ON THE EULER CHARACTERISTIC OF GENERALIZED KUMMER VARIETIES

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The aim of this article is to apply the Yau-Zaslow-Beauville method ([YZ], [B1]) to compute the Euler characteristic of the generalized Kummer varieties attached to a complex abelian surface (a calculation also done in [GS] by different methods). The argument is very geometric : given an ample line bundle L with $h^0(\mathrm{L}) = n$ on an abelian surface A, such that each curve in $|\mathrm{L}|$ is integral, we construct a projective symplectic $(2n-2)$-dimensional variety $\mathrm{J}^d(\mathrm{A})$ with a Lagrangian fibration $\mathrm{J}^d(\mathrm{A}) \to |\mathrm{L}|$ whose fiber over a point corresponding to a smooth curve C is the kernel of the Albanese map $\mathrm{J}^d\mathrm{C} \to \mathrm{A}$. The Yau-Zaslow-Beauville method shows that the Euler characteristic of $\mathrm{J}^d(\mathrm{A})$ is n times the number of genus 2 curves in $|\mathrm{L}|$, to wit $n^2\sigma(n)$ (where $\sigma(n) = \sum_{m|n} m$). The latter computation was also done in [G], where a general conjecture (proved in [BL1] for K3 surfaces and in [BL2] for abelian surfaces) is stated : if N_r^n is the number of genus $r+2$ curves in $|\mathrm{L}|$ passing through r general points of A, one should have

$$\sum_{n\in\mathbf{N}} \mathrm{N}_r^n q^n = \Big(\sum_{n\in\mathbf{N}} n\sigma(n)q^n\Big)^r \Big(\sum_{n\in\mathbf{N}} n^2\sigma(n)q^n\Big)$$

for L *sufficiently ample* (for example a sufficiently high power of an ample line bundle). It is remarkable that this identity should hold when L has type $(1,n)$ *for any* n ; it does not in general give the right number of genus 2 curves for other types (see remark 2.3).

Unlike the case of K3 surfaces, none of these varieties $\mathrm{J}^d(\mathrm{A})$ seem to be birationally isomorphic to the generalized Kummer variety $\mathrm{K}_{n-1}(\mathrm{A})$, a symplectic desingularization of a fiber of the sum morphism $\mathrm{A}^{(n)} \to \mathrm{A}$ introduced by Beauville in [B2]. However, using the Mukai-Fourier transform for sheaves on an abelian surface, a degeneration to the case when A is a product of elliptic curves and a result of Huybrechts, we prove that $\mathrm{K}_{n-1}(\mathrm{A})$ and $\mathrm{J}^{n-2}(\mathrm{A})$ are diffeomorphic, hence have the same Euler characteristic $n^3\sigma(n)$. I would like to thank Huybrechts very much for his help with theorem 3.4.

1. The symplectic variety $\mathrm{J}^d(\mathrm{A})$

Let A be a complex abelian surface with a polarization ℓ of type $(1,n)$. Assume that each curve with class ℓ is integral (this holds for generic (A,ℓ)). Let $\hat{\mathrm{A}}$ be the dual abelian surface. Let $\phi_\ell : \mathrm{A} \to \hat{\mathrm{A}}$ be the morphism associated with the polarization ℓ ; there exists a factorization $n\mathrm{Id}_\mathrm{A} : \mathrm{A} \xrightarrow{\phi_\ell} \hat{\mathrm{A}} \xrightarrow{\phi_{\hat\ell}} \mathrm{A}$, where $\hat\ell$ is a polarization on $\hat{\mathrm{A}}$ of type $(1,n)$.

We denote by $\mathrm{Pic}^\ell(\mathrm{A})$ the component of the Picard group of A corresponding to line bundles with class ℓ, by $\{\ell\}$ the component of the Hilbert scheme that parametrizes curves in A with class ℓ, by $\mathcal{C} \to \{\ell\}$ the universal family, and by $\bar{\mathcal{J}}\mathcal{C} \to \{\ell\}$ the compactified Picard scheme of this family ([AK]).

The variety $\bar{\mathcal{J}}\mathcal{C}$ splits as a disjoint union $\coprod_{d\in\mathbf{Z}} \bar{\mathcal{J}}^d\mathcal{C}$, where $\bar{\mathcal{J}}^d\mathcal{C}$ is a projective variety of dimension $2n+2$, which parameterizes pairs $(\mathrm{C},\mathcal{L})$ where C is a curve on A with class ℓ and $\mathcal{L}$ is a torsion free, rank 1 coherent sheaf on C of degree d (i.e. with

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$\chi(\mathcal{L}) = d + 1 - g(\mathrm{C}) = d - n$). According to Mukai ([M1], ex. 0.5), $\bar{\mathcal{J}}^d\mathcal{C}$ can be viewed as a connected component of the moduli space of simple sheaves $\mathcal{L}$ on A, and therefore is smooth, and admits a (holomorphic) symplectic structure. There is a natural morphism

$$\begin{array}{rccccc} \alpha : \bar{\mathcal{J}}^d\mathcal{C} & \longrightarrow & \{\ell\} & \longrightarrow & \mathrm{Pic}^\ell(\mathrm{A}) \\ (\mathrm{C}, \mathcal{L}) & \longmapsto & \mathrm{C} & \longmapsto & [\mathcal{O}_\mathrm{A}(\mathrm{C})] \end{array}$$

also defined by $\alpha(\mathcal{L}) = \det \mathcal{L}$. For each smooth curve C in $\{\ell\}$, the inclusion $\mathrm{C} \subset \mathrm{A}$ induces an Abel-Jacobi map $\mathrm{J}^d\mathrm{C} \to \mathrm{A}$; this defines a rational map

$$\beta : \bar{\mathcal{J}}^d\mathcal{C} \dashrightarrow \mathrm{A} \times \{\ell\} \longrightarrow \mathrm{A}$$

which is regular since A is an abelian variety and $\bar{\mathcal{J}}^d\mathcal{C}$ is normal. Let $\mathrm{J}^d(\mathrm{A})$ be a fiber of the map $(\alpha, \beta) : \bar{\mathcal{J}}^d\mathcal{C} \longrightarrow \mathrm{Pic}^\ell(\mathrm{A}) \times \mathrm{A}$ (they are all isomorphic). Note that $\mathrm{J}^d(\mathrm{A})$, $\mathrm{J}^{d+2n}(\mathrm{A})$ and $\mathrm{J}^{-d}(\mathrm{A})$ are isomorphic.

Proposition 1.1.– *The symplectic structure on $\bar{\mathcal{J}}^d\mathcal{C}$ induces a symplectic structure on the $(2n-2)$-dimensional variety $\mathrm{J}^d(\mathrm{A})$.*

Proof. Recall that there is a canonical isomorphism $\mathrm{T}_\mathcal{L}\bar{\mathcal{J}}^d\mathcal{C} \simeq \mathrm{Ext}^1(\mathcal{L}, \mathcal{L})$, and that the symplectic form ω is the pairing

$$\mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) \otimes \mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) \to \mathrm{Ext}^2(\mathcal{L}, \mathcal{L}) \xrightarrow{\mathrm{Tr}} \mathrm{H}^2(\mathrm{A}, \mathcal{O}_\mathrm{A}) \simeq \mathbf{C}$$

The map $\mathrm{T}_\mathcal{L}\alpha$ is the trace map $\mathrm{T} : \mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) \to \mathrm{H}^1(\mathrm{A}, \mathcal{O}_\mathrm{A})$, whereas the tangent map at the origin to the map $\iota : \mathrm{Pic}^0(\mathrm{A}) \to \bar{\mathcal{J}}^d\mathcal{C}$ defined by $\iota(\mathrm{P}) = \mathrm{P} \otimes \mathcal{L}$ is the dual $\mathrm{T}^* : \mathrm{H}^1(\mathrm{A}, \mathcal{O}_\mathrm{A}) \to \mathrm{Ext}^1(\mathcal{L}, \mathcal{L})$. Since $\alpha\iota$ is constant, $\mathrm{T} \circ \mathrm{T}^* = 0$; in particular

$$\mathrm{Ker}\, \mathrm{T} \supset \mathrm{Im}\, \mathrm{T}^* = (\mathrm{Ker}\, \mathrm{T})^\perp .$$

Note also that $\beta\iota\phi_\ell = n\mathrm{Id}_\mathrm{A}$ (use the Morikawa-Matsusaka endomorphism), hence $\mathrm{T}_\mathcal{L}\beta \circ \mathrm{T}^* = \mathrm{T}\phi_{\hat{\ell}}$ and $\mathrm{Ker}\, \mathrm{T}_\mathcal{L}\beta \cap \mathrm{Im}\, \mathrm{T}^* = \{0\}$. Since both $\mathrm{Ker}\, \mathrm{T}$ and $\mathrm{Ker}\, \mathrm{T}_\mathcal{L}\beta$ have codimension 2, this implies

$$\mathrm{Ker}\, \mathrm{T} = (\mathrm{Ker}\, \mathrm{T} \cap \mathrm{Ker}\, \mathrm{T}_\mathcal{L}\beta) \oplus (\mathrm{Ker}\, \mathrm{T})^\perp ,$$

and the restriction of ω to $\mathrm{Ker}\, \mathrm{T} \cap \mathrm{Ker}\, \mathrm{T}_\mathcal{L}\beta = \mathrm{T}_\mathcal{L}\mathrm{J}^d(\mathrm{A})$ is non-degenerate. ∎

The map α restricts to a morphism $\alpha : \mathrm{J}^d(\mathrm{A}) \to |\mathrm{L}|$ whose fiber $\mathrm{K}^d(\mathrm{C})$ over the point corresponding to a smooth curve C is the (connected) kernel of the Abel-Jacobi map $\beta : \mathrm{J}^d\mathrm{C} \to \mathrm{A}$; it is a Lagrangian fibration.

2. The Euler characteristic of $\mathrm{J}^d(\mathrm{A})$

We calculate the Euler characteristic of $\mathrm{J}^d(\mathrm{A})$ by using the Lagrangian fibration $\alpha : \mathrm{J}^d(\mathrm{A}) \to |\mathrm{L}|$, as in [B1].

Proposition 2.1.– *Let C be an integral element of $|\mathrm{L}|$. The Euler characteristic of $\mathrm{K}^d(\mathrm{C})$ is n if the normalization of C has genus 2, and 0 otherwise.*

Proof. Let $\eta : \widetilde{C} \to C$ be the normalization. There is a commutative diagram (as in §2 of [B1], we may restrict ourselves to the case $d = 0$ and drop the superscript d)

$$\begin{array}{ccccccc}
K(C) & \longrightarrow & \bar{J}C & \xrightarrow{\beta} & A & & \\
 & & \cup & & \uparrow \pi & & \\
 & & JC & \xrightarrow{\eta^*} & J\widetilde{C} & \to & 0 \\
 & & \cup & & \uparrow & & \\
 & & (\eta^*)^{-1}(\operatorname{Ker}\pi) & \longrightarrow & \operatorname{Ker}\pi & \to & 0
\end{array}$$

By lemma 2.1 of *loc.cit.*, the group JC acts freely on $\bar{J}C$. Note also that for M in JC and $\mathcal{L}$ in $\bar{J}C$,

$$\beta(M \otimes \mathcal{L}) = \beta(M) + \beta(\mathcal{L})$$

because this is true when $\mathcal{L}$ is invertible, and JC is dense in $\bar{J}C$. It follows that $(\eta^*)^{-1}(\operatorname{Ker}\pi)$ acts (freely) on $K(C)$. As in prop. 2.2 of *loc.cit.*, it follows that $e(K(C)) = 0$ if $\operatorname{Ker}\pi$ is infinite, that is if $g(\widetilde{C}) > 2$.

Assume now that $\widetilde{C}$ has genus 2. The situation here is much simpler than in *loc.cit.*, because the normalization η of C is *unramified* : it is the restriction to $\widetilde{C}$ of the isogeny $\pi : J\widetilde{C} \to A$. If $\check{C} \to C$ is the minimal unibranch partial normalization (*cf. loc.cit.*), it follows that $\widetilde{C} \to \check{C}$ is an unramified homeomorphism, hence an isomorphism ([Gr], 18.12.6).

There is a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Ker}\pi & \longrightarrow & J\widetilde{C} & \xrightarrow{\pi} & A \\
 & & \cap & & \cap\, \eta_* & & \| \\
0 & \longrightarrow & K(C) & \longrightarrow & \bar{J}C & \xrightarrow{\beta} & A
\end{array}$$

and an exact sequence

$$1 \to \mathcal{O}^*_{\widetilde{C}}/\mathcal{O}^*_C \longrightarrow JC \longrightarrow J\widetilde{C} \to 0 \,.$$

If one chooses a line bundle M on C corresponding to a point of $\mathcal{O}^*_{\widetilde{C}}/\mathcal{O}^*_C$ as in the proof of prop. 3.3 of *loc.cit.*, it acts on $\bar{J}C$, hence on $K(C)$. Beauville's reasoning proves that M acts *freely* on the complement of $\eta_* J\widetilde{C}$ in $\bar{J}C$, hence also on the complement of $\operatorname{Ker}\pi$ in $K(C)$. It follows that $e(K(C)) = e(\operatorname{Ker}\pi) = n$. ∎

As a corollary, we get, assuming that each curve with class ℓ is integral,

$$e(J^d(A)) = n \ \#\{\, C \in |L| \;\big|\; g(\widetilde{C}) = 2 \,\} \,.$$

It remains to count the number of (integral) genus 2 curves in $|L|$ or, which amounts to the same, the number of morphisms $f : \widetilde{C} \to A$ such that $f_*\widetilde{C} \in |L|$, where $\widetilde{C}$ is a smooth curve of genus 2, modulo automorphisms of $\widetilde{C}$. Let us do this calculation for any polarization ℓ of degree n on a simple abelian surface A. To f one associates an isogeny $\pi : J\widetilde{C} \to A$ such that $\pi_*\theta = \ell$ (where θ is the principal polarization on $J\widetilde{C}$), hence $\pi^*\ell = r\theta$, where r is the degree of π. It follows that $r = n$. Conversely, if $\pi : \widetilde{A} \to A$ is an isogeny such that $\pi^*\ell$ is n times a principal polarization, $\widetilde{A}$ is the Jacobian of a smooth genus 2 curve $\widetilde{C}$, and the image by π of any translate of $\widetilde{C}$ in $J\widetilde{C}$ has class ℓ. It follows that there are exactly $n^2 = \#\operatorname{Ker}\phi_\ell$ such translates whose image is in $|L|$.

The number of isomorphism classes of isogenies $\pi : \widetilde{\mathrm{A}} \to \mathrm{A}$ as above is also the number $\mathrm{N}(\ell)$ of isomorphism classes of isogenies $\hat{\pi}: \hat{\mathrm{A}} \to \widetilde{\mathrm{A}}$, where $\hat{\pi}^*\theta = \hat{\ell}$, hence also the number of subgroups G of $\mathrm{Ker}\,\phi_{\hat{\ell}}$ that are maximal totally isotropic for the Weil form. This kernel is (non-canonically) isomorphic to $\mathrm{H} \times \mathrm{H}^*$, where H is an abelian group of cardinal n and the Weil form is given by $e((x,x^*),(y,y^*)) = y^*(x) \cdot x^*(y)^{-1}$. If $\mathrm{K} = p_1(\mathrm{G}) \subset \mathrm{H}$ and $\mathrm{K}' = \mathrm{G} \cap (\{0\} \times \mathrm{H}^*)$, there exists a group homomorphism $u : \mathrm{K} \to \mathrm{H}^*/\mathrm{K}'$ such that

$$\mathrm{G} = \{\ (x,x^*) \in \mathrm{K} \times \mathrm{H}^* \mid u(x) = \overline{x^*}\ \}\ .$$

The fact that G is totally isotropic of cardinal n yields $\mathrm{K}' = \mathrm{K}^\perp$ and $u : \mathrm{K} \to \mathrm{K}^*$ *symmetric*. Note that the latter condition is empty if K is cyclic.

When ℓ has type $(1,n)$, the group H is cyclic and

$$\mathrm{N}(\ell) = \sum_{\mathrm{K} < \mathbf{Z}/n\mathbf{Z}} \#\,\mathrm{Hom}(\mathrm{K},\mathrm{K}^*) = \sum_{m|n} m = \sigma(n)\ .$$

Corollary 2.2.– *Assume each curve with class ℓ is integral; then $e(\mathrm{J}^d(\mathrm{A})) = n^3\sigma(n)$.*

Remark 2.3. Suppose ℓ is p times a principal polarization (p prime), so that $\mathrm{H} \simeq (\mathbf{Z}/p\mathbf{Z})^2$ and $\ell^2/2 = p^2$. The subgroups K are $(\mathbf{Z}/p\mathbf{Z})$-vector subspaces of H, hence

$$\mathrm{N}(\ell) = \sum_{\mathrm{K}<\mathrm{H}} \#\,\mathrm{Hom}^{\mathrm{sym}}(\mathrm{K},\mathrm{K}^*) = \sum_{e=0}^{2} \sum_{\mathrm{K},\dim \mathrm{K}=e} p^{e(e+1)/2} = 1 + p(p+1) + p^3 \neq \sigma(p^2)\ .$$

In this case, the formula given in the introduction does *not* give the right number of genus 2 curves in $|\mathrm{L}|$ (note that these curves are all integral and are interchanged by monodromy). In [V], Vainsencher defines a scheme Σ whose length, when finite, should count the number of nodes of curves in $|\mathrm{L}|$ with p^2-1 nodes (hence of genus 2); here however, Σ is not finite because $|\mathrm{L}|$ contains non-reduced curves.

3. A degeneration of $\mathrm{J}^{n-2}(\mathrm{A})$

Our aim is to relate the symplectic variety $\mathrm{J}^d(\mathrm{A})$ constructed above with the generalized Kummer variety $\mathrm{K}_{n-1}(\mathrm{A})$. Contrary to the case of K3 surfaces, these varieties do not seem to be birational for general A (except when $n=2$). Using the Mukai-Fourier transform, we will relate $\mathrm{J}^d(\mathrm{A})$ with a moduli space of rank $(n-d)$-sheaves on the dual surface $\hat{\mathrm{A}}$, then degenerate the situation to the case where A is a product of elliptic curves and $d = n-2$, where one can prove that this moduli space is birational to $\mathrm{K}_{n-1}(\mathrm{A})$.

For any sheaf F on A, we denote by $\mathcal{F}^\bullet\mathrm{F}$ the cohomology sheaves of the Mukai-Fourier transform of F (see [M2]). If only $\mathcal{F}^j\mathrm{F}$ is non-zero, we say that F has weak index j, and we write $\hat{\mathrm{F}} = \mathcal{F}^j\mathrm{F}$; in that case, $\hat{\mathrm{F}}$ has weak index $2-j$, and $\hat{\hat{\mathcal{F}}}\mathrm{F} \simeq (-1)^*\mathrm{F}$ (*loc.cit.*, cor. 2.4). If $\mathrm{H}^i(\mathrm{A},\mathrm{F}\otimes \mathrm{P}_{\hat{x}}) = 0$ for all $\hat{x} \in \hat{\mathrm{A}}$ and all $i \neq j$, we say that F has index j ; it implies that F has weak index j.

For any $\hat{x}$ in $\hat{\mathrm{A}}$, we denote by $\mathrm{P}_{\hat{x}}$ the corresponding line bundle on A ; we identify the dual of $\hat{\mathrm{A}}$ with A, so that, for any x in A, P_x is a line bundle on $\hat{\mathrm{A}}$.

Let C be a generic curve in $\{\ell\}$; the translate of the surface $\mathrm{Pic}^0(\mathrm{A})$ by a generic point $\mathcal{L}$ in $\mathrm{J}^d\mathrm{C}$ does not meet the d-dimensional subvariety $\mathrm{W}_d(\mathrm{C})$ of $\mathrm{J}^d\mathrm{C}$, as soon as $g(\mathrm{C}) > 2+d$, i.e. $d < n-1$. In that case, one has $\mathrm{H}^0(\mathrm{A},\mathcal{L}\otimes\mathrm{P}_{\hat{x}}) = 0$ for all $\hat{x}$ in $\hat{\mathrm{A}}$, so that $\mathcal{L}$ has index 1, and $\hat{\mathcal{L}}$ is a locally free simple sheaf on $\hat{\mathrm{A}}$ of rank $n-d$, first Chern class $\hat{\ell}$ and Euler characteristic 0.

Proposition 3.1.– *Let* (A,ℓ) *be a polarized abelian surface of type* $(1,n)$ *whose Néron-Severi group is generated by* ℓ. *For* $d<n-1$ *and* $\mathcal{L}$ *generic in* $\bar{\mathcal{J}}^d\mathcal{C}$, *the vector bundle* $\hat{\mathcal{L}}$ *on* $\hat{\mathrm{A}}$ *is* $\hat{\ell}$*-stable.*

Proof. We follow [FL] : assume $\hat{\mathcal{L}}$ is not stable, and look at torsion-free non-zero quotients of $\hat{\mathcal{L}}$ of smallest degree, and among those, pick one, Q, of smallest rank. Because $\mathrm{NS}(\hat{\mathrm{A}})=\mathbf{Z}\hat{\ell}$, the degree of Q is non-positive. The proofs of lemmes 2 and 3 of [FL] apply without change : Q has index 1 and if K be the kernel of $\hat{\mathcal{L}}\to\mathrm{Q}$, the sheaf $\mathcal{F}^2\mathrm{K}$ has finite support. Consider the exact sequence

$$0\to\mathcal{F}^1\mathrm{K}\to(-1)^*\mathcal{L}\to\hat{\mathrm{Q}}\to\mathcal{F}^2\mathrm{K}\to 0\ ;$$

since $c_1(\hat{\mathrm{Q}})\cdot\ell=c_1(\mathrm{Q})\cdot\hat{\ell}\le 0$ ([FL], lemme 1), the torsion sheaf $\hat{\mathrm{Q}}$ has finite support, hence index 0. But this index is also $2-\mathrm{ind}\,\mathrm{Q}=1$; this contradiction proves the proposition. ∎

For each $d<n-1$, we have constructed a birational rational map between $\bar{\mathcal{J}}^d\mathcal{C}$ and an irreducible component $\mathcal{M}^0_{\hat{\mathrm{A}}}(n-d,\hat{\ell},0)$ of the moduli space $\mathcal{M}_{\hat{\mathrm{A}}}(n-d,\hat{\ell},0)$ of $\hat{\ell}$-semi-stable sheaves on $\hat{\mathrm{A}}$ of rank $n-d$, first Chern class $\hat{\ell}$ and Euler characteristic 0. This map is a morphism if $d<0$. Let us interpret the maps α and β in this context. Let $(\mathrm{C},\mathcal{L})$ be a pair corresponding to a point of $\bar{\mathcal{J}}^d\mathcal{C}$; it follows from the exact sequence $0\to\mathcal{O}_{\mathrm{C}}\to\mathcal{O}_{\mathrm{C}}(x)\to\mathbf{C}_x\to 0$ that $\det\widehat{\mathcal{O}_{\mathrm{C}}(x)}\simeq\det\widehat{\mathcal{O}_{\mathrm{C}}}\otimes\mathrm{P}_{-x}$, hence

$$\det\hat{\mathcal{L}}\simeq\det\widehat{\mathcal{O}_{\mathrm{C}}}\otimes\mathrm{P}_{-\beta(\mathcal{L})}\simeq\det\widehat{\mathcal{O}_{\mathrm{A}}(-\mathrm{C})}\otimes\mathrm{P}_{-\beta(\mathcal{L})}\ .$$

Hence, the fibers of (α,β) are also the fibers of the map $\bar{\mathcal{J}}^d\mathcal{C}\to\mathrm{Pic}^{\ell}(\mathrm{A})\times\mathrm{Pic}^{-\hat{\ell}}(\hat{\mathrm{A}})$ which sends $\mathcal{L}$ to $(\det\mathcal{L},\det\mathcal{F}^\bullet\mathcal{L})$. Let $\Omega:\mathcal{M}^0_{\hat{\mathrm{A}}}(n-d,\hat{\ell},0)\to\mathrm{Pic}^{-\ell}(\mathrm{A})\times\mathrm{Pic}^{\hat{\ell}}(\hat{\mathrm{A}})$ be the map $\mathrm{E}\mapsto(\det\mathcal{F}^\bullet\mathrm{E},\det\mathrm{E})$, and let $\mathrm{M}_{n-d}(\hat{\mathrm{A}})$ be a fiber. We have proved the following.

Proposition 3.2.– *Let* (A,ℓ) *be a polarized abelian surface of type* $(1,n)$ *whose Néron-Severi group is generated by* ℓ. *For* $d<n-1$, *the Fourier-Mukai transform induces a birational isomorphism between* $\bar{\mathcal{J}}^d\mathcal{C}$ *and an irreducible component of* $\mathcal{M}_{\hat{\mathrm{A}}}(n-d,\hat{\ell},0)$ *which sends* $\mathrm{J}^d(\mathrm{A})$ *onto* $\mathrm{M}_{n-d}(\hat{\mathrm{A}})$.

We will now study the case where $n-d=2$ and A is the product of two general elliptic curves F and G, with ℓ of bidegree $(1,n)$. One has $\hat{\mathrm{A}}=\hat{\mathrm{F}}\times\hat{\mathrm{G}}$, and $\hat{\ell}$ has bidegree $(n,1)$. To avoid non-stable semi-stable sheaves, we will study the moduli space $\mathcal{M}'_{\hat{\mathrm{A}}}$ of rank 2 sheaves on $\hat{\mathrm{A}}$ with first Chern class $\hat{\ell}$ and Euler characteristic 0 which are semi-stable for the polarization $\hat{\ell}'$ of bidegree $(n+1,1)$, and call $\mathrm{M}'(\hat{\mathrm{A}})$ a fiber of the map $\Omega:\mathcal{M}'_{\hat{\mathrm{A}}}\to\mathrm{Pic}^{-\ell}(\mathrm{A})\times\mathrm{Pic}^{\hat{\ell}}(\hat{\mathrm{A}})$ defined above.

Proposition 3.3.– *In the above situation where* $\mathrm{A}=\mathrm{F}\times\mathrm{G}$, *the moduli space* $\mathcal{M}'_{\hat{\mathrm{A}}}$ *is smooth and birational to* $\hat{\mathrm{A}}^{(n)}\times\mathrm{A}$, *and the variety* $\mathrm{M}'(\hat{\mathrm{A}})$ *is smooth and birational to* $\mathrm{K}_{n-1}(\hat{\mathrm{A}})$.

Proof. Let E be an $\hat{\ell}'$-semi-stable rank 2 torsion free sheaf on $\hat{\mathrm{A}}$ with first Chern class $\hat{\ell}$ and Euler characteristic 0. Let $x\in\mathrm{A}$; by semi-stability of E^*, one has $\mathrm{H}^2(\hat{\mathrm{A}},\mathrm{E}\otimes\mathrm{P}_x^{-1})=0$, hence $h^0(\hat{\mathrm{A}},\mathrm{E}\otimes\mathrm{P}_x^{-1})=h^1(\hat{\mathrm{A}},\mathrm{E}\otimes\mathrm{P}_x^{-1})$. Since $\hat{\mathcal{F}}\mathrm{E}$ is non-zero, for at least one x, these numbers are non-zero and there is an inclusion $\mathrm{P}_x\hookrightarrow\mathrm{E}$; let K be the kernel of $\mathrm{E}\to\mathrm{E}/\mathrm{P}_x\to(\mathrm{E}/\mathrm{P}_x)/(\mathrm{E}/\mathrm{P}_x)_{\mathrm{tors}}$. There is an exact sequence

$$(*)\qquad 0\to\mathrm{K}\to\mathrm{E}\to\mathcal{I}_{\mathrm{Z}}\otimes\mathrm{K}'\to 0\ ,$$

where K' is a line bundle. The line bundle K has bidegree (a,b), with a and b non-negative and $b(n+1)+a\le(2n+1)/2$ (by $\hat\ell'$-semi-stability); hence $b=0$ and Z is a subscheme of $\hat{\mathrm{A}}$ of length $n-a$.

Set $\mathrm{M}=\mathrm{K}'\otimes\mathrm{K}^{-1}$. By Serre duality, $\mathrm{Ext}^1_{\hat{\mathrm{A}}}(\mathcal{I}_\mathrm{Z}\otimes\mathrm{K}',\mathrm{K})$ and $\mathrm{H}^1(\hat{\mathrm{A}},\mathcal{I}_\mathrm{Z}\otimes\mathrm{M})^*$ are isomorphic. Assume $\mathrm{H}^0(\hat{\mathrm{A}},\mathcal{I}_\mathrm{Z}\otimes\mathrm{M})=0$; one has

$$h^1(\hat{\mathrm{A}},\mathcal{I}_\mathrm{Z}\otimes\mathrm{M})=\mathrm{length}(\mathrm{Z})-\chi(\hat{\mathrm{A}},\mathrm{M})=a$$

and $a>0$ (otherwise $\mathcal{I}_\mathrm{Z}\otimes\mathrm{K}'$ would be a subsheaf of E with $\hat\ell'$-slope $2n+1$), and E depends on at most $2n+3-a$ parameters (2 for K, 2 for K', $2(n-a)$ for Z and $a-1$ for the extension). Since each component of $\mathcal{M}'_{\hat{\mathrm{A}}}$ has dimension $2n+2$, this forces $a=1$ for E generic. Let $\mathcal{M}^0$ be the subset of $\mathcal{M}'_{\hat{\mathrm{A}}}$ parametrized in this fashion.

Assume now $\mathrm{H}^0(\hat{\mathrm{A}},\mathcal{I}_\mathrm{Z}\otimes\mathrm{M})\neq0$; one checks (by projecting onto $|\mathrm{M}|$), that the set of pairs (Z,D) with $\mathrm{D}\in|\mathrm{M}|$ and $\mathrm{Z}\subset\mathrm{D}$, has dimension $\le n-2a-1+n-a$. Hence E depends on at most $2n-3a-1-\chi(\hat{\mathrm{A}},\mathcal{I}_\mathrm{Z}\otimes\mathrm{M})+4=2n-2a+3$ parameters. For E generic, this forces $a=0$, Z reduced and $h^0(\hat{\mathrm{A}},\mathcal{I}_\mathrm{Z}\otimes\mathrm{M})=1$. This yields a component of $\mathcal{M}'_{\hat{\mathrm{A}}}$ which can be parametrized as follows. Let $\mathrm{Z}=(\hat f_1,\hat g_1)+\ldots+(\hat f_n,\hat g_n)$ be generic in $\hat{\mathrm{A}}^{(n)}$, set $\mathrm{L}=\mathcal{O}_{\hat{\mathrm{F}}}(\hat f_1+\ldots+\hat f_n)$, and let $f\in\mathrm{F}$ and $\hat g\in\hat{\mathrm{G}}$. The vector space $\mathrm{Ext}^1_{\hat{\mathrm{A}}}(\mathcal{I}_\mathrm{Z}\otimes p^*_{\hat{\mathrm{F}}}\mathrm{L}\otimes p^*_{\hat{\mathrm{G}}}\mathcal{O}_{\hat{\mathrm{G}}}(\hat g),\mathcal{O}_{\hat{\mathrm{A}}})$ has dimension 1, hence there is a unique extension

$$0\to p^*_{\hat{\mathrm{F}}}\mathrm{P}_f\to\mathrm{E}\to\mathcal{I}_\mathrm{Z}\otimes p^*_{\hat{\mathrm{F}}}(\mathrm{L}\otimes\mathrm{P}_f)\otimes p^*_{\hat{\mathrm{G}}}\mathcal{O}_{\hat{\mathrm{G}}}(\hat g)\to0\ ,$$

where E is locally free (it satisfies the Cayley-Bacharach condition; see for example th. 5.1.1 of [HL]) and stable (the only thing to check is $\mathrm{H}^0(\hat{\mathrm{A}},\mathrm{E}\otimes p^*_{\hat{\mathrm{F}}}\mathrm{P}_{-f}\otimes p^*_{\hat{\mathrm{G}}}\mathcal{O}_{\hat{\mathrm{G}}}(-\hat g))=0$, and this is true because the extension is non-trivial). This yields a rational map

$$\phi:\hat{\mathrm{A}}^{(n)}\times\mathrm{F}\times\hat{\mathrm{G}}\dashrightarrow\mathcal{M}'_{\hat{\mathrm{A}}}$$

which is birational onto its image : given a locally free E as above, one recovers f and the $(\hat g_i-\hat g)$'s by noting that the set $\mathrm{C}_\mathrm{E}=\{\,x\in\mathrm{A}\mid\mathrm{H}^0(\hat{\mathrm{A}},\mathrm{E}\otimes\mathrm{P}_x)\neq0\,\}$ is

$$(\{-f\}\times\mathrm{G})\cup\bigcup_{i=1}^{n}(\mathrm{F}\times\{\,[\mathcal{O}_{\hat{\mathrm{G}}}(\hat g_i-\hat g)]\,\})\ ,$$

the $\hat f_i$'s because $\mathrm{Ext}^1_{\hat{\mathrm{A}}}(\mathcal{I}_\mathrm{Z}\otimes p^*_{\hat{\mathrm{F}}}\mathrm{L}\otimes p^*_{\hat{\mathrm{G}}}\mathcal{O}_{\hat{\mathrm{G}}}(\hat g),\mathcal{O}_{\hat{\mathrm{A}}})$ must be non-zero, and $\hat g$ by noting that $\det\mathrm{E}\simeq p^*_{\hat{\mathrm{F}}}(\mathrm{L}\otimes\mathrm{P}_{2f})\otimes p^*_{\hat{\mathrm{G}}}\mathcal{O}_{\hat{\mathrm{G}}}(\hat g)$. Because $\mathrm{H}^0(\hat{\mathrm{A}},\mathrm{E}\otimes p^*_{\hat{\mathrm{F}}}(\mathrm{P}_{-f}\otimes\mathcal{O}_{\hat{\mathrm{F}}}(-f_1))\otimes p^*_{\hat{\mathrm{G}}}\mathcal{O}_{\hat{\mathrm{G}}}(\hat g_1-\hat g))$ is non-zero, there exists an exact sequence $(*)$ with K of bidegree $(1,0)$. This proves that the set $\mathcal{M}^0$ defined above is contained in the image of ϕ, which must therefore be $\mathcal{M}'_{\hat{\mathrm{A}}}$.

Finally, E has weak index 1, $\mathcal{F}^1\mathrm{E}$ has support on C_E, and fixing $\det\mathcal{F}^1\mathrm{E}$ amounts to fixing $[\mathcal{O}_\mathrm{A}(\mathrm{C}_\mathrm{E})]$. It follows that taking a fiber of Ω amounts to fixing f, $\sum(\hat g_i-\hat g)$, $\sum\hat f_i$ and $\hat g$; hence $\mathrm{M}'(\hat{\mathrm{A}})$ is birational to $\mathrm{K}_{n-1}(\hat{\mathrm{A}})$. ■

The following proof is due to D. Huybrechts, and uses ideas from prop. 2.2 of [GH].

Theorem 3.4.– *Let (A,ℓ) be a polarized abelian surface of type $(1,n)$ whose Néron-Severi group is generated by ℓ. The symplectic varieties $\mathrm{J}^{n-2}(\mathrm{A})$, $\mathrm{M}_2(\hat{\mathrm{A}})$ and $\mathrm{K}_{n-1}(\hat{\mathrm{A}})$ are deformation equivalent. In particular, they are all irreducible symplectic.*

Proof. Let $f : \hat{\mathcal{A}} \to \mathrm{S}$ be a family of polarized abelian surfaces, where S is smooth connected quasi-projective, with a relative polarization $\hat{\mathcal{L}}$ of type $(1,n)$, such that the fiber over a point $0 \in \mathrm{S}$ is $\hat{\mathrm{F}} \times \hat{\mathrm{G}}$ with a polarization of bidegree $(n,1)$, and such that the Néron-Severi group of a very general fiber of f has rank 1. Let $g : \mathcal{M} \to \mathrm{S}$ be the (projective) relative moduli space of $\hat{\mathcal{L}}$-semi-stable sheaves of rank 2 with first Chern class $\hat{\ell}$ and Euler characteristic 0 on the fibers of f (*cf.* [HL], th. 4.3.7, p. 92).

Lemma 3.5.– *Under the hypothesis of the theorem, any rank 2 torsion free sheaf on $\hat{\mathrm{A}}$ with first Chern class $\hat{\ell}$ which is either simple or semi-stable is stable.*

Proof. Assume that a rank 2 torsion free sheaf E on $\hat{\mathrm{A}}$ with first Chern class $\hat{\ell}$ is not stable. There exists an exact sequence

$$0 \to \mathrm{K} \to \mathrm{E} \to \mathcal{I}_{\mathrm{Z}} \otimes \mathrm{K}' \to 0\ ,$$

where K and K′ are line bundles on $\hat{\mathrm{A}}$ with $c_1(\mathrm{K}) = k\hat{\ell}$, $c_1(\mathrm{K}') = (1-k)\hat{\ell}$ and $k > 0$. This proves that E is not semi-stable; moreover, $\mathrm{K} \otimes \mathrm{K}'^{-1}$ is ample, hence there exists a non-zero morphism $u : \mathrm{K}' \to \mathrm{K}$, which induces an endomorphism $\mathrm{E} \to \mathcal{I}_{\mathrm{Z}} \otimes \mathrm{K}' \xrightarrow{u} \mathrm{K} \to \mathrm{E}$ which is not a homothety, and E is not simple. ∎

By the lemma, the (closed) locus of non-stable points in $\mathcal{M}$ does not project onto S. By replacing S with an open subset, we may assume that there are no such points. Let now $\mathcal{S} \to \mathrm{S}$ be the (smooth) relative moduli space of simple sheaves on the fibers of f (see [AK]). There are embeddings $\mathcal{M} \subset \mathcal{S}$ and $\mathcal{M}'_{\hat{\mathrm{F}}\times\hat{\mathrm{G}}} \subset \mathcal{S}_0$ as closed and open subsets. Let $\mathcal{S}' = \mathcal{S} - (\mathcal{S}_0 - \mathcal{M}'_{\hat{\mathrm{F}}\times\hat{\mathrm{G}}})$; it is open in $\mathcal{S}$, hence smooth over S. Let $\mathcal{M}'$ be the closure of $g^{-1}(\mathrm{S} - \{0\})$ in $\mathcal{S}'$; the fibers of $g' : \mathcal{M}' \to \mathrm{S}$ are projective away from 0, and contained in $\mathcal{M}'_{\hat{\mathrm{F}}\times\hat{\mathrm{G}}}$ over 0. Norton's criterion ([N]) shows that points in $\mathcal{M}'_0$ are separated in the moduli space of simple sheaves on $\hat{\mathcal{A}}$ (because they are stable), hence in $\mathcal{S}$; therefore, $\mathcal{M}'$ is separated. By semi-continuity, $\mathcal{M}'_0$ is a closed subset of $\mathcal{M}'_{\hat{\mathrm{F}}\times\hat{\mathrm{G}}}$ of the same dimension, hence they are equal. Using the lemma, we get, after shrinking S again, a proper family $g' : \mathcal{M}' \to \mathrm{S}$ with projective irreducible smooth fibers which coincide with $g : \mathcal{M} \to \mathrm{S}$ away from 0.

For A general, $\mathrm{J}^{n-2}(\mathrm{A})$ is birationally isomorphic to $\mathrm{M}_2(\hat{\mathrm{A}})$ by prop. 3.2, and we just saw that the latter deforms to $\mathrm{M}'(\hat{\mathrm{F}} \times \hat{\mathrm{G}})$, itself birationally isomorphic to $\mathrm{K}_{n-1}(\hat{\mathrm{F}} \times \hat{\mathrm{G}})$ by prop. 3.3; in particular, these symplectic varieties are all irreducible symplectic. Since birationally isomorphic smooth projective irreducible symplectic varieties are deformation equivalent ([H], th. 10.12), the theorem is proved. ∎

Corollary 3.6.– *Let (A, ℓ) be a general polarized abelian surface of type $(1,n)$. The moduli space $\mathcal{M}_{\mathrm{A}}(2, \ell, 0)$ is smooth irreducible.*

Corollary ([GS]) 3.7.– *Let A be an abelian surface. The Euler characteristic of $\mathrm{K}_{n-1}(\mathrm{A})$ is $n^3\sigma(n)$.*

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