The diagonal property for abelian varieties

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Dedicated to Roy Smith on his 65th birthday.

Abstract. We study complex abelian varieties of dimension g that have a vector bundle of rank g with a section that vanishes at a single point, with multiplicity 1.

The authors of [**PSP**] study which varieties X (defined, say, over an algebraically closed field) satisfy what they call "the diagonal property": *there exists a vector bundle of rank* $\dim(X)$ *on* $X \times X$ *with a section whose zero-scheme is the diagonal.* This property is known to hold for all flag varieties SL_n/P ([**F1**]) and, using a variant of Serre's construction, they show that it holds for projective surfaces which have a cohomologically trivial line bundle. In particular, it holds for abelian surfaces.

The diagonal property implies the weaker "point property": *for each point* x *of* X*, there exists a vector bundle of rank* $\dim(X)$ *on* X *with a section whose zero-scheme is* x. When X is a group variety, these two properties are equivalent (it is even enough to have the point property for a single point x).

In this note, we study these (equivalent) properties when X is a complex abelian variety. We show in §1 that a general non-principally polarized abelian variety of a sufficiently high dimension does not have these properties. Using Picard bundles, we show in §2 that these properties hold when X is the Jacobian of a smooth curve (or a product of such). However, Lange pointed out that in any dimension, principally polarized abelian varieties that have the point property are dense in their moduli space (Remark 2.4). I do not know any principally polarized abelian variety that does not have the point property, although I would like to think that Jacobians of curves are the only principally polarized abelian varieties *with Picard number* 1 that have this property, thereby giving us another (partial) solution to the Schottky problem. In §3 and 4, we prove a necessary condition for the point property to hold, and use it to get restrictions on possible vector bundles.

We work over the complex numbers, although the results of §2 and 3 are valid over an algebraically closed field of arbitrary characteristic.

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1. Non-principally polarized abelian varieties

Let X be an abelian variety of dimension g. If $\mathscr{E}$ is a vector bundle of rank g on X with a section whose zero-scheme is the origin o, we have $c_g(\mathscr{E}) = [o]$ in the Chow group of X. We use simple-minded numerics coming from the fact that the number

$$\chi(X, \mathscr{E}) = \int_X \mathrm{ch}_g(\mathscr{E})$$

(this is the Hirzebruch-Riemann-Roch formula; [**F2**], Corollary 15.2.1) is an integer.

If (X, ℓ) is a very general polarized abelian variety of type $(\delta_1 \mid \cdots \mid \delta_g)$ ([**BL**], §3.1), we know by Mattuck's theorem ([**BL**], Theorem 17.4.1) that $c_i(\mathscr{E})$ is, in cohomology, a rational multiple of ℓ^i. Since $\frac{\ell^i}{\delta_1 \cdots \delta_i i!}$ is a nondivisible integral class in $H^{2i}(X, \mathbf{Z})$, we may write

$$c_i(\mathscr{E}) = a_i \frac{\ell^i}{\delta_1 \cdots \delta_i i!} \qquad \text{with } a_i \in \mathbf{Z}. \tag{1.1}$$

We obtain ([**Mc**], p. 20)

$$\chi(X, \mathscr{E}) = \delta_1 \cdots \delta_g \; D\Big(\frac{a_1}{\delta_1}, \dots, \frac{a_g}{\delta_1 \cdots \delta_g}\Big) \tag{1.2}$$

where

$$D(b_1, \dots, b_g) := \begin{vmatrix} b_1 & 1 & 0 & \cdots & \cdots & 0 \\ b_2 & b_1 & 1 & \ddots & & \vdots \\ b_3/2! & b_2/2! & b_1 & \ddots & \ddots & \vdots \\ b_4/3! & b_3/3! & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 \\ b_g/(g-1)! & b_{g-1}/(g-1)! & \cdots & b_3/3! & b_2/2! & b_1 \end{vmatrix} \tag{1.3}$$

PROPOSITION 1.1. *A very general* nonprincipally *polarized abelian variety of type* $(\delta_1 \mid \cdots \mid \delta_g)$ *whose dimension* g *is greater than some prime factor of* δ_g/δ_1 *does* not *have the point property.*

PROOF. We may assume $\delta_1 = 1$. Expanding the determinant (1.3), we see that the integer $\chi(X, \mathscr{E})$ can be written as $(-1)^{g-1} \frac{a_g}{(g-1)!} + \delta_g \frac{a}{(g-1)!}$ for some integer a. It follows that a_g is divisible by $\gcd(\delta_g, (g-1)!)$, hence the proposition. □

2. Jacobians of curves

Let C be a smooth connected projective curve of genus $g \geq 2$ and let JC be its Jacobian, endowed with its canonical principal polarization θ. Any element ξ of JC defines a numerically trivial line bundle P_ξ on JC.

Fixing a point c of C, we view the curve C as embedded in JC by sending a point x of C to the isomorphism class of $\mathscr{O}_C(x - c)$ and we define W_i as the i-fold sum $C + \cdots + C$, with the convention $W_0 = \{o\}$.

Let $\mathscr{P}$ be the Poincaré line bundle on $C \times JC$, uniquely defined by the properties

$$\mathscr{P}|_{\{c\} \times JC} \simeq \mathscr{O}_{JC} \quad \text{and} \quad \mathscr{P}|_{C \times \{\xi\}} \simeq P_\xi|_C \quad \text{for all } \xi \in JC.$$

Following [**S**], §2, Definition (see also [**Mk**], and [**Mu**], Definition 4.1), we define the *Picard bundle* by[1]

$$\mathscr{F} = R^1 q_*(\mathscr{P} \otimes p^* \mathscr{O}_C(-c))$$

where $p : C \times JC \to C$ and $q : C \times JC \to JC$ are the projections. By [**S**], *the sheaf* $\mathscr{F}$ *is locally free of rank* g *on* JC. Moreover, if ι is the involution $\xi \mapsto K_C - (2g-2)c - \xi$ of JC, the morphism[2] $\pi : \mathbf{P}(\iota^* \mathscr{F}) \to JC$ is isomorphic to the Abel-Jacobi map

$$\begin{array}{cccc} \alpha : & C^{(2g-1)} & \longrightarrow & JC \\ & x_1 + \cdots + x_{2g-1} & \longmapsto & \mathscr{O}_C(x_1 + \cdots + x_{2g-1} - (2g-1)c) \end{array}$$

and the divisor $C^{(2g-2)} + c$ in $C^{(2g-1)}$ represents the ample line bundle $\mathscr{O}_{\mathbf{P}(\iota^* \mathscr{F})}(1)$ ([**S**], Theorem 2). The Chern classes of $\mathscr{F}$ were computed by Mattuck in [**Mk**], §6, Corollary (see also [**S**], §4, and [**G**], Corollary 3 to Theorem 4); he obtains:

$$c_{g-i}(\mathscr{F}) = [W_i] \qquad \text{for all } i \in \{0, \dots, g\}, \tag{2.1}$$

in the Chow group of JC.

Theorem 2.1. *For all* $i \in \{0, \dots, g\}$, *we have*

$$h^i(JC, \mathscr{F} \otimes P_\xi) = \begin{cases} \binom{g-1}{i} & \textit{if } -\xi \in C; \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, the (scheme-theoretic) zero-locus of any nonzero section of $\mathscr{F}$ *is* $W_0 = \{o\}$.

The first statement of the theorem was also obtained in [**Mu**], Proposition 4.3, using the Fourier transform (see also [**G**], Corollary 2 to Theorem 4).

Proof. We have

$$\begin{aligned} H^i(JC, \mathscr{F} \otimes P_\xi) &\simeq H^i(JC, \iota^* \mathscr{F} \otimes \iota^* P_\xi) \\ &\simeq H^i(\mathbf{P}(\iota^* \mathscr{F}), \mathscr{O}_{\mathbf{P}(\iota^* \mathscr{F})}(1) \otimes \pi^* P_{-\xi}) \\ &\simeq H^i(C^{(2g-1)}, \mathscr{O}_{C^{(2g-1)}}(C^{(2g-2)} + c) \otimes \alpha^* P_{-\xi}) \\ &\simeq \wedge^i H^1(C, c - \xi) \otimes \operatorname{Sym}^{2g-1-i} H^0(C, c - \xi) \end{aligned}$$

by [**I**], §3.1. This space vanishes if $\xi \notin -C$, and if ξ represents $\mathscr{O}_C(c - c')$, it is isomorphic to $\wedge^i H^0(C, K_C - c')^\vee$.

In particular, the zero-locus of a nonzero section of $\iota^* \mathscr{F}$ is the set of points ξ' of JC such that $\alpha^{-1}(\xi') \subset C^{(2g-2)} + c$, i.e., c is a base-point for the linear system $|\mathscr{O}_C((2g-1)c) \otimes P_{\xi'}|$. This is $\{\iota(o)\}$. By (2.1), the order of vanishing at this point must be 1, and this proves the last statement of the theorem. □

Corollary 2.2. *The Jacobian of any smooth projective curve satisfies the point property.*

Of course, any (finite) product of varieties that satisfy the point property also satisfies this property.

The vector bundle $\mathscr{F}$ is known to be stable ([**K**]), and its Euler characteristic is 0 (by Theorem 2.1, or [**Mk**], §8).

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[1] This is the sheaf denoted by $\mathscr{F}_{-1}$ (or F_{-1}) in [**S**] and [**Mu**], and by P_{2g-1} in [**K**] and [**EL**]. In the terminology of [**Mu**], the sheaf $\mathscr{O}_C(-c)$, viewed as a (torsion) sheaf on JC, is IT_1 and its Fourier-Mukai transform is $\mathscr{F}$.

[2] As in [**S**], we use Grothendieck's convention for projectivization.

We may ask to what extent $\mathscr{F}$ is unique. In dimension 2, Mukai proved the following ([**Mu**], Theorem 5.4): *on a principally polarized abelian* surface (X,θ), *any stable vector bundle of rank* 2 *with first Chern class* θ *and second Chern class* 1 *is a translate of* $\mathscr{F}$ *tensored by a numerically trivial line bundle.* However, the vector bundle $\mathscr{E}$ constructed in [**PSP**] to prove the point property for an abelian surface X is different: it appears as an extension

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E} \longrightarrow P \otimes \mathscr{I}_o \longrightarrow 0 \tag{2.2}$$

where $P = \det(\mathscr{E})$ can be any nontrivial numerically trivial line bundle on X ($\mathscr{E}$ is of course not stable, only semistable).

Proposition 2.3. *The Picard bundle does not deform outside the closure of the Jacobian locus in the moduli space* $\mathscr{A}_g$ *of principally polarized abelian varieties of dimension* g.

Proof. In the terminology of [**Mu**], any small deformation $\mathscr{G}$ of $\mathscr{F}$ remains WIT_{g-1} ([**Mu**], Proposition 4.3), and $\widehat{\mathscr{G}}$ is WIT_1; since it is nonzero, its support must have dimension at least 1. Since $\widehat{\mathscr{F}}$ has support $-C$, the support of $\widehat{\mathscr{G}}$ has dimension at most 1 and must be a deformation of $-C$ (and, possibly, finitely many points). This can only happen if the deformed abelian variety is still a Jacobian (or a product of such) by Matsusaka's criterion ([**Ma**]). □

Remark 2.4 (Lange). For any $g > 0$, the set of principally polarized abelian varieties that satisfy the point property is *dense* in $\mathscr{A}_g$. This can be seen as follows: any abelian variety isogeneous to E^g, where E is an elliptic curve with complex multiplication, is *isomorphic* to the product of g elliptic curves ([**BL**], Exercise 5.6.(10)), hence has the point property. Moreover, the corresponding subset of $\mathscr{A}_g$ is dense ([**L**]). These varieties all have Picard number g^2. An explicit example of such a principally polarized abelian fourfold which is not a Jacobian can be found in [**D**], §5.

3. A necessary condition

Following ideas of [**PSP**], we get a necessary condition, on any smooth projective variety, for the point property to be satisfied.

Proposition 3.1. *Let* X *be a smooth projective variety of dimension* g *and let* $\mathscr{E}$ *be a vector bundle of rank* g *on* X *with a section* s *whose zero-scheme is a single point* x. *All sections of* $\det(\mathscr{E}) \otimes \omega_X$ *then vanish at* x.

Proof. Let $\mathscr{L}$ be the invertible sheaf $\det(\mathscr{E}) = \wedge^g \mathscr{E}$. The long exact sequence (Koszul complex)

$$0 \to \wedge^g \mathscr{E}^\vee \to \cdots \to \wedge^2 \mathscr{E}^\vee \to \mathscr{E}^\vee \xrightarrow{s^\vee} \mathscr{I}_x \longrightarrow 0$$

determines an element $[s]$ of $\operatorname{Ext}_X^{g-1}(\mathscr{I}_x, \mathscr{L}^\vee)$. The short exact sequence $0 \to \mathscr{I}_x \to \mathscr{O}_X \to \mathbf{C}_x \to 0$ induces another exact sequence

$$\operatorname{Ext}_X^{g-1}(\mathscr{I}_x, \mathscr{L}^\vee) \xrightarrow{\beta} \operatorname{Ext}_X^g(\mathbf{C}_x, \mathscr{L}^\vee) \to H^g(X, \mathscr{L}^\vee) \xrightarrow{\gamma} \operatorname{Ext}_X^g(\mathscr{I}_x, \mathscr{L}^\vee) \to 0$$

The image of $\beta([s])$ by the morphism

$$\operatorname{Ext}_X^g(\mathbf{C}_x, \mathscr{L}^\vee) \to H^0(X, \mathscr{E}xt^g_{\mathscr{O}_X}(\mathbf{C}_x, \mathscr{L}^\vee)) \simeq \mathbf{C} \tag{3.1}$$

is nonzero because $\mathscr{E}$ is locally free (compare with the proof of [**PSP**], Proposition 1). Since, by Serre duality, the vector space on the left-hand-side of (3.1) is also 1-dimensional, β is surjective, hence γ is bijective and its Serre-dual is

$$H^0(X, \mathscr{L} \otimes \omega_X \otimes \mathscr{I}_x) \xrightarrow{\sim} H^0(X, \mathscr{L} \otimes \omega_X)$$

In other words, all sections of $\mathscr{L} \otimes \omega_X$ vanish at x. □

Of course, $\det(\mathscr{E}) \otimes \omega_X$ might have no nonzero sections, in which case the proposition says nothing at all.

4. Principally polarized abelian varieties

Let now (X, θ) be a very general principally polarized abelian variety of dimension g and let $\mathscr{E}$ be a vector bundle of rank g on X. We may, as in (1.1), write in cohomology $c_i(\mathscr{E}) = a_i\theta^i/i!$, with $a_i \in \mathbf{Z}$, and $\chi(X, \mathscr{E}) = D(a_1, \ldots, a_g)$ (see (1.2)). This number is an integer, and this implies various congruences, one example of which is the following.

Proposition 4.1. *If* $p := g - 1$ *is* prime, *we have*

$$a_1 a_{g-1} \equiv a_g \pmod p$$

In particular, if $a_g = 1$, a_1 *is prime to* p.

Proof. Expanding the determinant (1.3) along its last row, we get

$$\chi(X, \mathscr{E}) = (-1)^{g-1} \frac{1}{(g-1)!}(a_g - a_1 a_{g-1}) + a$$

where a is a rational number with a denominator whose prime factors are all $< p$. Since $\chi(X, \mathscr{E})$ is an integer, this proves the proposition. □

Combining this result with Proposition 3.1, we obtain the following.

Corollary 4.2. *Let* (X, θ) *be a very general principally polarized abelian variety of dimension* g *and let* $\mathscr{E}$ *be a vector bundle of rank* g *on* X *with a section whose zero-scheme is the origin.*

If $\mathscr{E}$ *is* stable*, or if* $\mathscr{E}$ *is* semistable *and* $g - 1$ *is prime,* $c_1(\mathscr{E}) = \theta$.

In particular, if $g = 2$ and $\mathscr{E}$ is stable, it follows from the result of Mukai mentioned in §2 that $\mathscr{E}$ is a translate of the vector bundle $\mathscr{F}$ constructed there, tensored by a numerically trivial line bundle.

Proof. Write $c_1(\mathscr{E}) = a_1\theta$. If $\mathscr{E}$ is stable (resp. semistable), we have $a_1 \geq 1$ (resp. $a_1 \geq 0$). On the other hand, by Proposition 3.1, the linear system $|\det(\mathscr{E})|$ has a base-point, hence $a_1 \leq 1$ by the Lefschetz Theorem. Finally, by Proposition 4.1, if $g - 1$ is prime, $a_1 \neq 0$. This proves the corollary. □

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