Proceedings of Symposia in Pure Mathematics
Volume 49 (1989), Part 1

The Trisecant Conjecture for Pryms

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Introduction. We examine below the following conjecture of Welters [We]: An indecomposable principally polarized abelian variety (ppav in the sequel) is a Jacobian if and only if its Kummer variety has a trisecant line.

In particular, we prove first that *the family $\mathcal{J}_g$ of Jacobians is an irreducible component of the locus of ppav's satisfying the above trisecant property* (joint work with A. Beauville).

Secondly, we prove that *if a* (generalized) *indecomposable Prym variety has the trisecant property, then it is a Jacobian.* This proves Welters' conjecture in dimension ≤ 5.

As a by-product of our methods, we get some results on 4-dimensional ppav's with a given number of vanishing thetaconstants, the most striking of them being that *there is only one indecomposable* 4*-dimensional ppav with* 10 *vanishing thetaconstants* (apart from hyperelliptic Jacobians). This particular ppav was discovered earlier by R. Varley in [Va].

1. The Schottky problem. The Schottky problem is the problem of characterizing Jacobians among all ppav's. Up to now, there have been three principal ways of attacking this question.

(1) One can use *Schottky-Jung relations* to try to find equations for $\overline{\mathcal{J}_g}$ in the moduli space $\mathcal{A}_g$ of all ppav's. These relations involve the so-called thetaconstants. The interested reader should consult B. Van Geemen's thesis [V] for more details.

(2) One can use, after *Andreotti and Mayer*, the singularities of the theta divisor of Jacobians. Namely, any Jacobian (JC, Θ) satisfies

$$\dim \operatorname{Sing} \Theta \geq \dim JC - 4$$

and these two authors proved in [A-M] that $\overline{\mathcal{J}_g}$ is an irreducible component of

$$\mathcal{N}_{g-4} = \{(A, \Theta) \in \mathcal{A}_g | \dim \operatorname{Sing} \Theta \geq g - 4\}.$$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14K99, 14H40.

Unfortunately, already in dimension 4, this set has other components. Beauville proved in [B] that

$$\mathcal{N}_0 = \overline{\mathcal{J}}_4 \cup \mathcal{O}_{\text{null}},$$

where

$$\mathcal{O}_{\text{null}} = \{(A, \Theta) \in \mathcal{A}_4 | \Theta \text{ symmetric} \quad \text{and} \quad \operatorname{Sing}\Theta \cap {}_2A \neq \varnothing\},$$

these two sets being irreducible. The situation gets even worse as g gets large, as we will see later. Therefore, one needs additional properties to characterize Jacobians.

(3) *Trisecants to the Kummer variety.* A. Weil noticed in [W] that on a Jacobian (JC, Θ) one has, for any points p, q, r, s of C,

$$\Theta \cap \Theta_{p-q} \subset \Theta_{p-r} \cup \Theta_{s-q}. \tag{1}$$

The existence of such an inclusion has an interpretation in terms of the Kummer variety [**M 1, M 2, We**].

Recall that for any ppav (A, Θ), there is a commutative diagram

$$\begin{array}{ccc} & & |2\Theta|^* \simeq \mathbb{P}^{2^g-1} \\ & \overset{\varphi}{\nearrow} & \\ A & & \wr| \text{ linear isomorphism} \\ & \underset{\psi}{\searrow} & \\ & & |2\Theta| \end{array}$$

where φ is the morphism associated to the base point-free linear system $|2\Theta|$ and ψ is defined by $\psi(x) = \Theta_x + \Theta_{-x} \in |2\Theta|$ (Theorem of the square).

The *Kummer variety* of (A, Θ) is the image of either φ or ψ. If (A, Θ) is indecomposable, it is isomorphic to the quotient of A by the involution $x \mapsto -x$. This is a singular g-dimensional variety.

Its importance for us stems from the following proposition.

PROPOSITION. *Let (A, Θ) be an indecomposable ppav and a, b, c, d nonzero elements of A such that $a \neq c, d$ and $a + b = c + d = x$. Then the following properties are equivalent*:

(i) $\Theta \cdot \Theta_a \subset \Theta_c \cup \Theta_d$ *(scheme-theoretically).*

(ii) $\exists \lambda, \mu, \nu \in \mathbb{C}^*$ $\lambda\theta\theta_x + \mu\theta_a\theta_b + \nu\theta_c\theta_d = 0$, *where for any $z \in A$, θ_z denotes a generator of $H^0(A, \Theta_z)$.*

(iii) *For any $y \in A$ such that $2y = x$, the points $\psi(y)$, $\psi(y-a)$ and $\psi(y-c)$ are on a line.*

The reader will notice that (ii) $\Rightarrow$ (i) is trivial and that (ii) $\Rightarrow$ (iii) follows from translating the equation by $(-y)$.

It follows that the Kummer variety of a Jacobian has a 4-dimensional family of trisecants (obtained by varying the points p, q, r, s on C).

On the other hand, since we know of no ppav enjoying the property that its Kummer variety has *one* trisecant, which is not a Jacobian, Welters has conjectured that, if

$$\mathcal{T}ri_g = \{(A, \Theta) \text{ indecomposable such that its Kummer variety has } one \text{ trisecant line}\},$$

then

Trisecant conjecture:

$$\mathcal{J}_g = \mathcal{T}ri_g$$

REMARK. One can give a meaning to each of the interpretations (i), (ii), (iii) of the inclusion (1) when p, q, r, s converge to the same point of C. The limiting equation (ii) is equivalent to the K-P equation. Therefore, the above conjecture can be seen as a discrete analogue of the Novikov conjecture proved by T. Shiota in [S].

The first result toward the conjecture is the following theorem, obtained in collaboration with A. Beauville, which links the last two approaches to the Schottky problem.

THEOREM [**B-D**]. *Let* (A, Θ) *be an indecomposable ppav satisfying one of the conditions* (i), (ii), (iii) *above. Then*

$$\dim \operatorname{Sing} \Theta \geq \dim A - 4.$$

One deduces immediately from this theorem and the theorem of Andreotti and Mayer mentioned above that

COROLLARY. $\mathcal{J}_g$ *is a component of* $\mathcal{T}ri_g$.

We will only prove the corollary. More precisely, we show that if a ppav (A, Θ) is such that

$$\begin{cases} \mathrm{NS}(A) = \text{divisors/algebraic equivalence} = \mathbb{Z}[\Theta], \\ \exists a \neq 0 \quad \Theta \cap \Theta_a \text{ reducible,} \end{cases}$$

then $\dim \operatorname{Sing} \Theta \geq \dim A - 4$.

Since the Neron-Severi group of a generic Jacobian satisfies the above property, the corollary will follow.

Suppose $\dim \operatorname{Sing} \Theta < \dim A - 4$. Then, by Samuel's conjecture [**G**, Exp. XI, Corollary 3.14] Θ is locally factorial. If we write $\Theta \cap \Theta_a = D + D'$, the Weil divisors D and D' of Θ are therefore Cartier. Again, the Lefschetz theorem à la Grothendieck [**G**, Exp. XII, Corollary 3.6] yields an isomorphism $\operatorname{Pic}(A) \xrightarrow{\sim} \operatorname{Pic}(\Theta)$: the line bundles $\mathcal{O}(D)$ and $\mathcal{O}(D')$ come from line bundles L and L' on A. By hypothesis, we have: $\exists m, m' \in \mathbb{N}^* \ L \sim m\Theta$, $L' \sim m'\Theta$. Since $L \otimes L' = \mathcal{O}(\Theta_a)$, one has $m + m' = 1$. Contradiction. □

2. The trisecant conjecture for Pryms. Our first theorem states that $\mathcal{T}ri_g \subset \mathcal{N}_{g-4}$. Unfortunately, the only thing known about the set $\mathcal{N}_{g-4}$ is its intersection with the Prym locus $\mathcal{P}_g$ [**M 3**, **B**].

Recall that to any double étale cover $\pi \colon \tilde{C} \to C$ of smooth connected curves, one associates a ppav $P = J\tilde{C}/\pi^* JC$, its *Prym variety*.

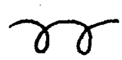
Beauville has extended this definition to certain double covers of stable curves **[B]**. The corresponding family $\mathscr{P}_g$ ($g = g(C) - 1$) in $\mathscr{A}_g$ is closed and

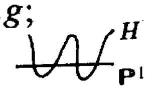
$$\begin{array}{ll} \mathscr{P}_g = \mathscr{A}_g & \text{for } g \leq 5, \\ \dim \mathscr{P}_g = 3g & \text{for } g \geq 5, \\ \mathscr{J}_g \subset \mathscr{P}_g & \text{for any } g. \end{array}$$

Using Beauville's list of double covers for which the Prym variety is in $\mathscr{N}_{g-4}$ and Donagi's tetragonal construction **[Do]**, we show:

PROPOSITION **[D1]**. *The irreducible components of* $\mathscr{P}_g^{\text{ind}} \cap \mathscr{N}_{g-4}$ *(indecomposable Pryms which are in* $\mathscr{N}_{g-4}$*) are, for* $g \geq 5$:

$\mathscr{J}_g$ *of dimension* $3g-3$;

$\mathscr{E}_g^2$ $=\{$*Pryms of double covers of* $C =$  *with normalization of* C *hyperelliptic of genus* $g-1\}$ *of dimension* $2g$;

$\mathscr{E}_{1,g-1}^2$ $=\{$*Pryms of double covers of* $C =$ 

with H *hyperelliptic of genus* $g-2\}$ *of dimension* $2g-1$;

$\mathscr{P}_{t,g-t}^2$ *for* $2 \leq t \leq g/2$ $=\{$*Pryms of double covers of* $C =$ 

with $g(C') = t-1$ *and* $g(C'') = g-t-1\}$ *of dimension* $3g-4$.

Although the following result will not be used in the sequel, we can also prove

THEOREM **[D1]**. *For* $g \geq 5$, $\overline{\mathscr{J}}_g$, $\overline{\mathscr{E}_g^2}$ *and* $\overline{\mathscr{E}_{1,g-1}^2}$ *are irreducible components of* $\mathscr{N}_{g-4}$.

For $2 \leq t \leq g/2$, $g \geq 5$, $\mathscr{P}_{t,g-t}^2$ *is contained in an irreducible component* $\mathscr{A}_{t,g-t}^2$ *of* $\mathscr{N}_{g-4}$, *of codimension* $t(g-t)$ *in* $\mathscr{A}_g$.

Let us describe now $\mathscr{A}_{t,g-t}^2$. Suppose (A, Θ) is a ppav which contains an abelian subvariety X' of dimension t such that $\deg \Theta|_{X'} = 2$. Then there exists another abelian subvariety X'' of A of dimension $g-t$, satisfying also $\deg \Theta|_{X''} = 2$, such that the inclusions $X' \subset A$ and $X'' \subset A$ induce an isogeny:

$$\pi : X' \times X'' \to A$$

with kernel $(\mathbf{Z}/2)^2$, compatible with the polarizations. Moreover, there exist bases

$$\{s', t'\} \quad \text{for } H^0\left(X', \Theta|_{X'}\right), \qquad \{s'', t''\} \quad \text{for } H^0\left(X'', \Theta|_{X''}\right)$$

such that

$$\pi^* \Theta = \operatorname{div}(s's'' + t't'').$$

Setting

$$F' = \{s' = t' = 0\} \subset X', \qquad F'' = \{s'' = t'' = 0\} \subset X'',$$

one sees immediately that

$$\pi(F' \times F'') \subset \operatorname{Sing} \Theta.$$

If $g-t$, $t \geq 2$, F' and F'' are nonempty; hence $(A, \Theta) \in \mathscr{N}_{g-4}$.

DEFINITION. $\mathscr{A}_{t,g-t}^2$ is the family of all such (A, Θ)'s.

REMARK. For any "type" $\delta = (d_1|d_2|\cdots|d_r)$ of polarization, a similar construction yields families $\mathscr{A}^{\delta}_{t,g-t}$, which are contained in $\mathscr{N}_{g-2\deg\delta}$ for $t, g-t \geq \deg\delta$.

In this way, we get irreducible components of $\mathscr{N}_{g-6}$ for $g \geq 7$ (with $\delta = (3)$) and of $\mathscr{N}_{g-8}$ for $g \geq 9$ (with $\delta = (2|2)$) **[D 1]**.

COROLLARY. *In dimension* 5, *the irreducible components of* $\mathscr{N}_1$ *are*

$$\overline{\mathscr{J}_5}, \quad \overline{\mathscr{E}^2_5}, \quad \overline{\mathscr{E}^2_{1,4}}, \quad \mathscr{A}^2_{2,3}, \quad \text{and} \quad \mathscr{A}_1 \times \mathscr{A}_4.$$

The respective dimensions are 12, 10, 9, 9, 11.

Getting back to Pryms, a careful examination of the elements of each component of $\mathscr{P}^{\text{ind}}_g \cap \mathscr{N}_{g-4}$, plus a proof by induction on the dimension (starting in dimension 4 by using results of Z. Ran **[R]**) yields

THEOREM **[D 2]**. $\mathscr{P}_g \cap \mathscr{T}ri_g = \mathscr{J}_g$ *for* $g \geq 4$.

COROLLARY. *The trisecant conjecture is true in dimension* ≤ 5.

3. Vanishing thetaconstants on 4-dimensional ppav's. The preceding analysis can be extended to the case $g = 4$. Recall that we have **[B]**

$$\mathscr{N}_0 = \overline{\mathscr{J}_4} \cup \theta_{\text{null}},$$

where θ_{null} is the locus of ppav's of dimension 4 for which any symmetric representative of the theta divisor has a singular point that is of order 2.

This is equivalent to the vanishing of a thetaconstant $\theta\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](0, \tau)$ (with $\varepsilon, \varepsilon' \in (\mathbb{Z}/2)^g$, $\varepsilon \cdot \varepsilon' \equiv 0 \pmod 2$) at a point τ of the Siegel upper half-space $\mathscr{H}_4$ corresponding to the ppav.

That is why one says that a thetaconstant vanishes. Let us introduce

$$\theta^{(p)}_{\text{null}} = \{(A, \Theta) \in \mathscr{A}_4 \text{ with at least } p \text{ vanishing thetaconstants}\}.$$

Since each new vanishing thetaconstant corresponds to the vanishing of a function, one would expect that $\operatorname{codim}_{\mathscr{A}_4}(\theta^{(p)}_{\text{null}}) = p$. This is true in a sense, as shown by the following theorem.

THEOREM **[D3]**. (1) θ_{null} *is irreducible*, 9*-dimensional, and a generic element has exactly* 1 *vanishing thetaconstant.*

(2) $\theta^{(2)}_{\text{null}}$ *is irreducible*, 8*-dimensional, and a generic element has exactly* 2 *vanishing thetaconstants* (*in the preceding notations*, $\theta^{(2)}_{\text{null}}$ *is* $\mathscr{E}^2_4$).

(3) $\theta^{(3)}_{\text{null}}$ *is purely* 7*-dimensional and has* 3 *components*: $\overline{\mathscr{H}}_4$, *closure of the locus of hyperelliptic Jacobians* (*generically* 10 *vanishing thetaconstants*); $\mathscr{A}_1 \times \mathscr{A}_3$ (*generically* 28 *vanishing thetaconstants*) *and* $\mathscr{D}$ (*generically* 3 *vanishing thetaconstants*).

(4) $\theta^{(4)}_{\text{null}}$ *has a* 6*-dimensional component contained in* $\mathscr{D}$, *which is* $\mathscr{A}^2_{2,2}$ (*with generically* 4 *vanishing thetaconstants*).

(5) $\theta^{(9),\text{ind}}_{\text{null}} - \mathscr{H}_4$ *is irreducible* 1*-dimensional. Its elements are isogenous to a product* E^4, *where* E *is an elliptic curve.*

(6) $\theta_{\mathrm{null}}^{(10),\mathrm{ind}} - \mathscr{H}_4$ *has exactly one element which we will call* A_{10} *and which corresponds to the case where* E *has complex multiplication by* i $(j(E) = 1728)$.

The ppav A_{10} was studied in great detail by R. Varley in **[Va]**. A. Beauville has pointed out to me the following alternative construction of A_{10}. Let E_i be the elliptic curve with complex multiplication by i. As an abelian variety, A_{10} is isomorphic to E_i^4; therefore, giving an indecomposable principal polarization on A_{10} is equivalent to giving an indecomposable unimodular positive hermitian form on the $\mathbb{Z}[i]$-module $\mathbb{Z}[i]^4$. Let Γ_8 be the lattice of roots of the root system E_8 and let Q be the corresponding quadratic form. There exists, up to conjugation, a unique element J of the Weyl group $W(E_8)$ with square (-1_{Γ_8}) (cf. **[C]**). This element endows Γ_8 with the structure of a free $\mathbb{Z}[i]$-module of rank 4; the hermitian form H defined on Γ_8 by $H(x,y) = Q(x,y) + iQ(Jx,y)$ yields the desired polarization.

It follows from **[C]** that the automorphism group of A_{10}, which is identified with the centralizer of J in $W(E_8)$, has order 46080.

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