ADDENDUM TO “ON THE EULER CHARACTERISTIC OF GENERALIZED KUMMER VARIETIES”

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Much progress has been made since the publication of [D]. I will describe some improvements that are now known.

Let A be an abelian surface with a nondivisible ample line bundle L with first Chern class ℓ and let $\widehat{A}$ be its dual abelian surface. The Mukai lattice of A is the even integral cohomology $H^\star(A,\mathbf{Z}) = H^0(A,\mathbf{Z}) \oplus H^2(A,\mathbf{Z}) \oplus H^4(A,\mathbf{Z})$ endowed with the quadratic form

$$(a_0, a_2, a_4) \cdot (b_0, b_2, b_4) := a_2 \cdot b_2 - a_0 b_4 - a_4 b_0$$

(see [Y, Section 1.1]). For $\mathbf{v} \in H^\star(A,\mathbf{Z})_{\mathrm{alg}}$ primitive, we denote by $M_{A,L}(\mathbf{v})$ the projective moduli space of L-semistable sheaves on A with Chern character $\mathbf{v}$. We assume $\mathbf{v}^2 \geq 6$, that $\mathbf{v}$ is positive in the sense of [Y, Definition 0.1], and that every L-semistable sheaf in $M_{A,L}(\mathbf{v})$ is L-stable. In that case, $M_{A,L}(\mathbf{v})$ is a smooth projective variety with surjective Albanese map $M_{A,L}(\mathbf{v}) \to A \times \widehat{A}$; all the fibers are isomorphic to the same hyperkähler variety $K_{A,L}(\mathbf{v})$ of dimension $\mathbf{v}^2 - 2$ which is of generalized Kummer type ([Y, Theorem 0.2]). When $\mathbf{v} = (1, 0, -m)$, with $m \geq 2$, the variety $K_{A,L}(\mathbf{v})$ is isomorphic to the generalized Kummer variety $K_{m-1}(A)$ (of dimension $2m - 2$) for any L.

Take $\mathbf{v} = (0, \ell, m)$ and $\mathbf{w} = (r, \ell, 0)$ with $m, r \geq 2$. These vectors are positive and, if every curve in the linear system $|L|$ is integral, every L-semistable sheaf in $M_{A,L}(\mathbf{v})$ or $M_{A,L}(\mathbf{w})$ is L-stable. Assume $2n := \ell^2 \geq 6$. The variety $K_{A,L}(\mathbf{v})$ is the variety $\mathrm{J}^{m+n}(A)$ of [D] and the variety $K_{A,L}(\mathbf{w})$ is the variety $\mathrm{M}_r(A)$ of [D]. By what we explained above, these varieties are all of Kummer type of the same dimension $2n - 2$ hence are all deformation equivalent. This is a vast generalization of [D, Theorem 3.4].

Assume moreover $\mathrm{NS}(A) = \mathbf{Z}[L]$. By [Y, Proposition 3.39], the variety $\mathrm{J}^d(A)$ is birationally isomorphic to a generalized Kummer variety of some abelian surface if and only if $d \equiv n \pm 1 \pmod{2n}$ and the variety $\mathrm{M}_r(A)$ is birationally isomorphic to a generalized Kummer variety of some abelian surface if and only if $r \equiv \pm 1 \pmod{2n}$ (this agrees with [D, Proposition 3.2]).

References

[D] Debarre, O., On the Euler characteristic of generalized Kummer varieties, *Amer. J. Math.* **121** (1999), 577–586.

[Y] Yoshioka, K., Bridgeland's stability and the positive cone of the moduli spaces of stable objects on an abelian surface, in *Development of moduli theory, Kyoto 2013,* 473–537, Adv. Stud. Pure Math. **69**, Math. Soc. Japan, Tokyo, 2016.

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