HYPER-KÄHLER FOURFOLDS AND GRASSMANN GEOMETRY

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Abstract. We construct a new 20-dimensional family of projective hyper-Kähler fourfolds and prove that they are deformation-equivalent to the second punctual Hilbert scheme of a K3 surface of genus 12.

1. Introduction

An irreducible hyper-Kähler manifold is a compact Kähler manifold whose space of holomorphic 2-forms is generated by an everywhere nondegenerate form. It is known, as a consequence of the Kodaira embedding theorem and the study of the period map, that algebraic hyper-Kähler manifolds form a countable union of hypersurfaces in the local universal deformation space of any hyper-Kähler manifold.

Beauville described, in each dimension $2n$, two families of such varieties ([B]):

(1) the n-th punctual Hilbert scheme $S^{[n]}$ of a K3 surface S;
(2) the fiber at the origin of the Albanese map of the $(n+1)$-st punctual Hilbert scheme of an abelian surface.

All of the irreducible hyper-Kähler manifolds constructed later on have been proved to be deformation-equivalent to one of Beauville's examples, with the exception of two sporadic families of examples constructed by O'Grady in dimensions 6 ([O1]) and 10 ([O2]).

Beauville's examples all have, in dimension at least 4, Picard number ≥ 2, while a very general algebraic deformation has Picard number 1, hence is not of the same type. There are very few explicit geometric descriptions for these deformations. More precisely, there are, to our knowledge, only three such families that are explicitly described, each of which is 20-dimensional and parametrizes general polarized deformations of the second punctual Hilbert scheme of a K3 surface:

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(1) Beauville and Donagi proved in [BD] that the variety $F(X)$ of lines on a smooth cubic hypersurface $X \subset \mathbf{P}^5$ is an algebraic hyper-Kähler fourfold. This gives a 20-dimensional moduli space of fourfolds, and along an explicitly described hypersurface in this moduli space (corresponding to "Pfaffian" cubics), $F(X)$ is isomorphic to the second punctual Hilbert scheme of a general K3 surface S of genus 8.

(2) Iliev and Ranestad proved in [IR1] that the variety $V(X)$ of sum of powers of a general cubic $X \subset \mathbf{P}^5$ as above is another algebraic hyper-Kähler fourfold, with 20 moduli. Along another hypersurface in the moduli space (corresponding to "apolar" cubics), $V(X)$ is also isomorphic to $S^{[2]}$. While the Hodge structure on $H^2(V(X), \mathbf{Z})$ is presumably isogenous to that of $H^2(F(X), \mathbf{Z})$ (a fact which is not known), it is shown in [IR2] that the polarization on $V(X)$ is in general numerically different from the Plücker polarization on $F(X)$. This guarantees that we have two different families of deformations of $S^{[2]}$.

(3) O'Grady constructed in [O3] a 20-parameter family of hyper-Kähler algebraic fourfolds. They are quasi-étale double covers of certain sextic hypersurfaces constructed by Eisenbud, Popescu, and Walter, and are deformations of the second punctual Hilbert scheme of a general K3 surface of genus 6.

Our purpose in this paper is to construct and study another family of hyper-Kähler fourfolds, which is close in spirit to the Beauville-Donagi family: it is related to the geometry of Grassmannians, and there is an associated Fano hypersurface which will play the role of the cubic hypersurface in [BD]. The Grassmannian considered here is $G(6, V_{10})$, which parametrizes vector subspaces of dimension 6 of a fixed vector space V_{10} of dimension 10. Our starting point, which came to us following a discussion with Peskine, is a 3-form $\sigma \in \bigwedge^3 V_{10}^*$. A dimension count shows that the moduli space of such σ is 20-dimensional.

We associate with σ two varieties: a hypersurface F_σ in $G(3, V_{10})$, and a fourfold Y_σ in $G(6, V_{10})$. Our first result is the following.

Theorem 1.1. *There is a natural correspondence $G_\sigma \subset Y_\sigma \times F_\sigma$, which is of relative dimension 9 over Y_σ. When Y_σ and F_σ are smooth of the expected dimension, this correspondence induces an isomorphism of rational Hodge structures:*

$$H^{20}(F_\sigma, \mathbf{Q})_{\mathrm{van}} \simeq H^2(Y_\sigma, \mathbf{Q})_{\mathrm{van}}.$$

The Hodge structure on the left-hand side has Hodge numbers $h^{9,11} = h^{11,9} = 1$ and $h^{10,10} = 20$, the other Hodge numbers being 0.

As a consequence, we conclude that Y_σ is an irreducible hyper-Kähler fourfold with second Betti number 23. Although the construction of Y_σ allows us to construct explicit hypersurfaces in the moduli space where its Picard number jumps to 2 (see sections 2, 3, and 5), we have not been able to identify an explicit hypersurface in the moduli space where Y_σ is isomorphic to the second punctual Hilbert scheme of a K3 surface. We prove however that Y_σ is a deformation of such a Hilbert scheme.

Theorem 1.2. *The varieties Y_σ, endowed with the Plücker line bundle, are deformation-equivalent to the second punctual Hilbert scheme $S^{[2]}$ of a K3 surface S of genus 12, endowed with the line bundle whose pull-back to $\widetilde{S \times S}$ is $(\mathscr{O}_S(1) \boxtimes \mathscr{O}_S(1))^{10}(-33\widetilde{E})$.*

In this theorem, $\widetilde{S \times S} \to S \times S$ is the blow-up of the diagonal, $\widetilde{E}$ is the exceptional divisor, and the pull-back is via the canonical double cover $\widetilde{S \times S} \to S^{[2]}$. The proof of this result is closely related to that of the main result of [Hu], where Huybrechts proved that birational equivalence implies deformation equivalence for irreducible hyper-Kähler manifolds. However, we are in a situation where only a singular degeneration of Y_σ is birationally equivalent to the second punctual Hilbert scheme of a K3 surface, to which we cannot apply directly Huybrechts' theorem.

To close this introduction, we would like to explain how our construction fits into the general results of [GHS] on the Kodaira dimensions of certain modular varieties dominated by moduli spaces of hyper-Kähler manifolds (we would like to thank Hulek for pointing this out to us). We quickly review some of the relevant results of [GHS].

Let S be a K3 surface and let L be the rank-23 lattice $H^2(S^{[2]}, \mathbf{Z})$, equipped with the Beauville-Bogomolov quadratic form q ([B]). Depending on the positive integer d, there are, under the action of the stable orthogonal group of L, either one or two orbits of primitive vectors h of L with $q(h) = 2d$: one is called *of split type* and the other *of nonsplit type* (it occurs if and only if $d \equiv -1 \pmod 4$). Polarized hyper-Kähler manifolds which are deformation equivalent to $(S^{[2]}, h)$ admit a quasi-projective coarse moduli space $\mathscr{M}_h$ which is finite over a dense open subset of a locally symmetric modular variety $\mathscr{S}_h$. When h is of split type, it is proved in [GHS] that $\mathscr{S}_h$ (hence also every component of $\mathscr{M}_h$) is of general type for $d \geq 12$ and of nonnegative Kodaira dimension for $d = 9$ or 11.

On the other hand, for the class h of the line bundle mentioned in Theorem 1.2, we have

$$q(h) = 100 \times 22 + (33)^2 \times (-2) = 22,$$

hence $d = 11$. Furthermore, h is of nonsplit type, and our construction proves that one component of $\mathscr{M}_h$ (hence also $\mathscr{S}_h$) is unirational.

Remark 1.3. Part of the results of this paper (and particularly those concerning the Hodge theory of the hypersurface F_σ) are related to those of [KMM], where hypersurfaces or complete intersections in homogeneous varieties with a Hodge structure on middle cohomology of 3-dimensional Calabi-Yau type are exhibited and studied.

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Notation. If V is a complex vector space, we denote by $G(d, V)$ the Grassmannian of vector subspaces of V of dimension d, by $\mathscr{S}_d$ the rank-d tautological vector subbundle on $G(d, V)$, and by $\mathscr{E}_d$ its dual.

2. The hypersurface F_σ and the fourfold Y_σ

Let V_{10} be a (complex) vector space of dimension 10 and let σ be a general element in $\bigwedge^3 V_{10}^*$. The 3-form σ determines a Plücker hyperplane section

$$F_\sigma \subset G(3, V_{10}) \subset \mathbf{P}(\bigwedge^3 V_{10})$$

consisting of 3-dimensional vector subspaces of V_{10} on which σ vanishes.

On the other hand, σ determines a subvariety

$$Y_\sigma \subset G(6, V_{10})$$

defined as the set of 6-dimensional vector subspaces of V_{10} on which σ vanishes identically. It is the zero-set of a general section of $\bigwedge^3 \mathscr{E}_6$. As $\mathscr{E}_6$ is generated by global sections, Y_σ is smooth of codimension $\mathrm{rk}(\bigwedge^3 \mathscr{E}_6) = 20$. Using a Koszul resolution and Bott's theorem, one shows that it is connected.

We denote by $\mathscr{O}_{G(6,V_{10})}(1) = \det(\mathscr{E}_6)$ the Plücker line bundle on $G(6, V_{10})$. As $\omega_{G(6,V_{10})} = \mathscr{O}_{G(6,V_{10})}(-10)$ and $\det(\bigwedge^3 \mathscr{E}_6) = \mathscr{O}_{G(6,V_{10})}(10)$, we conclude by adjunction that Y_σ is a smooth fourfold with trivial canonical bundle.

Next we observe that there is a natural correspondence between F_σ and Y_σ. Namely, each point of Y_σ determines a 6-dimensional vector subspace $W_6 \subset V_{10}$ on which σ vanishes identically, hence an inclusion $G(3, W_6) \subset F_\sigma$. Putting this together in a family gives us a variety

$$G_\sigma = \{([W_3], [W_6]) \in G(3, V_{10}) \times G(6, V_{10}) \mid W_3 \subset W_6,\ \sigma|_{W_6} = 0\},$$

with two projections

$$Y_\sigma \xleftarrow{\ q\ } G_\sigma \xrightarrow{\ p\ } F_\sigma. \tag{1}$$

The fibers of q are the 9-dimensional Grassmannians $G(3, W_6)$. There is thus an induced cohomological correspondence

$$q_*p^* : H^{20}(F_\sigma, \mathbf{Q}) \to H^2(Y_\sigma, \mathbf{Q}),$$

whose restriction to vanishing cohomology will be denoted by

$$(q_*p^*)_{\mathrm{van}} : H^{20}(F_\sigma, \mathbf{Q})_{\mathrm{van}} \to H^2(Y_\sigma, \mathbf{Q}), \tag{2}$$

where, if we denote by j the inclusion $F_\sigma \hookrightarrow G(3, V_{10})$,

$$H^{20}(F_\sigma, \mathbf{Q})_{\mathrm{van}} := \mathrm{Ker}\big(H^{20}(F_\sigma, \mathbf{Q}) \xrightarrow{j_*} H^{22}(G(3, V_{10}), \mathbf{Q})\big).$$

Our aim in this section is to investigate the geometry of Y_σ and of the correspondence introduced above. We will show the following.

Theorem 2.1. *The variety Y_σ is an irreducible hyper-Kähler fourfold with $b_2 = 23$.*

This means by definition that Y_σ has an everywhere nondegenerate holomorphic 2-form, unique up to multiplication by a nonzero scalar, and, because Y_σ has trivial canonical bundle, this is equivalent by [B] and [Bo] to $h^{2,0}(Y_\sigma) \neq 0$ and no finite cover of Y_σ is a product of two algebraic K3 surfaces, or the product of an abelian surface by an algebraic K3 surface, or an abelian fourfold.

The first step in the proof is the following result concerning the geometry of F_σ.

Theorem 2.2. *1) The only nonzero Hodge numbers of the Hodge structure on $H^{20}(F_\sigma, \mathbf{Q})_{\mathrm{van}}$ are*

$$h^{9,11}(F_\sigma) = h^{11,9}(F_\sigma) = 1 \quad \text{and} \quad h^{10,10}(F_\sigma)_{\mathrm{van}} = 20.$$

2) For σ very general, the Hodge structure on $H^{20}(F_\sigma, \mathbf{Q})_{\mathrm{van}}$ is simple.

3) The morphism of Hodge structures $(q_*p^*)_{\mathrm{van}}$ in (2) is injective.

Proof. 1) This is a consequence of Griffiths' description of the Hodge structure on the vanishing cohomology of an ample hypersurface (see [G] and [V1], 6.1.2). Let $U := G(3,V_{10}) - F_\sigma$. We have first of all the following.

Lemma 2.3. *The restriction map*

$$H^{20}(G(3,V_{10}),\mathbf{Q}) \to H^{20}(U,\mathbf{Q})$$

is zero.

Proof. The cohomology of $G(3,V_{10})$ is generated as an algebra by the classes $\ell = c_1(\mathscr{S}_3)$, $c_2 = c_2(\mathscr{S}_3)$, and $c_3 = c_3(\mathscr{S}_3)$, where ℓ is proportional to the class of F_σ, hence vanishes on U. On the other hand, consider the projective bundle $\mathbf{P}(\mathscr{S}_3)$ on $G(3,V_{10})$. It admits a natural map α to $\mathbf{P}(V_{10})$ and its cohomology is generated by $h = \alpha^* c_1(\mathscr{O}_{\mathbf{P}(V_{10})}(1))$ as an algebra over $H^\bullet(G(3,V_{10}),\mathbf{Q})$, with the sole relation

$$h^3 + h^2\ell + hc_2 + c_3 = 0.$$

Modulo ℓ, hence in $H^\bullet(U)$, this relation becomes

$$h^3 + hc_2 + c_3 = 0.$$

Together with the vanishing $h^{10} = 0$, this yields the following equalities in $H^\bullet(U,\mathbf{Q})$:

$$c_2^4 = 3c_2c_3^2,\ c_3^3 = 4c_3c_2^3,\ c_2^2c_3^2 = 0.$$

But the only polynomials of weighted degree 10 in c_2 and c_3 are c_2^5 and $c_2^2c_3^2$, and they vanish by the relations above. □

This lemma and the Thom exact sequence ([V1], 6.1.1) show that the residue map is an isomorphism

$$H^{21}(U,\mathbf{Q}) \simeq H^{20}(F_\sigma,\mathbf{Q})_{\mathrm{van}}.$$

Now we apply Griffiths' theory ([G]; see also [V1], 6.1.2), which describes the Hodge filtration on the cohomology $H^{21}(U,\mathbf{C})$ (which up to a shift of -1 corresponds to the Hodge filtration on $H^{20}(F_\sigma,\mathbf{Q})_{\mathrm{van}}$). The only assumption we need is the vanishing

$$H^i(G(3,V_{10}),\Omega^j_{G(3,V_{10})}(k)) = 0 \quad \text{for all } k>0,\ i>0,\ j\geq 0,$$

which we get from Bott's theorem. It follows that

$$F^pH^{20}(F_\sigma,\mathbf{C})_{\mathrm{van}} = F^{p+1}H^{21}(U,\mathbf{C})$$

is generated by residues

$$\mathrm{Res}_{F_\sigma}\frac{\alpha}{\sigma^{21-p}},$$

where α runs through the space of sections of $\omega_{G(3,V_{10})}(21-p) = \mathscr{O}_{G(3,V_{10})}(11-p)$. We immediately get the vanishing of $F^{12}H^{20}(F_\sigma, \mathbf{C})_{\rm van}$, hence of $h^{p,20-p}(F_\sigma, \mathbf{C})_{\rm van}$ for $p \geq 12$.

For $p = 11$, we get a 1-dimensional vector space generated by $\mathrm{Res}_{F_\sigma} \frac{\alpha}{\sigma^{10}}$, where α is a nowhere vanishing section of $\omega_{G(3,V_{10})}(10) = \mathscr{O}_{G(3,V_{10})}$. For $p = 10$, we find that $H^{10,10}(F_\sigma, \mathbf{C})_{\rm van}$ is generated by the residues

$$\mathrm{Res}_{F_\sigma} \frac{\alpha}{\sigma^{11}},$$

where α runs through the space of sections of $\omega_{G(3,V_{10})}(11) = \mathscr{O}_{G(3,V_{10})}(1)$. Finally, we recall the analysis (adapted from [G]; see also [V1], 6.1.3, where the case of hypersurfaces in a projective space is treated) of the kernel of the maps

$$\begin{aligned} H^0(G(3,V_{10}), \mathscr{O}_{G(3,V_{10})}) \simeq H^0(F_\sigma, \mathscr{O}_{F_\sigma}) \quad &\to \quad H^{11,9}(F_\sigma) \\ \alpha \quad &\mapsto \quad \mathrm{Res}_{F_\sigma} \frac{\alpha}{\sigma^{10}} \end{aligned}$$

and

$$\begin{aligned} H^0(G(3,V_{10}), \mathscr{O}_{G(3,V_{10})}(1))/\mathbf{C}\sigma \simeq H^0(F_\sigma, \mathscr{O}_{F_\sigma}(1)) \quad &\to \quad H^{10,10}(F_\sigma) \\ \alpha \quad \mapsto \quad \mathrm{Res}_{F_\sigma} \frac{\alpha}{\sigma^{11}} \quad &\pmod{H^{11,9}(F_\sigma)} \end{aligned}$$

induced by the residue maps. The same analysis as in the case of hypersurfaces in a projective space shows that the kernels are Jacobian ideals obtained respectively from sections of $T_{G(3,V_{10})}(-1)$ and sections of $T_{G(3,V_{10})}$ via the natural maps

$$H^0(G(3,V_{10}), T_{G(3,V_{10})}(l)) \to H^0(F_\sigma, \mathscr{O}_{F_\sigma}(l+1)),$$

for $l \in \{-1, 0\}$, induced by the normal bundle exact sequence of F_σ.

Now $H^0(G(3,V_{10}), T_{G(3,V_{10})}(-1)) = 0$, whereas the vector space $H^0(G(3,V_{10}), T_{G(3,V_{10})})$ has dimension 99 and injects into $H^0(F_\sigma, \mathscr{O}_{F_\sigma}(1))$. Hence we conclude $h^{10,10}(F_\sigma)_{\rm van} = 119 - 99 = 20$.

2) The simplicity of a polarized Hodge structure of weight 20 with Hodge numbers $h^{11,9} = 1$, and $h^{i,20-i} = 0$ for $i > 11$, is equivalent to the fact that there are no Hodge classes in $H^{10,10}$ (here we use the polarization to say that any nontrivial Hodge substructure has $h^{11,9} = 0$, hence consists of Hodge classes, or its orthogonal complement has $h^{11,9} = 0$, hence consists of Hodge classes). So it suffices to prove that for σ very general, there are no Hodge classes in $H^{20}(F_\sigma, \mathbf{Q})_{\rm van}$. This is a Noether-Lefschetz type theorem which is proved by the classical Lefschetz monodromy argument (see [V1], 3.2.3).

3) By simplicity, the morphism of Hodge structures $(q_*p^*)_{\rm van}$ is either 0 or injective. It thus suffices to prove that it is not 0. Equivalently, it

suffices to prove that the morphism

$$p_*q^* : H_2(Y_\sigma, \mathbf{Q}) \to H_{20}(F_\sigma, \mathbf{Q})$$

has rank at least 2. Indeed, since $H_2(G(6, V_{10}), \mathbf{Q})$ has dimension 1, denoting by $i_\sigma : Y_\sigma \to G(6, V_{10})$ the inclusion, we find that p_*q^* has rank at least 2 if and only if its restriction

$$p_*q^*|_{\operatorname{Ker} i_{\sigma*}} : \operatorname{Ker} i_{\sigma*} \to H_{20}(F_\sigma, \mathbf{Q})$$

has rank at least 1. But this morphism takes its values in $H_{20}(F_\sigma, \mathbf{Q})_{\mathrm{van}}$ and its dual is the morphism $(q_*p^*)_{\mathrm{van}}$ composed with the inclusion of $(\operatorname{Ker} i_{\sigma*})^*$ into $H^2(Y_\sigma, \mathbf{Q})$.

In order to see that the rank of p_*q^* is at least 2, we make the following construction. Consider subspaces $V_4 \subset V_7 \subset V_{10}$, where the subscripts indicate the dimension, and choose $\sigma \in \bigwedge^3 V_{10}^*$ satisfying

$$\sigma|_{V_7} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \quad \text{and} \quad V_4 = \{\alpha_1 = \alpha_2 = \alpha_3 = 0\}.$$

One verifies that one can choose such a σ keeping Y_σ and F_σ smooth.

In this situation, Y_σ contains a line (with respect to the Plücker embedding); namely, choosing any V_5 such that $V_4 \subset V_5 \subset V_7$, and observing that σ vanishes on any hyperplane of V_7 containing V_4, we find that the line

$$C = \{[W_6] \mid V_5 \subset W_6 \subset V_7\}$$

is contained in Y_σ.

Let $Z = p(q^{-1}(C))$. Observe that the class $z \in H_{20}(F_\sigma, \mathbf{Q})$ of Z is equal to p_*q^*c, where c is the class of C. Furthermore, the classes so obtained are in the same orbit under the monodromy action (this is because the set of triples $(\sigma, [V_4], [V_7])$ as above is irreducible, hence the corresponding classes are all obtained one from the other by parallel transport).

We will now specialize σ further in two ways, asking that Y_σ contain two curves C and C' as above (but of course with different cohomology classes in Y_σ).

A) We choose $V_4 \subset V_7 \subset V_{10}$ and $V_4' \subset V_7' \subset V_{10}$ in such a way that the intersection $V_7 \cap V_7'$ is transverse, and $V_4 \cap V_4' = \{0\}$. In a suitable basis $(e_1, \ldots, e_{10})$ of V_{10}, we take

$$V_7 = \langle e_1, \ldots, e_7\rangle,\ V_4 = \langle e_2, \ldots, e_5\rangle,\ V_7' = \langle e_4, \ldots, e_{10}\rangle,\ V_4' = \langle e_6, \ldots, e_9\rangle.$$

Then,

$$\sigma|_{V_7} = e_1^* \wedge e_6^* \wedge e_7^* \quad \text{and} \quad \sigma|_{V_7'} = e_4^* \wedge e_5^* \wedge e_{10}^*,$$

and this is compatible, because on the intersection

$$V_7 \cap V_7' = \langle e_4, \ldots, e_7\rangle,$$

the two 3-forms $e_1^* \wedge e_6^* \wedge e_7^*$ and $e_4^* \wedge e_5^* \wedge e_{10}^*$ vanish. One verifies that for a general choice of σ as above, Y_σ and F_σ are smooth.

B) We choose $V_4 \subset V_7 \subset V_{10}$ and $V_4' \subset V_7' \subset V_{10}$ in such a way that the intersection $V_7 \cap V_7'$ is transverse, but $V_4 \cap V_4'$ is 1-dimensional. In a suitable basis $(e_1, \ldots, e_{10})$ of V_{10}, we take

$$V_7 = \langle e_1, \ldots, e_7 \rangle, \; V_4 = \langle e_1, \ldots, e_4 \rangle,$$
$$V_7' = \langle e_4, \ldots, e_{10} \rangle, \; V_4' = \langle e_4, e_8, e_9, e_{10} \rangle.$$

Then,

$$\sigma|_{V_7} = e_5^* \wedge e_6^* \wedge e_7^* \quad \text{and} \quad \sigma|_{V_7'} = e_5^* \wedge e_6^* \wedge e_7^*$$

are obviously compatible and we indeed have a 1-dimensional intersection $V_4 \cap V_4'$, generated by e_4.

One checks that for a general choice of σ as above, Y_σ and F_σ are smooth.

The proof of the theorem is then concluded by the following lemma.

Lemma 2.4. *The classes z, $z' \in H_{20}(F_\sigma, \mathbf{Q})$ constructed above satisfy*

$$z \cdot z' = 0$$

in situation A), and

$$z \cdot z' = 1$$

in situation B).

Indeed, if $p_* q^*$ has rank at most 1, the classes z and z' must be proportional. As they are in the same orbit of the monodromy action, they are equal. This contradicts the fact that they satisfy $z \cdot z' = 0$ or $z \cdot z' = 1$ according to the configuration. □

Proof of Lemma 2.4. Note that in both cases, the (singular) variety Z is described as follows:

$$Z = \{W_3 \subset V_7 \mid \dim(W_3 \cap V_5) \geq 2\}.$$

In situation A), we may choose V_5 and V_5' transverse, so that $V_5 \cap V_5' = \{0\}$. But then,

$$Z \cap Z' = \{W_3 \subset V_7 \cap V_7' \mid \dim(W_3 \cap V_5) \geq 2, \; \dim(W_3 \cap V_5') \geq 2\}$$

is clearly empty.

In situation B), we may choose V_5 and V_5' so that they meet along the 1-dimensional vector space $\langle e_4 \rangle = V_4 \cap V_4'$. Then

$$Z \cap Z' = \{W_3 \subset V_7 \cap V_7' \mid \dim(W_3 \cap V_5) \geq 2, \; \dim(W_3 \cap V_5') \geq 2\},$$

and denoting by $V_{5,0}$ (resp. $V'_{5,0}$) the 2-dimensional intersection$V_5 \cap V'_7$ (resp. $V'_5 \cap V_7$), we find

$$Z \cap Z' = \{W_3 \subset V_7 \cap V'_7 \mid \dim(W_3 \cap V_{5,0}) \geq 2,\ \dim(W_3 \cap V'_{5,0}) \geq 2\}.$$

As $V_{5,0}$ and $V'_{5,0}$ are 2-dimensional, one must have for such a W_3:

$$V_{5,0} = W_3 \cap V_{5,0} \quad \text{and} \quad V'_{5,0} = W_3 \cap V'_{5,0},$$

and finally, $W_3 = V_{5,0} + V'_{5,0}$.

Thus the intersection $Z \cap Z'$ consists of one point, namely the point $[V_{5,0} + V'_{5,0}]$ of $G(3, V_{10})$, and it follows that $z \cdot z'$ is nonzero in this case. To prove $z \cdot z' = 1$, one notes that Z and Z' are smooth at the above point, and one checks that the intersection is transverse. □

Remark 2.5. The hyper-Kähler manifolds Y_σ containing a line as above are very similar to the Fano varieties of lines in a cubic fourfold ([BD]) containing a plane ([V2]). Indeed, the V_5 introduced in the construction of the line C varies in the plane $\mathbf{P}(V_7/V_4)$. Furthermore, the subset of Y_σ swept out by the curves C is the dual plane $\mathbf{P}((V_7/V_4)^*) \subset Y_\sigma$ parametrizing hyperplanes of V_7 containing V_4. This, as noticed in [V2], is a Lagrangian plane in Y_σ.

Proof of Theorem 2.1. Theorem 2.2 implies $h^{2,0}(Y_\sigma) \neq 0$. In order to show that Y_σ is an irreducible hyper-Kähler variety, it thus suffices to show that no finite étale cover of Y_σ is an abelian fourfold, the product of an abelian surface and an algebraic K3 surface, or the product of two algebraic K3 surfaces. But this follows again from Theorem 2.2. Indeed, this theorem implies that for very general σ, the Hodge structure on $H^2(Y_\sigma, \mathbf{Q})$ contains an irreducible Hodge substructure with $h^{1,1} = 20$. If such a covering existed, this irreducible Hodge structure would inject into the transcendental part of the H^2 of an abelian fourfold, an abelian surface, or an algebraic K3 surface, where "transcendental" means "orthogonal to the set of Hodge classes in the Poincaré dual cohomology group." But the Hodge structures on the transcendental part of the H^2 of an abelian fourfold, an abelian surface, or an algebraic K3 surface all have $h^{1,1}_{\mathrm{tr}} \leq 19$.

To conclude the proof of the theorem, we need to show $b_2(Y_\sigma) = 23$. We already know $b_2(Y_\sigma) \geq 23$: indeed, the image of $(q_* p^*)_{\mathrm{van}}$ has rank 22 and it is not the whole of $H^2(Y_\sigma, \mathbf{Q})$ because it does not contain any Hodge class for a very general σ. As Y_σ is an irreducible hyper-Kähler fourfold, the equality $b_2(Y_\sigma) = 23$ then follows from [Gu], where it is proved that 23 is the maximal possible second Betti number. □

Remark 2.6. It is also possible (and even shorter) to prove that Y_σ is a hyper-Kähler variety by showing

$$\chi(Y_\sigma, \mathscr{O}_{Y_\sigma}) = \sum_{i=0}^{20} (-1)^i \chi\Big(G(6, V_{10}), \bigwedge^i (\bigwedge^3 \mathscr{S}_6)\Big) = 3,$$

using for example Macaulay. Alternatively, as shown to us by Manivel and Han, using the Koszul resolution of $\mathscr{O}_{Y_\sigma}$, Bott's theorem, and properties of the irreducible representations that occur in $\bigwedge^i(\bigwedge^3 V_6)$ (or, alternatively, the program LiE), one can prove directly $h^2(Y_\sigma, \mathscr{O}_{Y_\sigma}) = 1$. However, the proof above is more geometric and explains where the holomorphic 2-form comes from.

To conclude this section, note that Theorem 2.1 allows us in turn to refine Theorem 2.2 as follows. Consider again the inclusion i_σ of Y_σ into $G(6, V_{10})$ and define the vanishing cohomology $H^2(Y_\sigma, \mathbf{Q})_{\mathrm{van}}$ as the kernel of

$$i_{\sigma *} : H^2(Y_\sigma, \mathbf{Q}) \to H^{42}(G(6, V_{10}), \mathbf{Q}) \simeq H_6(G(6, V_{10}), \mathbf{Q}).$$

Corollary 2.7. *The morphism $i_{\sigma *}$ has rank 1. The morphism of Hodge structures $(q_* p^*)_{\mathrm{van}}$ defined in (2) takes values in $H^2(Y_\sigma, \mathbf{Q})_{\mathrm{van}}$ and induces an isomorphism*

$$H^{20}(F_\sigma, \mathbf{Q})_{\mathrm{van}} \simeq H^2(Y_\sigma, \mathbf{Q})_{\mathrm{van}}.$$

Proof. The composition

$$i_{\sigma *} \circ (q_* p^*)_{\mathrm{van}} : H^{20}(F_\sigma, \mathbf{Q})_{\mathrm{van}} \to H^{42}(G(6, V_{10}), \mathbf{Q})$$

vanishes, because the Hodge structure on the right-hand side is trivial, while the Hodge structure on the left-hand side is nontrivial and generically simple. Thus $(q_* p^*)_{\mathrm{van}}$ takes values in $H^2(Y_\sigma, \mathbf{Q})_{\mathrm{van}}$. We know that this morphism is injective and that the left-hand side has dimension 22. Hence all the statements follow from the equality $b_2(Y_\sigma) = 23$, so that $\dim(\operatorname{Ker} i_{\sigma *}) \leq 22$, with equality if and only if $\operatorname{rk}(i_{\sigma *}) = 1$ and $(q_* p^*)_{\mathrm{van}}$ surjects onto $H^2(Y_\sigma, \mathbf{Q})_{\mathrm{van}}$. □

3. Singular hypersurfaces F_σ

In this section, we are interested in those $\sigma \in \bigwedge^3 V_{10}^*$ for which the hypersurface $F_\sigma \subset G(3, V_{10})$ is *singular.*

Proposition 3.1. *The dual variety $G(3, V_{10})^* \subset \mathbf{P}(\bigwedge^3 V_{10}^*)$ is an irreducible hypersurface. For σ general in $G(3, V_{10})^*$, the corresponding hyperplane section F_σ of $G(3, V_{10})$ has a unique singular point. It corresponds to a 3-dimensional vector subspace $W \subset V_{10}$ such that $\sigma|_{\bigwedge^2 W \wedge V_{10}} = 0$.*

Proof. The fact that the dual variety $G(3,V_{10})^* \subset \mathbf{P}(\bigwedge^3 V_{10}^*)$ is a hypersurface follows for example from [L], §3, which proves that its degree is 640. Then it is classical that this hypersurface is irreducible, and that a general point corresponds to a hyperplane tangent to $G(3,V_{10})$ at a single point.

Let $[W]$ be a point of $G(3,V_{10})$. The embedding of

$$T_{G(3,V_{10}),[W]} \simeq \mathrm{Hom}(W, V_{10}/W)$$

into

$$T_{\mathbf{P}(\bigwedge^3 V_{10}),[\bigwedge^3 W]} \simeq \mathrm{Hom}(\bigwedge^3 W, \bigwedge^3 V_{10} \Big/ \bigwedge^3 W)$$

is given by

$$u \mapsto \big(w_1\wedge w_2\wedge w_3 \mapsto u(w_1)\wedge w_2\wedge w_3 + w_1\wedge u(w_2)\wedge w_3 + w_1\wedge w_2\wedge u(w_3)\big).$$

Therefore, the hyperplane section $F_\sigma \subset G(3,V_{10})$ defined by $\sigma \in \bigwedge^3 V_{10}^*$ is singular at $[W]$ if and only if $\sigma(w_1\wedge w_2\wedge v)=0$ for all w_1, w_2 in W and all $v \in V_{10}$. □

We will henceforth assume that σ corresponds to a general point of the discriminant hypersurface $G(3,V_{10})^*$ and we denote by $[W]$ the unique singular point of F_σ. By Proposition 3.1, we have

$$\sigma|_{\bigwedge^2 W\wedge V_{10}} = 0.$$

For each $d \in \{0,1,2,3\}$, let Y_σ^d be the closure in Y_σ of the set of points $[W_6]$ such that $\dim(W\cap W_6)=d$.

Proposition 3.2. *1) The variety Y_σ^3 is a general K3 surface of genus 12.*

2) The varieties Y_σ^1 and Y_σ^2 are either empty or smooth of dimension 2.

3) The variety Y_σ^0 is a smooth and irreducible fourfold.

4) The variety Y_σ is a normal and irreducible fourfold.

Proof. 1) Choose a decomposition $V_{10} = W \oplus Q$. Since σ vanishes on $\bigwedge^2 W \wedge V_{10}$, we can write $\sigma = \sigma_1 + \sigma_2$ with $\sigma_1 \in W^* \otimes \bigwedge^2 Q^*$ and $\sigma_2 \in \bigwedge^3 Q^*$. The projection $V_{10} \to Q$ induces an isomorphism between Y_σ^3 and

$$S = \{[W'] \in G(3,Q) \mid [W\oplus W'] \in Y_\sigma\}. \tag{3}$$

This variety is defined by the vanishing of σ_2, viewed as a section of $\mathscr{O}_{G(3,Q)}(1)$, and of σ_1, viewed as a 3-dimensional space of sections of $\bigwedge^2 \mathscr{S}_3^*$. Since σ_1 and σ_2 are general, S is a general K3 surface of genus 12 ([M], Theorem 10).

2) This follows from a parameter count, which we will only do for Y^2_σ, the case of Y^1_σ being completely analogous. The dimension of the set of $([\sigma],[W],[W_1],[W_2],[W_4],[W'_3])$ such that $W = W_1 \oplus W_2$, $V_{10} = W \oplus W_4 \oplus W'_3$, and both $\sigma|_{\bigwedge^2 W \wedge V_{10}}$ and $\sigma|_{\bigwedge^3 (W_2 \oplus W_4)}$ vanish, i.e., with the notation above,

$$\begin{aligned} \sigma_1 &\in \Big(W_2 \otimes \big((W_4 \otimes W'_3) \oplus \bigwedge^2 W'_3\big)\Big)^* \oplus \Big(W_1 \otimes \bigwedge^2 (W_4 \oplus W'_3)\Big)^* \\ \sigma_2 &\in \Big(\bigwedge^2 W_4 \otimes W'_3\Big)^* \oplus \Big(W_4 \otimes \bigwedge^2 W'_3\Big)^* \oplus \Big(\bigwedge^3 W'_3\Big)^*, \end{aligned}$$

is $30 + 21 + 18 + 12 + 1 - 1 = 81$ for the choice of $[\sigma]$, plus $9 + 16 + 24 + 21 = 70$ for the choices of W_1, W_2, W_4, and W'_3, hence 151. The set of $([\sigma],[W],[W_6])$ such that F_σ is singular at $[W]$ and $[W_6] \in Y^2_\sigma$, is therefore smooth of dimension 151 minus $2 + 8 + 21$ for the choices of W_1, W_2, and W'_3, hence 120. For $[\sigma]$ general in the 118-dimensional hypersurface $G(3, V_{10})^*$, it follows by generic smoothness that Y^2_σ is either empty, or smooth of dimension 2.

3) Similarly, we consider the set of $([\sigma],[W],[W_6],[W_1])$ such that $V_{10} = W \oplus W_6 \oplus W_1$, and both $\sigma|_{\bigwedge^2 W \wedge V_{10}}$ and $\sigma|_{\bigwedge^3 W_6}$ vanish. It is smooth, hence so is the (122-dimensional) set of $([\sigma],[W],[W_6])$ such that F_σ is singular at $[W]$, and $[W_6] \in Y^0_\sigma$. By generic smoothness, so is the general, 4-dimensional fiber Y^0_σ of the projection $([\sigma],[W],[W_6]) \mapsto ([\sigma],[W])$.

4) Since Y_σ has everywhere dimension at least 4, the variety Y^0_σ is dense in Y_σ, which has therefore dimension 4. It is moreover a local complete intersection, hence is connected in codimension 1 ([H]). It is also connected and, its singular locus being contained in the surface $Y^1_\sigma \sqcup Y^2_\sigma \sqcup Y^3_\sigma$, it is irreducible and normal. □

Let $p : V_{10} \to V_{10}/W$ be the canonical projection. The K3 surface S of (3) is defined more canonically as

$$S = \{[W'] \in G(3, V_{10}/W) \mid [p^{-1}(W')] \in Y_\sigma\}. \tag{4}$$

We now prove the main result of this section.

Theorem 3.3. *There is a birational isomorphism*

$$\phi : S^{[2]} \dashrightarrow Y_\sigma$$

defined as follows: let $[W']$ and $[W'']$ be general points of S; then $\phi([W'],[W''])$ is the only element $[W_6]$ of Y^0_σ such that $p(W_6) = W' \oplus W''$.

Proof. We first show that the map ϕ^{-1} is well-defined at a general point $[W_6]$ of Y_σ^0. We will show that there are exactly two points $[W']$ of S such that $W' \subset p(W_6)$.

Choose as above a decomposition $V_{10} = W \oplus Q$ with $W_6 \subset Q$ and identify V_{10}/W with Q. Let W' be a 3-dimensional vector subspace of W_6. Since σ vanishes on $\bigwedge^2 W \wedge V_{10}$ and on $\bigwedge^3 W_6$, the condition $[W \oplus W'] \in Y_\sigma$ is equivalent to the vanishing of σ on $W \otimes \bigwedge^2 W'$. This means that $[W'] \in G(3, W_6)$ is in the zero-locus of 3 sections of $\bigwedge^2 \mathscr{S}_3^*$. Since $c_3(\bigwedge^2 \mathscr{S}_3^*)^3 = 2$, this zero-locus consists of either two or infinitely many points. As in the proof of Proposition 3.2 above, one sees that it consists in fact of two points $[W']$ and $[W'']$ and that moreover, $W' + W''$ has dimension 6, hence is equal to W_6. In other words, ϕ^{-1} is well-defined at the point $[W_6]$, which it maps to the unordered pair $([W'], [W''])$.

Conversely, let $[W']$ and $[W'']$ be general points of S. Choose again a splitting $V_{10} \simeq W \oplus V_{10}/W$. Write a 6-dimensional vector subspace W_6 of $W \oplus W' \oplus W''$ such that $W \cap W_6 = \{0\}$ as the graph

$$\{u(w', w'') + w' + w'' \mid w' \in W',\ w'' \in W''\}$$

of some linear map $u : W' \oplus W'' \to W$. The condition that σ vanish on W_6 is then equivalent to the vanishing of the form $(\mathrm{Id}_{W' \oplus W''}, u)^* \sigma \in \bigwedge^3 (W' \oplus W'')^*$. Since σ vanishes on $\bigwedge^3 (W \oplus W')$ and on $\bigwedge^3 (W \oplus W'')$, this form is actually in $\left(\bigwedge^2 W' \otimes W''\right)^* \oplus \left(W' \otimes \bigwedge^2 W''\right)^*$ and depends in an affine way on u. In other words, $[W_6] \in Y_\sigma$ if and only if u is in the inverse image of 0 by the *affine* map $f_{W',W''} : u \mapsto (\mathrm{Id}_{W' \oplus W''}, u)^* \sigma$. For later use, we denote by

(5)
$$\begin{array}{rcl} \vec{f}_{W',W''} : \mathrm{Hom}(W' \oplus W'', W) & \to & \left(\bigwedge^2 W' \otimes W''\right)^* \oplus \left(W' \otimes \bigwedge^2 W''\right)^* \\ u & \mapsto & (\mathrm{Id}, u)^* \sigma_1 \end{array}$$

the associated linear map.

It follows that the set of elements $[W_6]$ of Y_σ^0 such that $W_6 \subset W \oplus W' \oplus W''$ are (possibly empty) affine spaces.

The graph of ϕ^{-1} has dimension 4 and dominates $S^{[2]}$, and we just proved that the fibers are affine spaces. It follows that this projection is birational, hence ϕ^{-1} (and ϕ) are birational isomorphisms. □

We end this section with the computation of the line bundle $\phi^* \mathscr{O}_{Y_\sigma}(1)$ on $S^{[2]}$. Recall that, if

$$\varepsilon : \widetilde{S \times S} \to S \times S$$

is the blow-up of the diagonal, $S^{[2]}$ can be seen as the quotient of $\widetilde{S \times S}$ by the involution exchanging the two factors. We denote by

$$r : \widetilde{S \times S} \to S^{[2]}$$

the quotient map, by $\widetilde{E} \subset \widetilde{S \times S}$ the exceptional divisor of ε, and by E its image in $S^{[2]}$.

For $i \in \{1, 2\}$, let $p_i : \widetilde{S \times S} \to S$ be the i-th projection. Given coherent sheaves $\mathscr{F}$ and $\mathscr{G}$ on S, we define coherent sheaves on $\widetilde{S \times S}$ by setting

$$\mathscr{F} \boxtimes \mathscr{G} := p_1^*\mathscr{F} \otimes p_2^*\mathscr{G} \quad \text{and} \quad \mathscr{F} \boxplus \mathscr{G} := p_1^*\mathscr{F} \oplus p_2^*\mathscr{G}.$$

Proposition 3.4. *The pull-back to $\widetilde{S \times S}$ of $\phi^*\mathscr{O}_{Y_\sigma}(1)$ is isomorphic to $(\mathscr{O}_S(1) \boxtimes \mathscr{O}_S(1))^{10}(-33\widetilde{E})$.*

Proof. There are two natural vector bundles of rank 6 on the open subset U_1 of $S^{[2]}$ where ϕ defines a morphism $\phi_{U_1} : U_1 \to Y_\sigma \subset G(6, V_{10})$: the pull-back $\phi_{U_1}^*(\mathscr{E}_6|_{Y_\sigma})$ and $\mathscr{F}_6|_{U_1}$, where

$$\mathscr{F}_6 := r_* p_1^*(\mathscr{E}_3|_S).$$

Recalling from Theorem 3.3 the definition of ϕ, observe that there is a natural morphism

$$P : \mathscr{F}_6|_{U_1} \to \phi_{U_1}^*(\mathscr{E}_6|_{Y_\sigma})$$

induced by the dual of the projection $p : V_{10} \to V_{10}/W$. This implies

$$\phi^*(\mathscr{O}_{Y_\sigma}(1)) = \det(\mathscr{E}_6|_{Y_\sigma}) = (\det \mathscr{F}_6)(D), \tag{6}$$

where D is the divisor defined by the vanishing of the determinant of P. Next, as the pull-back of $\mathscr{F}_6$ to $\widetilde{S \times S}$ fits into the exact sequence

$$0 \to r^*\mathscr{F}_6 \to \mathscr{E}_3|_S \boxplus \mathscr{E}_3|_S \to \varepsilon_{\widetilde{E}}^* \mathscr{E}_3|_S \to 0,$$

where $\varepsilon_{\widetilde{E}} : \widetilde{E} \to S$ is induced by the blow-up map ε, we get

$$\det(r^*\mathscr{F}_6) = (\mathscr{O}_S(1) \boxtimes \mathscr{O}_S(1))(-3\widetilde{E}). \tag{7}$$

It remains to analyze D. We first compute the class of the divisor D' where the morphism (5), suitably defined, is not of maximal rank. It has the same support as D. Indeed, on the complement of D, the map $\vec{f}_{W',W''}$ defined in (5) is an isomorphism (otherwise, positive-dimensional fibers of ϕ^{-1} would appear in codimension 1). We will next compute the respective multiplicities of D and D'.

Lemma 3.5. *The pull-back to $\widetilde{S \times S}$ of the divisor D' is in the linear system $|(\mathscr{O}_S(1) \boxtimes \mathscr{O}_S(1))^6(-20\widetilde{E})|$.*

Proof. In order to compute the full class of D' as a determinant, we need first to extend the definition of $\vec{f}_{W',W''}$ at a general point of E.

The rational map
$$S^{[2]} \dashrightarrow G(6, V_{10}/W)$$
defined by the global sections of $\mathscr{F}_6$ is well-defined on an open subset U_2 of $S^{[2]}$ whose complement has codimension ≥ 2. At a point $z \in U_2$, we may consider the fiber $\mathscr{F}^*_{6,z}$ as a hyperplane in V_{10}/W which, when z is a general point $([W'],[W''])$, is just $W' \oplus W''$.

On the other hand, there is a natural restriction map
$$R : \bigwedge^3 \mathscr{F}_6 \to \mathscr{L}_2,$$
where $\mathscr{L}_2$ is the rank-2 vector bundle $r_* p_1^*(\bigwedge^3 \mathscr{E}_3)$ on $S^{[2]}$. The fiber of $\mathscr{L}_2$ at a pair $([W'],[W''])$ away from E is the direct sum $(\bigwedge^3 W' \oplus \bigwedge^3 W'')^*$.

At the point z, the map $\vec{f}_{W',W''} : u \mapsto (\mathrm{Id}_{W'\oplus W''}, u)^*\sigma_1$ defined in (5) is now a map $\mathrm{Hom}(\mathscr{F}^*_{6,z}, W) \to \bigwedge^3 \mathscr{F}_{6,z}$ which is linear and takes values in $\operatorname{Ker} R_z$ away from E, and this still remains true along E (this is due to the fact that for $[W']$, $[W''] \in S$, $\sigma_1|_{\bigwedge^3(W\oplus W'\oplus W'')}$ belongs to $W^* \otimes W'^* \otimes W''^*$). Hence, we have extended the definition of the map (5) over U_2 as the fiber of a map
$$\vec{f} : \mathscr{H}om(\mathscr{F}^*_6, W \otimes \mathscr{O}_{U_2}) \to \mathscr{K}er R|_{U_2}$$
between two vector bundles of rank 18. One checks that R is surjective in codimension 1. It follows that the vanishing of $\det(\vec{f})$ gives us a divisor in the linear system
$$|\big(\det(\bigwedge^3 \mathscr{F}_6) \otimes (\det \mathscr{L}_2)^{-1} \otimes (\det \mathscr{F}_6)^{-3}| = |(\det \mathscr{L}_2)^{-1} \otimes (\det \mathscr{F}_6)^7|.$$
Using (7) and the fact that, analogously, the determinant of $\mathscr{L}_2$ pulls back to $(\mathscr{O}_S(1) \boxtimes \mathscr{O}_S(1))(-\widetilde{E})$ on $\widetilde{S \times S}$, we see that this line bundle pulls back to the line bundle
$$(\mathscr{O}_S(1) \boxtimes \mathscr{O}_S(1))^6(-20\widetilde{E})$$
on $\widetilde{S \times S}$, and this concludes the proof of the lemma. □

We can conclude the proof of Proposition 3.4 with the following lemma.

Lemma 3.6. *1) The map ϕ contracts D' to Y^3_σ, so that the rank of P along D' is* 3.

2) The divisor D' has everywhere multiplicity 2.

Indeed, as P has rank 3 along D', the reduced divisor $D'_{\rm red}$ underlying D' appears with multiplicity at least 3 in the divisor defined by $\det(P)$. Using the explicit description of D' given in the proof of item 2) of Lemma 3.6 and the computation of the differential $dP : \operatorname{Ker} P \to \operatorname{Coker} P$ in the normal direction to D', one checks that the multiplicity is exactly 3.

By Lemmas 3.5 and 3.6, $r^*D'_{\rm red}$ belongs to the linear system $|(\mathscr{O}_S(1)\boxtimes \mathscr{O}_S(1))^3(-10\widetilde{E})|$. Hence we get, using (6) and (7):

$$\begin{aligned} r^*\phi^*\mathscr{O}_{Y_\sigma}(1) &= r^*\big((\det \mathscr{F}_6)(3D'_{\rm red})\big) \\ &= (\mathscr{O}_S(1)\boxtimes\mathscr{O}_S(1))(-3\widetilde{E})\otimes(\mathscr{O}_S(1)\boxtimes\mathscr{O}_S(1))^9(-30\widetilde{E}) \\ &= (\mathscr{O}_S(1)\boxtimes\mathscr{O}_S(1))^{10}(-33\widetilde{E}), \end{aligned}$$

which is the content of the proposition. □

Proof of Lemma 3.6. 1) By definition of D', a point z in $U_1\cap U_2$ has the property that the point $\phi(z)$ of Y_σ corresponds to a vector subspace $W_6\subset V_{10}$ such that $p|_{W_6}: W_6\to V_{10}/W$ is not of maximal rank. In other words, with the notation of Proposition 3.2, $\phi(z)$ belongs to Y^i_σ, for some $i\geq 1$. Furthermore, the rank of P at z is equal to the rank of $p|_{W_6}$. By Proposition 3.2, we have $\dim Y^i_\sigma\leq 2$ for $i\geq 1$, thus 1) is equivalent to the following.

Claim. *If σ is general (in the hypersurface parametrizing singular F_σ), no divisor of $S^{[2]}$ is contracted to Y^1_σ or Y^2_σ.*

Let us first consider the case of Y^1_σ. On U_2, the fiber of ϕ over a point $[W_6]\in Y^1_\sigma$ is contained in the set of $([W'],[W''])\in S^{(2)}$ such that $W_6\subset p^{-1}(W'\oplus W'')$. As $[W_6]\in Y^1_\sigma$, the space $p(W_6)$ has dimension 5. Let $p(W_6)^\perp\subset(V_{10}/W)^*$ be the 2-dimensional space of linear forms vanishing on $p(W_6)$. As $p(W_6)$ has codimension 1 in $W'\oplus W''$, we may assume that $p(W_6)\cap W'$ has codimension 1 in W', and the rank at $[W']$ of the evaluation map

$$\mathrm{ev}: p(W_6)^\perp\otimes\mathscr{O}_S\to\mathscr{E}_3$$

is 1. If the fiber $\phi^{-1}([W_6])$ has positive dimension, the set of $[W']$ as above contains a curve, and the saturation $(\operatorname{Im}\mathrm{ev})_{\rm sat}$ has rank 2 and nontrivial effective determinant. But S is very general, hence its Picard group is cyclic, generated by $\det\mathscr{E}_3=\mathscr{O}_S(1)$, so the curve above has to be in a linear system $|\mathscr{O}_S(l)|$, for some $l>0$. We get a contradiction from the fact that the cokernel $\mathscr{E}_3/(\operatorname{Im}\mathrm{ev})_{\rm sat}$ is a rank-1 torsion-free sheaf with determinant equal to $\mathscr{O}_S(1-l)$, with $l\geq 1$; this would imply that $\mathscr{E}_3^*(1-l)$ has a nonzero section for some $l\geq 1$, which is absurd.

We now turn to the case of Y_σ^2. A point $[W_6]$ in Y_σ^2 is such that $W_2 := W \cap W_6$ has dimension 2 and $W_4 := p(W_6)$ has dimension 4. We want to show that the set of $([W'],[W''])\in S^{(2)}$ with $W_4 \subset W' \oplus W''$ is finite.

We count parameters as in the proof of Proposition 3.2.2), whose notation we keep. We want to compute the dimension of the set of $([\sigma],[W],[W_1],[W_2],[W_4],[W_3'],[W'],[W''])$ such that $W = W_1 \oplus W_2$, $V_{10} = W \oplus W_4 \oplus W_3'$, $W_4 \subset W' \oplus W'' \subset W_4 \oplus W_3'$, and, in addition to the conditions

$$\sigma|_{\bigwedge^2 W \wedge V_{10}} = \sigma|_{\bigwedge^3 (W_2 \oplus W_4)} = 0 \tag{8}$$

of that proof, such that

$$\sigma|_{\bigwedge^3 (W \oplus W')} = \sigma|_{\bigwedge^3 (W \oplus W'')} = 0.$$

This means that the forms σ_1 and σ_2 must satisfy

$$\sigma_1|_{W \otimes \bigwedge^2 W'} = \sigma_1|_{W \otimes \bigwedge^2 W''} = \sigma_2|_{\bigwedge^3 W'} = \sigma_2|_{\bigwedge^3 W''} = 0. \tag{9}$$

Observe that we may assume $\dim(W' \cap W_4) = \dim(W'' \cap W_4) = 1$, as the case where one of these dimensions is ≥ 2 can be ruled out by the method used in the proof above. Then one checks that the $9+9+1+1 = 20$ conditions (9) are transverse to the conditions (8).

Therefore, using the numbers from the proof of Proposition 3.2.2), there are $70+2+9+9 = 90$ parameters for the choice of W_1, W_2, W_4, W_3', W', and W'', and, $81-20 = 61$ parameters for the choice of $[\sigma]$. It follows that the set of $([\sigma],[W],[W_6],[W'],[W''])$ such that F_σ is singular at $[W]$ and the point $([W'],[W''])$ of $S^{[2]}$ is mapped to the point $[W_6]$ of Y_σ^2 has the same dimension 120 as the set of $([\sigma],[W],[W_6])$. It follows that the corresponding projection is generically finite, which proves the claim.

2) By the proof of 1), we now have another set-theoretic description of the divisor D': on U_2, it is the set of pairs $([W'],[W''])\in S^{[2]}$ such that there exists $[W_6]\in Y_\sigma^3$ with

$$W \subset W_6 \subset W \oplus W' \oplus W''.$$

This locus has another determinantal description as follows. A W_6 as above is determined by its 3-dimensional projection W_3 in $W' \oplus W''$, and $[W_3]$ must be an element of S. Write W_3 as the graph of a map $v : W' \to W''$ (the nontransverse cases cannot fill in a divisor, by arguments as above). Recall that S is defined by a 3-dimensional space of 2-forms $\sigma_1 \in W^* \otimes \bigwedge^2 (V_{10}/W)^*$ and a 3-form $\sigma_2 \in \bigwedge^3 (V_{10}/W)^*$. Since σ_1 vanishes on $W \otimes \bigwedge^2 W'$ and $W \otimes \bigwedge^2 W''$, its restriction to $W \otimes (W' \oplus W'')$ belongs to $W^* \otimes W'^* \otimes W''^*$, so that the vanishing of

$(\mathrm{Id}, v)^*\sigma_1$ provides 9 linear equations on v. The existence of a nonzero solution v is thus equivalent to the nonindependence of these linear equations. We have a morphism (only defined on $U_2 - E$, but globally defined on the double cover $\widetilde{S \times S}$)

$$\begin{array}{rcl} \beta : \mathscr{H}om(\mathscr{S}_1, \mathscr{S}_2) & \to & W^* \otimes \bigwedge^2 p_1^*\mathscr{E}_3 \\ v & \mapsto & (\mathrm{Id}, v)^*\sigma_1. \end{array}$$

At a point $([W'], [W''])$ where β does not have maximal rank, $\beta^{-1}_{W',W''}(0)$ contains a line V_1, and we now have to impose a supplementary condition on $v \in V_1$ in order that the corresponding $W_3 = \mathrm{Im}(\mathrm{Id}, v)$ be in S, namely

$$(\mathrm{Id}, v)^*\sigma_2 = 0.$$

Observe that this last equation is quadratic (inhomogeneous) in v, and vanishes at $v = 0$. Hence there is in fact a unique $W_3 \subset W' \oplus W''$ for a general point $([W'], [W''])$ in the divisor D'' defined by $\det(\beta)$.

In order to conclude, we have to prove the following.

Claim. *For general σ, the divisor D'' is reduced and $D' = 2D''$.*

The first fact is elementary and left to the reader. As for the second one, it follows from the observation that at a point $([W'], [W''])$ of $S^{[2]}$, the morphism $\vec{f}_{W',W''}$ defined in (5) is nothing but the transpose of the direct sum of the morphism

$$\begin{array}{rcl} \beta_{W',W''} : \mathrm{Hom}(W', W'') & \to & \left(W \otimes \bigwedge^2 W'\right)^* \\ v & \mapsto & (\mathrm{Id}, v)^*\sigma_1 \end{array}$$

introduced above, and its counterpart

$$\beta_{W'',W'} : \mathrm{Hom}(W'', W') \to \left(W \otimes \bigwedge^2 W''\right)^*,$$

obtained by exchanging W' and W'' (here we identify W'^* with $\bigwedge^2 W'$ and similarly for W''). It follows from our discussion above that $\det(\beta_{W',W''})$ and $\det(\beta_{W'',W'})$ both vanish simply along D'_{red}. Hence $\det(\vec{f}_{W',W''})$ vanishes with multiplicity 2 along D'_{red}. □

4. The fourfold Y_σ as a deformation of $\mathrm{Hilb}^2(K3)$

It follows from Theorem 2.1 and [Gu] that the Hodge numbers of Y_σ are the same as those of the Hilbert scheme of pairs of points on a K3 surface. We prove in this section the following more precise result.

Theorem 4.1. *The variety Y_σ with its Plücker polarization is a deformation of $(S^{[2]}, L)$, where S is the K3 surface of genus 12 introduced in* *the previous section, and L is the line bundle on $S^{[2]}$ whose pull-back to $\widetilde{S\times S}$ is $(\mathscr{O}_S(1)\boxtimes\mathscr{O}_S(1))^{10}(-33\widetilde{E})$.*

We want to use the degeneration described in the previous section, but we have to be careful, as the central fiber is very singular and only birationally equivalent to $S^{[2]}$. In particular, L is *not* ample on $S^{[2]}$. We will borrow part of the arguments of [Hu].

The proof of Theorem 4.1 will follow from a computation of Hilbert polynomials and from the following variant of Huybrechts' theorem saying that birationally equivalent hyper-Kähler manifolds are deformation equivalent.

We start from the following more general situation: X is an irreducible hyper-Kähler manifold of dimension $2n$, Y is a normal projective variety, and $\phi : X \dashrightarrow Y$ is a birational map. We will assume that Y is a projective degeneration of irreducible hyper-Kähler manifolds, which means that there is an ample line bundle H on Y and a flat projective family

$$(\mathscr{Y},\mathscr{H})\to\Delta,\quad \text{where}\quad \mathscr{H}\in\operatorname{Pic}(\mathscr{Y}),$$

with central fiber (Y,H) and general fiber (Y_t,H_t), where H_t is ample on Y_t, an irreducible hyper-Kähler manifold. Note that this implies in particular that the canonical bundle of Y is trivial on its smooth locus Y_{reg}.

Proposition 4.2. *Assume that the line bundle $L:=\phi^*H$ on X has the following property:*

$$\forall k\in\mathbf{Z}\quad \chi(X,L^k)=\chi(Y,H^k). \tag{10}$$

Then a general small deformation of (X,L) is isomorphic to a small (smooth) deformation of (Y,H).

Proof. Let $T\subset Y$ be the union of the singular locus of Y and the indeterminacy locus of ϕ^{-1}, and let $D\subset X$ be the union of $\phi^{-1}(T)$ and of the indeterminacy locus of ϕ. (Note that in the case where Y is not smooth, D may have divisorial components.)

Lemma 4.3. *The map ϕ induces an isomorphism*

$$X - D\simeq Y - T.$$

Proof. By construction, ϕ induces a morphism $\phi_D : X - D\to Y - T\subset Y_{\mathrm{reg}}$, and ϕ^{-1} a morphism $\phi_T^{-1} : Y - T\to X$. Let η_X be a generator of the (1-dimensional) space of holomorphic 2-forms on X and let η_Y be its pull-back by ϕ_T^{-1}. It is a nonzero holomorphic 2-form on $Y - T$. The form η_Y^n is nonzero on $Y - T$, hence it does not vanish there, because the canonical bundle of $Y - T$ is trivial and $Y - T$ has no nonconstant holomorphic functions. In other words, η_Y is nondegenerate on $Y - T$. Since $(\phi_T^{-1})^*(\eta_X) = \eta_Y$, we conclude that ϕ_T^{-1} is étale.

Let us show that ϕ_D is surjective. Let $y \in Y - T$; then ϕ^{-1} is defined at y, and ϕ^{-1} is étale at y. It follows that ϕ is defined at $\phi^{-1}(y)$ and $y = \phi(\phi^{-1}(y))$. As $y \notin T$ and ϕ is defined at $\phi^{-1}(y)$, we conclude $\phi^{-1}(y) \notin D$.

Finally, we have $\phi_D^*(\eta_Y) = \eta_X|_{X - D}$; in particular, since η_X is nondegenerate, we obtain as above that ϕ_D is étale.

Hence we have proved that ϕ_D is an étale surjective birational morphism between the smooth varieties $X - D$ and $Y - T$. It is therefore an isomorphism and the lemma is proved. □

Lemma 4.4. *Under the assumptions of Proposition 4.2, for all integers k, the map ϕ^* induces an isomorphism*

$$H^0(Y, H^k) \simeq H^0(X, L^k).$$

Proof. Consider the following composition of maps:

$$H^0(Y, H^k) \to H^0(Y - T, H^k) \xrightarrow{\phi^*} H^0(X - D, L^k).$$

The first map is bijective by the normality of Y. The second map is an isomorphism by Lemma 4.3. It follows that we get an isomorphism

$$H^0(Y, H^k) \to H^0(X - D, L^k)$$

which obviously factors as

$$H^0(Y, H^k) \xrightarrow{\phi^*} H^0(X, L^k) \longrightarrow H^0(X - D, L^k),$$

where the last map is the restriction map, which is injective. Hence the map $\phi^* : H^0(Y, H^k) \to H^0(X, L^k)$ is also bijective. □

We can now prove Proposition 4.2. Indeed, consider a deformation $\pi : (\mathscr{X}, \mathscr{L}) \to \Delta$ of the pair (X, L), such that for $t \in \Delta$ very general, the group $\mathrm{Pic}(X_t)$ has rank 1.

We claim that for $t \in \Delta$ general, the line bundle L_t is ample on X_t. (This is the only place where we will use the fact that (Y, H) is a projective degeneration of an irreducible hyper-Kähler manifold.) Indeed, its Hilbert polynomial is equal to the Hilbert polynomial of L on X, hence of H on Y by our main assumption (10), or equivalently of H_s on Y_s for general s. Its terms of degree $2n$ and $2n - 2$ are therefore equal to those of H_s on Y_s. First, the terms of degree $2n$ are positive multiples of $q_X(L)^n$ and $q_{Y_s}(H_s)^n$ respectively, where q_X is the Beauville-Bogomolov quadratic form on $H^2(X, \mathbf{Z})$ ([B]) and similarly for q_{Y_s}. Next, the terms of degree $2n-2$ are multiples of $q_X(L)^{n-1}$ and

$q_{Y_s}(H_s)^{n-1}$, the signs of the coefficients being the same. This indeed follows from the Riemann-Roch formula and the fact (which we can apply to X and Y_s) that for any $2n$-dimensional irreducible hyper-Kähler manifold Z, and any degree-2 class α on Z,

$$q_Z(\alpha)^{n-1} = \mu_Z c_2(T_Z)\alpha^{2n-2},$$

with $\mu_Z > 0$. In conclusion, $q_X(L)^n$ and $q_{Y_s}(H_s)^n$ have the same sign (and are nonzero), and so do $q_X(L)^{n-1}$ and $q_{Y_s}(H_s)^{n-1}$. Hence $q_X(L)$ and $q_{Y_s}(H_s)$ have the same sign. As $q_{Y_s}(H_s) > 0$, we get $q_X(L) > 0$. By [Hu], this implies now that X_t is projective. For $t \in \Delta$ very general, since $\mathrm{Pic}(X_t)$ is cyclic, either L_t or L_t^{-1} is ample. The second case is impossible because $H^0(X, L^k) = 0$ for $k < 0$, hence by semi-continuity, $H^0(X_t, L_t^k) = 0$ for $k < 0$. By openness of ampleness, the claim is proved.

But then, we have

$$\forall k \gg 0 \quad \chi(X, L^k) = \chi(X_t, L_t^k) = h^0(X_t, L_t^k).$$

On the other hand, we have by Lemma 4.4

$$h^0(X, L^k) = h^0(Y, H^k) = \chi(Y, H^k)$$

for k large enough, and the last term equals by assumption $\chi(X, L^k)$. Hence we get

$$\forall k \gg 0 \quad h^0(X, L^k) = h^0(X_t, L_t^k),$$

and it follows by the semi-continuity and base change theorems that on a neighborhood Δ' of 0 in Δ, the sheaf $\pi_*(\mathscr{L}^k)$ is locally free and has for fiber $H^0(X_t, L_t^k)$ at t. But then, we get a flat projective family $\mathscr{Y}$ over Δ' by the formula

$$\mathscr{Y} = \mathscr{P}roj\Big(\bigoplus_{k\geq 0} \pi_*(\mathscr{L}^k)\Big).$$

By the above base change result and Lemma 4.4, the fiber of this family over 0 is isomorphic to Y, endowed with the line bundle H, while the fiber over $t \neq 0$ is X_t endowed with the line bundle L_t. □

Theorem 4.1 will be obtained as a consequence of Proposition 4.2, applied to the birational map constructed in the previous section between $X = S^{[2]}$ and $Y = Y_\sigma$, with $H = \mathscr{O}_{Y_\sigma}(1)$ and $L = \phi^*\mathscr{O}_{Y_\sigma}(1)$.

As we know that the singular variety Y_σ is normal (Proposition 3.2) and is a projective degeneration of an irreducible hyper-Kähler fourfold by Theorem 2.1, in order to apply Proposition 4.2, we only need to check the assumptions concerning the Hilbert polynomials, and this is done in the following lemma.

Lemma 4.5. *The Hilbert polynomials of $\mathscr{O}_{Y_\sigma}(1)$ and L coincide.*

Proof. Let us first compute the Hilbert polynomial of $\mathscr{O}_{Y_\sigma}(1)$. We claim that for any integer k,

$$\chi(Y_\sigma, \mathscr{O}_{Y_\sigma}(k)) = 3 + \frac{55}{2}k^2 + \frac{121}{2}k^4.$$

Indeed, we may do the computation for Y_σ smooth, in which case the Hilbert polynomial is given by the Riemann-Roch formula. Let us denote by c_i the Chern classes of the vector bundle $\mathscr{S}_6^*|_{Y_\sigma}$, so in particular $c_1 = c_1(\mathscr{O}_{Y_\sigma}(1))$. Recalling that the class of Y_σ in $G(6, V_{10})$ is $c_{20}(\bigwedge^3 \mathscr{E}_6)$, Macaulay gives us the following intersection numbers on Y_σ:

$$c_1c_3 = 330,\ c_4 = 105,\ c_1^2c_2 = 825,\ c_2^2 = 477,\ c_1^4 = 1452. \tag{11}$$

As Y_σ is a hyper-Kähler variety, its odd-degree Chern classes $c_1(T_{Y_\sigma})$ and $c_3(T_{Y_\sigma})$ vanish. Hence the Riemann-Roch formula takes the following very simple form:

$$\chi(Y_\sigma, \mathscr{O}_{Y_\sigma}(k)) = \chi(Y_\sigma, \mathscr{O}_{Y_\sigma}) + \frac{c_2(T_{Y_\sigma})c_1^2}{24}k^2 + \frac{c_1^4}{24}k^4. \tag{12}$$

The first term of the sum equals 3 by Theorem 2.1. According to (11), the last term equals $\frac{121}{2}k^4$. For the middle term, we need to compute $c_2(T_{Y_\sigma})c_1^2$. This is a tedious but straightforward computation. The tangent bundle T_{Y_σ} appears in the normal bundle sequence:

$$0 \to T_{Y_\sigma} \to T_{G(6,V_{10})}|_{Y_\sigma} \to \bigwedge^3 \mathscr{E}_6|_{Y_\sigma} \to 0.$$

Using the equality $T_{G(6,V_{10})} = \mathscr{E}_6 \otimes \big((V_{10} \otimes \mathscr{O}_{G(6,V_{10})})/\mathscr{S}_6\big)$, we now compute

$$c_2(T_{Y_\sigma}) = 5c_1^2 - 8c_2$$

which, together with (11), gives

$$c_2(T_{Y_\sigma})c_1^2 = 660.$$

Thus the claim is proved.

We now turn to the computation of the Hilbert polynomial of the line bundle L on $S^{[2]}$.

This is an explicit and standard computation. As $S^{[2]}$ is hyper-Kähler, formula (12) applies as well to $S^{[2]}$ and L:

$$\chi(S^{[2]}, L^k) = \chi(S^{[2]}, \mathscr{O}_{S^{[2]}}) + \frac{c_2(T_{S^{[2]}})c_1(L)^2}{24}k^2 + \frac{c_1(L)^4}{24}k^4.$$

The first number in the sum is 3. It thus suffices to show the equalities

$$c_1(L)^4 = 1452 \quad \text{and} \quad c_2(T_{S^{[2]}})c_1(L)^2 = 660.$$

By Proposition 3.4, the pull-back of L on $\widetilde{S\times S}$ is

$$(\mathscr{O}_S(1)\boxtimes\mathscr{O}_S(1))^{10}(-33\widetilde{E}).$$

Letting

$$\ell_i = p_i^* c_1(\mathscr{O}_S(1)) \quad \text{and} \quad e=[\widetilde{E}]$$

on $\widetilde{S\times S}$, we need to show

$$\begin{aligned}(10(\ell_1+\ell_2)-33e)^4 &= 2904\\ (10(\ell_1+\ell_2)-33e)\cdot r^*c_2(T_{S^{[2]}}) &= 1320. \qquad (13)\end{aligned}$$

The first equality follows from

$$\ell_i^2=22,\ \ell_i^3=0,\ \ell_i\ell_je^2=-22,\ e^4=24, \qquad (14)$$

together with the vanishing of the contributions of any odd power of e.

As to the second equality, note the two exact sequences, which compare $r^*\Omega_{S^{[2]}}$ and $\varepsilon^*\Omega_{S\times S}$:

$$0\to r^*\Omega_{S^{[2]}}\to\Omega_{\widetilde{S\times S}}\to\mathscr{O}_{\widetilde{E}}(-\widetilde{E})\to 0,$$

$$0\to \varepsilon^*\Omega_{S\times S}\to\Omega_{\widetilde{S\times S}}\to\mathscr{O}_{\widetilde{E}}(2\widetilde{E})\to 0.$$

This gives us the following formula for the total Chern class of $r^*\Omega_{S^{[2]}}$:

$$r^*c(\Omega_{S^{[2]}})=\varepsilon^*c(\Omega_{S\times S})c(\mathscr{O}_{\widetilde{E}}(2\widetilde{E}))c(\mathscr{O}_{\widetilde{E}}(-\widetilde{E}))^{-1}. \qquad (15)$$

For $i\in\{1,2\}$, let o_i be the class of a fiber of the projection $p_i:\widetilde{S\times S}\to S$; we have

$$\varepsilon^*c(\Omega_{S\times S})=(1+24o_1)(1+24o_2)$$

and we deduce from (15)

$$r^*c_2(\Omega_{S^{[2]}})=24o_1+24o_2-3e^2.$$

Equality (13) then follows from (14) together with $o_1\ell_2^2=o_2\ell_1^2=22$, $o_i\ell_i=0$, and $o_ie^2=-1$. □

At this point, we have shown that a small smooth deformation of $(Y_\sigma,\mathscr{O}_{Y_\sigma}(1))$ is isomorphic to a small deformation of $(S^{[2]},L)$. In order to conclude the proof of Theorem 4.1, it only remains to prove the following lemma (and use item 4) of Proposition 3.2).

Lemma 4.6. *Whenever Y_σ has dimension 4, any small deformation of $(Y_\sigma,\mathscr{O}_{Y_\sigma}(1))$ is given by a deformation of σ.*

Proof. Let Z be a local complete intersection projective scheme and let L be a line bundle on Z. The first Chern class of L, seen as an element of $H^1(Z,\Omega_Z)$, defines an extension

$$0 \to \Omega_Z \to \mathscr{P}_{Z,L} \to \mathscr{O}_Z \to 0 \tag{16}$$

and first-order deformations of the pair (Z,L) are parametrized by $\mathrm{Ext}^1_Z(\mathscr{P}_{Z,L},\mathscr{O}_Z)$.

In our situation, Y_σ is the zero-set of the section σ of the vector bundle $\mathscr{F} = \bigwedge^3 \mathscr{E}_6$ on $G := G(6,V_{10})$. The discussion above applies to both $(G,\mathscr{O}_G(1))$ and $(Y_\sigma,\mathscr{O}_{Y_\sigma}(1))$. Since the normal bundle to Y_σ in G is $\mathscr{F}|_{Y_\sigma}$, we obtain an exact sequence

$$0 \to \mathscr{F}^*|_{Y_\sigma} \to \mathscr{P}_{G,\mathscr{O}_G(1)}|_{Y_\sigma} \to \mathscr{P}_{Y_\sigma,\mathscr{O}_{Y_\sigma}(1)} \to 0.$$

from which we deduce an exact sequence
(17)

$$H^0(Y_\sigma,\mathscr{F}|_{Y_\sigma}) \xrightarrow{\beta} \mathrm{Ext}^1_{Y_\sigma}(\mathscr{P}_{Y_\sigma,\mathscr{O}_{Y_\sigma}(1)},\mathscr{O}_{Y_\sigma}) \to \mathrm{Ext}^1_{Y_\sigma}(\mathscr{P}_{G,\mathscr{O}_G(1)}|_{Y_\sigma},\mathscr{O}_{Y_\sigma}).$$

We need to show that the composition

$$H^0(G,\mathscr{F}) \xrightarrow{\alpha} H^0(Y_\sigma,\mathscr{F}|_{Y_\sigma}) \xrightarrow{\beta} \mathrm{Ext}^1_{Y_\sigma}(\mathscr{P}_{Y_\sigma,\mathscr{O}_{Y_\sigma}(1)},\mathscr{O}_{Y_\sigma})$$

is surjective. We will prove that both α and β are surjective.

Using the Koszul resolution for $\mathscr{O}_{Y_\sigma}$, we see that α is surjective if

$$H^i(G,\mathscr{F}\otimes \bigwedge^i \mathscr{F}^*) = 0$$

for all $i > 0$, a fact that can be checked using Bott's theorem and the program LiE, as explained in Remark 2.6.

To show the surjectivity of β, it is enough by (17) to show that $\mathrm{Ext}^1_{Y_\sigma}(\mathscr{P}_{G,\mathscr{O}_G(1)}|_{Y_\sigma},\mathscr{O}_{Y_\sigma})$ vanishes. Consider the exact sequence

$$H^1(Y_\sigma,\mathscr{O}_{Y_\sigma}) \to \mathrm{Ext}^1_{Y_\sigma}(\mathscr{P}_{G,\mathscr{O}_G(1)}|_{Y_\sigma},\mathscr{O}_{Y_\sigma}) \to H^1(Y_\sigma,T_G|_{Y_\sigma}) \xrightarrow{\gamma} H^2(Y_\sigma,\mathscr{O}_{Y_\sigma})$$

obtained from (16). Again, as in Remark 2.6, one shows using the Koszul resolution and Bott's theorem that $H^1(Y_\sigma,\mathscr{O}_{Y_\sigma})$ vanishes. The map

$$\gamma : H^1(Y_\sigma,T_G|_{Y_\sigma}) \to H^2(Y_\sigma,\mathscr{O}_{Y_\sigma})$$

is given by cup-product with $c_1(\mathscr{O}_{Y_\sigma}(1))$. Using the Koszul resolution again, it is injective if the cup-product maps

$$\gamma_i : H^{i+1}(G,T_G\otimes \bigwedge^i \mathscr{F}^*) \to H^{i+2}(G,\bigwedge^i \mathscr{F}^*)$$

by $c_1(\mathscr{O}_G(1))$ are injective for all $i \geq 0$. The tangent bundle T_G is isomorphic to $\mathscr{Q}_4\otimes\mathscr{S}_6^*$, hence appears in the exact sequence

$$0 \to \mathscr{S}_6\otimes\mathscr{S}_6^* \to V_{10}\otimes\mathscr{S}_6^* \to T_G \to 0, \tag{18}$$

whose extension class is

$$c_1(\mathscr{O}_G(1)) \in H^1(G, \Omega_G) \simeq \mathrm{Ext}^1_G(T_G, \mathscr{O}_G) \subset \mathrm{Ext}^1_G(T_G, \mathscr{S}_6 \otimes \mathscr{S}_6^*).$$

Proceeding as above, one can show $H^{i+1}(G, \mathscr{S}_6^* \otimes \bigwedge^i \mathscr{F}^*) = 0$ for all $i \geq 0$. In the long exact sequence in cohomology associated with (18), we deduce that the edge map

$$H^{i+1}(G, T_G \otimes \bigwedge^i \mathscr{F}^*) \xrightarrow{\gamma_i} H^{i+2}(G, \bigwedge^i \mathscr{F}^*) \hookrightarrow H^{i+2}(G, \mathscr{S}_6 \otimes \mathscr{S}_6^* \otimes \bigwedge^i \mathscr{F}^*)$$

is bijective. This proves this injectivity of γ_i, hence the lemma. □

5. Further comments and questions

The geometric invariant theory of the 3-vectors $\sigma \in \bigwedge^3 V_{10}^*$ does not seem to have been studied. We introduced in section 2 a natural hypersurface in the moduli space

$$\mathbf{P}(\bigwedge^3 V_{10}^*) \mathbin{/\!/} \mathrm{PGL}(V_{10}^*).$$

It parametrizes those Y_σ containing a line in the Plücker embedding. Section 3 was devoted to another hypersurface in this moduli space, parametrizing singular F_σ.

There is a third natural hypersurface in this moduli space: it is the set of σ for which F_σ contains a 10-dimensional Grassmannian $G(2,7) \subset G(3, V_{10})$. Here we choose a $V_8 \subset V_{10}$ together with a nonzero x in V_8, and we see $G(2,7)$ as the set of $W_3 \subset V_{10}$ such that $x \in W_3 \subset V_8$. The fact that this $G(2,7)$ is contained in F_σ is equivalent to the fact that the 2-form $\mathrm{Int}_x \sigma$ vanishes on V_8. That the existence of such a subvariety of F_σ is a divisorial condition on σ follows from the equality $h^{9,11}(F_\sigma) = 1$ and the semi-regularity of the embedding $G(2,7) \subset F_\sigma$, which tells us that deforming F_σ preserving $G(2,7)$ is equivalent to deforming F_σ preserving the Hodge class $[G(2,7)]$ (see [Bl]).

A related question concerns the existence of a hypersurface in the moduli space where Y_σ is actually isomorphic to $S^{[2]}$ for some $K3$ surface S. This should hold along a hypersurface where the Picard number of Y_σ jumps (or equivalently, by Corollary 2.7, where the dimension of the space of degree-20 Hodge classes on F_σ jumps).

There are two families of $K3$ surfaces which are natural candidates, namely those of genus 16 and those of genus 21. Indeed, the first ones admit a rigid rank-2 vector bundle with 10 independent sections, so that their second Hilbert schemes carry a rigid rank-4 vector bundle with 10 independent sections, which embeds them into $G(4,10) \simeq G(6,10)$. Similarly, $K3$ surfaces of genus 21 admit a rigid rank-3 vector bundle with 10 independent sections, so that their second Hilbert schemes carry a rigid rank-6 vector bundle with 10 independent sections, which embeds them into $G(6,10)$. In both cases and surprisingly enough, the degree of the hyper-Kähler subvarieties of $G(6,10)$ that one obtains is 1452, which is the degree of Y_σ. However the other Chern numbers of the tautological vector bundle (see (11)) do not coincide.

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