates on $\mathbf{P}^2_W$. The strict transform $\widetilde{W}_O \subset \widetilde{\mathbf{P}}^6_O$ of W_O is defined by the equations

$$a_{12}x_{34} - a_{13}x_{24} + a_{23}x_{14} = a_{12}x_{35} + a_{13}x_{14} - a_{23}x_{34} = 0. \tag{12}$$

It follows that $\widetilde{W}_O \to W_O$ is an isomorphism over $W_O - C_O$ and a $\mathbf{P}^1$-bundle over C_O. Furthermore, the projection $\widetilde{W}_O \to \mathbf{P}^2_W$ is a $\mathbf{P}^2$-bundle, hence $\widetilde{W}_O$ is smooth.

9.3. **The $\mathbf{P}^2$-bundles $\mathbf{P}(\mathscr{M}_O) \to \mathbf{P}^2_O$ and $\widetilde{W}_O \to \mathbf{P}^2_W$ are isomorphic.** If $V_4 \subset V_5$ is a hyperplane that contains V_O and is defined by the equation $b_1x_1 + b_2x_2 + b_3x_3 = 0$, the vector space $M_{V_4} = \wedge^2 V_4 \cap V_8$ defined in §3.2 has equations

$$b_1x_{14} + b_2x_{24} + b_3x_{34} = -b_1x_{34} - b_2x_{14} + b_3x_{35} = 0$$

and

$$b_2x_{12} + b_3x_{13} = b_1x_{12} - b_3x_{23} = b_1x_{13} + b_2x_{23} = 0$$

in V_8. It is therefore equal to the fiber of $\widetilde{W}_O \to \mathbf{P}^2_W$ at the point $a = (b_3, -b_2, b_1)$ (see (11) and (12)). This isomorphism $\mathbf{P}^2_O \simeq \mathbf{P}^2_W$ induces an isomorphism $\mathbf{P}^2_O \times \mathbf{P}^6_O \simeq \mathbf{P}^2_W \times \mathbf{P}^6_O$, hence an isomorphism between the $\mathbf{P}^2$-bundles $\mathbf{P}(\mathscr{M}_O) \to \mathbf{P}^2_O$ and $\widetilde{W}_O \to \mathbf{P}^2_W$.

References

[B1] Beauville, A., Variétés de Prym et jacobiennes intermédiaires, *Ann. Sci. Éc. Norm. Supér.* **10** (1977), 309–391.

[B2] Beauville, A., Determinantal hypersurfaces, *Michigan Math. J.* **48** (2000), 39–64.

[B3] Beauville, A., Sous-variétés spéciales des variétés de Prym, *Compos. Math.* **45** (1982), 357–383.

[C1] Catanese, F., Generic invertible sheaves of 2-torsion and generic invertible theta-characteristics on nodal plane curves, *Algebraic geometry, Sitges (Barcelona), 1983*, 58–70, Lecture Notes in Math. **1124**, Springer-Verlag, Berlin, 1985.

[C2] Catanese, F., Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications, *Invent. Math.* **63** (1981), 433–465.

[C3] Catanese, F., Homological algebra and algebraic surfaces, *Algebraic geometry—Santa Cruz 1995*, 3–56, Proc. Sympos. Pure Math. **62**, Part 1, Amer. Math. Soc., Providence, RI, 1997.

[Co] Collino, A., Footnote to a paper of Griffiths, *Boll. Un. Mat. Ital.* **17** (1980), 748–756.

[D] Debarre, O., Sur les variétés de Prym des courbes tétragonales, *Ann. Sci. Éc. Norm. Supér.* **21** (1988), 545–559.

[DIM] Debarre, O., Iliev, A., Manivel, L., On the period map for prime Fano threefolds of degree 10, eprint `arXiv:0812.3670`.

[Di] Dixon, A.C., Note on the Reduction of a Ternary Quantic to a Symmetrical Determinant, *Math. Proc. Cambridge Philos. Soc.* **11** (1902), 350–351.

[G] Grothendieck, A., *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA2)*, Masson & North-Holland, Paris-Amsterdam, 1968.

[H] Harris, J., Theta-characteristics on algebraic curves, *Trans. Amer. Math. Soc.* **271** (1982), 611–638.

[HP] Hodge, W. V. D., Pedoe, D., *Methods of algebraic geometry. Vol. II.* Book III: General theory of algebraic varieties in projective space. Book IV: Quadrics and Grassmann varieties. Reprint of the 1952 original. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1994.

[IP] Iskovskikh, V. A., Prokhorov, Yu., Fano varieties, *Algebraic geometry, V*, 1–247, Encyclopaedia Math. Sci. **47**, Springer-Verlag, Berlin, 1999.

[Lo] Logachev, D., Fano threefolds of genus 6, eprint `arXiv:math/0407147`.

[M] Mori, S., Threefolds whose canonical bundles are not numerically effective, *Ann. of Math.* **116** (1982), 133–176.

[Mu] Mumford, D., On the Kodaira dimension of the Siegel modular variety, *Algebraic geometry—open problems* (Ravello, 1982), 348–375, Lecture Notes in Math. **997**, Springer, Berlin, 1983.

[PV] Piontkowski, J., Van de Ven, A., The automorphism group of linear sections of the Grassmannian $G(1, N)$, *Doc. Math.* **4** (1999), 623–664.

[V] Verra, A., The Prym map has degree two on plane sextics, *The Fano Conference,* 735–759, Univ. Torino, Turin, 2004.

Département de Mathématiques et Applications – UMR CNRS 8553, École Normale Supérieure, 45 rue d'Ulm, 75230 Paris cedex 05, France
E-mail address: Olivier.Debarre@ens.fr

Institute of Mathematics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., bl. 8, 1113 Sofia, Bulgaria
E-mail address: ailiev@math.bas.bg

Institut Fourier, Université de Grenoble I et CNRS, BP 74, 38402 Saint-Martin d'Hères, France
E-mail address: Laurent.Manivel@ujf-grenoble.fr