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The main purpose of this article is to describe all effective algebraic cycles with minimal cohomology class in the Jacobian of a complex curve. More precisely, let (A,θ) be a complex principally polarized abelian variety of dimension g. For $0\le d\le g$, the cohomology class $\theta_d=\theta^d/d!$ is *minimal*, i.e. non-divisible, in $\mathrm{H}^{2d}(\mathrm{A},\mathbf{Z})$. When (A,θ) is isomorphic to the Jacobian (JC,θ) of a curve C of genus g, the image of the symmetric product $\mathrm{C}^{(g-d)}$ by any Abel-Jacobi map is a subvariety $\mathrm{W}_{g-d}(\mathrm{C})$ of JC with class θ_d ([ACGH], p. 25).

Our main result (theorem 5.1) implies that *any effective algebraic cycle in* JC *with class* θ_d *is a translate of either* $\mathrm{W}_{g-d}(\mathrm{C})$ *or* $-\mathrm{W}_{g-d}(\mathrm{C})$.

For $1<d<g$, I know of only one other family of principally polarized abelian varieties with an effective algebraic cycle with minimal class: in the 5-dimensional intermediate Jacobian JT of a cubic threefold T in $\mathbf{P}^4$, the image by any Abel-Jacobi map of the Fano surface of lines contained in T is a surface with class θ_3 ([CG], [B]).

I suspect that these should be the only examples of effective algebraic cycles with class θ_d on abelian varieties of dimension g, when $1<d<g$. This holds for any g and $d=g-1$ by Matsusaka's criterion ([M]), and for $g=4$ and $d=2$ by a result of Ran ([R1]).

In a second part, the main theorem is used to prove a weak version of this conjecture: *for* $1<d<g$, *the Jacobian locus (resp. the locus of intermediate Jacobians of cubic threefolds) is an irreducible component of the set of principally polarized abelian varietiesof dimension* g *for which* θ_d *(resp.* θ_3 *) is the class of an effective algebraic cycle.* This result was first proved by Barton and Clemens ([BC]) for $g=4$ and $d=2$; a slighty weaker form of it appeared later in [R1] (corollary III.1), but the proof is incomplete.

Throughout this article, we will be working over the field of complex numbers. A variety will be a reduced projective (complex) scheme of finite type.

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I. Subvarieties With Minimal Classes In Jacobians

Let (JC,θ) be the Jacobian of a smooth curve C of genus g. The aim of this part is to prove that any effective algebraic cycle in JC with class θ_d is a translate of either $\mathrm{W}_{g-d}(\mathrm{C})$ or $-\mathrm{W}_{g-d}(\mathrm{C})$.

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1. Non-degenerate subvarieties

This notion was introduced by Ran ([R1], [R2]). Let A be an abelian variety of dimension g and let W be a subvariety of A of pure dimension d. Let W_{reg} be the smooth part of W. We say that W is *non-degenerate* if the restriction map $H^0(A,\Omega^d_A) \longrightarrow H^0(W_{reg},\Omega^d_{W_{reg}})$ is injective. By [R1], lemma II.1, this is equivalent to each one of the following properties:

- the cup-product map $\cdot[W] : H^{d,0}(A) \longrightarrow H^{g,g-d}(A)$ is injective, where $[W] \in H^{g-d,g-d}(A)$ is the cohomology class of W,
- the contraction map $\cdot\{W\} : H^{g,d}(A) \longrightarrow H^{g-d,0}(A)$ is injective, where $\{W\} \in H_{d,d}(A)$ is the homology class of W.

This allows the extension of the definition to effective algebraic cycles. Note that on a principally polarized abelian variety, an effective algebraic cycle with class a multiple of a minimal class is non-degenerate.

2. The property $(\mathcal{P})$

Let V and W be two irreducible subvarieties of an abelian variety A and let $f : V \times W \to A$ be the addition map. We will say that V *has property* $(\mathcal{P})$ *with respect to* W if, for v generic in V, the only irreducible subvariety of $f^{-1}(v+W)$ which dominates both $v+W$ via f, and W via the second projection, is $\{v\} \times W$.

(2.1) Note that this implies that $\{v\} \times W$ is a component of $f^{-1}(v+W)$, hence that the latter has the same dimension as W at some point, hence that f is generically finite onto its image; in particular, $\dim(V)+\dim(W) \leq \dim(A)$. Note also that if V has property $(\mathcal{P})$ with respect to W, then any translate of V has property $(\mathcal{P})$ with respect to any translate of W.

Example 2.2. If C is a curve of genus g, it is not difficult to check that $W_d(C)$ has property $(\mathcal{P})$ with respect to $W_e(C)$ whenever $d+e \leq g$.

Lemma 2.3. – *Let* V *and* W *be two irreducible subvarieties of an abelian variety* A *and let* $g : W \times W \to A$ *be the subtraction map. Then,* V *has property* $(\mathcal{P})$ *with respect to* W *if and only if, for* v *generic in* V, *the only irreducible subvariety of* $g^{-1}(V-v)$ *which dominates both factors* W, *is the diagonal.*

Proof. Let $f : V \times W \to A$ be the addition map. Let v be generic in V and consider the automorphism h of $A \times A$ defined by $h(x,y) = (v+x-y, y)$. One checks that $h\big(g^{-1}(V-v)\big) = f^{-1}(v+W)$. The proposition then follows from the fact that, if Z is a variety contained in $g^{-1}(V-v)$, then $f\big(h(Z)\big) = v + p_1(Z)$ and $p_2\big(h(Z)\big) = p_2(Z)$. ∎

Proposition 2.4. – *Let* V *and* W *be two irreducible subvarieties of an abelian variety* A. *Assume that* V *has property* $(\mathcal{P})$ *with respect to* W. *Then:*

- *the variety* $(-V)$ *has property* $(\mathcal{P})$ *with respect to* W, *of* V *which meets* Ω *, then* U *has property* $(\mathcal{P})$ *with respect to* W *.*

Proof. The first point follows from lemma 2.3. Let $g : \mathrm{W} \times \mathrm{W} \to \mathrm{A}$ be the difference map. Let Ω be the open set of points v in V such that the only irreducible subvariety of $g^{-1}(\mathrm{V} - v)$ which dominates both factors is the diagonal. By the lemma, Ω is dense in V. Take u in $\Omega \cap \mathrm{U}$; since $\mathrm{U} - u \subset \mathrm{V} - u$, any irreducible subvariety of $g^{-1}(\mathrm{U} - u)$ which dominates both factors is also a subvariety of $g^{-1}(\mathrm{V} - u)$, hence is equal to the diagonal. By lemma 2.3, this proves the second point. ∎

Although the following result can be checked directly, it is an easy consequence of lemma 2.3.

Proposition 2.5. – *Let* C *be a curve of genus* g *and let* V *be an irreducible subvariety of* JC *. Assume that* V *has property* $(\mathcal{P})$ *with respect to some* $\mathrm{W}_d(\mathrm{C})$*. Then* V *also has property* $(\mathcal{P})$ *with respect to any* $\mathrm{W}_e(\mathrm{C})$ *for* $0 \leq e \leq d$*.*

Proof. Let $g : \mathrm{W}_d(\mathrm{C}) \times \mathrm{W}_d(\mathrm{C}) \to \mathrm{JC}$ and $h : \mathrm{W}_e(\mathrm{C}) \times \mathrm{W}_e(\mathrm{C}) \to \mathrm{JC}$ be the difference maps. Let v be generic in V, and let Z be an irreducible subvariety of $h^{-1}(v - \mathrm{V})$ which dominates both factors. Then $\{\, z + (w, w) \mid z \in \mathrm{Z} \,,\ w \in \mathrm{W}_{d-e}(\mathrm{C}) \,\}$ is an irreducible subvariety of $g^{-1}(v - \mathrm{V})$ which dominates both factors. By lemma 2.3, it is the diagonal of $\mathrm{W}_d(\mathrm{C}) \times \mathrm{W}_d(\mathrm{C})$, hence Z is the diagonal of $\mathrm{W}_e(\mathrm{C}) \times \mathrm{W}_e(\mathrm{C})$. By lemma 2.3 again, this proves that V has property $(\mathcal{P})$ with respect to $\mathrm{W}_e(\mathrm{C})$. ∎

3. Ran's theorem

Let V and W be two subvarieties of A, of respective pure codimensions d and $g - d$. Assume that $s = \mathrm{V} \cdot \mathrm{W} > 0$. The addition map $\mathrm{V} \times \mathrm{W} \to \mathrm{A}$ is then surjective, hence generically étale. It follows that for x generic in A, the varieties V and $x - \mathrm{W}$ meet transversally at distinct smooth points $v_1, \ldots, v_s$. Let $\mathrm{P}_i : \mathrm{T}_0\mathrm{A} \to \mathrm{T}_0\mathrm{A}$ be the projector with image $\mathrm{T}_{v_i}\mathrm{V}$ and kernel $\mathrm{T}_{x-v_i}\mathrm{W}$. Let $c(\mathrm{V}, \mathrm{W})$ be the endomorphism $\sum_{i=1}^{s} \bigwedge^{g-d} \mathrm{P}_i$ of $\bigwedge^{g-d} \mathrm{T}_0\mathrm{A}$; Ran proves ([R1], theorem 2) that the transposed endomorphism ${}^t c(\mathrm{V}, \mathrm{W})$ of $\bigwedge^{g-d} \mathrm{T}_0^*\mathrm{A} \simeq \mathrm{H}^{g-d,0}(\mathrm{A})$ is equal to the composition:

$$\mathrm{H}^{g-d,0}(\mathrm{A}) \xrightarrow{\cdot[\mathrm{V}]} \mathrm{H}^{g,d}(\mathrm{A}) \xrightarrow{\cdot\{\mathrm{W}\}} \mathrm{H}^{g-d,0}(\mathrm{A}) \,.$$

In particular, when V and W are non-degenerate, $c(\mathrm{V}, \mathrm{W})$ is an automorphism.

The following result of Ran on non-degenerate subvarieties with minimal intersection number motivates the somewhat abstruse definition of "property $(\mathcal{P})$".

Theorem 3.1. (Ran) – *If* V *and* W *are non-degenerate subvarieties of a* g*-dimensional principally polarized abelian variety* (A, θ)*, of respective pure codimensions* d *and* $g - d$*, then* $\mathrm{V} \cdot \mathrm{W} \geq \binom{g}{d}$*. If moreover* W *is irreducible and* $\mathrm{V} \cdot \mathrm{W} = \binom{g}{d}$*, then* V *is irreducible and has property* $(\mathcal{P})$ *with respect to* W *.*

$$\binom{g}{d} = \operatorname{rank}\big(c(\mathrm{V},\mathrm{W})\big) \leq \mathrm{V}\cdot\mathrm{W}\ .$$

If there is equality, it follows from (7.4) in [R2] that W meets at most one component of V. But by [R1], corollary II.6, W meets each component of V. This implies that V is irreducible. To prove property $(\mathcal{P})$, we keep the same notation as above and proceed as in the proof of [R1], theorem 5. If α_i spans the line $\bigwedge^{g-d}\mathrm{T}_{v_i}\mathrm{V}$ in $\bigwedge^{g-d}\mathrm{T}_0\mathrm{A}$, then $\{\alpha_1,\ldots,\alpha_s\}$ (with $s=\binom{g}{d}$) is a basis for $\bigwedge^{g-d}\mathrm{T}_0\mathrm{A}$. The choice of an identification $\bigwedge^n\mathrm{T}_0\mathrm{A}\simeq\mathbf{C}$ induces an isomorphism $\bigwedge^d\mathrm{T}_0\mathrm{A}\simeq\bigwedge^{g-d}\mathrm{T}_0^*\mathrm{A}$. Let β_i be an element of $\bigwedge^d\mathrm{T}_{x-v_i}\mathrm{W}$ such that $\beta_i(\alpha_i)=1$. Then $c(\mathrm{V},\mathrm{W})=\sum_{i=1}^s\alpha_i\otimes\beta_i$. In particular, for $i\neq j$:

$$\beta_i\big(c(\mathrm{V},\mathrm{W})^{-1}(\alpha_j)\big) = 0\ .$$

Fix $v=v_1$ in V and let x vary in $v+\mathrm{W}$, so that $v\in\mathrm{V}\cap(x-\mathrm{W})$. We get:

$$\bigwedge^{d}\mathrm{T}_{x-v_i}\mathrm{W}\wedge c(\mathrm{V},\mathrm{W})^{-1}\big(\bigwedge^{g-d}\mathrm{T}_v\mathrm{V}\big) = 0$$

for $i>1$. Since W is non-degenerate, the points $x-v_1,\ldots,x-v_s$ must therefore describe a proper subvariety of W. This proves the last part of the theorem, since $p_2f^{-1}(x)=\mathrm{W}\cap(x-\mathrm{V})=\{x-v,x-v_1,\ldots,x-v_s\}$. ■

For example, if C is a curve of genus g, the theorem implies that $\mathrm{W}_d(\mathrm{C})$ has property $(\mathcal{P})$ with respect to $\mathrm{W}_{g-d}(\mathrm{C})$ for $d\leq g$. Together with proposition 2.5, this proves the claim of example 2.2.

4. An auxiliary result

Let C be a smooth curve of genus g and let $n>1$. We define a subset $\mathcal{T}_n$ of $\mathrm{C}^{(n)}$ as follows: consider all surjective morphisms $\mathrm{C}\to\mathrm{C}'$ of degree $r>1$, where C' is a smooth irrational curve. Such a morphism induces a map $\psi:\mathrm{C}'\to\mathrm{C}^{(r)}$; let $\mathcal{T}_n$ be the union of all $\psi(\mathrm{C}')+\mathrm{C}^{(n-r)}$, for $r\leq n$, obtained in this way, of the inverse image in $\mathrm{C}^{(n)}$ of $\mathrm{W}^1_n(\mathrm{C})$ and of the diagonal $2\mathrm{C}+\mathrm{C}^{(n-2)}$. For $n\leq g$, it is a proper closed subvariety of $\mathrm{C}^{(n)}$.

Proposition 4.1. – *Let* (JC,θ) *be the Jacobian of a smooth curve* C *of genus* $g\geq 2$ *and let* Z *be a subvariety of* $\mathrm{C}^{(n)}$ *of pure codimension* m. *Then, for any* E *in* $\mathrm{C}^{(n-m)}$, *the variety* Z *meets* $\mathrm{E}+\mathrm{C}^{(m)}$. *Moreover, if there exists* E *in* $\mathrm{C}^{(n-m)}$ *such that* $\mathrm{Z}\cdot(\mathrm{E}+\mathrm{C}^{(m)})=1$, *then either* Z *is contained in* $\mathcal{T}_n$, *or there exist points* $c_1,\ldots,c_m$ *of* C *such that* $\mathrm{Z}=c_1+\ldots+c_m+\mathrm{C}^{(n-m)}$.

Proof. The first part follows from the fact that the cohomology class of $\mathrm{E}+\mathrm{C}^{(m)}$ is the $(n-m)$–fold self-intersection of the cohomology class of the ample divisor $\mathrm{C}^{(n-1)}$ in $\mathrm{C}^{(n)}$ ([ACGH], pp. 309, 310).

Assume now that $\mathrm{Z}\cdot(\mathrm{E}+\mathrm{C}^{(m)})=1$. We first do the case when Z is a curve. We may assume that Z is not contained in $x+\mathrm{C}^{(n-1)}$ for any x in C. Since $\mathrm{Z}\cdot(x+\mathrm{C}^{(n-1)})=1$

that $Z = \{x + \tau(x) \mid x \in C\}$. Let Γ be the curve $\{(x, \tau(x)) \mid x \in C\}$ in $C \times C^{(n-1)}$. If the induced morphism $\phi : \Gamma \to Z$ is not birational, Z is contained in $\mathcal{T}_n$.

Since Z is smooth, we are left with the case where ϕ is an isomorphism. If τ is constant, the proof is over. Assume therefore that $\tau(C)$ is a curve. If $n = 2$, we get $Z = \{x + \tau(x) \mid x \in C\}$, where τ is an involution of C. In particular, Z is contained in $\mathcal{T}_2$. We assume $n \geq 3$ and proceed by induction. Let us show that the curve $\tau(C)$ satisfies the same property as Z. Let x be a point of C and assume that $\tau(y) = x + D$ and $\tau(y') = x + D'$ are both in $x + C^{(n-1)}$ for some y and y' on C. Then, $y + x + D$ and $y' + x + D'$ are on Z hence the hypothesis on Z implies $y + D = y' + D'$. Therefore, either $y = y'$ and $D = D'$, or there exists an element E of $C^{(n-3)}$ such that $D = y' + E$ and $D' = y + E$. Then, both $(y, \tau(y))$ and $(y', \tau(y'))$ are sent by addition to $x + y + y' + E$ hence, since ϕ is an isomorphism, we get again $y = y'$ and $D = D'$. It follows that $\tau(C)$ and $x + C^{(n-2)}$ have a single common point and one checks that the intersection is transverse. We can therefore apply the induction hypothesis to $\tau(C)$: since we have assumed that Z is not contained in any $x + C^{(n-1)}$, the curve $\tau(C)$ is in $\mathcal{T}_{n-1}$. It follows that Z is contained in $C + \mathcal{T}_{n-1}$, hence in $\mathcal{T}_n$. This finishes the proof of the lemma when Z is a curve.

We now do the general case, by induction on the dimension of Z. Let Z of dimension $n - m > 1$ satisfy the hypothesis. For x generic in C, the inverse image of Z by the map $C^{(n-1)} \to C^{(n)}$ which sends D to $x + D$ satisfies the same property. Therefore, either Z is contained in $C + \mathcal{T}_{n-1}$, hence in $\mathcal{T}_n$, or there exists a morphism $\tau : C \to C^{(m)}$ such that:

$$Z = \{\ x + \tau(x) + C^{(n-m-1)} \mid x \in C\ \}.$$

But τ must then be constant. This finishes the proof of the proposition. ∎

5. The main theorem

We now prove our main result.

THEOREM 5.1. – *Let* (JC, θ) *be the Jacobian of a smooth curve* C *of genus* g *and let* V *be an effective non-degenerate algebraic* $(g-d)$*-cycle in* JC *such that* $V \cdot W_d(C) = \binom{g}{d}$. *Then* V *is a translate of either* $W_{g-d}(C)$ *or* $-W_{g-d}(C)$.

We refer to § 1 for the definition of a non-degenerate cycle. Recall that any effective algebraic cycle with class θ_d satisfies the hypotheses of the theorem.

Proof. By replacing any multiple component of V by a sum of translates, we may assume that V is a subvariety. Ran's theorem 3.1 implies that V is irreducible and has property $(\mathcal{P})$ with respect to $W_d(C)$. We first prove by induction on $g - d$ that any $(g-d)$–dimensional irreducible subvariety V of JC which has property $(\mathcal{P})$ with respect to $W_d(C)$ is a translate of some $W_r(C) - W_{g-d-r}(C)$, with $0 \leq r \leq g - d$. This is obvious for $g - d = 0$, hence we assume $g - d > 0$.

For any positive integer e, we define the addition map $f_e^+ : V \times W_e(C) \to JC$ and the subtraction map $f_e^- : V \times W_e(C) \to JC$. When $e \leq d$, proposition 2.5 implies that V

from (2.1) that f_e^+ and f_e^- are finite onto their images.

PROPOSITION 5.2. – *Let* V *be an irreducible subvariety of* JC *of codimension* $d < g$ *having property* $(\mathcal{P})$ *with respect to* $\mathrm{W}_d(\mathrm{C})$. *Then there exists an irreducible subvariety* U *of* JC *such that either* $\mathrm{V} = \mathrm{U} + \mathrm{C}$ *or* $\mathrm{V} = \mathrm{U} - \mathrm{C}$.

Proof. The maps $f_d^{\pm}$ are not birational ([R1], lemma II.15). Let n be the smallest integer such that either f_n^+ or f_n^- is not birational onto its image. Since both the hypotheses and conclusion of the lemma hold for V if and only if they do for $-\mathrm{V}$ (proposition 2.4), we may assume that f_n^+ is not birational onto its image. For v generic in V, there exists an irreducible component Γ of $(f_n^+)^{-1}(v + \mathrm{W}_n(\mathrm{C}))$, distinct from $\{v\} \times \mathrm{W}_n(\mathrm{C})$, which dominates $v + \mathrm{W}_n(\mathrm{C})$. Since V has property $(\mathcal{P})$ with respect to $\mathrm{W}_n(\mathrm{C})$ (proposition 2.5), the projection $p_2(\Gamma)$ is a subscheme G of $\mathrm{W}_n(\mathrm{C})$ of dimension $m < n$. For D generic in G, there exists a subvariety $\mathrm{S_D}$ of V of dimension $n - m$ such that:

$$\mathrm{S_D} + \mathrm{D} \subset v + \mathrm{W}_n(\mathrm{C}) \ .$$

Let E be generic in $\mathrm{W}_m(\mathrm{C})$. By proposition 4.1, the subvariety $\mathrm{E} + \mathrm{W}_{n-m}(\mathrm{C})$ of $\mathrm{W}_n(\mathrm{C})$ meets G at some point $\mathrm{D} = \mathrm{E} + \mathrm{E}'$. For the same reason, the subvariety $\mathrm{S_D} + \mathrm{D} - v$ of $\mathrm{W}_n(\mathrm{C})$ meets $\mathrm{E}' + \mathrm{W}_m(\mathrm{C})$ at r points $\mathrm{E}' + \mathrm{E}_1, \ldots, \mathrm{E}' + \mathrm{E}_r$ with $r > 0$, i.e. there exist points $v_1, \ldots, v_r$ of $\mathrm{S_D} \subset \mathrm{V}$ such that:

$$v_i + (\mathrm{E} + \mathrm{E}') - v = \mathrm{E}' + \mathrm{E}_i \ , \qquad \text{i.e.} \qquad v_i - \mathrm{E}_i = v - \mathrm{E} \ .$$

Since $m < n$, the map $f_m^- : \mathrm{V} \times \mathrm{W}_m(\mathrm{C}) \to \mathrm{JC}$ is birational onto its image. Therefore, since v and E are generic, we must have $\mathrm{E}_i = \mathrm{E}$ for all i and $v \in \mathrm{S_D}$. In other words:

$$(\mathrm{S_D} + \mathrm{D} - v) \cap \big(\mathrm{E}' + \mathrm{W}_m(\mathrm{C})\big) = \{\mathrm{D}\}$$

as sets. This is actually a scheme-theoretic equality; if ϵ is a tangent vector such that:

$$\epsilon \in \mathrm{T_D}(\mathrm{S_D} + \mathrm{D} - v) = \mathrm{T}_v\mathrm{S_D} \subset \mathrm{T}_v\mathrm{V}$$
$$\epsilon \in \mathrm{T_D}\big(\mathrm{E}' + \mathrm{W}_m(\mathrm{C})\big) = \mathrm{T_E}\mathrm{W}_m(\mathrm{C}) \ ,$$

then (ϵ, ϵ) is in the kernel of the differential of f_m^- at (v, E), hence is 0.

By proposition 4.1, $\mathrm{E}' + \mathrm{W}_m(\mathrm{C})$ meets each component of $\mathrm{S_D} + \mathrm{D} - v$, which must therefore be irreducible. Recall that since $f_n^+(\Gamma) = v + \mathrm{W}_n(\mathrm{C})$, the subvarieties $\mathrm{S_D} + \mathrm{D} - v$ cover $\mathrm{W}_n(\mathrm{C})$ as D varies in G. In particular, we may assume that $\mathrm{S_D} + \mathrm{D} - v$ is not contained in the image of $\mathcal{T}_n$ (defined in §4) in $\mathrm{W}_n(\mathrm{C})$. It then follows from proposition 4.1 applied to the strict transform of $\mathrm{S_D} + \mathrm{D} - v$ in $\mathrm{C}^{(n)}$ that there exists an effective divisor $\mathrm{E_D}$ of degree m such that:

$$\mathrm{S_D} + \mathrm{D} - v = \mathrm{W}_{n-m}(\mathrm{C}) + \mathrm{E_D} \ .$$

and if x is any point of C, the point $v-c+x$ is in $\mathrm{S_D}$ hence in V. Since v is generic, this implies that some irreducible component T of the scheme:

$$\{\ (v,c)\in \mathrm{V}\times \mathrm{C}\ \big|\ v-c+\mathrm{C}\subset \mathrm{V}\ \}$$

dominates V. Note that $\mathrm{U}=f_1^-(\mathrm{T})$ satisfies $\mathrm{U}+\mathrm{C}\subset\mathrm{V}$. In particular, $\dim \mathrm{U}<\dim \mathrm{V}\leq \dim \mathrm{T}$. In general, if f_1^- contracts a subvariety F of $\mathrm{V}\times\mathrm{C}$ to a point, the projection $\mathrm{F}\to\mathrm{C}$ is an isomorphism. This implies that $\mathrm{U}=f_1^-(\mathrm{T})$ has dimension $\dim\mathrm{T}-1$, and that $\mathrm{V}=p_1(\mathrm{T})\subset\mathrm{U}+\mathrm{C}$, hence $\mathrm{V}=\mathrm{U}+\mathrm{C}$. ∎

Replacing V with $-\mathrm{V}$ if necessary, we may assume that $\mathrm{V}=\mathrm{U}+\mathrm{C}$. The following proposition shows that U satisfies the induction hypothesis.

PROPOSITION 5.3. – *Let* V *be an irreducible subvariety of* JC *of codimension* $d<g$ *having property* $(\mathcal{P})$ *with respect to* $\mathrm{W}_d(\mathrm{C})$. *Assume that* $\mathrm{V}=\mathrm{U}+\mathrm{C}$. *Then* U *has property* $(\mathcal{P})$ *with respect to* $\mathrm{W}_{d+1}(\mathrm{C})$.

Proof. By proposition 2.4, U has property $(\mathcal{P})$ with respect to $\mathrm{W}_d(\mathrm{C})$, hence also with respect to C (proposition 2.5).

Let u be generic in U and let $\phi^+_{d+1}:\mathrm{U}\times\mathrm{W}_{d+1}(\mathrm{C})\to\mathrm{JC}$ be the addition map. Let Γ_0 be an irreducible subvariety of $\mathrm{U}\times\mathrm{W}_{d+1}(\mathrm{C})$ such that $\phi^+_{d+1}(\Gamma_0)=u+\mathrm{W}_{d+1}(\mathrm{C})$ and $p_2(\Gamma_0)=\mathrm{W}_{d+1}(\mathrm{C})$. Let Γ be an irreducible subvariety of $\mathrm{U}\times\mathrm{C}^{(d+1)}$ which maps onto Γ_0 by the natural map:

$$\Pi_\mathrm{U}=\mathrm{Id}_\mathrm{U}\times\pi_{d+1}:\mathrm{U}\times\mathrm{C}^{(d+1)}\to\mathrm{U}\times\mathrm{W}_{d+1}(\mathrm{C})\ .$$

We have $\phi^+_{d+1}\Pi_\mathrm{U}(\Gamma)=u+\mathrm{W}_{d+1}(\mathrm{C})$ and $p_2(\Gamma)=\mathrm{C}^{(d+1)}$. Moreover, since $\phi^+_{d+1}\Pi_\mathrm{U}$ is generically finite onto its image, the dimension of Γ is $d+1$. We need to show that $\Gamma=\{u\}\times\mathrm{C}^{(d+1)}$.

For any c in C, each component of $\Gamma\cap(\phi^+_{d+1}\Pi_\mathrm{U})^{-1}\big(u+c+\mathrm{W}_d(\mathrm{C})\big)$ that dominates $u+c+\mathrm{W}_d(\mathrm{C})$ has dimension d. Since $p_2(\Gamma)=\mathrm{C}^{(d+1)}$, at least one of these components, say Γ_c, projects onto a divisor Z_c in $\mathrm{C}^{(d+1)}$. We use the elementary:

LEMMA 5.4. – *Let* Z *be an irreducible divisor in* $\mathrm{C}^{(d+1)}$ *and let* $g:\mathrm{C}\times\mathrm{C}^{(d)}\to\mathrm{C}^{(d+1)}$ *be the natural map. Then, there exists an irreducible divisor* Z' *in* $\mathrm{C}\times\mathrm{C}^{(d)}$ *such that* $g(\mathrm{Z}')=\mathrm{Z}$ *and* $p_2(\mathrm{Z}')=\mathrm{C}^{(d)}$.

Proof. By proposition 4.1, $g^*\mathrm{Z}\cdot(\mathrm{C}\times\{\mathrm{D}\})=\mathrm{Z}\cdot(\mathrm{C}+\mathrm{D})$ is non-zero for any point D in $\mathrm{C}^{(d)}$, hence $p_2\big(g^{-1}(\mathrm{Z})\big)=\mathrm{C}^{(d)}$. Some component Z' of $g^{-1}(\mathrm{Z})$ must satisfy the same property. Its dimension is then $\geq d$, hence it must map onto Z by the finite map g. ∎

Pick such a divisor Z'_c for Z_c and let Γ'_c be a component of $\mathrm{Z}'_c\times_{\mathrm{Z}_c}\Gamma_c$ that projects

$$
\begin{array}{ccccccc}
\Gamma'_c & \subset & \mathrm{U}\times\mathrm{C}\times\mathrm{C}^{(d)} & \xrightarrow{\mathrm{Id_U}\times g} & \mathrm{U}\times\mathrm{C}^{(d+1)} & \supset & \Gamma_c \\
\downarrow & & \downarrow{\scriptstyle p_{23}} & & \downarrow{\scriptstyle p_2} & & \downarrow \\
\mathrm{Z}'_c & \subset & \mathrm{C}\times\mathrm{C}^{(d)} & \xrightarrow{g} & \mathrm{C}^{(d+1)} & \supset & \mathrm{Z}_c \\
\downarrow & & \downarrow{\scriptstyle p_2} & & & & \\
\mathrm{C}^{(d)} & = & \mathrm{C}^{(d)} & & & &
\end{array}
$$

Since $\mathrm{Id_U}\times g$ is finite, it maps Γ'_c onto Γ_c. Set $\phi^+_{1,d} = \phi^+_1 \times \mathrm{Id}_{\mathrm{C}^{(d)}}$ and consider:

$$\mathrm{U}\times\mathrm{C}\times\mathrm{C}^{(d)} \xrightarrow{\phi^+_{1,d}} \mathrm{V}\times\mathrm{C}^{(d)} \xrightarrow{\Pi_\mathrm{V}} \mathrm{V}\times\mathrm{W}_d(\mathrm{C}) \xrightarrow{\phi^+_d} \mathrm{JC}\ .$$

Then:

$$\phi^+_d \Pi_\mathrm{V} \phi^+_{1,d}(\Gamma'_c) = \phi^+_{d+1}\Pi_\mathrm{U}(\Gamma_c) = u + c + \mathrm{W}_d(\mathrm{C})$$
$$p_2 \Pi_\mathrm{V} \phi^+_d(\Gamma'_c) = \pi_d p_3(\Gamma'_c) = \mathrm{W}_d(\mathrm{C})\ .$$

Since V has property $(\mathcal{P})$ with respect to $\mathrm{W}_d(\mathrm{C})$, this implies:

$$\Pi_\mathrm{V}\phi^+_{1,d}(\Gamma'_c) = \{u+c\}\times\mathrm{W}_d(\mathrm{C})$$

for c generic. Therefore:

$$\Gamma_c = \{u'\}\times(c'+\mathrm{C}^{(d)}) \qquad \text{and} \qquad \mathrm{Z}_c = c' + \mathrm{C}^{(d)}\ ,$$

for some points u' of U and c' of C such that $u'+c' = u+c$. The union as c varies in C of all Z_c's must be $\mathrm{C}^{(d+1)}$. It follows that c' must describe the whole of C. Recall that U has property $\mathcal{P}$ with respect to C, hence that the only component of $(\phi^+_1)^{-1}(u+\mathrm{C})$ which dominates both $u+\mathrm{C}$ via ϕ^+_1 and C via p_2 is $\{u\}\times\mathrm{C}$. This implies $\Gamma_c = \{u\}\times(c+\mathrm{C}^{(d)})$ for c generic, hence the proposition. ■

It then follows from our induction hypothesis that U is a translate of some $\mathrm{W}_{r-1}(\mathrm{C}) - \mathrm{W}_{g-d-r}(\mathrm{C})$, hence that V is a translate of $\mathrm{W}_r(\mathrm{C}) - \mathrm{W}_{g-d-r}(\mathrm{C})$.

When C is hyperelliptic, $-\mathrm{C}$ is a translate of C, hence $-\mathrm{W}_{g-d-r}(\mathrm{C})$ is a translate of $\mathrm{W}_{g-d-r}(\mathrm{C})$, and V is a translate of $\mathrm{W}_{g-d}(\mathrm{C})$. To conclude the proof of the theorem, we may therefore assume that C is non-hyperelliptic.

hyperelliptic. Then:

$$\big(\mathrm{W}_r(\mathrm{C}) - \mathrm{W}_{g-d-r}(\mathrm{C})\big) \cdot \mathrm{W}_d(\mathrm{C}) = \binom{g-d}{r}\binom{g}{d} .$$

Proof. The subtraction map $\mathrm{W}_r(\mathrm{C}) \times \mathrm{W}_{g-d-r}(\mathrm{C}) \to \mathrm{JC}$ is birational onto its image. In fact, if (D,E) and $(\mathrm{D}',\mathrm{E}')$ are two distinct elements of $\mathrm{W}_r(\mathrm{C}) \times \mathrm{W}_{g-d-r}(\mathrm{C})$ such that $\mathrm{D} - \mathrm{E} \equiv \mathrm{D}' - \mathrm{E}'$, then:

- either $h^0(\mathrm{C}, \mathrm{D}+\mathrm{E}') = 1$, in which case (D,E) belongs to the divisor $\{(\mathrm{D},\mathrm{E}) \mid \mathrm{Supp}(\mathrm{D}) \cap \mathrm{Supp}(\mathrm{E}) \neq \emptyset\}$,
- or $h^0(\mathrm{C}, \mathrm{D}+\mathrm{E}') > 1$, in which case (D,E) belongs to:

$$\{\ (\mathrm{D},\mathrm{E}) \mid \exists \mathrm{L} \in \mathrm{W}^1_{g-d}(\mathrm{C}) \quad \mathrm{D},\mathrm{E} \leq |\mathrm{L}|\ \} .$$

This locus has dimension $\max_{a>0}\big(\dim \mathrm{W}^a_{g-d}(\mathrm{C}) + 2a\big)$, which is less than $g-d$ if C is non-hyperelliptic (Martens' theorem, [ACGH], p. 191).

If $*$ denotes the Pontryagin product on $\mathrm{H}^\bullet(\mathrm{JC}, \mathbf{Q})$, it follows that the cohomology class of $\big(\mathrm{W}_r(\mathrm{C}) - \mathrm{W}_{g-d-r}(\mathrm{C})\big)$ is:

$$\theta_{g-r} * \theta_{d+r} = \binom{g-d}{r}\theta_d .$$

Its intersection number with $\mathrm{W}_d(\mathrm{C})$ is therefore:

$$\binom{g-d}{r}\theta_d \cdot \theta_{g-d} = \binom{g-d}{r}\binom{g}{d} .$$

This proves the lemma. ∎

But this intersection number is equal to $\mathrm{V} \cdot \mathrm{W}_d(\mathrm{C})$, hence to $\binom{g}{d}$. It follows that either $r = g-d$, or $r = 0$, hence either V is a translate of $\mathrm{W}_{g-d}(\mathrm{C})$, or it is a translate of $-\mathrm{W}_{g-d}(\mathrm{C})$. ∎

6. Preliminaries

For any positive integer g, let $\mathcal{A}_g$ be the moduli space of complex principally polarized abelian varietiesof dimension g, let $\mathcal{J}_g$ be the closure in $\mathcal{A}_g$ of the subvariety which corresponds to Jacobians of smooth curves of genus g, and let $\mathcal{CT}_5$ be the closure in $\mathcal{A}_5$ of the subvariety which corresponds to intermediate Jacobians of smooth cubic threefolds. For $0 < d \leq g$, the subset $\mathcal{C}_{g,d}$ of $\mathcal{A}_g$ which corresponds to principally polarized abelian varieties (A, θ) for which θ_d is the class of an effective algebraic cycle (so that $\mathcal{C}_{g,1} = \mathcal{C}_{g,g} = \mathcal{A}_g$), is *closed* in $\mathcal{A}_g$. Indeed, let $\mathcal{X} \to \mathrm{S}$ be a versal family of abelian varieties of dimension g with a relatively ample line bundle $\mathcal{L}$ on $\mathcal{X}$ which induces principal polarizations on the fibers, such that the classification morphism $\mathrm{S} \to \mathcal{A}_g$ is a finite cover. Fix an embedding of $\mathcal{X}$ in some projective space $\mathbf{P}^{\mathrm{N}}_{\mathrm{S}}$, using for example the sections of $\mathcal{L}^{\otimes 3}$. Chow coordinates show that the family of effective cycles in $\mathcal{X}$ whose fibers over S have codimension d and degree $\theta_d(3\theta)^{g-d}$, is projective over S. In particular, so is its closed subset which parametrizes effective cycles with class θ_d in the fibers. Consequently, its image in S is also closed, hence so is its image $\mathcal{C}_{g,d}$ in $\mathcal{A}_g$.

7. Degenerations of abelian varieties

Let $\Delta = \{t \in \mathbf{C} \bigm| |t| < 1\}$. A degeneration of principally polarized abelian varietiesof dimension $g+1$ will be a proper family $\mathcal{X} \to \Delta$, with $\mathcal{X}$ smooth, whose fibers over Δ^* are principally polarized abelian varietiesof dimension $g+1$ and whose central fiber X is a projective variety whose normalization is a $\mathbf{P}^1$–bundle $\mathbf{P}$ over a principally polarized abelian variety (A, θ) of dimension g. More precisely, there exists an element a of A such that $\mathbf{P} = \mathbf{P}\big(\mathcal{O}_{\mathrm{A}} \oplus \mathcal{O}_{\mathrm{A}}(\widehat{a})\big)$, where $\widehat{a}$ is the image of a by the canonical isomorphism $\phi_\theta : \mathrm{A} \to \mathrm{Pic}^0(\mathrm{A})$. The bundle $p : \mathbf{P} \to \mathrm{A}$ has two disjoint sections $\mathbf{P}_0$ and $\mathbf{P}_\infty$, and X is obtained from $\mathbf{P}$ by identifying any point x on $\mathbf{P}_0$ with the point $x - a$ on $\mathbf{P}_\infty$. The singular locus X_s of X will always be identified with A via the isomorphism $\mathrm{X}_s \to \mathbf{P}_0 \to \mathrm{A}$. By [N], theorem 16.1, we may also assume the existence of a relatively ample line bundle $\mathcal{L}$ on $\mathcal{X}$ which induces the principal polarization on the smooth fibers and which restricts on X to an ample line bundle whose pull-back to $\mathbf{P}$ is $\mathcal{O}_{\mathbf{P}}(p^*\Theta + \mathbf{P}_0) \simeq \mathcal{O}_{\mathbf{P}}(p^*\Theta_a + \mathbf{P}_\infty)$, for a suitable representative Θ of the polarization θ. Another (more accessible) reference for the existence of $\mathcal{L}$ is [HW], proposition 4.1.3 (the proof is given in the surface case, but works in general).

The projective varieties X with their polarizations correspond to the boundary points of $\mathcal{A}_{g+1}$ in a suitable partial compactification $\bar{\mathcal{A}}_{g+1}$, where they form a divisor $\partial\mathcal{A}_{g+1}$. There is a surjective map $q : \partial\mathcal{A}_{g+1} \to \mathcal{A}_g$ which, in the above notation, sends X with its polarization to (A, θ), and whose generic fiber is isomorphic to $\mathrm{A}/\pm 1$. For any subvariety $\mathcal{B}$ of $\mathcal{A}_{g+1}$, we will denote by $\partial\mathcal{B}$ the intersection of $\partial\mathcal{A}_{g+1}$ with the closure of $\mathcal{B}$ in $\bar{\mathcal{A}}_{g+1}$.

Proposition 7.1. – *For $1 < d < g+1$, the image by q of the boundary $\partial\mathcal{C}_{g+1,d}$ is*

Proof. Keeping the same notation, we assume moreover that the smooth fibers of $\mathcal{X} \to \Delta$ correspond to elements of $\mathcal{C}_{g+1,d}$. Then, there exists a subvariety $\mathcal{Z}$ of $\mathcal{X}$ such that the cycles associated with the fibers of $\mathcal{Z} \to \Delta$ over Δ^* are sums of effective cycles, each with class θ_d. Over the open interval $(0,1)$, the scheme $\mathcal{Z}$ contains a subscheme whose fibers have class θ_d. Its topological closure meets the central fiber X in a union $\mathrm{Z} = \mathrm{Z}_1 \cup \cdots \cup \mathrm{Z}_s$ of irreducible algebraic subvarieties of codimension d. For $i = 1, \ldots, s$, let Y_i be the closure in $\mathbf{P}$ of the inverse image of Z_i by the bijection $\mathbf{P} - \mathbf{P}_\infty \to \mathrm{X}$. Then:

$$\sum_{i=1}^{s} m_i[\mathrm{Y}_i] = \left(p^*\theta + [\mathbf{P}_0]\right)^d / d! = p^*\theta_d \ + \ [\mathbf{P}_0] \cdot p^*\theta_{d-1} \ , \qquad (7.2)$$

for some positive integers $m_1, \ldots, m_s$. It follows that the image of $\sum_{i=1}^{s} m_i \mathrm{Y}_i$ in A has class θ_{d-1}, and that its intersection with $\mathbf{P}_\infty$ has class θ_d. This proves the proposition. ∎

8. Proof of the theorem

THEOREM 8.1. – *For $1 < d < g$, the locus $\mathcal{J}_g$ is a component of $\mathcal{C}_{g,d}$. Moreover, $\mathcal{CT}_5$ is a component of $\mathcal{C}_{5,3}$ and is not contained in $\mathcal{C}_{5,2}$.*

Proof. The proof of the first part is by induction on g. Since this property is empty for $g = 2$, we assume that it holds for some $g \geq 2$ and show that it then holds in dimension $g+1$. Let $1 < d < g+1$ and let $\mathcal{F}$ be a component of $\mathcal{C}_{g+1,d}$ which contains either $\mathcal{J}_{g+1}$ or $\mathcal{CT}_5$ (if $g = 4$). It is known that $q(\partial\mathcal{J}_{g+1}) = \mathcal{J}_g$ and $q(\partial\mathcal{CT}_5) = \mathcal{J}_4$. It follows from proposition 7.1 and our induction hypothesis that $\mathcal{J}_g$ is a component of $q(\partial\mathcal{F})$. With the notation of the proof of proposition 7.1, we may therefore assume that (A, θ) is the Jacobian (JC, θ) of a generic smooth curve C of genus g. Then, $\partial\mathcal{J}_g \cap q^{-1}\{(\mathrm{JC}, \theta)\}$ corresponds to the image of the surface $\mathrm{C} - \mathrm{C}$ in $\mathrm{JC}/\pm 1$. For $g = 4$, the curve C has two pencils g_3^1 and h_3^1 of degree 3, and $\partial\mathcal{CT}_5 \cap q^{-1}\{(\mathrm{JC}, \theta)\}$ corresponds to the point $\pm(g_3^1 - h_3^1)$ of $\mathrm{JC}/\pm 1$ ([C]).

We will show that $\partial\mathcal{C}_{g+1,d} \cap q^{-1}\{(\mathrm{JC}, \theta)\}$ is equal to $(\mathrm{C} - \mathrm{C}) \cup \{\pm(g_3^1 - h_3^1)\}$ for $g = 4$ and $d = 3$, and to $\mathrm{C} - \mathrm{C}$ otherwise.

Keeping the notation of the proof of proposition 7.1, the image in JC of the effective cycle $\mathrm{Y} = \sum_{i=1}^{s} m_i \mathrm{Y}_i$ has class θ_{d-1}. By theorem 5.1, it is a translate of $\pm\mathrm{W}_{g-d+1}(\mathrm{C})$ which, possibly after translation and action of $(-1)_\mathrm{X}$, we may assume to be $\mathrm{W}_{g-d+1}(\mathrm{C})$. Consequently, after reindexing if necessary, we have $m_1 = 1$, the morphism $\mathrm{Y}_1 \to p(\mathrm{Y}_1)$ is birational, and $\mathrm{Y}_i = p^{-1}\big(p(\mathrm{Y}_i)\big)$ for $i > 1$. Identity (7.2) and theorem 5.1 then imply that:

Case 1: either $\mathrm{Y} = \mathrm{Y}_1$ maps birationally onto $\mathrm{W}_{g-d+1}(\mathrm{C})$ and meets $\mathbf{P}_0$ and $\mathbf{P}_\infty$ along subvarieties which are translates of $\pm\mathrm{W}_{g-d}(\mathrm{C})$,

Case 2: or $\mathrm{Y} = \mathrm{Y}_1 + \mathrm{Y}_2$, where Y_1 has class $[\mathbf{P}_0] \cdot p^*\theta_{d-1}$ and maps isomorphically onto $\mathrm{W}_{g-d+1}(\mathrm{C})$, and Y_2 is the pull-back by p of a translate V of $\pm\mathrm{W}_{g-d}(\mathrm{C})$. In this case, lar locus X_s of X, which may well happen, contrary to what is asserted in [BC] on p. 64), or it meets neither $\mathbf{P}_0$ nor $\mathbf{P}_\infty$. In the latter case, the line bundle $\mathcal{O}_{\mathbf{P}}(\mathbf{P}_0 - \mathbf{P}_\infty) = p^*\mathcal{O}_{JC}(\widehat{a})$ is trivial on Y_1. Since the restriction $\mathrm{Pic}^0(JC) \to \mathrm{Pic}^0\big(W_{g-d+1}(C)\big)$ is injective, this implies that a is 0 hence is in $C - C$. We may therefore assume that Y_1 is contained in $\mathbf{P}_0$.

The following result makes the structure of the limit cycle more precise.

LEMMA 8.2. – *Let z be a point of $Z \cap X_s$. Then either z belongs to a component of Z contained in X_s, or each of the two branches of X contains a local component of Z at z.*

Proof. Let $\varepsilon : \widetilde{\mathcal{X}} \to \mathcal{X}$ be the blow-up of the smooth subscheme X_s. The strict transform $\widetilde{X}$ of X can be identified with $\mathbf{P}$. Let $\widetilde{\mathcal{Z}}$ be the strict transform of $\mathcal{Z}$. If no component of Z through z is contained in X_s, the scheme $Z \cap X_s = \mathcal{Z} \cap X_s$ has pure codimension 2 in $\mathcal{Z}$ at z. It follows that its ideal in $\mathcal{Z}$ is not invertible at z, hence that the rational curve $\varepsilon^{-1}(z)$ is contained in $\widetilde{\mathcal{Z}}$. This curve meets $\widetilde{X}$ at two points (one on $\mathbf{P}_0$, the other on $\mathbf{P}_\infty$) which correspond to the two branches of X at z. It follows that the intersection of $\widetilde{\mathcal{Z}}$ with the Cartier divisor $\widetilde{X}$ is non-empty, hence has dimension $\dim \widetilde{\mathcal{Z}} - 1 = g + 1 - d$, at each of these two points. Consequently, the intersection of Z with each of the two branches of X has dimension $g + 1 - d$ at z, and this proves the lemma. ∎

In case 1, this means that $V = Y \cap \mathbf{P}_0$ and $Y \cap \mathbf{P}_\infty$ must be identified through the glueing process, hence that $Y \cap \mathbf{P}_\infty = V - a$. In particular, both V and $V - a$ are contained in $W_{g-d+1}(C)$.

In case 2, both V and $V - a$ must be again contained in $W_{g-d+1}(C)$.

Recall that in both cases, V is a translate of $\pm W_{g-d}(C)$.

LEMMA 8.3. – *Let C be a smooth curve of genus g and let a be an element of JC. Assume that some translate of $W_{m+1}(C)$ contains $\varepsilon W_m(C)$ and $\varepsilon W_m(C) - a$, for some $\varepsilon = \pm 1$ and $0 \le m \le g - 2$. Then either $a \in C - C$, or $\varepsilon = -1$, $g = 4$ and a is the difference of two g^1_3's.*

Proof. Let L_a be the line bundle of degree 0 on C associated with a. If $\varepsilon = 1$, there exists a line bundle L on C of degree 1 such that:

$$h^0\big(C, L(c_1 + \cdots + c_m)\big) > 0 \qquad \text{and} \qquad h^0\big(C, L \otimes L_a^{-1}(c_1 + \cdots + c_m)\big) > 0\ ,$$

for any points $c_1, \ldots, c_m$ of C. Riemann-Roch implies that both L and $L \otimes L_a^{-1}$ have a non-zero-section, hence that $a \in C - C$.

If $\varepsilon = -1$, we may assume that C is non-hyperelliptic and $0 < m \le g - 3$ (otherwise, $-W_{m+1}(C)$ is a translate of $W_{m+1}(C)$). There exists a line bundle L on C of degree $2m + 1$ such that:

$$h^0\big(C, L(-c_1 - \cdots - c_m)\big) > 0 \qquad \text{and} \qquad h^0\big(C, L \otimes L_a^{-1}(-c_1 - \cdots - c_m)\big) > 0\ ,$$

g^m_{2m+1}'s. By [ACGH], exercise B-7, p. 138, this implies under our assumptions that $m = g-3$, and that the residual series of both L and $\mathrm{L} \otimes \mathrm{L}_a^{-1}$ are g^1_3's. If $g > 4$, there is at most one g^1_3 and $a = 0$. If $g = 4$, there are at most two g^1_3's and a is either 0 or their difference. ■

This proves that $\partial\mathcal{C}_{g+1,d}$ coincides over $\mathcal{J}_g$ with $\partial(\mathcal{J}_{g+1} \cup \mathcal{CT}_5)$ for $g = 4$ and $d = 3$, and with $\partial\mathcal{J}_{g+1}$ otherwise. In particular:

$$\dim \partial\mathcal{F} \leq 2 + \dim \mathcal{J}_g = 3g - 1 \qquad \text{if} \quad \mathcal{F} \supset \mathcal{J}_{g+1} \ ,$$
$$\dim \partial\mathcal{F} \leq \dim \mathcal{J}_4 = \dim \mathcal{CT}_5 - 1 \qquad \text{if} \quad \mathcal{F} \supset \mathcal{CT}_5 \ .$$

Since $\partial\mathcal{F}$ is the intersection of $\mathcal{F}$ with the Cartier divisor $\partial\mathcal{A}_{g+1}$, we have $\dim \mathcal{F} \leq \dim \partial\mathcal{F} + 1$. It follows that either $\mathcal{F} = \mathcal{J}_{g+1}$ or $\mathcal{F} = \mathcal{CT}_5$, which finishes the proof of the induction step.

To conclude, note that if C is non-hyperelliptic of genus 4, then $g^1_3 - h^1_3$ is not in $\mathrm{C} - \mathrm{C}$ if it is non-zero. Therefore, $\mathcal{CT}_5$ is not contained in $\mathcal{C}_{5,2}$. ■

Note that we only needed for $\mathrm{C} - \mathrm{C}$ to be a component of $\partial\mathcal{C}_{g+1,d} \cap q^{-1}\{(\mathrm{JC}, \theta)\}$ for the proof. Our more precise result, combined with the fact that $\mathcal{J}_4 = \mathcal{C}_{4,3} = \mathcal{C}_{4,2}$, proves:

$$\partial\mathcal{C}_{5,2} = \partial\mathcal{J}_5 \qquad \text{and} \qquad \partial\mathcal{C}_{5,3} = \partial(\mathcal{J}_5 \cup \mathcal{CT}_5) \ .$$

This makes the following conjecture plausible, at least in dimension 5:

Conjecture – For $1 < d < g$ and $(g, d) \neq (5, 3)$, then $\mathcal{C}_{g,d} = \mathcal{J}_g$; furthermore, $\mathcal{C}_{5,3} = \mathcal{J}_5 \cup \mathcal{CT}_5$.

A more tractable version would be the weaker:

Conjecture' – For $0 < d < g$, then $\mathcal{C}_{g,d} \cap \mathcal{C}_{g,g-d} = \mathcal{J}_g$.

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