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REPRESENTATIONS OF $SO(k, \mathbf{C})$ ON HARMONIC POLYNOMIALS ON A NULL CONE

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ABSTRACT. The linear action of the group $SO(k, \mathbf{C})$ on the vector space $\mathbf{C}^{n\times k}$ extends to an action on the algebra of polynomials on $\mathbf{C}^{n\times k}$. The polynomials that are fixed under this action are called $SO(k, \mathbf{C})$-invariant. The $SO(k, \mathbf{C})$-harmonic polynomials are common solutions of the $SO(k, \mathbf{C})$-invariant differential operators. The ideal of all $SO(k, \mathbf{C})$-invariants without constant terms, the null cone of this ideal, and the orbits of $SO(k, \mathbf{C})$ on this null cone are studied in great detail. All irreducible holomorphic representations of $SO(k, \mathbf{C})$ are concretely realized on the space of $SO(k, \mathbf{C})$-harmonic polynomials.

1. INTRODUCTION

Let G be a linear algebraic reductive subgroup of the group $\mathrm{GL}(E)$ of all invertible linear transformations on a finite dimensional complex vector space E. If $S(E^*)$ is the symmetric algebra of all polynomial functions on E then the action of G on E induces an action of G on $S(E^*)$, denoted by $g \cdot p$, for $g \in G$ and $p \in S(E^*)$. We say that $p \in S(E^*)$ is *G-invariant* if $g \cdot p = p$ for all $g \in G$. The G-invariant polynomial functions form a subalgebra $J(E^*)$ of $S(E^*)$. Given $X \in E$, let ∂_X denote the differential operator defined by

$$[\partial_X f](Y) = \left\{ \frac{d}{dt} f(Y + tX) \right\}_{t=0}, \qquad t \in \mathbf{R},$$

for all smooth functions f on E. The map $X \to \partial_X$ induces an isomorphism of the algebra $S(E^*)$ onto the algebra of all differential operators on E with constant coefficients. The image of an element $p \in S(E^*)$ under this isomorphism is denoted by $p(D)$. If $J_+(E^*)$ is the subset of all elements in $J(E^*)$ without constant term, then an element $f \in S(E^*)$ is said to be *G-harmonic* if $p(D)f = 0$ for all $p \in J_+(E^*)$. The subspace of all G-harmonic polynomial functions in $S(E^*)$ is denoted by $H(E^*)$. The study of $H(E^*)$ and the decomposition $S(E^*) = J(E^*)H(E^*)$ (the "separation of variable" theorem) was

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initiated by H. Maass in [M1] and [M2] and was extensively developed by S. Helgason in [H1] and by B. Kostant in [K]. Several authors have investigated the representation theory for specific types of Lie groups G on $H(E^*)$. A non-exhaustive list of publications on this subject includes [L], [G], [S], [T1], [T2], [K-O], [K-V], [G-K], and [G-P-R].

It was shown in [T1] that, up to isomorphism, all irreducible holomorphic representations of a Lie group G of type B_l or D_l can be concretely realized as G-submodules of $H(E^*)$ except for the case of the "mirror-conjugate representations" of D_l which was left unsettled (see [Z, Chapter XVI, §114] for the definition of the "mirror-conjugate representations"). In this paper we will settle this special case in conjunction with the description, in both cases B_l and D_l, of the ideal $J_+(E^*)S(E^*)$ and a detailed description of the orbit structure of the G action in the null cone P of the common zeros of polynomial functions in $J_+(E^*)S(E^*)$.

2. Description of the ideal $J_+(E^*)S(E^*)$ and its null cone

In this article E denotes $\mathbf{C}^{n\times k}$ and G is $SO(k, \mathbf{C})$. Then G acts linearly on E by right multiplication and leaves the nondegenerate symmetric bilinear form $(X, Y) \to \operatorname{tr}(XY^t)$, $X, Y \in E$, invariant. It follows that the function X^* defined by $X^*(Y) = \operatorname{tr}(XY^t)$ is an element of E^*, the dual of E. It was shown in [T1] that the algebra $S(E^*)$ can be equipped with the inner product

$$\langle p_1, p_2\rangle = p_1(D)\overline{p_2(\overline{X})}|_{X=0}, \qquad p_1, p_2 \in S(E^*),$$

which is invariant under the restriction of the action of G to $G_0 = SO(k)$.

A slight modification of the techniques in [H2, Chapter III], leads to the following results concerning $S(E^*)$ and $H(E^*)$.

The algebra $S(E^*)$ is decomposed into an orthogonal direct sum with respect to the inner product given above as $S(E^*) = J_+(E^*)S(E^*) \oplus H(E^*)$. If $H_1(E^*)$ denotes the subspace of $H(E^*)$ spanned by the polynomial functions of the form $(X^*)^m$, $X \in P$, $m = 0, 1, 2, \ldots$ and if $H_2(E^*)$ denotes the subspace of $H(E^*)$ of all polynomial functions which vanish on P then we have the orthogonal direct sum decomposition $H(E^*) = H_1(E^*) \oplus H_2(E^*)$. Moreover, the linear subspace $J_+(E^*)S(E^*) \oplus H_2(E^*)$ is the ideal in $S(E^*)$ of all polynomial functions which vanish on P, i.e., the ideal $\sqrt{J_+(E^*)S(E^*)}$.

We will now study this ideal $J_+(E^*)S(E^*)$. Recall ([W], [D-P, Theorem 5.6ii]) that it is generated by the $n(n+1)/2$ polynomials

$$p_{ij}(X) = \sum_{s=1}^{k} X_{is}X_{js}, \qquad 1 \le i \le j \le n,$$

together with the $(k \times k)$-minors of the matrix X (which are 0 when $k > n$). We also derive geometric properties of the *null cone* P of E defined by $J_+(E^*)S(E^*)$.

The following theorem sums up our results, which extend earlier results of [T1] (case $k > 2n$) and [H] (case k odd, $k < 2n$).

Theorem 2.1

(i) *For $k > 2n$, the ideal $J_+(E^*)S(E^*)$ is prime. The scheme P is a complete intersection, with one open dense orbit.*

(ii) *For $k = 2n$, the ideal $J_+(E^*)S(E^*)$ is the intersection of two prime ideals, hence it is radical. The scheme P is a complete intersection, with two open orbits.*

(iii) *For $k < 2n$, the ideal $J_+(E^*)S(E^*)$ is not radical, except for $k \leq 2$. The orbits are nowhere dense, except for $k = 1$. For k odd > 1, P is irreducible and nowhere reduced. For k even, P has two irreducible components and is generically reduced (but not reduced except for $k = 2$).*

Proof. As a set, the scheme P is $\{X \in \mathbf{C}^{n\times k} | XX^t = 0 \text{ and } \mathrm{Rank}(X) < k\}$. Therefore, a matrix X is in P if and only if the image $\Lambda(X)$ of the morphism $X^t : \mathbf{C}^n \to \mathbf{C}^k$ is totally isotropic for the quadratic form $\sum_{1\leq i\leq k} Y_i^2$ (such a space is automatically of dimension $\leq k/2$). The space $\Sigma_{r,k}$ of all r-dimensional totally isotropic spaces for a non-degenerate quadratic form has a long history. We borrow the following facts from [G-H, pp. 735–739]:

(a) $\Sigma_{r,k}$ is empty for $r > k/2$.

(b) $\Sigma_{r,k}$ has dimension $r(k - (3r + 1)/2)$ if $r \leq k/2$. It is irreducible for $r < k/2$ but has two irreducible components for $k = 2r$.

Now, by Witt's theorem [A, Chapter III], two elements X_1 and X_2 of P are in the same G-orbit if and only if $\Lambda(X_1)$ and $\Lambda(X_2)$ are of the same rank r and are in the same component of $\Sigma_{r,k}$. We get the following description of the subspaces P_r of P consisting of matrices of rank r:

(2.1) P_r is empty for $r > n$ or $r > k/2$. For $r \leq n$ and $r \leq k/2$, it has dimension

$$\dim\{\text{surjections } \mathbf{C}^n \to \mathbf{C}^r\} + \dim \Sigma_{r,k} = r(n + k - (3r + 1)/2).$$

(2.2) For $r < k/2$ and $r \leq n$, P_r is covered by the following (non-disjoint) G-orbits: where A is any $(n - r) \times r$ matrix.

$$\begin{array}{c} \\ r\left\{\vphantom{\begin{matrix}1\\ \ddots \\ 1\end{matrix}}\right. \\ n-r\{ \end{array} \begin{array}{c} \begin{array}{ccc} \quad r \quad & \quad r \quad & k-2r \end{array} \\ \left[\begin{array}{c|c|c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \begin{matrix}i & & \\ & \ddots & \\ & & i\end{matrix} & 0 \\ \hline A & iA & \end{array}\right] \end{array} \cdot G,$$

(2.3) For $k = 2r$ and $r \leq n$, P_r is covered by the following two families of (non-disjoint) G-orbits:

$$\begin{array}{cc} & \begin{array}{cc} \quad r \quad & \quad r \quad \end{array} \\ \begin{array}{c} r\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ n-r\{ \end{array} & \left[\begin{array}{ccc|ccc} 1 & & & i & & \\ & \ddots & & & \ddots & \\ & & 1 & & & i \\ \hline & A & & & iA & \end{array}\right] \cdot G \end{array}$$

and

$$\begin{array}{cc} & \begin{array}{cc} \quad r \quad & \quad r \quad \end{array} \\ \begin{array}{c} r\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ n-r\{ \end{array} & \left[\begin{array}{ccc|cccc} 1 & & & i & & & \\ & \ddots & & & \ddots & & \\ & & & & & i & \\ & & 1 & & & & -i \\ \hline & A & & & A' & & \end{array}\right] \cdot G, \end{array}$$

where, again, A is any $(n-r) \times r$ matrix and A' is the matrix obtained by switching signs in the last column of iA.

It follows easily from (2.2) that P_r is contained in the closure of P_{r+1} whenever the latter is non-empty. This yields the following geometric description of P:

(2.4) For $k \geq 2n$, the maximal r for which P_r is non-empty is n. The stratum P_n is dense in P, which therefore has dimension $nk - n(n+1)/2$ or codimension $n(n+1)/2$ in E (by (2.1)). It follows moreover, from (2.2) and (2.3), that

(a) For $k > 2n$, P_n is just one orbit.

$$\left[\begin{array}{ccc|ccc|c} 1 & & & i & & & \\ & \ddots & & & \ddots & & 0 \\ & & 1 & & & i & \end{array}\right] \cdot G$$

(b) For $k = 2n$, P_n is the union of 2 orbits:

$$P_n^+ = \left[\begin{array}{ccc|ccc} 1 & & & i & & \\ & \ddots & & & \ddots & \\ & & 1 & & & i \end{array}\right] \cdot G$$

and

$$P_n^- = \left[\begin{array}{ccc|cccc} 1 & & & i & & & \\ & \ddots & & & \ddots & & \\ & & & & & i & \\ & & 1 & & & & -i \end{array}\right] \cdot G.$$

(2.5) For $k < 2n$, write $k = 2k' + \varepsilon$ with $\varepsilon = 0$ or 1, i.e., k' is the rank of the group $SO(k, \mathbf{C})$. The maximal r for which P_r is non-empty is k'. The stratum $P_{k'}$ is dense in P, which therefore has dimension $k'(n + k - (3k' + 1)/2)$ or codimension $k'n - k'(k' - 1)/2 + (n - k')\varepsilon$ in E (by (2.1)). Moreover, by (2.2) and (2.3), P has 1 irreducible component when k is odd, 2 when k is even. As suspected in [T1, Remark 2.10 (1)], $P_{k'}$ contains, in general, infinitely many orbits. More precisely, each such orbit is determined by the $(n - k')$-dimensional linear subspace $\operatorname{Ker} X^t$ of $\mathbf{C}^n$, hence $P_{k'}/G$ is isomorphic to the $k'(n - k')$-dimensional Grassmannian $G(k', n - k')$ and is infinite for $k > 1$.

We now turn our attention to the *scheme structure* of P, given by the ideal $I = J_+(E^*)S(E^*)$.

Case $k \geq 2n$. By (2.4), P has codimension equal to the number of its defining equations. In other words, P is the *complete intersection* of the p_{ij}'s. Since the Unmixedness Theorem holds in the polynomial ring $S(E^*)$ [M, p. 107 and Theorem 31, p. 108], to check that P is reduced (i.e., that I is radical), it is enough to find one point on each component of P at which P is reduced. But this follows from [T1, Lemma 2.9], where it is shown, using the Jacobian criterion, that P is smooth on the dense stratum P_n.

Case $k < 2n$. The situation is very different. Recall that we wrote $k = 2k' + \varepsilon$, where $\varepsilon = 0$ or 1. By (2.5), the codimension of P is $(k'n - k'(k' - 1)/2 + (n - k')\varepsilon)$, which is strictly less than the number of its defining equations, except for $k = 2n - 1 > 1$. Except for this case, where P is still the complete intersection of the p_{ij}'s and I is primary, P may well have embedded primes (and it does, at least for k even > 2).

The same argument used in the above mentioned Lemma 2.9 of [T1] shows that, on the dense stratum $P_{k'}$, the rank of the Jacobian matrix of the p_{ij}'s is $(k'n - k'(k' - 1)/2)$. If $k' < k - 1$, that is if $k > 2$, the derivatives of the $(k \times k)$-minors are 0 on $P_{k'}$. It follows that, for k odd > 1, P is nowhere reduced, hence I is not radical but that, for k even, P is generically smooth!

This is therefore not enough to conclude in the case k even and we will use a direct computation, based on (2.1) only, to show that I is not radical whenever $2 < k < 2n$.

Recall that any element of P has rank $\leq k'$. Thus, if $X' = (X_{ij})_{1 \leq i, j \leq k'+1}$, $(\det X')$ vanishes on P, hence is in the radical of I (recall that $k' + 1 \leq n$). Suppose $(\det X') \in I$. Substitute 0 for all the X_{ij}'s except for X_{12}, X_{21} and X_{ii} for $1 \leq i \leq k' + 1$. When $k > k' + 1$, that is when $k > 2$, all the $(k \times k)$-minors of X vanish and the only non-zero p_{ij}'s left are $p_{11} = X_{11}^2 + X_{12}^2$, $p_{12} = X_{11}X_{21} + X_{12}X_{22}$, $p_{22} = X_{21}^2 + X_{22}^2$ and $p_{ii} = X_{ii}^2$ for $3 \leq i \leq k' + 1$.

Modulo $(X_{33}^2, \ldots, X_{k'+1,k'+1}^2)$, we get a congruence of the type

$$(X_{11}X_{22} - X_{12}X_{21})X_{33}\cdots X_{k'+1,k'+1} = \sum_{1\leq i\leq j\leq 2} q_{ij}p_{ij}.$$

By comparing coefficients of $X_{33}\cdots X_{k'+1,k'+1}$, we get $X_{11}X_{22} - X_{12}X_{21} \in (p_{11}, p_{12}, p_{22})$, which is untrue. Therefore, $(\det X')$ is not in I when $k \geq 3$, and I is not radical. Direct calculations show that, for $k = 1$, $I = (X_{11}, \ldots, X_{n1})$ is prime and that, for $k = 2$, I is radical. □

It follows immediately from Theorem 2.1 and the remarks made earlier that, for the case $k \geq 2n$, the ideal $J_+(E^*)S(E^*)$ is radical (prime when $k > 2n$) and hence $H_2(E^*) = \{0\}$. Thus we have the following result which extends an earlier result in [T1, Theorem 2.5].

Corollary 2.2 ("Separation of variables" theorem for $S(E^*)$, $E = \mathbf{C}^{n\times k}$, $k \geq 2n$).

(i) *The algebra of all polynomial functions on* $\mathbf{C}^{n\times k}$, $k \geq 2n$, *can be decomposed as*

$$S(E^*) = J_+(E^+)S(E^*) \oplus H(E^*)(\textit{orthogonal direct sum})$$
$$S(E^*) = J(E^*)H(E^*).$$

(ii) *The space* $H(E^*)$ *is generated by all powers of all polynomial functions* f *satisfying*

$$f(X) = \sum_{i,j} A_{ij}X_{ij} \text{ with } AA^t = 0, \qquad 1 \leq i \leq n,\ 1 \leq j \leq k.$$

(iii) *The space* $H_2(E^*)$ *of all G-harmonic polynomial functions that vanish on the null cone* P *of the common zeros of polynomial functions in* $J_+(E^*)S(E^*)$ *is zero.*

For the case $k = 2n$, Corollary 2.2 will play a crucial role in the proof of Theorem 3.1 of the next section.

3. The harmonic representations of $SO(k, \mathbf{C})$

In [T1] it was shown that all irreducible holomorphic representations of $G = SO(k, \mathbf{C})$ can be explicitly realized as G-submodules of $H(E^*)$ with the exception of the "mirror-conjugate representations" of G. In this section we will show that these representations too can be realized on $H(E^*)$ and have a very interesting characterization on the orbits of G in P, which was studied in Theorem 2.1. Since "the mirror-conjugate representations" can only occur when k is even we shall assume henceforth that $k = 2n$.

Let

$$\gamma = \frac{1}{\sqrt{2}} \left[\begin{array}{ccc|ccc} 1 & & & & & 1 \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & 1 & & \\ \hline -i & & & & & i \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & -i & i & & \end{array}\right] \begin{array}{l} \\ n \\ \\ \\ n \\ \\ \end{array}$$

Then

$$\gamma^{-1} = \frac{1}{\sqrt{2}} \left[\begin{array}{ccc|ccc} 1 & & & i & & \\ & \ddots & & & \ddots & \\ & & 1 & & & i \\ \hline & & 1 & & & -i \\ & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & \\ 1 & & & -i & & \end{array}\right],$$

where the entries not exhibited are 0. Then

$$(\gamma^{-1})(\gamma^{-1})^t = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} = \sigma.$$

If $\widetilde{G} = \gamma^{-1} G \gamma$, then it can be easily verified that

$$\widetilde{G} = \{\tilde{g} \in \mathrm{GL}(2n, \mathbf{C}) : \tilde{g}\sigma(\tilde{g})^t = \sigma, \det \tilde{g} = 1\}.$$

In general we shall denote, by $\tilde{g}$, the image of g under the conjugation $g \to \gamma^{-1} g \gamma$, and vice versa. It follows that $\widetilde{G}$ is the connected component of the identity in the group of linear transformations which preserve the symmetric bilinear form $\mathrm{tr}(x\sigma y^t) = x_1 y_{2n} + x_2 y_{2n-1} + \cdots + x_{2n} y_1$, for all $x = (x_1, \ldots, x_{2n})$ and $y = (y_1, \ldots, y_{2n})$ in $\mathbf{C}^{2n}$. It is well-known (cf. [Z, Chapter XVI, §114]) that $\widetilde{G}$ has the Gauss decomposition induced by $\mathrm{GL}(2n, \mathbf{C})$

$$\widetilde{G} = \overline{\widetilde{Z}_- \widetilde{D} \widetilde{Z}_+}, \tag{3.1}$$

where the components $\widetilde{Z}_-$, $\widetilde{D}$, $\widetilde{Z}_+$ are the intersections of $\widetilde{G}$ with the subgroups $Z_-(2n)$, $D(2n)$, and $Z_+(2n)$ of all lower triangular unipotent, of all diagonal, and of all upper triangular unipotent matrices of $\mathrm{GL}(2n, \mathbf{C})$, respectively. It follows that $\widetilde{B} = \widetilde{Z}_- \widetilde{D}$ is a Borel subgroup of $\widetilde{G}$ which consists of all lower triangular matrices of the form

$$\tilde{b} = \begin{bmatrix} b_{11} & & & & & \\ & \ddots & & & & \\ & & b_{nn} & & & \\ & & & b_{nn}^{-1} & & \\ & & & & \ddots & \\ & & & & & b_{11}^{-1} \end{bmatrix}. \tag{3.2}$$

Let $(m)^+$ denote an n-tuple of positive integers $(m_1, m_2, \ldots, m_n)$ satisfying the condition $m_1 \geq m_2 \geq \cdots \geq m_n > 0$, and set $(m)^- = (m_1, \ldots, m_{n-1}, -m_n)$. Define the holomorphic characters $\pi^{(m)^+}$ and $\pi^{(m)^-}$ on $\widetilde{B}$ by

$$\pi^{(m)^+}(\tilde{b}) = b_{11}^{m_1} \cdots b_{nn}^{m_n} \quad \text{and} \quad \pi^{(m)^-}(\tilde{b}) = b_{11}^{m_1} \cdots b_{nn}^{-m_n}$$

for all $\tilde{b} \in \widetilde{B}$. Set

$$V^{(m)^+} = \{f : \widetilde{G} \to \mathbf{C} : f \text{ holomorphic and } f(\tilde{b}\tilde{g}) = \pi^{(m)^+}(\tilde{b})f(\tilde{g}) \quad \forall (\tilde{b}, \tilde{g}) \in \widetilde{B} \times \widetilde{G}\}$$

and

(3.3)

$$V^{(m)^-} = \{f : \widetilde{G} \to \mathbf{C} : f \text{ holomorphic and } f(\tilde{b}\tilde{g}) = \pi^{(m)^-}(\tilde{b})f(\tilde{g}) \quad \forall (\tilde{b}, \tilde{g}) \in \widetilde{B} \times \widetilde{G}\}.$$

Let $\widetilde{R}_\pi^{(m)^+}$ (respectively $\widetilde{R}_\pi^{(m)^-}$) denote the representation of $\widetilde{G}$ obtained by right translation on $V^{(m)^+}$ (respectively $V^{(m)^-}$). Then, by the Borel-Weil theorem, $\widetilde{R}_\pi^{(m)^+}$ (respectively $\widetilde{R}_\pi^{(m)^-}$) is irreducible with signature $(m)^+$ (respectively $(m)^-$). These representations are termed "mirror-conjugate representations" of G in [Z, Chapter XVI]. Moreover, if $\tilde{g} \in \widetilde{G}$ then, in the Gauss decomposition of $\widetilde{G}$, $\tilde{g} = \tilde{b}[\tilde{g}]\tilde{z}[\tilde{g}]$ with $\tilde{b}[\tilde{g}] \in \widetilde{B}$ and $\tilde{z}[\tilde{g}] \in \widetilde{Z}_+$, and $(\tilde{b}[\tilde{g}])_{ii} = \Delta_i(\tilde{g})/\Delta_{i-1}(\tilde{g})$, where $\Delta_i(\tilde{g})$ is the ith principal minor of $\tilde{g}$, $\Delta_0(\tilde{g}) = 1$, $1 \leq i \leq n$, so that the highest weight vector of $V^{(m)^+}$ (respectively $V^{(m)^-}$) is given by

$$f^{(m)^+}(\tilde{g}) = \pi^{(m)^+}(\tilde{b}[\tilde{g}]) = \Delta_1^{m_1-m_2}(\tilde{g})\Delta_2^{m_2-m_3}(\tilde{g}) \cdots \Delta_n^{m_n}(\tilde{g}) \tag{3.4}$$

and

$$f^{(m)^-}(\tilde{g}) = \pi^{(m)^-}(\tilde{b}[\tilde{g}]) = \Delta_1^{m_1-m_2}(\tilde{g})\Delta_2^{m_2-m_3}(\tilde{g}) \cdots \Delta_n^{-m_n}(\tilde{g})$$

for all $\tilde{g} \in \widetilde{G}$.

For the same n-tuple, $(m_1, m_2, \ldots, m_n) = (m)^+$, define a holomorphic character of $B \equiv B(n)$, the lower triangular subgroup of $\mathrm{GL}(n, \mathbf{C})$, by

$$\xi^{(m)}(b) = b_{11}^{m_1} \cdots b_{nn}^{m_n}, \qquad b \in B.$$

Let $H(E^*, (m))$ denote the subspace of all G-harmonic polynomial functions p which also satisfy the covariant condition

$$p(bx) = \xi^{(m)}(b)p(X), \qquad \forall (b, X) \in B \times E.$$

Let

$$X_0^+ = \left[\begin{array}{ccc|ccc} 1 & & & i & & \\ & \ddots & & & \ddots & \\ & & 1 & & & i \end{array}\right]$$

and

$$X_0^- = \left[\begin{array}{ccc|ccc} 1 & & & i & & \\ & \ddots & & & \ddots & \\ & & 1 & & & i \\ & & & & & -i \end{array}\right].$$

Then it follows from Equation (2.7) of Theorem 2.1 that P_n is the union of two orbits, $P_n^+ = X_0^+ \cdot G$ and $P_n^- = X_0^- \cdot G$. Let $H^+(E^*, (m))$ (respectively $H^-(E^*(m)))$ denote the subspace of all functions in $H(E^*, (m))$ which vanish on the orbit P_n^- (respectively P_n^+). Then we have

Theorem 3.1. (i) *If $H(E^*, (m))$ is the subspace of $H(E^*)$ consisting of all G-harmonic polynomial functions p which also satisfy the covariant condition $p(bX) = \xi^{(m)}(b)p(X)$ for all $(b, X) \in B \times E$, then $H(E^*, (m))$ is decomposed into an orthogonal direct sum as*

$$H(E^*, (m)) = H^+(E^*, (m)) \oplus H^-(E^*, (m)).$$

(ii) *If $R^{(m)}$ denotes the representation of G obtained by right translation on $H(E^*, (m))$, then the restriction $R^{(m)^+}$ (respectively $R^{(m)^-}$) of $R^{(m)}$ to $H^+(E^*, (m))$ (respectively $H^-(E^*, (m)))$ is irreducible with signature $(m)^+ = (m_1, \dots, m_n)$ (respectively $(m)^- = (m_1, \dots, m_{n-1}, -m_n)$).*

Proof. We define a representation $R_\pi^{(m)}G$ on $V^{(m)^+} \oplus V^{(m)^-}$ by

$$[R_\pi^{(m)}(g_0)(f^+ + f^-)](\tilde{g}) = f^+(\tilde{g}\tilde{g}_0) + f^-(\tilde{g}\tilde{g}_0),$$

for all $f^+ \in V^{(m)^+}$, $f^- \in V^{(m)^-}$, $g_0 \in G$, $\tilde{g} \in \widetilde{G}$. Using the "Weyl's unitarian trick" (cf. [V, §4.11]), we may equip $V^{(m)^+} \oplus V^{(m)^-}$ with an inner product which is invariant under the compact real form $G_0 = SO(k)$ of G, and, using Schur's orthogonality relations, we can show that since $V^{(m)^+}$ and $V^{(m)^-}$ are inequivalent G_0-simple modules they form an orthogonal direct sum relative to the G_0-invariant inner product. Set

$$\begin{array}{r} n-1\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ S_0 = 2\left\{\begin{array}{c}\\ \\ \end{array}\right. \\ n-1\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \end{array} \left[\begin{array}{ccc|cc|ccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ \hline & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ \hline & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right], \qquad \Pi = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \end{array}\right]\Big\}\, n,$$

$$\begin{array}{ccc} n-1 & 2 & n-1 \end{array} \qquad\qquad \begin{array}{cc} n & n \end{array}$$

and define a linear map $\Lambda : H(E^*.(m)) \to V^{(m)^+} \oplus V^{(m)^-}$ by

$$\Lambda p = \Lambda^+ p + \Lambda^- p, \quad \text{for all } p \in H(E^*, (m)),$$

where

$$\Lambda^+ p(\tilde{g}) = p(\Pi \tilde{g} \gamma^{-1})$$

and

$$\Lambda^- p(\tilde{g}) = p(\Pi s_0 \tilde{g} \gamma^{-1}), \quad \text{for all } \tilde{g} \in \widetilde{G}.$$

Then

$$\Lambda^+ p(\tilde{g}) = p\left(\frac{1}{\sqrt{2}} X_0^+ g\right) \quad \text{and} \quad \Lambda^- p(\tilde{g}) = p\left(\frac{1}{\sqrt{2}} X_0^- g\right), \tag{3.5}$$

for all $g = \gamma \tilde{g} \gamma^{-1} \in G$. Let us verify that Λ^+ (respectively Λ^-) does indeed map $H(E^*, (m))$ into $V^{(m)^+}$ (respectively $V^{(m)^-}$). If

$$\tilde{b} = \begin{bmatrix} b_{11} & & & & & \\ & \ddots & & & & \\ & & b_{nn} & & & \\ & * & & b_{nn}^{-1} & & \\ & & & & \ddots & \\ & & & & & b_{11}^{-1} \end{bmatrix}$$

is an element of $\widetilde{B}$ then $\Pi \tilde{b} = b \Pi$ with

$$b = \begin{bmatrix} b_{11} & & \\ & \ddots & \\ * & & b_{nn} \end{bmatrix}$$

in $B(n)$. So

$$\begin{aligned} \Lambda^+ p(\tilde{b} \tilde{g}) &= p(\Pi \tilde{b} \tilde{g} \gamma^{-1}) = p(b \Pi \tilde{g} \gamma^{-1}) \\ &= \xi^{(m)}(b) p(\Pi \tilde{g} \gamma^{-1}) = \pi^{(m)^+}(\tilde{b}) \Lambda^+ p(\tilde{g}) \end{aligned}$$

and obviously $\Lambda^+ p$ is a holomorphic function on $\widetilde{G}$. So Λ^+ maps $H(E^*, (m))$ into $V^{(m)^+}$. Similarly, $\Pi s_0 \tilde{b} = \Pi (s_0 \tilde{b} s_0) s_0$ since $s_0^{-1} = s_0$, and

$$s_0 \tilde{b} s_0 = \begin{bmatrix} b_{11} & & & & & & \\ & \ddots & & & & & \\ & & b_{n-1,n-1} & & & & \\ & & & b_{nn}^{-1} & & & \\ & & & & b_{nn} & & \\ & & & & & \ddots & \\ & & & & & & b_{11}^{-1} \end{bmatrix},$$

$\Pi s_0 \tilde{b} s_0 = b^- \Pi$, with

$$b^- = \begin{bmatrix} b_{11} & & & \\ & \ddots & & \\ & & b_{n-1,n-1} & \\ & & & b_{n,n}^{-1} \end{bmatrix}$$

in $B(n)$. Therefore,

$$\begin{aligned}\Lambda^- p(\tilde{b}\tilde{g}) &= p(\Pi s_0 \tilde{b}\tilde{g}\gamma^{-1}) \\ &= p(\Pi(s_0\tilde{b}s_0)s_0\tilde{g}\gamma^{-1}) \\ &= p(b^- \Pi s_0 \tilde{g}\gamma^{-1}) \\ &= b_{11}^{m_1}\cdots b_{n-1,n-1}^{m_{n-1}} b_{nn}^{-m_n} p(\Pi s_0\tilde{g}\gamma^{-1}) \\ &= \pi^{(m)^-}(\tilde{b})\Lambda^- p(\tilde{g}).\end{aligned}$$

So Λ^- maps $H(E^*, (m))$ into $V^{(m)^-}$. Now Λ is an intertwining operator since

$$\begin{aligned}[\Lambda(R^{(m)}(g_0)p)](\tilde{g}) &= [R^{(m)}(g_0)p](\Pi\tilde{g}\gamma^{-1}) + [R^{(m)}(g_0)p](\Pi s_0\tilde{g}\gamma^{-1}) \\ &= p(\Pi\tilde{g}\gamma^{-1}g_0) + p(\Pi s_0\tilde{g}\gamma^{-1}g_0) \\ &= p(\Pi\tilde{g}\tilde{g}_0\gamma^{-1}) + p(\Pi s_0\tilde{g}\tilde{g}_0\gamma^{-1}) \\ &= \Lambda^+ p(\tilde{g}\tilde{g}_0) + \Lambda^- p(\tilde{g}\tilde{g}_0) \\ &= [R_\pi^{(m)}(g_0)\Lambda p](\tilde{g})\end{aligned}$$

for all $g_0 \in G$ and $\tilde{g} \in \widetilde{G}$.

Let $p_\xi(X) = \Delta_1^{m_1-m_2}(X)\Delta_2^{m_2-m_3}(X)\ldots\Delta_{n-1}^{m_n-m_{n-1}}(X)\Delta_n^{m_n}(X)$ and set $p_0^{(m)^+}(X) = p_\xi(X\gamma)$, $p_0^{(m)^-}(X) = p_\xi(X\gamma s_0)$. Then an easy computation analogous to the one in Lemma 3.4 of [T1] shows that both $p_0^{(m)^+}$ and $p_0^{(m)^-}$ belong to $H(E^*, (m))$. Moreover, from Equation (3.4) it follows that

$$\begin{aligned}\Lambda^+ p_0^{(m)^+}(\tilde{g}) &= p_0^{(m)^+}(\Pi\tilde{g}\gamma^{-1}) = p_\xi(\Pi\tilde{g}\gamma^{-1}\gamma) \\ &= p_\xi(\Pi\tilde{g}) = f^{(m)^+}(\tilde{g}).\end{aligned}$$

Now it can be shown that the conjugation $\tilde{g} \to s_0\tilde{g}s_0$ preserves the Gauss decomposition (see [Z, Chapter XVI, §114]) and, if

$$\tilde{b}[\tilde{g}] = \begin{bmatrix} b_{11} & & & & & & \\ & \ddots & & & & & \\ & & b_{n-1,n-1} & & & & \\ & & & b_{nn} & & & \\ & * & & & b_{nn}^{-1} & & \\ & & & & & \ddots & \\ & & & & & & b_{11}^{-1} \end{bmatrix},$$

then

$$\tilde{b}[s_0\tilde{g}s_0] = \begin{bmatrix} b_{11} & & & & & & \\ & \ddots & & & & & \\ & & b_{n-1,n-1} & & & & \\ & & & b_{n,n}^{-1} & & & \\ & & & & b_{n,n} & & \\ & & * & & & b_{n-1,n-1}^{-1} & \\ & & & & & & b_{11}^{-1} \end{bmatrix}$$

so that

$$\begin{aligned} \Lambda^- p_0^{(m)}(\tilde{g}) &= p_0^{(m)^-}(\Pi s_0\tilde{g}\gamma^{-1}) = p_\xi(\Pi s_0\tilde{g}\gamma^{-1}\gamma s_0) \\ &= p_\xi(\Pi s_0\tilde{g}s_0) \\ &= p_\xi(\Pi\tilde{b}[s_0\tilde{g}s_0]\tilde{z}[s_0\tilde{g}s_0]) \\ &= p_\xi(b^v\Pi\tilde{z}[s_0\tilde{g}s_0]) \\ &= \xi^{(m)}(b^v)p_\xi(\Pi\tilde{z}[s_0\tilde{g}s_0]) \\ &= f^{(m)^-}(\tilde{g}), \end{aligned}$$

for

$$b^v = \begin{bmatrix} b_{11} & & & \\ & \ddots & & \\ & & b_{n-1,n-1} & \\ * & & & b_{n,n}^{-1} \end{bmatrix}$$

and $p(\Pi\tilde{z}[s_0\tilde{g}s_0]) = 1$. Also,

$$\begin{aligned} \Lambda^+ p_0^{(m)^-}(\tilde{g}) &= p_0^{(m)^-}(\Pi\tilde{g}\gamma^{-1}) = p_\xi(\Pi\tilde{g}\gamma^{-1}\gamma s_0) \\ &= p_\xi(\Pi\tilde{g}s_0) = p_\xi(\Pi b[\tilde{g}]\tilde{z}[\tilde{g}]s_0) \\ &= p_\xi(b\Pi\tilde{z}[\tilde{g}]s_0) \\ &= \xi^{(m)}(b)p_\xi(\Pi\tilde{z}[\tilde{g}]s_0) \\ &= \xi^{(m)}(b)p_\xi(\Pi s_0(s_0\tilde{z}[\tilde{g}]s_0)). \end{aligned}$$

Since the conjugation $\tilde{g} \to s_0\tilde{g}s_0$ preserves the Gauss decomposition of $\widetilde{G}$, $s_0\tilde{z}[\tilde{g}]s_0$ is of the form

$$\left[\begin{array}{cccc|ccc} 1 & & & & & & \\ & 1 & & * & & * & \\ & & \ddots & & & & \\ & & & 1 & & & \\ \hline & & & & 1 & & \\ & & 0 & & & \ddots & \\ & & & & & & 1 \end{array}\right]$$

and

$$\Pi s_0(s_0\tilde{z}[\tilde{g}]\tilde{s}_0) = \left[\begin{array}{cccc|c} 1 & & & & \\ & 1 & & * & \\ & & \ddots & & * \\ & & & 0 & \end{array}\right]$$

so that $\Delta_n(\Pi s_0(s_0\tilde{z}[\tilde{g}]s_0)) = 0$ and hence $p_\xi(\Pi s_0(s_0\tilde{z}[\tilde{g}]s_0)) = 0$. It follows that $\Lambda^+ p_0^{(m)^-} = 0$. Similarly,

$$\begin{aligned} \Lambda^- p_0^{(m)^+}(\tilde{g}) &= p_0^{(m)^+}(\Pi s_0\tilde{g}\gamma^{-1}) = p_\xi(\Pi s_0\tilde{g}) \\ &= p_\xi(\Pi s_0\tilde{b}[\tilde{g}]\tilde{z}[\tilde{g}]) \\ &= p_\xi(\Pi(s_0\tilde{b}[\tilde{g}]s_0)s_0\tilde{z}[\tilde{g}]). \end{aligned}$$

Again, since the conjugation $\tilde{g} \to s_0\tilde{g}s_0$ preserves the Gauss decomposition of $\widetilde{G}$, $s_0\tilde{b}[\tilde{g}]s_0$ is of the form

$$\begin{bmatrix} b_{11} & & & & & \\ & \ddots & & & & \\ & & b_{n-1,n-1} & & & \\ & & & b_{n,n}^{-1} & & \\ & & & & b_{n,n} & \\ & & * & & & b_{11}^{-11} \end{bmatrix}$$

so that $\Pi(s_0\tilde{b}[\tilde{g}]s_0) = b^v\Pi$. It follows that

$$p_\xi(\Pi(s_0\tilde{b}[\tilde{g}]s_0)s_0\tilde{z}[g]) = \xi^{(m)}(b^v)p_\xi(s_0\tilde{z}[g]).$$

As above we see that $p_\xi(s_0\tilde{z}[g]) = 0$ and infer that $\Lambda^- p_0^{(m)^+} = 0$. Since $\Lambda p_0^{(m)^+} = \Lambda^+ p_0^{(m)^+} = f^{(m)^+}$ (resp. $\Lambda p_0^{(m)^-} = \Lambda^- p_0^{(m)^-} = f^{(m)^-}$) is a cyclic vector of the simple G-module $V^{(m)^+}$ (resp. $V^{(m)^-}$) and Λ is an intertwining operator it follows that Λ is a G-module epimorphism. If $p \in H(E^*, (m))$ and $\Lambda p = \Lambda^+ p + \Lambda^- p = 0$ then Equation (3.5) implies that $p((1/\sqrt{2})X_0^+ g) = p((1/\sqrt{2})X_0^- g) = 0$ for all $g \in G$, that is, p vanishes on both P_n^+ and P_n^-. Since $\overline{P_n^+ \cup P_n^-} = P$, it follows that $p = 0$ on P and, by Corollary 2.2(iii), p is the 0 polynomial function. Thus Λ is a G-module monomorphism, and, hence, a G-module isomorphism. Let $H^+(E^*, (m)) = \operatorname{Ker}\Lambda^-$ (resp. $H^-(E^*, (m)) = \operatorname{Ker}\Lambda^+$) be the subspace of all elements of $H(E^*(m))$ which vanish on P_n^- (resp. P_n^+). Then clearly $H^+(E^*, (m))$ and $H^-(E^*(m))$ are G-submodules of $H(E^*, (m))$ and $\Lambda_{|H^+(E^*,(m))} = \Lambda^+$, $\Lambda_{|H^-(E^*,(m))} = \Lambda^-$. Moreover, $p_0^{(m)^+} \in H^+(E^*, (m))$ and $p_0^{(m)^-} \in H^-(E^*, (m))$. So both $H^+(E^*(m))$ and $H^-(E^*, (m))$ are nonzero, and it follows that $\Lambda^+ : H^+(E^*, (m)) \to V^{(m)^+}$ and $\Lambda^- : H^-(E^*, (m)) \to V^{(m)^-}$ are isomorphisms of simple G-modules. The fact that the inner product $\langle\cdot,\cdot\rangle$ defined earlier on $S(E^*)$ is invariant under the restriction of the action of G to $G_0 = SO(k)$ and that $H^+(E^*, (m))$ and $H^-(E^*(m))$ are inequivalent simple G-modules implies immediately that $H(E^*, (m)) = H^+(E^*, (m)) \oplus H^-(E^*, (m))$ is an orthogonal direct sum. □

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