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TRISECANT LINES AND JACOBIANS

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Let X be a principally polarized complex abelian variety. Symmetric representatives Θ of the polarization differ by translations by points of order 2, hence the linear system $|2\Theta|$ is well defined. It is base point free and defines a morphism $K\colon X \to |2\Theta|^*$, whose image is the *Kummer variety* of X. When X is the Jacobian of an algebraic curve, there are infinitely many *trisecants* to $K(X)$, i.e., lines in the projective space $|2\Theta|^*$ meeting $K(X)$ in at least three points.

Welters conjectured in [9] that the existence of *one* trisecant line to the Kummer variety should characterize Jacobians among all indecomposable principally polarized abelian varieties. This conjecture should be thought of as a discrete analog of Novikov's conjecture (now proved by Shiota in [8]). It is known to hold if X is a Prym variety [6] and, in particular, *it holds in dimension* ≤ 5.

The existence of a line meeting $K(X)$ at three distinct points $K(a)$, $K(b)$, $K(c)$ is equivalent to the scheme-theoretic inclusion:

$$\Theta_a \cdot \Theta_b \subset \Theta_c \cup \Theta_{-c}, \tag{0.1}$$

where, for a point x of X, we write Θ_x for $\Theta + x$. This forces the intersection $\Theta_a \cdot \Theta_b$ to be reducible.

We prove here a weak version of Welters' conjecture: *an indecomposable principally polarized complex abelian variety X is a Jacobian if and only if there exist points a, b, c of X with $2a \neq 2b$, such that $K(a)$, $K(b)$, $K(c)$ are distinct and collinear and $\Theta_a \cdot \Theta_b$ is reduced and has two irreducible components* (Theorem 3.1). The assumptions on $\Theta_a \cdot \Theta_b$ can be replaced by: *the points $b-a$, $c-b$, $a-c$ generate X and* $\dim \operatorname{Sing}\Theta \leq \dim X - 4$ (Theorem 3.5).

Our method consists in proving first that, under the above geometrical assumptions, inclusion (0.1) implies:

$$\Theta_a \cdot \Theta_b \subset \Theta_c \cup \Theta_{a+b-c}. \tag{0.2}$$

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This inclusion is equivalent to the existence of a line tangent to $K(X)$ at some point and meeting $K(X)$ at some other point: a degenerate trisecant! It is also equivalent to the fact that the scheme

$$V_{a,b,c} = 2\{\zeta \in X \mid K(\zeta+a),\, K(\zeta+b),\, K(\zeta+c) \text{ are collinear}\}$$

has length ≥ 2 at $(-a-b)$.

As in [1] and [2], our aim is to prove that $\dim_{-a-b} V_{a,b,c} > 0$ and to apply Welters' criterion [9] to conclude that X is a Jacobian. This is done in §2, assuming now that (0.2) holds and that $\Theta_a \cdot \Theta_b$ is only reduced (Theorems 2.1 and 2.2). The method is one that Arbarello used in [1]: first translate Welters' criterion into an infinite set of differential equations (this is done in §1), then show that each of these equations follows from the first one, which is itself equivalent to (0.2).

1. A reformulation of Welters' criterion

Let X be a complex indecomposable principally polarized abelian variety, let Θ be a symmetric representative of the polarization, and let $K\colon X \to |2\Theta|^*$ be the Kummer map. Let a, b, c be distinct points of X.

Riemann's addition formula [7, p. 336] tells us that there exist a basis $\{\theta\}$ for $H^0(X, \Theta)$ and a basis $\{\theta_n\}_n$ for $H^0(X, 2\Theta)$ such that

$$\forall z, \zeta \in X \quad \theta(z+\zeta)\theta(z-\zeta) = \sum_n \theta_n(z)\theta_n(\zeta). \tag{1.1}$$

The subscheme $V_{a,b,c}$ of X defined in the introduction is the set of points 2ζ of X such that

$$\exists \alpha, \beta, \gamma \in \mathbb{C} \ \forall n \quad \alpha\theta_n(\zeta+a) + \beta\theta_n(\zeta+b) + \gamma\theta_n(\zeta+c) = 0.$$

Formula (1.1) implies that these equations are equivalent to: $\exists \alpha, \beta, \gamma \in \mathbb{C}\ \forall z \in X$

$$\alpha\theta(z+\zeta+a)\theta(z-\zeta-a) + \beta\theta(z+\zeta+b)\theta(z-\zeta-b) + \gamma\theta(z+\zeta+c)\theta(z-\zeta-c) = 0. \tag{1.2}$$

Welters' criterion [9] states that X *is a Jacobian if and only if there exist distinct points* a, b, c *of* X *such that* $\dim V_{a,b,c} > 0$.

Since K takes the same value at opposite points, $(-a-b)$ always belongs to $V_{a,b,c}$. The aim of this section is to translate the condition $\dim_{-a-b} V_{a,b,c} > 0$ into an infinite set of equations.

This condition is equivalent to the existence of a formal curve:

$$2\zeta(\varepsilon) = -a-b+D(\varepsilon), \quad \text{with } D(\varepsilon) = \sum_{i\geq 1} D_i\varepsilon^i,$$

contained in $V_{a,b,c}$. The D_i's are tangent vectors to X at the origin, or constant vector fields. We will look for a smooth germ $(D_1 \neq 0)$.

This is in turn equivalent to a relation of the type:

$$\forall z \in X \quad \alpha(\varepsilon)\theta(z+\zeta(\varepsilon)+a)\theta(z-\zeta(\varepsilon)-a) + \beta(\varepsilon)\theta(z+\zeta(\varepsilon)+b)\theta(z-\zeta(\varepsilon)-b) + \gamma(\varepsilon)\theta(z+\zeta(\varepsilon)+c)\theta(z-\zeta(\varepsilon)-c) = 0, \tag{1.3}$$

where $\alpha(\varepsilon)$, $\beta(\varepsilon)$, and $\gamma(\varepsilon)$ are relatively prime elements of $\mathbb{C}[[\varepsilon]]$. It is convenient to set $u = a - \frac{1}{2}(a+b)$ and $v = c - \frac{1}{2}(a+b)$ in (1.3), which becomes

$$\alpha(\varepsilon)\theta(z+u+\tfrac{1}{2}D(\varepsilon))\theta(z-u-\tfrac{1}{2}D(\varepsilon)) + \beta(\varepsilon)\theta(z-u+\tfrac{1}{2}D(\varepsilon))\theta(z+u-\tfrac{1}{2}D(\varepsilon)) + \gamma(\varepsilon)\theta(z+v+\tfrac{1}{2}D(\varepsilon))\theta(z-v-\tfrac{1}{2}D(\varepsilon)) = 0. \tag{1.4}$$

Write $P(z, \varepsilon) = \sum_{s\geq 0} P_s(z)\varepsilon^s$ for the left-hand side of (1.4). Formula (1.1) implies, as in [2], that *P_s is a section of $\mathcal{O}_X(2\Theta)$*.

For any point x of X, write θ_x for the translate of θ by x. We have

$$P_0 = (\alpha(0)+\beta(0))\theta_{-u}\cdot\theta_u + \gamma(0)\theta_{-v}\cdot\theta_v.$$

Since a, b, c are distinct and Θ is irreducible, P_0 vanishes if and only if $\alpha(0)+\beta(0) = \gamma(0) = 0$. It follows that $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ are units and that, by dividing by $-\beta(\varepsilon)$, we may assume:

$$\alpha(\varepsilon) = 1 + \sum_{i\geq 1}\alpha_i\varepsilon^i, \quad \beta(\varepsilon) = -1, \quad \gamma(\varepsilon) = \sum_{i\geq 1}\gamma_i\varepsilon^i.$$

One has

$$P_1 = \alpha_1\theta_{-u}\cdot\theta_u + D_1\theta_{-u}\cdot\theta_u - \theta_{-u}\cdot D_1\theta_u + \gamma_1\theta_{-v}\cdot\theta_v. \tag{1.5}$$

Since D_1 is assumed to be nonzero and $2u$ is nonzero, γ_1 has to be nonzero if P_1 vanishes. Allowing linear changes of the D_i's, we may finally assume $\gamma(\varepsilon) = \varepsilon$.

We can state the following translation of Welters' criterion:

Theorem 1.6. *The abelian variety X is a Jacobian if and only if there exist complex numbers $\alpha_1, \alpha_2, \ldots$ and constant vector fields $D_1, D_2, \ldots$ on X with $D_1 \neq 0$, such that the sections P_s vanish for all positive integers s.*

Following [2], we can go further. Notice that P_s only depends on $\alpha_1, \cdots, \alpha_s$ and $D_1, \cdots, D_s$. Moreover, one can write P_s as

$$P_s = Q_s + \alpha_s\theta_{-u}\cdot\theta_u + D_s\theta_{-u}\cdot\theta_u - \theta_{-u}\cdot D_s\theta_u, \tag{1.7}$$

where Q_s does not depend on α_s nor on D_s. Notice that Q_s, which corresponds to vanishing α_s and D_s, is also a section of $\mathcal{O}_X(2\Theta)$.

It follows from (1.7) that the restriction of P_s to $\Theta_u \cdot \Theta_{-u}$ is *independent of* α_s *and* D_s. Moreover, one has:

Lemma 1.8. *The section* P_s *of* $\mathcal{O}_X(2\Theta)$ *vanishes for some choice of* α_s *and* D_s *if and only if the restriction of* Q_s *to* $\Theta_u \cdot \Theta_{-u}$ *does.*

Proof of Lemma 1.8. The cohomology sequence of the exact sequence

$$0 \to \mathcal{O}_{\Theta_u}(\Theta_u) \xrightarrow{\cdot\theta_{-u}} \mathcal{O}_{\Theta_u}(2\Theta) \to \mathcal{O}_{\Theta_u \cdot \Theta_{-u}}(2\Theta) \to 0$$

implies that Q_s vanishes on $\Theta_u \cdot \Theta_{-u}$ if and only if there exists a vector field D such that

$$Q_s|_{\Theta_u} = D\theta_u \cdot \theta_{-u}.$$

The section $(Q_s - D\theta_u \cdot \theta_{-u} + \theta_u \cdot D\theta_{-u})$ of $\mathcal{O}_X(2\Theta)$ then vanishes on Θ_u and one concludes with the cohomology sequence of the exact sequence

$$0 \to \mathcal{O}_X(\Theta_{-u}) \xrightarrow{\cdot\theta_u} \mathcal{O}_X(2\Theta) \to \mathcal{O}_{\Theta_u}(2\Theta) \to 0. \quad \text{q.e.d.}$$

In particular, P_1, given by (1.5), will vanish for some choice of α_1 and D_1 if and only if

(1.9) $$\Theta_u \cdot \Theta_{-u} \subset \Theta_v \cup \Theta_{-v}.$$

2. Characterizations of Jacobians by a degenerate trisecant

We start from an indecomposable principally polarized abelian variety X and points u and v of X such that (1.9) holds. In geometric terms, this is equivalent to the existence of a line passing through $K(v)$ and tangent to the Kummer variety $K(X)$ at $K(u)$: a degenerate trisecant!

In the notation of the last section, (1.9) translates into: $P_1 = 0$ for some choice of α_1 and D_1. Our aim is to prove that this forces X to be a Jacobian by showing, by induction, that the sections P_s all vanish. Unfortunately, we need an extra hypothesis:

Theorem 2.1. *Let* X *be a complex indecomposable principally polarized abelian variety and let* Θ *be a symmetric representative of the polarization. Then* X *is a Jacobian if and only if there exist points* u *and* v *of* X *with* $u, -u, v, -v$ *all distinct, such that* $\Theta_u \cdot \Theta_{-u} \subset \Theta_v \cup \Theta_{-v}$ *and* $\Theta_u \cdot \Theta_{-u}$ *is reduced.*

Proof of Theorem 2.1. Suppose X is the Jacobian of a curve C. For distinct points p, q, r, s on C, one has

$$\Theta \cdot \Theta_{p-q} = V_p \cup W_q \subset \Theta_{p-r} \cup \Theta_{s-q},$$

where V_q and W_q are integral, depend only on p and q respectively, and are distinct one from another for general choices of p and q. Taking $r = s$, $u = \frac{1}{2}(p+q) - q$, and $v = \frac{1}{2}(p+q) - r$ proves one direction of the theorem.

To prove the other direction, it is enough to show, by (1.6) and (1.8), that for any integer $s \geq 2$, one has

$$P_1 = \cdots = P_{s-1} = 0 \Rightarrow P_s \text{ (or } Q_s) \text{ vanishes on } \Theta_u \cdot \Theta_{-u}.$$

It is convenient to slightly change the setting. Write

$$R(z, \varepsilon) = P(z + \tfrac{1}{2}D(\varepsilon), \varepsilon) = \sum_{s \geq 1} R_s(z)\varepsilon^s.$$

Assume $P_1 = \cdots = P_{s-1} = 0$. As in [1], it is elementary to check that $R_1 = \cdots = R_{s-1} = 0$ and that $P_s = R_s$. Therefore, it is enough to prove that R_s vanishes on $\Theta_u \cdot \Theta_{-u}$. Notice that

$$R(z, \varepsilon) = \alpha(\varepsilon)\theta(z + D(\varepsilon) + u)\theta(z - u) - \theta(z + D(\varepsilon) - u)\theta(z + u)$$
$$+ \varepsilon\theta(z + D(\varepsilon) + v)\theta(z - v),$$

whose restriction to $\Theta_u \cdot \Theta_{-u}$ is just $\varepsilon\theta(z + D(\varepsilon) + v)\theta(z - v)$.

Writing $e^{D(\varepsilon)} = \sum_{s \geq 0} \Delta_s \varepsilon^s$, we get

$$R_s|_{\Theta_u \cdot \Theta_{-u}} = (\Delta_{s-1}\theta_{-v}) \cdot \theta_v.$$

Now set

$$T(z, \varepsilon) = P(z - \tfrac{1}{2}D(\varepsilon), \varepsilon) = \sum_{s \geq 1} T_s(z)\varepsilon^s$$

and $e^{-D(\varepsilon)} = \sum_{s \geq 0} \Delta_s^- \varepsilon^s$. Again, we get

$$T_s|_{\Theta_u \cdot \Theta_{-u}} = \theta_{-v} \cdot (\Delta_{s-1}^- \theta_v).$$

Moreover,

$$T(z, \varepsilon) = R(z - D(\varepsilon), \varepsilon) = e^{-D(\varepsilon)} R(z, \varepsilon),$$

so that

$$T_s = \sum_{i=0}^{s} (\Delta_{s-i}^- R_i) = R_s,$$

since $R_1, \cdots, R_{s-1}$ vanish. It follows that, on $\Theta_u \cdot \Theta_{-u}$, one has

$$R_s^2 = R_s T_s = (\Delta_{s-1}\theta_{-v}) \cdot \theta_v \cdot \theta_{-v} \cdot (\Delta_{s-1}^- \theta_v) = 0.$$

Since $\Theta_u \cdot \Theta_{-u}$ is reduced, R_s vanishes on $\Theta_u \cdot \Theta_{-u}$ and the proof is then completed. q.e.d.

We now give another version of Theorem 2.1, in which some assumptions on Θ and u will automatically insure that $\Theta_u \cdot \Theta_{-u}$ is reduced.

Theorem 2.2. *Let X be a principally polarized complex abelian variety and let Θ be a symmetric representative of the polarization. Suppose that there exist points u and v of X with $u, -u, v, -v$, all distinct, such that:*

(1.9) $$\Theta_u \cdot \Theta_{-u} \subset \Theta_v \cup \Theta_{-v},$$

(2.3) *the closed subgroup of X generated by u has dimension ≥ 2,*

(2.4) $$\dim \operatorname{Sing} \Theta \leq \dim X - 4.$$

Then X is the Jacobian of a nonhyperelliptic curve.

Remarks 2.5. (1) Notice the similarity between the hypothesis of this theorem and those of the theorem on p. 60 of [1].

(2) Hypothesis (2.4) is quite strong, since (1.9) implies $\dim \operatorname{Sing} \Theta \geq \dim X - 4$ [3, Théorème 2.3].

Theorem 2.2 follows immediately from Theorem 2.1 and the following proposition.

Proposition 2.6. *Let X be a principally polarized complex abelian variety. Let Θ be a representative of the polarization and suppose* (2.4) *holds. Let w be a point of X such that the closed subgroup it generates has dimension ≥ 2. Then $\Theta \cdot \Theta_w$ is reduced.*

Proof of Proposition 2.6. Suppose $\Theta \cdot \Theta_w$ is not reduced and pick a nonreduced component Z. Let $n\colon \widetilde{Z} \to Z_{\text{red}}$ be the normalization. The argument of the proof of Proposition 3 of [4] shows that (2.4) implies that w is in the kernel of the composed homomorphism

$$X \xrightarrow{\varphi} \operatorname{Pic}^0 X \xrightarrow{n^*} \operatorname{Pic}^0 \widetilde{Z},$$

where φ is the isomorphism associated with the principal polarization of X. The transpose of n^* is the Albanese morphism $\operatorname{Alb}(\widetilde{Z}) \to X$, whose image contains a translate of Z_{red}, hence has codimension ≤ 2. Using (2.3), this implies that the abelian variety A generated by w (i.e.,the neutral component of the closed subgroup generated by w) has dimension 2, Z_{red} is a translate of an abelian subvariety B of X, and the addition map $\pi\colon A \times B \to X$ is an isogeny. According to Proposition (9.1) of [5], the polarization $\pi^*\Theta$ splits as $L_A \boxtimes L_B$, and there exists a basis $\{s_i\}_{1 \leq i \leq N}$ (resp. $\{t_i\}_{1 \leq i \leq N}$) for $H^0(A, L_A)$ (resp. $H^0(B, L_B)$) such that $\sum_{i=1}^N s_i t_i$ is an equation for $\pi^*\Theta$. Write $Z_{\text{red}} = z_0 + B$ with $z_0 \in A$. The kernel of

$n^*\colon \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(B)$ is connected; hence the kernel of $n^* \circ \varphi$ is A and $w \in A$. We get

$$\forall z \in B \quad \sum_{i=1}^{N} s_i(z_0)t_i(z) = \sum_{i=1}^{N} s_i(z_0 - w)t_i(z) = 0.$$

This implies that both z_0 and $z_0 - w$ are in the base locus F_A of L_A. But F_A is stable by translation by a subgroup $H(L_A)$ of A of order N^2. Since w is not in $H(L_A)$ because it generates A, we get $2N^2$ distinct points of F_A. But $L_A^2 = 2N$; hence either $N = 1$ and X is decomposable, which would contradict (2.4), or F_A is contained in a translate of an elliptic curve, which would contradict the fact that w, which is the difference of two points of F_A, generates A.

Hence $\Theta \cdot \Theta_w$ is reduced. q.e.d.

3. Characterizations of Jacobians by an honest trisecant

In this section, we will show how, under further geometric assumptions, the existence of one honest trisecant implies the existence of a degenerate trisecant. The results of §2 will then yield two new characterizations of Jacobians. The first one is:

Theorem 3.1. *Let X be a complex indecomposable principally polarized abelian variety and let Θ be a representative of the polarization. Then X is a Jacobian if and only if there exist points a, b, c on X with $2a \neq 2b$, such that:*

(3.2) *The points $K(a)$, $K(b)$, $K(c)$ of the Kummer variety of X are distinct and collinear.*

(3.3) *$\Theta_a \cdot \Theta_b$ is reduced and has two irreducible components.*

Proof of Theorem 3.1. We may assume Θ is symmetric. It follows from (1.2) that (3.2) implies $\Theta_a \cdot \Theta_b \subset \Theta_c \cup \Theta_{-c}$.

Write $\Theta_a \cdot \Theta_b = V \cup W$, with V and W integral, $V \subset \Theta_c$, and $W \subset \Theta_{-c}$. Since a and b are different from c and from $-c$, one has

$$V \not\subset \Theta_{-c} \quad \text{and} \quad W \not\subset \Theta_c. \tag{3.4}$$

Notice that $(a + b - V)$ is either V or W. In the latter case, one has $W = a + b - V \subset \Theta_{a+b-c}$ and we can conclude with Theorem 2.1 (where, as before, $u = \frac{1}{2}(a + b) - a$ and $v = \frac{1}{2}(a + b) - c$). We can therefore assume that $V = a+b-V$ and $W = a+b-W$. On the other hand, by (1.2) again, (3.2) also implies

$$\Theta_a \cdot \Theta_c \subset \Theta_b \cup \Theta_{-b}.$$

The scheme $\Theta_a \cdot \Theta_c$ contains V, hence $a+c-V$, which is equal to $a+c-(a+b-V) = V+c-b$, which is not contained in Θ_{-b} by (3.4). It follows that $a+c-V$ is contained in $\Theta_b \cdot \Theta_a \cdot \Theta_c$, whose only component of this dimension is V. Therefore $V = a+c-V = V+c-b$. By switching the roles of a and b, we get $V = V+c-a$; hence $V = V+a-b$. By switching c and $-c$, one also gets $W = W+a-b$, from which it follows that $\Theta_a \cdot \Theta_b$, hence also $\Theta \cdot \Theta_{a-b}$, is invariant by translation by $(a-b)$. This is possible only if $2(a-b) = 0$, which we assumed did not hold. Therefore, one cannot have $V = a+b-V$ and the proof is complete. q.e.d.

We conclude with another characterization of nonhyperelliptic Jacobians in the spirit of theorem 2.2 and the theorem on p. 60 of [1].

Theorem 3.5. *Let X be a principally polarized complex abelian variety and let Θ be a representative of the polarization. Then X is a nonhyperelliptic Jacobian if and only if there exist points a, b, c on X such that:*

(3.6) *The points $K(a)$, $K(b)$, and $K(c)$ of the Kummer variety are distinct and collinear.*

(3.7) *The points $a-c$, $b-a$, $c-b$ generate closed subgroups of X of dimensions at least 2, 2, and $(\dim X - 1)$, respectively.*

(3.8) $$\dim \operatorname{Sing} \Theta \leq \dim X - 4.$$

By working a bit harder, it is possible to weaken the hypothesis on $(c-b)$ by asking only that the subgroup it generates have codimension ≤ 2.

Proof of Theorem 3.5. We may assume Θ is symmetric and $\dim X \geq 4$. It follows from Proposition 2.6 that $\Theta_a \cdot \Theta_b$, $\Theta_b \cdot \Theta_c$, and $\Theta_c \cdot \Theta_a$ are reduced. We begin with:

Lemma 3.9. *Under the hypothesis of the theorem, the components of $\Theta_a \cdot \Theta_b$ which are contained in Θ_c and Θ_{-c} are also contained in Θ_{-a} and Θ_{-b} (same statement with a, b, c permuted).*

Proof of Lemma 3.9. Let Z be a component of $\Theta_a \cdot \Theta_b$ contained in $\Theta_c \cdot \Theta_{-c}$. By differentiating the trisecant relation

$$\alpha\theta_a \cdot \theta_{-a} + \beta\theta_b \cdot \theta_{-b} + \gamma\theta_c \cdot \theta_{-c} = 0,$$

we get that the section $(\alpha D\theta_a \cdot \theta_{-a} + \beta D\theta_b \cdot \theta_{-b})$ vanishes on Z for any constant vector field D.

Since $\Theta_a \cdot \Theta_b$ is reduced, Z is not contained in its singular locus, from which it follows that θ_{-a} and θ_{-b} vanish on Z. q.e.d.

Now let $\mathscr{V}$ be the set of components Z of $\Theta_a \cdot \Theta_b$ such that neither Z nor $(a+b-Z)$ is contained in Θ_{-c}.

Lemma 3.10. *Assume $\mathscr{V}$ is nonempty and let V be the union of the components of $\Theta_a \cdot \Theta_b$ which belong to $\mathscr{V}$. Then $V = V + b - c$.*

Proof of Lemma 3.10. It is enough to show that, for any Z in $\mathscr{V}$, $Z+b-c$ is also in $\mathscr{V}$. Since Z is contained in $\Theta_a \cdot \Theta_b$ but not in Θ_{-c}, it is contained in Θ_c, hence in $\Theta_a \cdot \Theta_c$. Therefore $(a+c-Z) \subset \Theta_a \cdot \Theta_c \subset \Theta_b \cup \Theta_{-b}$. Since $(a+b-Z) \not\subset \Theta_{-c}$, one has

$$(a+c-Z) \not\subset \Theta_{-b}, \tag{3.11}$$

and therefore $(a+c-Z) \subset \Theta_b$, i.e., $(Z+b-c) \subset \Theta_a$.

Furthermore, $Z \subset \Theta_c$ implies $(Z+b-c) \subset \Theta_b$.

Since $Z \not\subset \Theta_{-c}$, Lemma 3.9 implies $Z \not\subset \Theta_{-b}$, that is, $(Z+b-c) \not\subset \Theta_{-c}$. Finally, (3.11) and Lemma 3.9 imply that $(a+c-Z) \not\subset \Theta_{-c}$, that is, $(a+b-(Z+b-c)) \not\subset \Theta_{-c}$. This proves that $(Z+b-c) \in \mathscr{V}$. q.e.d.

The abelian subvariety A of X generated by $(b-c)$ has codimension ≤ 1 by (3.7); hence we get a contradiction. Therefore $\mathscr{V}$ is empty and one has

$$\Theta_a \cdot \Theta_b \subset \Theta_{-c} \cup \Theta_{a+b+c}.$$

Theorem 2.1 implies that X is a Jacobian (set $u = \frac{1}{2}(a+b) - a$ and $v = \frac{1}{2}(a+b) + c$).

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