Appendix to "Irreducibility, Brill–Noether' loci, and Vojta's inequality" by Thomas J. Tucker

On a curve C with no g^1_d such that $W_d(C)$ contains finitely many elliptic curves

Olivier Debarre

The aim of this appendix is to complement a construction from [DF] (and to correct an error in the proof of prop. 5.7 of this article). The setup is the following: let E be a complex elliptic curve and let S be its second symmetric product, with $s : S \to E$ the sum map. To avoid confusion between addition of divisors and addition of points on E, we write (x) for the divisor associated with a point x of E. We define two divisors on S by setting $F = s^{-1}(\mathbf{o})$ and $H = \{(\mathbf{o}) + (x) \mid x \in E\}$. For any divisor D on S and any $x \in E$, we set $D_x = D + s^*((x) - (\mathbf{o}))$. We denote par $\sim$ numerical equivalence for divisors.

Fix an integer $d \geq 4$ and a point x_0 on E, and set $L = \mathcal{O}_S((d+1)H - F_{x_0})$. A general curve C in $|L|$ is smooth of genus $\binom{d}{2} + 1$ ([DF]).

1. On the Kawamata locus of $W_d(C)$

Our aim is to prove that the Kawamata locus of $W_d(C)$, i.e., the union of all (translated) non-zero abelian varieties in $W_d(C)$, is 1-dimensional. We will analyze pencils of low degree on C, using as in [DF] the following result of Reider ([R]).

Theorem (I. Reider).— *Let L be a nef line bundle on a smooth projective surface S and let C be a smooth curve in $|L|$. Let A be a base-point-free g^1_δ on C such that $\delta < L^2/4$. There exists a divisor D on S such that:*

a) $h^0(S, D) \geq 2$;
b) $h^0(C, D - A) > 0$;
c) $C \cdot D < 2\delta$;
d) $(C - D) \cdot D \leq \delta$.

For any divisor D on S, we denote by $\overline{D}$ its restriction to C.

Corollary. — *Let A be a base-point-free g^1_δ on C with $\delta \leq 2d$ and $\delta \leq 3d - 10$. One of the following possibilities occur:*

a) *there exist $x \in E$ and an effective divisor B on C such that $A + B \equiv 2\overline{H}_x$, and $\delta \geq 2d - 4$;*
b) *there exist $x \in E$ and an effective divisor B on C such that $A + B \equiv 3\overline{H}_x - \overline{F}$, and $\delta \geq 2d - 4$;*
c) *one has $A = 4\overline{H} - 2\overline{F}$ and $\delta = 2d - 2$.*

Proof. A case-by-case inspection shows that the divisor D in Reider's theorem satisfies one of the following: $D \sim 2H$ and $\delta \geq 2d-4$, $D \sim 3H - F$ and $\delta \geq 2d-4$, or $D \equiv 4H - 2F$ and $\delta \geq 2d-2$. In the first (resp. second) (resp. last) case, $|D|$ induces a base-point-free g^2_{2d} (resp. a g^1_{2d-1} with at most 3 distinct base points by [DF], prop. 4.2) (resp. a base-point-free g^1_{2d-2}). Therefore, we are in case a) (resp. b)) (resp. c)). ∎

As noted in [DF], the variety $W_d(C)$ contains a translate E_0 of E, to wit the image of the morphism $\psi : x \mapsto \overline{H}_x$.

Theorem.— *Assume C is general in $|L|$ and $d \geq 4$. The Kawamata locus of $W_d(C)$ is 1-dimensional. More precisely,*

a) *$W_{d-1}(C)$ contains no non-zero abelian varieties;*
b) *the only non-zero abelian varieties contained in $W_d(C)$ are translates of E_0 by torsion points;*
c) *for $d \geq 9$, the only non-zero abelian variety contained in $W_d(C)$ is E_0.*

Proof. Let A be a non-zero (translated) abelian variety in $W_\delta(C)$, with $\delta \leq d$. By a theorem of Mori (see proof of prop. 5.4 of [DF]), JC/s^*JE is simple hence A maps to a point in this quotient. It follows that A is a translate of E. Take δ minimal, so that a linear system corresponding to a general point a of A has only one element, which we will denote by D_a, and so that the divisors D_a have no common point. Since E_0 does not come from a morphism, one shows as in the proof of Lemma 5 of [AH] that $E_0 + A \subset W^2_{d+\delta}(C)$.

Assume $d \geq 9$ and let $y \in E$, $a \in A$ and $p \in C$; the corollary of Reider's theorem applied to $\overline{H}_y + D_a - p$ yields that

a) either there exist $x \in E$ and an effective divisor B on C such that, for all $e \in E$, one has $\overline{H}_{y-e} + D_{a+e} - p + B \equiv 2\overline{H}_x$;
b) or there exist $x \in E$ and an effective divisor B on C such that, for all $e \in E$, one has $\overline{H}_{y-e} + D_{a+e} - p + B \equiv 3\overline{H}_x - \overline{F}$;
c) or $\delta = d-1$ and, for all $e \in E$, one has $\overline{H}_{y-e} + D_{a+e} - p \equiv 4\overline{H} - 2\overline{F}$.

Case c) cannot occur because p is fixed in $|4\overline{H} - 2\overline{F} + p|$ by Riemann–Roch. For the same reason $B' = B - p$ is effective in cases a) and b). In case b), we get $D_{a+e} + B' \equiv 2\overline{H}_{x-y+e} - \overline{F}$. But this is impossible since $H^0(C, 2\overline{H}_{x-a+e} - 2\overline{F}) = 0$. Therefore, we are in case a), and $\overline{H}_{x-y+e} = D_{a+e} + B'$. When e varies, the left-hand side varies; the $\overline{H}_{x-y+e}$'s having no fixed point, we must have $B' = 0$ and $\overline{H}_{x-y+e} = D_{a+e}$. It follows that $A = E_0$.

Assume now $d \geq 4$, and set

$$\Gamma = \{\alpha \in JC \mid E_0 + \alpha \subset W_d(C)\} .$$

We will use theorem 2 of [AH]; one should however be careful: first one needs to add to the hypotheses of this theorem that the embedding $A \subset W_d(C)$ does not come from a morphism (this is true in our case by prop. 5.14 of [DF]). Second, the proof of the theorem given in [AH] is incomplete: the proof of lemma 6 on which it relies is wrong when $\dim(A) > 1$ (this fortunately does not concern us) and the case $r_2 = 3$ and $\dim(A) = 1$ needs a separate treatment (which was provided by Abramovich in a private communication). The conclusion of the theorem is then $g(C) \leq \binom{\delta}{2} + 1$, which implies for example $\delta = d$. However, we get more from the proof, to wit that if there is equality, then $r_k = \binom{k+1}{2} - 1$ for $2 \leq k \leq d$. The same reasoning shows in our case that the same holds for the sum of k generic elements of $E_0 + \alpha_1, \ldots, E_0 + \alpha_k$, where $\alpha_1, \ldots, \alpha_k \in \Gamma$. In particular, for a_1 generic in $E_0 + \alpha_1$, for a_2 generic in $E_0 + \alpha_2$ and for x generic in E, we have

$$h^0(C, (k-2)\overline{H}_x + D_{a_1} + D_{a_2}) = \binom{k+1}{2}$$

for $2 \leq k \leq d$. Since $K_C \equiv (d-2)\overline{H} + \overline{H}_{-x_0}$, Riemann–Roch implies

$$h^0(C, 2\overline{H}_x - D_{a_1}) = 1 \quad \text{and} \quad h^0(C, 3\overline{H}_x - D_{a_1} - D_{a_2}) = 1 \ ,$$

i.e., the curves $E_0 - \alpha_1$ and $E_0 - \alpha_1 - \alpha_2$ are contained in $W_d(C)$. This proves that Γ is a closed (proper) subgroup of JC hence is a translate of E by a finite group. ■

2. Erratum for [DF]

The following proposition corrects the part of the proof of prop. 5.7 of [DF] which is incomplete (to wit the case $d = 4$).

Proposition.— *Assume $d = 4$ and C general in $|L|$. Then C has no g^1_4.*

Proof. Let A be a base-point-free g^1_δ on C, with $\delta \leq 4$. Following [DF] (5.12), one constructs a rank 2 vector bundle T on S that fits into an exact sequence

$$(*) \qquad 0 \to H^0(A)^* \otimes \mathcal{O}_S \to T \to \mathcal{O}_C(C - A) \to 0 \ .$$

Note that $H^2(T) = 0$ and $\chi(T) = 10 - \delta$ by Riemann–Roch. If $h = h^0(T) > 6$, the kernel of the map $\wedge^2 H^0(T) \to H^0(\wedge^2 T) \simeq H^0(S, C) \simeq \mathbf{C}^{10}$ meets the $(2h-3)$-dimensional set of decomposable vectors off the origin. One proceeds as in [DF] (where the numbers at the top of page 246 are all wrong) to show that there exists a divisor D on S that fits into an exact sequence

$$0 \to \mathcal{O}_S(D) \to T \to \mathcal{I}_Z(C - D) \to 0 \ ,$$

where Z is a finite subscheme of S; moreover, either $D \sim 2H$ or $D \sim 3H - F$. Then, $h^0(T) \leq h^0(D) + h^0(C - D) = 5$, which is a contradiction (this remark avoids the lengthy proof in [DF]).

It follows that $h^0(T) = 6$, $h^1(T) = 0$ and $\delta = 4$. Assume first that there is a non-zero morphism $u : T \to T \otimes \omega_S$. We argue as in [L]: since $H^0(\omega_S^2) = 0$, the morphism $\wedge^2 u$ vanishes hence u drops rank everywhere. Then $N = (\mathrm{Im}\, u)^{**}$ is a line bundle on S which is a subsheaf of $T \otimes \omega_S$; there is a morphism $T \to N$ which is surjective off a finite subset of S. Note that by Riemann–Roch, one has $h^0(\mathcal{O}_C(C - A)) \geq 5 > 2 = h^1(H^0(A)^* \otimes \mathcal{O}_S)$, hence the exact sequence $(*)$ shows that T is generated by global sections off a finite subset of S, hence so is N. It follows that either $h^0(S, N) \geq 2$, or $N \simeq \mathcal{O}_S$; but the latter cannot occur since $\mathrm{Hom}(T, \mathcal{O}_S) = 0$. Tensoring $(*)$ by $\omega_S \otimes N^*$, we see that $H^0(T \otimes \omega_S \otimes N^*) \neq 0$ implies

$H^0(\omega_C \otimes N^* \otimes \mathcal{O}_C(-A)) \neq 0$ and in particular $(5H - F) \cdot (3H - N) \geq 4$. Furthermore, there is an exact sequence

$$0 \to N \to T \otimes \omega_S \to \mathcal{I}_Z(\omega_S^{\otimes 2} \otimes N^*(C)) \to 0 \ ,$$

which implies $N \cdot (H + F - N) \leq c_2(T \otimes \omega_S) = 0$. A case-by-case analysis shows that the only possibility is $N \sim 2H$; but then $H^0(\omega_C \otimes N^*(-A)) \neq 0$ implies $A \equiv \overline{H}_x$, which is not a pencil.

Hence $\mathrm{Hom}(T, T \otimes \omega_S)$ vanishes, and so does $H^2(\mathrm{End}\, T)$ by duality. Dualizing $(*)$ yields

$$0 \to T^* \to H^0(A) \otimes \mathcal{O}_S \to \mathcal{O}_C(A) \to 0 \ .$$

Tensoring by T, we get $H^1(T \otimes A) = 0$. We now follow another construction of [L], where a moduli space P is constructed which parametrizes triples (C, A, l), where C is a smooth curve in $|L|$, A is a base-point-free g^1_4 on C, and l is a surjective morphism $H \otimes_{\mathbf{C}} \mathcal{O}_S \to A$ which induces an isomorphism on global sections, two such morphisms being identified if they differ by multiplication by a non-zero scalar. Let $\pi : P \to |L|$ be the forgetful morphism. The tangent space to P at (C, A, l) is identified with the kernel $\tilde{H}^0(T \otimes A)$ of the map $H^0(T \otimes A) \to H^1(\mathrm{End}\, T) \xrightarrow{\mathrm{Tr}} H^1(\mathcal{O}_S)$; the tangent space to $|L|$ at C is identified with the kernel $\tilde{H}^0(C, L)$ of the map $H^0(C, L) \to H^1(\mathcal{O}_S)$. There is an exact sequence ([L], page 304)

$$\tilde{H}^0(T \otimes A) \xrightarrow{T_{(C,A,l)}\pi} \tilde{H}^0(C, L) \longrightarrow (\mathrm{Ker}\, \mu)^* \longrightarrow \tilde{H}^1(T \otimes A)$$

where $\mu : H^0(A) \otimes H^0(\omega_C \otimes A^*) \to H^0(\omega_C)$ is the Petri map. By the base-point-free pencil trick, its kernel is isomorphic to $H^0(\omega_C \otimes (A^{\otimes 2})^*)$, which has by Riemann–Roch dimension at least $h^0(A^{\otimes 2}) - 2 > 0$. Since $H^1(T \otimes A)$ vanishes, $T_{(C,A,l)}\pi$ is not surjective, hence neither is π by generic smoothness. This shows that there is no g^1_4 on a generic C in $|L|$, and finishes the proof of the proposition. ∎

REFERENCES

[AH] Abramovich, D., Harris, J., Abelian varieties and curves in $W_d(C)$, *Comp. Math.* **78** (1991), 227–238.

[DF] Debarre, O., Fahlaoui, R., Abelian Varieties in $W^r_d(C)$, *Comp. Math.* **88** (1993), 235–249.

[L] Lazarsfeld, R., Brill–Noether–Petri without Degenerations, *J. Diff. Geom.* **23** (1986), 299–307.

[R] Reider, I., Vector Bundles of Rank 2 and Linear Systems on Algebraic Surfaces, *Ann. of Math.* **127** (1988), 309–316.