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Fulton-Hansen and Barth-Lefschetz theorems for subvarieties of abelian varieties

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Sommese showed that a large part of the geometry of a *smooth* subvariety of a complex abelian variety depends on "how ample" its normal bundle is (see § 1 for more details). Unfortunately, the only known way of measuring this ampleness uses rather strong properties of the ambiant abelian variety.

We show that a notion of non-degeneracy due to Ran is a good substitute for ampleness of the normal bundle. It can be defined as follows: *an irreducible subvariety V of an abelian variety X is geometrically non-degenerate if for any abelian variety Y quotient of X, the image of V in Y either is Y or has same dimension as V.* This property does not require V to be smooth; for smooth subvarieties, it is (strictly) weaker than ampleness of the normal bundle.

Our main result is a Fulton-Hansen type theorem for an irreducible subvariety V of an abelian variety: *the dimension of the "secant variety" of V along a subvariety S* (defined as $V - S$), *and that of its "tangential variety" along S* (defined in the smooth case as the union of the projectivized tangent spaces to V at points of S, translated at the origin) *differ by* 1. Corollaries include a new proof of the finiteness of the Gauss map and an estimate on the ampleness of the normal bundle of a smooth geometrically non-degenerate subvariety.

We also complement Sommese's work with a new Barth-Lefschetz theorem for subvarieties of abelian varieties whose proof is based on an idea of Schneider and Zintl. Let C be a smooth curve in an abelian variety X; we apply this result to give an estimate on the dimension of the singular locus of $C + \cdots + C$ in X.

We work over the field of complex numbers.

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1. Geometrically non-degenerate subvarieties

Recall ([S1]) that a line bundle L on an irreducible projective variety V is k-ample if, for some $m > 0$, the line bundle L^m is generated by its global sections and the fibers of the associated map $\phi_{L^m}: V \to \mathbf{P}^N$ are all of dimension $\leqq k$. A vector bundle E on V is k-ample if the line bundle $\mathcal{O}_{\mathbf{P}E^*}(1)$ is k-ample. Ordinary ampleness coincides with 0-ampleness.

Let X be an abelian variety and let V be an irreducible variety with a morphism $f: V \to X$. Let V^0 be the open set of smooth points of V at which f is unramified. Define the normal bundle to f as the vector bundle on V^0 quotient of $f^*(TX)_{|V^0}$ by TV^0.

For any $x \in X$, let τ_x be the translation by x. For any $v \in V$, the differential of the map $\tau_{-f(v)} f$ at v is a linear map $T_v V \to T_0 X$, which we will simply denote by f_*.

Proposition 1.1. *Under the above assumptions, let S be a complete irreducible subvariety of V^0. The following properties are equivalent:*

(i) *the restriction to S of the normal bundle to f is k-ample;*

(ii) *for any hyperplane H in $T_0 X$, the set $\{s \in S \mid f_*(T_s V) \subset H\}$ has dimension $\leqq k$.*

Proof. Let N be the restriction to S of the normal bundle to f and let

$$\iota: \mathbf{P}N^* \to \mathbf{P}f^*(T^*X)_{|S}$$

be the canonical injection. The morphism

$$\phi: \mathbf{P}N^* \xrightarrow{\iota} \mathbf{P}f^*(T^*X)_{|S} \simeq \mathbf{P}T_0^*X \times S \xrightarrow{\mathrm{pr}_1} \mathbf{P}T_0^*X$$

satisfies

$$\phi^* \mathcal{O}_{\mathbf{P}T_0^*X}(1) = \iota^* \mathcal{O}_{\mathbf{P}f^*(T^*X)_{|S}}(1) = \mathcal{O}_{\mathbf{P}N^*}(1)\,.$$

It follows that N is k-ample if and only if the fibers of ϕ have dimension $\leqq k$ ([S1], prop. 1.7). The proposition follows, since the restriction of the projection $\mathbf{P}N^* \to S$ to any fiber of ϕ is injective. □

When X is simple, the normal bundle to any smooth subvariety of X is ample ([H]). More generally, the normal bundle to any smooth subvariety of X is k-ample, where k is the maximum dimension of a proper abelian subvariety of X ([S1], prop. 1.20).

Following Ran, we will say that a d-dimensional irreducible subvariety V of X is *geometrically non-degenerate* if the kernel of the restriction $H^0(X, \Omega_X^d) \to H^0(V_{\mathrm{reg}}, \Omega_{V_{\mathrm{reg}}}^d)$ contains no non-zero decomposable forms. This property holds if and only if for any abelian variety Y quotient of X, the image of V in Y either is Y or has same dimension as V ([R1], lemma II.12).

Examples. (1) A divisor is geometrically non-degenerate if and only if it is ample; a curve is geometrically non-degenerate if and only if it generates X. Any geometrically non-degenerate subvariety of positive dimension generates X, but the converse is false in general. However, any irreducible subvariety of a *simple* abelian variety is geometrically non-degenerate.

(2) If ℓ is a polarization on X and V is an irreducible subvariety of X with class a rational multiple of ℓ^c, it follows from [R1], cor. II.2 and II.3 that V is non-degenerate in the sense of [R1], II, hence geometrically non-degenerate. In particular, the subvarieties $W_d(C)$ of the Jacobian of a curve C are geometrically non-degenerate; it can be checked that their normal bundle is ample when they are smooth (use prop. 1.1).

We generalize this notion as follows. Let k be a non-negative integer.

Definition 1.2. An irreducibe subvariety V of an abelian variety X is k-geometrically non-degenerate if and only if for any abelian variety Y quotient of X, the image of V in Y either is Y or has dimension $\geqq \dim(V) - k$.

Proposition 1.3. *In an abelian variety, any smooth irreducible subvariety with k-ample normal bundle is k-geometrically non-degenerate.*

Proof. Let $\pi : X \to Y$ be a quotient of X such that $\pi(V) \neq Y$. The tangent spaces to V along a smooth fiber of $\pi_{|V}$ are all contained in a fixed hyperplane, hence general fibers of $\pi_{|V}$ have dimension $\leqq k$ by prop. 1.1. □

The converse is not true, as the construction sketched below shows, but a partial converse will be obtained in 2.3. Roughly speaking, if Y is a quotient of X, and if the image W of V in Y is not Y, k-geometrical non-degeneracy requires that the general fibers of $V \to W$ be of dimension $\leqq k$, whereas k-ampleness of the normal bundle requires that every fiber of $V \to W$ be of dimension $\leqq k$.

Let L_E be an ample line bundle on an elliptic curve E, with linearly independent sections s_1, s_2 defining a morphism $E \to \mathbf{P}^1$ with ramification points $(e_1, 1), \dots, (e_4, 1) \in \mathbf{P}^1$. Let L_Y be an ample line bundle on a simple abelian variety Y of dimension $\geqq 3$, with linearly independent sections t_1, t_2, t_3 such that $\mathrm{div}(t_3)$, $F = \mathrm{div}(t_1) \cap \mathrm{div}(t_2) \cap \mathrm{div}(t_3)$ and $\mathrm{div}(e_i t_1 + t_2) \cap \mathrm{div}(t_3)$ are smooth for $i = 1, \dots, 4$ (such a configuration can be constructed using results from [D2]). Set $X = E \times Y$ and define a subvariety V of X by the equations $s_1 t_1 + s_2 t_2 = t_3 = 0$; then V is smooth of codimension 2, geometrically non-degenerate, but its normal bundle is not ample (for all $e \in E$ and $f \in F$, one has $T_{(e,f)} V \subset T_e E \oplus T_f(\mathrm{div}(t_3))$), only 1-ample (cor. 2.3).

Proposition 1.4. *Let X be an abelian variety and let V and W be irreducible subvarieties of X. Define a morphism $\phi : V^r \times W \to X^r$ by $\phi(v_1, \dots, v_r, w) = (v_1 - w, \dots, v_r - w)$. If V is k-geometrically non-degenerate,*

$$\dim \phi(V^r \times W) \geqq \min\big(r \dim(X), r \dim(V) + \dim(W) - k\big).$$

Proof. Assume first $r \dim(V) + \dim(W) - k \geqq r \dim(X)$. Let $\pi : X \to X/K$ be a quotient of X. I claim that $r \dim \pi(V) + \dim \pi(W) \geqq r \dim(X/K)$. If $\pi(V) = X/K$, this is obvious; otherwise, we have $\dim \pi(V) \geqq \dim(V) - k_0$, where $k_0 = \min(k, \dim(K))$, hence

$$\begin{aligned} r \dim \pi(V) + \dim \pi(W) &\geqq r\big(\dim(V) - k_0\big) + \dim(W) - \dim(K) \\ &\geqq r \dim(X) + k - r k_0 - \dim(K) \geqq r \dim(X/K). \end{aligned}$$

It follows that $(V, \dots, V, W)$ (where V is repeated r times) fills up X in the sense of [D1], (1.10); th. 2.1 of loc.cit. then implies that ϕ is onto.

Assume now $s = r\,\mathrm{codim}(V) - \dim(W) + k > 0$; let C be an irreducible curve in X that generates X. Let W' be the sum of W and s copies of C; then

$$r\dim(V) + \dim(W') - k = r\dim(X)$$

and the first case shows that the sum of the image of ϕ and s curves is X^r. The proposition follows. □

We obtain a nice characterization of k-geometrically non-degenerate varieties.

Corollary 1.5. *An irreducible subvariety V of an abelian variety X is k-geometrically non-degenerate if and only if it meets any subvariety of X of dimension* $\geqq \mathrm{codim}(V) + k$.

Proof. Assume that V meets any subvariety of X of dimension $\geqq \mathrm{codim}(V) + k$ and let $\pi : X \to Y$ be a quotient of X. If $\pi(V) \neq Y$, there exists a subvariety W of Y of dimension $\dim(Y) - \dim \pi(V) - 1$ that does not meet $\pi(V)$. Since V does not meet $\pi^{-1}(W)$,

$$\mathrm{codim}(V) + k > \dim \pi^{-1}(W) = \dim(X) - \dim \pi(V) - 1$$

hence $\dim \pi(V) \geqq \dim(V) - k$ and V is k-geometrically non-degenerate. Conversely, assume V is k-geometrically non-degenerate; let W be an irreducible subvariety of X of dimension $\geqq \mathrm{codim}(V) + k$. Proposition 1.4 shows that $V - W = X$, hence V meets W. □

2. A Fulton-Hansen-type result

Fulton and Hansen proved in [FH] (cf. also [FL1], [Z1], [Z2]) a beautiful result that relates the dimension of the tangent variety and that of the secant variety of a subvariety of a projective space. We prove an analogous result for a subvariety of an abelian variety.

Let X be an abelian variety and let V be a variety with a morphism $f: V \to X$. Recall that f is unramified along a subvariety S of V if $\Delta_S = \{(v, s) \in V \times S \mid v = s\}$ is an open subscheme of $V \times_X S$. Following [FL1], we will say that f is *weakly unramified* along S if Δ_S is a connected component of $V \times_X S$, ignoring scheme structures. In that case, if $p: V \times S \to X$ is the morphism defined by $p(v, s) = f(v) - f(s)$ and $\varepsilon : \tilde{X} \to X$ is the blow-up of the origin, there exists a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{p}} & \tilde{X} \\ \alpha\downarrow & & \varepsilon\downarrow \\ V \times S & \xrightarrow{p} & X \end{array}$$

where α is the blow-up of $V \times_X S$. Let E be the exceptional divisor above $\Delta_S \subset V \times S$ and set $T(V, S) = \tilde{p}(E)$. It is a subscheme of $\mathbf{P}T_0 X$ equal to $\bigcup_{s \in S} \mathbf{P}f_*(T_s V)$ when V is smooth along S and f is unramified along S. Loosely speaking, $T(V, S)$ is the set of limits in $\tilde{X}$

of $(f(v)-f(s))$, as $v \in V$ and $s \in S$ converge to the same point. Obviously, $\dim T(V,S) < \dim(f(V)-f(S))$.

Theorem 2.1. *Let X be an abelian variety and let V be an irreducible projective variety with a morphism $f: V \to X$. Let S be a complete irreducible subvariety of V along which f is weakly unramified. Then* $\dim T(V,S) = \dim(f(V)-f(S)) - 1$.

We begin with a lemma.

Lemma 2.2. *Let C be an irreducible projective curve with a morphism $g: C \to X$ such that $g(C)$ is a smooth curve through the origin. Assume that g is unramified at some point $c_0 \in C$ with $g(c_0) = 0$ and that $\mathbf{P}T_0 g(C) \notin T(V,S)$. The morphism $h: V \times C \to X$ defined by $h(v,c) = f(v) - g(c)$ is weakly unramified along $S \times \{c_0\}$ and $T(V \times C, S \times \{c_0\})$ is contained in the cone over $T(V,S)$ with vertex $\mathbf{P}T_0 g(C)$.*

One can prove that $T(V \times C, S \times \{c_0\})$ is actually equal to the cone.

Proof. Let Γ be a smooth irreducible curve, let γ_0 be a point on Γ and let

$$q = (q_1, q_2, q_3) : \Gamma \to (V \times C) \times_X S$$

be a morphism with $q(\gamma_0) = (s_0, c_0, s_0)$. We need to prove that $q(\Gamma) \subset \Delta'_S$, where

$$\Delta'_S = \{(s, c_0, s) \mid s \in S\};$$

since Δ_S is a connected component of $V \times_X S$, it suffices to show that q_2 is constant. Suppose the contrary; then (q_1, q_3) lifts to a morphism $\tilde{q}_{13} : \Gamma \to \tilde{Y}$ and g to a morphism $\tilde{g} : C \to \tilde{X}$. Since $p(q_1, q_3) = g q_2$, one has $\tilde{p}\tilde{q}_{13} = \tilde{g} q_2$ hence $\tilde{g}(c_0) = \tilde{p}(\tilde{q}_{13}(\gamma_0)) \in T(V,S)$. This contradicts the hypothesis since $\tilde{g}(c_0)$ is the point $\mathbf{P}T_0 g(C)$ of $\mathbf{P}T_0 X$. This proves the first part of the lemma.

The second part is similar: let $\tilde{Z} \to (V \times C) \times S$ be the blow-up of $(V \times C) \times_X S$, let Γ be a smooth irreducible curve with a point $\gamma_0 \in \Gamma$ and let $\tilde{q} : \Gamma \to \tilde{Z}$ be a morphism such that $\tilde{q}(\gamma_0)$ is in the exceptional divisor above Δ'_S. Write $q = \alpha\tilde{q} = (q_1, q_2, q_3)$ and keep the same notation as above. Then $pq(\Gamma)$ is contained in the surface $pq_{13}(\Gamma) - g(C)$ hence $\tilde{p}\tilde{q}(\gamma_0)$ belongs to the line in $\mathbf{P}T_0 X$ through $\tilde{p}(\tilde{q}_{13}(\gamma_0))$ and $\tilde{g}(c_0) = \mathbf{P}T_0 g(C)$. This proves the lemma. □

Proof of the theorem. We proceed by induction on the codimension of $f(V) - f(S)$. Assume $f(V) - f(S) = X$; if $T(V,S) \neq \mathbf{P}T_0 X$, pick a point $u \notin T(V,S)$ and a smooth projective curve C' in X tangent to u at 0, and such that the restriction induces an injection $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(C')$. Let C be a smooth curve with a connected ramified double cover $g: C \to C'$ unramified at a point c_0 above 0; the map $\mathrm{Pic}^0(C') \to \mathrm{Pic}^0(C)$ induced by g is injective.

Since p is surjective, C' generates X and C is smooth, th. 3.6 of [D1] implies that $(V \times S) \times_X C$ is connected. If $h: V \times C \to X$ is defined by $h(v,c) = f(v) - g(c)$, it follows that $(V \times C) \times_X S$ is also connected. On the other hand, the lemma implies that the set $\{(s, c_0, s) \mid s \in S\}$ is a connected component of, hence is equal to, $(V \times C) \times_X S$. It follows that $h^{-1}(f(S)) = S \times \{c_0\}$. Since $g^{-1}(0)$ consists of 2 distinct points, this is absurd, hence $T(V, S) = \mathbf{P}T_0 X$ and the theorem holds in this case.

Assume now $f(V) - f(S) \neq X$. Take a curve C' as above; by the lemma, the morphism $f' : V \times C' \to X$ defined by $f'(v, c') = f(v) + c'$ is weakly unramified along $S \times \{0\}$, and $\dim T(V \times C', S \times \{0\}) \leqq \dim T(V, S) + 1$. It follows from the induction hypothesis that

$$\dim T(V, S) \geqq \dim\big(f(V) + C' - f(S)\big) - 2 = \dim\big(f(V) - f(S)\big) - 1 ,$$

which proves the theorem. □

The following corollary provides a partial converse to prop. 1.3.

Corollary 2.3. *Let X be an abelian variety of dimension n and let V be an irreducible projective variety of dimension d with a morphism $f: V \to X$ such that $f(V)$ is k-geometrically non-degenerate. Let V^0 be the open set of smooth points of V at which f is unramified. The restriction of the normal bundle to f to any complete irreducible subvariety S of V^0 is $(n - d - 1 + k)$-ample.*

Proof. By prop. 1.1, we must show that for any hyperplane H in $T_0 X$, any irreducible component S_H of $\{s \in S \mid f_*(T_v V) \subset H\}$ has dimension $\leqq n - d - 1 + k$. But $T(V, S_H)$ is contained in $\mathbf{P}H$ and the theorem gives $f(V) - f(S_H) \neq X$. Since f is unramified along S_H and $f(V)$ is k-geometrically non-degenerate, prop. 1.4 implies that $f(V) - f(S_H)$ has dimension $\geqq d + \dim(S_H) - k$; this proves the corollary. □

It should be noted that the corollary also follows from the main result of [Z3] (cor. 1), whose proof is unfortunately so sketchy (to say the least) that I could not understand it.

Corollary 2.4. *Let X be an abelian variety and let V be an irreducible projective variety with a morphism $f: V \to X$. Let L be a linear subspace of $T_0 X$ and let S be a complete irreducible subvariety of V along which f is unramified. Assume that*

$$\dim\big(f_*(T_s V) \cap L\big) < m$$

for all $s \in S$, and let $\Delta_{f(S)}$ be the small diagonal in $f(S)^m$. Then

$$\dim\big(f(V)^m - \Delta_{f(S)}\big) < m \dim(X) - \dim(L) + m .$$

In particular, if $m \leqq \dim(L)$ and $f(V)$ is k-geometrically non-degenerate,

$$m \dim(V) + \dim(S) < m \dim(X) - \dim(L) + m + k .$$

Proof. Let $r = \dim(L)$; the variety $N = \{[t_1, \dots, t_m] \in \mathbf{P}(L^m) \mid t_1 \wedge \cdots \wedge t_m = 0\}$ has codimension $r - m + 1$ in $\mathbf{P}(L^m)$. Consider the morphism $f^m : V^m \to X^m$ and the subvariety Δ_S of V^m. The hypothesis implies that in $\mathbf{P}(T_0 X^m)$, the intersection of $T(V^m, \Delta_S)$ and $\mathbf{P}(L^m)$ is contained in N. It follows that

$$\dim T(V^m, \Delta_S) \leqq \dim(N) + \dim \mathbf{P}(T_0 X^m) - \dim \mathbf{P}(L^m)$$
$$= \dim \mathbf{P}(T_0 X^m) - (r - m + 1).$$

The first inequality of the corollary follows from th. 2.1, and the second from prop. 1.4. □

3. Applications to the Gauss map

We keep the same setting: X is an abelian variety and V an irreducible projective variety of dimension d with a morphism $f: V \to X$. Let V^0 be the open set of smooth points of V at which f is unramified; define the *Gauss map* $\gamma: V^0 \to G(d, T_0 X)$ by $\gamma(v) = f_*(T_v V)$. The following result was first proved by Ran ([R2]), and by Abramovich ([A]) in all characteristics.

Proposition 3.1. *Let X be an abelian variety and let V be an irreducible projective variety with a morphism $f: V \to X$. If S is a complete irreducible variety contained in a fiber of the Gauss map, $f(V)$ is stable by translation by the abelian variety generated by $f(S)$. In particular, the Gauss map of a smooth projective subvariety of X invariant by translation by no non-zero abelian subvariety of X is finite.*

Proof. Under the hypothesis of the proposition, $T(V, S)$ has dimension $\dim(V) - 1$; th. 2.1 implies that $f(V) - f(S)$ is a translate of $f(V)$, hence the proposition. □

For any linear subspace L of $T_0 X$ and any integer $m \leqq \dim(L)$, let $\Sigma_{L,m}$ be the Schubert variety $\{M \in G(d, T_0 X) \mid \dim(L \cap M) \geqq m\}$; its codimension in $G(d, T_0 X)$ is $m(\operatorname{codim}(L) - d + m)$.

Proposition 3.2. *Let X be an abelian variety and let V be an irreducible projective variety of dimension d with a morphism $f: V \to X$ such that $f(V)$ is k-geometrically non-degenerate. Let $\gamma: V^0 \to G(d, T_0 X)$ be the Gauss map, let L be a linear subspace of $T_0 X$ and let m be an integer $\leqq \dim(L)$. Any complete subvariety S of V^0 of dimension $\geqq \operatorname{codim} \Sigma_{L,m} + (m-1)(\dim(L) - m) + k$ meets $\gamma^{-1}(\Sigma_{L,m})$.*

Proof. Apply cor. 2.4. □

The hypothesis could probably be weakened to $\dim(S) \geqq \operatorname{codim} \Sigma_{L,m} + k$ (see next proposition); the proposition gives that for $m = 1$ or $\dim(L)$. The corresponding Schubert varieties are $\Sigma_{L,1} = \{M \in G(d, T_0 X) \mid L \cap M \notin 0\}$ and $\Sigma_{L,\dim(L)} = \{M \in G(d, T_0 X) \mid L \subset M\}$.

More generally, a result of Fulton and Lazarsfeld imposes strong restrictions on the image of the Gauss map of smooth subvarieties with ample normal bundle which I believe should also hold for geometrically non-degenerate subvarieties.

Proposition 3.3. *Let X be an abelian variety and let V be a smooth irreducible projective variety of dimension d with an unramified morphism $f: V \to X$ and Gauss map $\gamma: V \to G(d, T_0 X)$. Assume that the normal bundle to f is ample; any subvariety S of $\gamma(V)$ meets any subvariety of $G(d, T_0 X)$ of codimension $\leqq \dim(S)$.*

Proof. If Q is the universal quotient bundle on $G(d, T_0 X)$, the pull-back $\gamma^*(Q)$ is isomorphic to the normal bundle f^*TX/TV, hence is ample. It follows from [FL2] that for each Schubert variety Σ_λ of codimension m in $G(d, T_0 X)$ and each irreducible subvariety S of V of dimension m, one has $\int_S \gamma^*[\Sigma_\lambda] > 0$. Now the class of any irreducible subvariety Z of $G(d, T_0 X)$ of codimension m is a linear combination with non-negative coefficients (not all zero) of the Schubert classes; this implies $\int_S \gamma^*[Z] > 0$, hence $S \cap \gamma^{-1}(Z) \neq \emptyset$. □

Regarding the Gauss map of a smooth subvariety of an abelian variety, Sommese and Van de Ven also proved in [SV] a strong result for higher relative homotopy groups of pullbacks of *smooth* subvarieties of the Grassmannian.

4. A Barth-Lefschetz-type result

Sommese has obtained very complete results on the homotopy groups of *smooth* subvarieties of an abelian variety. For example, he proved in [S2] that if V is a smooth subvariety of dimension d of an abelian variety X, with k-ample normal bundle, $\pi_q(X, V) = 0$ for $q \leqq 2d - n - k + 1$. For arbitrary subvarieties, we have the following:

Theorem 4.1. *Let X be an abelian variety and let V be a k-geometrically non-degenerate normal subvariety of X of dimension $> \frac{1}{2}(\dim(X) + k)$. Then $\pi_1^{\mathrm{alg}}(V) \simeq \pi_1^{\mathrm{alg}}(X)$.*

Proof. The case $k = 0$ is cor. 4.2 of [D1]. The general case is similar, since the hypothesis implies that the pair (V, V) satisfies condition $(*)$ of [D1]. □

Going back to smooth subvarieties, I will give an elementary proof of (a slight improvement of) the cohomological version of Sommese's theorem, based on the following vanishing theorem ([LP]) and the ideas of [SZ].

Vanishing theorem 4.2 (Le Potier, Sommese). *Let E be a k-ample rank r vector bundle on a smooth irreducible projective variety V of dimension d. Then*

$$H^q(V, E^* \otimes \Omega_V^p) = 0 \quad \textit{for} \quad p + q \leqq d - r - k\,.$$

Recall also the following elementary lemma from [SZ]:

Lemma 4.3. *Let $0 \to F \to E_0 \to E_1 \to \cdots \to E_k \to 0$ be an exact sequence of sheaves on a scheme V. Assume $H^s(V, E_i) = 0$ for $0 \leqq i < k$ and $s \leqq q$; then $H^q(V, F) \simeq H^{q-k}(V, E_k)$.*

Theorem 4.4. *Let V be a smooth irreducible subvariety of dimension d of an abelian n-fold X and let $\mathscr{L}$ be a nef line bundle on V. Assume that the normal bundle N of V in X is a direct sum $\bigoplus N_i$, where N_i is k_i-ample of rank r_i. For $j > 0$,*

$$H^q(V, S^j N^* \otimes \mathscr{L}^{-1}) = 0 \quad \textit{for} \quad q \leqq d - \max(r_i + k_i)\,.$$

Proof. Since $S^j N^*$ is a direct summand of $S^{j-1} N^* \otimes N^*$, it is enough to show, by induction on j, that $H^q(V, S^j N^* \otimes N_i^* \otimes \mathscr{L}^{-1})$ vanishes for $j \geqq 0$ and $q \leqq d - r_i - k_i$.

Since $N_i \otimes \mathscr{L}$ is k_i-ample, the case $j=0$ follows from Le Potier's theorem. For $k \geqq 1$, tensor the exact sequence

$$0 \to S^j N^* \to S^{j-1} N^* \otimes \Omega^1_{X|V} \to \cdots \to \Omega^j_{X|V} \to \Omega^j_V \to 0$$

by $N_i^* \otimes \mathscr{L}^{-1}$. Since Ω^1_X is trivial, the induction hypothesis and the lemma give

$$H^q(V, S^j N^* \otimes N_i^* \otimes \mathscr{L}^{-1}) \simeq H^{q-j}(V, \Omega^j_V \otimes N_i^* \otimes \mathscr{L}^{-1}),$$

and this group vanishes for $q \leqq d - r_i - k_i$ by Le Potier's theorem. □

We are now ready to prove our version of Sommese's result.

Theorem 4.5. *Let V be a smooth irreducible subvariety of dimension d of an abelian n-fold X. Assume that its normal bundle is a direct sum $\bigoplus N_i$, where N_i is k_i-ample of rank r_i. Then:*

(a) $H^q(X, V; \mathbb{C}) = 0$ *for* $q \leqq d - \max(r_i + k_i) + 1$;

(b) *for all nonzero elements P of* $\mathrm{Pic}^0(V)$, *the cohomology groups $H^q(V, P)$ vanish for* $q \leqq d - \max(r_i + k_i)$.

Remarks 4.6. (1) It is likely that (a) should hold for cohomology with integral coefficients.

(2) If the normal bundle is k-ample, we get $H^q(X, V; \mathbb{C}) = 0$ for $q \leqq 2d - n - k + 1$. If the normal bundle is a sum of ample line bundles, $H^q(X, V; \mathbb{C}) = 0$ for $q \leqq d$; in particular, the restriction $H^0(X, \Omega^d_X) \to H^0(V, \Omega^d_V)$ is injective and V is non-degenerate in the sense of [R1], II, hence also geometrically non-degenerate.

(3) By [GL], $H^q(V, P) = 0$ for P outside of a subset of codimension $\geqq d - q$ of $\mathrm{Pic}^0(V)$. By [S], this subset is a union of translates of abelian subvarieties of X by torsion points.

Proof of the theorem. For (a), it is enough by Hodge theory to study the maps

$$H^i(X, \Omega^j_X) \to H^i(V, \Omega^j_{X|V}) \xrightarrow{\psi} H^i(V, \Omega^j_V).$$

Since Ω^j_X is trivial, we only need to look at $\phi: H^i(X, \mathcal{O}_X) \to H^i(V, \mathcal{O}_V)$ and ψ. We begin with ψ. We may assume $j > 0$. Let M_j be the kernel of the surjection

$$\Omega^j_{X|V} \to \Omega^j_V \to 0.$$

The long exact sequence of th. 4.4 gives

$$0 \to S^j N^* \to S^{j-1} N^* \otimes \Omega^1_{X|V} \to \cdots \to N^* \otimes \Omega^{j-1}_{X|V} \to M_j \to 0.$$

The lemma and the theorem then yield

$$H^i(V, M_j) \simeq H^{i+j-1}(V, S^j N^*) = 0$$

for $i+j-1 \leqq d - \max(k_i + r_i)$, since $j > 0$. This implies that ψ has the required properties.

For $i = 0$, the map ψ is $H^0(X, \Omega_X^j) \to H^0(V, \Omega_V^j)$. By Hodge symmetry, this proves that ϕ also has the required properties, hence the first point.

For (b), we may assume $d - \max(k_i + r_i) \geqq 1$, in which case the first point implies $\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^0(V)$. Let $P \in \mathrm{Pic}^0(X)$ be non-zero; the same proof as above yields

$$H^0(V, \Omega_V^q \otimes P_{|V}) = 0 \quad \text{for} \quad q \leqq d - \max(k_i + r_i)\,.$$

The theorem follows from the existence of an anti-linear isomorphism

$$H^0(V, \Omega_V^q \otimes P_{|V}) \simeq H^q(V, P_{|V}^*)$$

([GL]). □

I will end this section with an amusing consequence of th. 4.5. If C is a curve in an abelian variety X, write C_d for the subvariety $C + \cdots + C$ (d times) of X. Recall that if C is general of genus n and $d < n$, the singular locus of $C_d = W_d(C)$ in the Jacobian JC has dimension $2d - n - 2$.

Proposition 4.7. *Let X be an abelian variety of dimension n and let C be a smooth irreducible curve in X. Assume that C generates X and that its Gauss map is birational onto its image. For $d < n$, the singular locus of C_d has dimension $\geqq 2d - n - 1$ unless X is isomorphic to the Jacobian of C and C is canonically embedded in X.*

Proof. Let $\gamma : C_{\mathrm{reg}} \to \mathbf{P}T_0 X$ be the Gauss map and let $\pi : C^{(d)} \to C_d$ be the sum map. The image of the differential of π at the point $(c_1 \bullet \cdots \bullet c_d)$ is the linear subspace of $T_0 X$ generated by $\gamma(c_1), \ldots, \gamma(c_d)$. Since C generates X, the curve $\gamma(C)$ is non-degenerate; it follows that for $c_1, \ldots, c_d$ general, the points $\gamma(c_1), \ldots, \gamma(c_d)$ span a $(d-1)$-plane whose intersection with the curve $\gamma(C)$ consists only of these points. Thus π is birational. Moreover, if $x = c_1 + \cdots + c_d$ is smooth on C_d, then $\gamma(c_i) \in \tau_x^*(T_x C_d) \cap \gamma(C)$ hence $\pi^{-1}(x)$ is finite. By Zariski's main theorem, π induces an isomorphism between $\pi^{-1}((C_d)_{\mathrm{reg}})$ and $(C_d)_{\mathrm{reg}}$.

Let s be the dimension of the singular locus of C_d and assume $-1 \leqq s \leqq 2d - n - 2$. Let L be a very ample line bundle on X; the intersection W of C_d with $(s+1)$ general elements of $|L|$ is smooth of dimension $\geqq 2$ and contained in $(C_d)_{\mathrm{reg}}$. If H is a hyperplane in $T_0 X$ and $x = c_1 + \cdots + c_d \in W$, the inclusion $T_x C_d \subset H$ implies $\gamma(c_i) \in \mathbf{P}H \cap \gamma(C)$; the restriction of $N_{C_d/X}$ to W is ample by prop. 1.1. Since $N_{W/X}$ is the direct sum of this restriction and of $(s+1)$ copies of L, the restriction $H^1(X, \mathcal{O}_X) \to H^1(W, \mathcal{O}_W)$ is bijective by th. 4.5. On the other hand, the line bundle $\pi^* L$ is nef and big on $C^{(d)}$, hence the Kawamata-Viehweg vanishing theorem ([K], [V]) implies

$$H^1(C^{(d)}, \mathcal{O}_{C^{(d)}}) \subset H^1(\pi^{-1}(W), \mathcal{O}_{\pi^{-1}(W)}) \simeq H^1(W, \mathcal{O}_W)\,.$$

Since $H^1(C, \mathcal{O}_C) \simeq H^1(C^{(d)}, \mathcal{O}_{C^{(d)}})$ ([M]), we get $h^1(C, \mathcal{O}_C) \leqq h^1(X, \mathcal{O}_X)$ and there must be equality because C generates X. Thus, the inclusion $C \subset X$ factors through an isogeny $\phi : JC \to X$. Since π is birational, the inverse image $\phi^{-1}(C_d)$ is the union of $\deg(\phi)$ translates of $W_d(C)$. But any two translates of $W_d(C)$ meet along a locus of dimension $\geqq 2d - n > s$, hence ϕ is an isomorphism. □

References

[A] *D. Abramovich*, Subvarieties of semiabelian varieties, Comp. Math. **90** (1994), 37–52.

[D1] *O. Debarre*, Théorèmes de connexité et variétés abéliennes, Am. J. Math. **117** (1995), 787–805.

[D2] *O. Debarre*, Sur les variétés abéliennes dont le diviseur thêta est singulier en codimension 3, Duke Math. J. **56** (1988), 221–273.

[FH] *W. Fulton, J. Hansen*, A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings, Ann. Math. **110** (1979), 159–166.

[FL1] *W. Fulton, R. Lazarsfeld*, Connectivity and its Applications in Algebraic Geometry, in: Algebraic Geometry, Proceedings of the Midwest Algebraic Geometry Conference, Chicago 1980, Springer Lecture Notes **862**.

[FL2] *W. Fulton, R. Lazarsfeld*, Positive polynomials for ample vector bundles, Ann. Math. **118** (1983), 35–60.

[GL] *M. Green, R. Lazarsfeld*, Deformation Theory, Generic Vanishing Theorems and Some Conjectures of Enriques, Catanese and Beauville, Invent. Math. **90** (1987), 389–407.

[H] *R. Hartshorne*, Ample Vector Bundles on Curves, Nagoya Math. J. **43** (1971), 73–89.

[K] *Y. Kawamata*, A Generalisation of Kodaira-Ramanujan's Vanishing Theorem, Math. Ann. **261** (1982), 43–46.

[LP] *J. Le Potier*, Annulation de la cohomologie à valeurs dans un fibré vectoriel holomorphe positif de rang quelconque, Math. Ann. **218** (1975), 35–53.

[M] *I. Macdonald*, Symmetric Products of an Algebraic Curve, Topology **1** (1962), 319–343.

[R1] *Z. Ran*, On subvarieties of abelian varieties, Invent. Math. **62** (1981), 459–479.

[R2] *Z. Ran*, The structure of Gauss-like maps, Comp. Math. **52** (1984), 171–177.

[SZ] *M. Schneider, J. Zintl*, The theorem of Barth-Lefschetz as a consequence of Le Potier's vanishing theorem, Manuscr. Math. **80** (1993), 259–263.

[S] *C. Simpson*, Subspaces of Moduli Spaces of Rank One Local Systems, Ann. Sc. Ecole Norm. Sup. **26** (1993), 361–401.

[S1] *A. Sommese*, Submanifolds of Abelian Varieties, Math. Ann. **233** (1978), 229–256.

[S2] *A. Sommese*, Complex Subspaces of Homogeneous Complex Manifolds II. Homotopy Results, Nagoya Math. J. **86** (1982), 101–129.

[SV] *A. Sommese, A. Van de Ven*, Homotopy groups of pullbacks of varieties, Nagoya Math. J. **102** (1986), 79–90.

[V] *E. Viehweg*, Vanishing Theorems, J. reine angew. Math. **335** (1982), 1–8.

[Z1] *F. Zak*, Structure of Gauss Maps, Funct. Anal. Appl. **21** (1987), 32–41.

[Z2] *F. Zak*, Tangents and Secants of Algebraic Varieties, Transl. Math. Monogr. **127**, A.M.S., Providence 1993.

[Z3] *F. Zak*, Theorems on Tangents to Subvarieties of Complex Tori, Math. Notes Acad. Sc. USSR **43** (1988), 219–225.

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