CHAPTER 28

ON HIGHEST WHITTAKER MODELS AND INTEGRAL STRUCTURES

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Abstract. We show that the integral functions in a highest Whittaker model of an irreducible integral $\overline{\mathbf{Q}}_\ell$-representation of a p-adic reductive connected group form an integral structure.

Introduction. This work is motivated by a question of E. Urban (March 2001) for the group $Sp(4)$. The fact that the integral Whittaker functions form an integral structure is an ingredient at the nonarchimedean places for deducing congruences between Eisenstein series and cuspidal automorphic forms from congruences between special values of L-functions using the theory of Langlands-Shahidi. Many fundamental and deep theorems in the theory of Whittaker models and of L-functions attached to automorphic representations of reductive groups with arithmetical applications are due to Joseph Shalika and his collaborators, or inspired by him. Whittaker models and their generalizations as the Shalika models have become a basic tool to study automorphic representations and they may become soon a basic tool for studying congruences between them.

Let (F, G, ℓ) be the triple formed by a local nonarchimedean field F of residual characteristic p, the group G of rational points of a reductive connected F-group, a prime number ℓ different from p. We denote by $\overline{\mathbf{Q}}_\ell$ an algebraic closure of the field $\mathbf{Q}_\ell$ of ℓ-adic numbers, $\overline{\mathbf{Z}}_\ell$ the ring of its integers, Λ the maximal ideal, $\overline{\mathbf{F}}_\ell = \overline{\mathbf{Z}}_\ell/\Lambda\overline{\mathbf{Z}}_\ell$ the residual field (an algebraic closure of the finite field $\mathbf{F}_\ell$ with ℓ-elements), $\mathrm{Mod}_{\overline{\mathbf{Q}}_\ell} G$ the category of $\overline{\mathbf{Q}}_\ell$-representations of G, $\mathrm{Irr}_{\overline{\mathbf{Q}}_\ell} G$ the subset of irreducible representations. All representations (π, V) of G are smooth: the stabilizer of any vector $v \in V$ is open. The dimension of a representation of G is usually infinite.

However, a reductive p-adic group tries very hard to behave like a finite group. A striking example of this principle is the strong Brauer-Nesbitt theorem:

Theorem 1. *Let (π, V) be a $\overline{\mathbf{Q}}_\ell$-representation of G of finite length, which contains a G-stable free $\overline{\mathbf{Z}}_\ell$-submodule L. Then the $\overline{\mathbf{Z}}_\ell G$-module L is finitely generated, $L/\Lambda L$ has finite length and the semi-simplification of $L/\Lambda L$ is independent of the choice of L.*

This is a stronger version of the Brauer-Nesbitt theorem in [V II.5.11.b] because the hypotheses (loc. cit.) contained the property that the $\overline{\mathbf{Z}}_\ell G$-module L is finitely

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generated and $\overline{\mathbf{Z}}_\ell$-free. Here we prove that the $\overline{\mathbf{Z}}_\ell$-freeness of L implies that L is $\overline{\mathbf{Z}}_\ell G$-finitely generated.

A representation $(\pi, V) \in \mathrm{Mod}_{\overline{\mathbf{Q}}_\ell} G$ is called *integral* when the vector space V contains a G-stable free $\overline{\mathbf{Z}}_\ell$-submodule L containing a $\overline{\mathbf{Q}}_\ell$-basis, and L is called an *integral structure*.

There is not yet a standard notation for the Whittaker models. Our notation is the following. A Whittaker $\overline{\mathbf{Q}}_\ell$-representation of G is associated to a pair (Y, μ) where Y is a nilpotent element of $\mathrm{Lie}\, G$ and μ is a cocharacter of G related by $\mathrm{Ad}\,\mu(x)Y = x^{-2}Y$ for all $x \in F^*$. The Whittaker $\overline{\mathbf{Q}}_\ell$-representation of G is an induced representation $\mathrm{Ind}_N^G \Omega$, where N is the unipotent subgroup of G defined by the cocharacter μ and Ω is an admissible irreducible representation (character or an infinite dimensional metaplectic representation) of N defined by the nilpotent element Y [MW]. The contragredient $(N, \tilde{\Omega})$ of (N, Ω) is associated to $(-Y, \mu)$. When $Y = 0$, Ω is the trivial character of N. When Y is regular, i.e., the dimension $d(Y)$ of the nilpotent orbit $\mathcal{O} = \mathrm{Ad}\, G, Y$ is maximal among the dimensions of the nilpotent orbits of $\mathrm{Lie}\, G$, N is a maximal unipotent subgroup and Ω is a generic character of N; the corresponding Whittaker $\overline{\mathbf{Q}}_\ell$-representation of G is called generic. We need the assumption that the characteristic of F is 0 and $p \neq 2$ in order to refer to [MW]. It is clear that a generic Whittaker $\overline{\mathbf{Q}}_\ell$-representation of G can be defined without any assumption on F.

Let $\pi \in \mathrm{Mod}_{\overline{\mathbf{Q}}_\ell} G$, which may fail to be irreducible. A Whittaker model of π associated to (Y, μ) is a subrepresentation of $\mathrm{Ind}_N^G \Omega$ isomorphic to π, if there exists one. If π has a model in a generic Whittaker $\overline{\mathbf{Q}}_\ell$-representation of G, then π is called generic and the model is called a generic Whittaker model. The "highest Whittaker models" of π are the Whittaker models of π associated to (Y, μ) when the nilpotent orbit $\mathcal{O}$ is maximal among the nilpotent orbits of $\mathrm{Lie}\, G$ associated to the Whittaker models of π, when π has a Whittaker model. When π is irreducible and generic, the generic Whittaker models are the highest Whittaker models of π.

When π is irreducible, the characteristic of F is 0 and $p \neq 2$, a Whittaker model with our definition which is called a degenerate Whittaker model in [MW]; the set of Whittaker models of π is not empty [MW].

We relate now the Whittaker models with the integral structures. The representation Ω has a natural integral structure L_Ω but the induction does not respect integral structures: in general, the $\overline{\mathbf{Z}}_\ell G$-submodule $\mathrm{Ind}_H^G L_\Omega$ is not $\overline{\mathbf{Z}}_\ell$-free and does not generate $\mathrm{Ind}_N^G \Omega$, and the Whittaker representation $\mathrm{Ind}_H^G \Omega$ is not integral. However, we have the following remarkable property.

THEOREM 2. *Let $\pi \in \mathrm{Mod}_{\overline{\mathbf{Q}}_\ell} G$ admissible and let $V \subset \mathrm{Ind}_N^G \Omega$ be a highest Whittaker model of π. Then the two following properties (1) and (2) are equivalent:*

(1) *π is integral.*

(2) *The functions in V with values in L_Ω form an integral structure of π.*

Under the restriction on (F, π), the characteristic of F is 0 and $p \neq 2$, π is irreducible.

When V is a generic Whittaker model of π, the equivalence is true without restriction on (F, π).

As (2) implies clearly (1), the key point is to show that (1) implies (2). We prove that (1) implies (2) iff any element v of V has a denominator, i.e., the values of a multiple of v belong to L_Ω (II.5), and we give two general criteria A, B for this property (II.6 and II.7).

Criterion A given in (II.6) is that (π, V) contains an integral structure L such that the Ω-coinvariant $p_\Omega L$ is $\overline{\mathbf{Z}}_\ell N$-finitely generated. This is an integral version of the fact that the Ω-coinvariant $p_\Omega V$ is finite dimensional (Moeglin and Waldspurger) when V is a highest Whittaker model of $\pi \in \mathrm{Irr}_{\overline{\mathbf{Q}}_\ell} G$ attached to (N, Ω). To explain the method due to Rodier, let us suppose that Ω is a character. One approximates (N, Ω) by characters χ_n of open compact pro-p-subgroups K_n of G. The key point is to prove that the projection p_Ω on the (N, Ω)-coinvariants restricts to an isomorphism $e_n V \simeq p_\Omega V$, where e_n is the projector on the (K_n, χ_n)-invariants, when n is big enough. Recall that V is admissible, hence $p_\Omega V$ is finite dimensional. The tool to prove the isomorphism is the expansion of the trace of π around 1. As $e_n L$ is a lattice of $e_n V$ for any integral structure L of (π, V), criterion A is satisfied if p_Ω restricts to an isomorphism $e_n L \simeq p_\Omega L$. This is proved in Section III by a careful analysis of the proof of [MW].

Compact induction behaves well for integral structures. A compact Whittaker representation $\mathrm{ind}_H^G \Omega$ is integral and $\mathrm{ind}_H^G L_\Omega$ is an integral structure. The Whittaker representation $\mathrm{Ind}_N^G \Omega$ is the contragredient of the compact Whittaker representation $\mathrm{ind}_N^G \tilde{\Omega}$, where $\tilde{\Omega}$ is the contragredient of Ω because Ω is admissible and N unimodular. The criterion B given in (II.7) is a property of the K-invariants of $\mathrm{ind}_N^G \tilde{L}_\Omega$ as a right module for the Hecke algebra of (G, K) when K is an open compact subgroup of G. It is an integral version of a finiteness theorem: the component of $\mathrm{ind}_N^G \tilde{\Omega}$ in any Bernstein block is finitely generated. In the generic case and without restriction on (F, π), this has been recently proved by Bushnell and Henniart [BH 7.1]. Their proof is well adapted to criterion B and one can, after some simplifications, obtain that a generic compact Whittaker representation satisfies criterion B. This is done in Section IV.

For a generic irreducible representation with the restriction on F, we get two very different proofs of the Theorem 2, using criteria A and B. For $GL(n, F)$ with no restriction on F, when the representation is also cuspidal, a third proof was known and showed that modulo homotheties, the Kirillov model is the unique integral structure [V4]. The Kirillov integral model was used for $GL(2, F)$ to prove that the semi-simple local Langlands correspondence modulo ℓ is uniquely defined by equalities between ϵ factors [V6]. The characterization of the local Langlands correspondence modulo ℓ in the general case $n > 2$ by L and ϵ factors remains open. Probably the case $n = 3$ is accessible.

As noticed by Jacquet and Shalika for $GL(n, F)$, the Whittaker models of representations induced from tempered irreducible representations are useful. Being aware of future applications, we did not consider only integral models of irreducible representations. The criteria A, B, as well as Theorem 1 and the generic case of Theorem 2 are given for representations that may fail to be irreducible.

In the appendix, we compare, for a representation V of G over an algebraically closed field R of characteristic $\neq p$, the three properties:

(i) the right $\mathcal{H}_R(G, K)$-modules V^K are finitely generated for the open compact subgroups K of G;

(ii) the components of V in the blocks of $\mathrm{Mod}_R\, G$ are finitely generated;

(iii) the irreducible quotients of V have finite multiplicity.

The criterion A is an integral version of (iii), the criterion B is an integral version of (i). The property (i) is equivalent (ii) in the complex case [BH] and it is clear that (ii) implies (iii) but is not equivalent. We give a proof of the equivalence between (i) and (ii) in the modular case, and in the complex case we give certain properties of V and of its Jacquet functors implying the equivalence between (ii) and (iii). For instance, the complex representation of $GL(2, F)$ compactly induced from a character of a maximal (split or not split) torus and its coinvariants by a unipotent subgroup satisfy this properties. This representation introduced by Waldspurger and studied also by Tunnel, plays a role in the arithmetic theory of automorphic forms.

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I. Proof of the strong Brauer-Nesbitt theorem. Let (π, V) be a finite length $\overline{\mathbf{Q}}_\ell$-representation of G which contains a G-stable free $\overline{\mathbf{Z}}_\ell$-submodule L. We will prove that the $\overline{\mathbf{Z}}_\ell G$-module L is finitely generated. The rest of the theorem follows from the Brauer-Nesbitt theorem proved in [V II.5.11.b].

The proof uses an unrefined theory of types for G as in [V II.5.11.b]. One can take either the mottes [V1] or the more sophisticated Moy-Prasad types.

The subrepresentation π' of π generated by L has finite length and we may suppose that $\pi = \pi'$ is generated by L.

One may replace $\overline{\mathbf{Z}}_\ell$ by the ring of integers O_E of a finite extension E of $\mathbf{Q}_\ell$ as in [V II.4.7]. What is important is that O_E is a principal local ring. Let p_E be a generator of the maximal ideal, let $k_E := O_E/p_E O_E$ be the residual field.

The theory of unrefined types shows that $L/p_E L \in \mathrm{Mod}_{k_E}\, G$ has finite length because it contains only finitely many unrefined minimal types modulo G-conjugation ([V II.5.11.a], where the condition $O_E G$-finitely generated is useless).

Let m be the length of $L/p_E L$. We will prove the $\overline{\mathbf{Z}}_\ell G$-module L is generated by m elements.

We cannot conclude immediately because the free O_E-module L is usually not of finite rank. As $\ell \neq p$, the open compact pro-p-subgroups K of G form a fundamental system of neighborhoods of 1. The finite length $\overline{\mathbf{Q}}_\ell$-representation (π, V) of G is admissible: for any open compact pro-p-subgroup K of G, the E-dimension of the vector space V^K is finite. The O_E-modules L^K are free of finite rank. By smoothness we have $L = \cup_K L^K$.

The $k_E G$-module $L/p_E L$ is generated by m elements $w_1, \ldots, w_m$. We lift these elements arbitrarily to $v_1, \ldots, v_m$ in L and we consider the $O_E G$-submodule L' of L that they generate. As O_E is principal and L is O_E-free, the O_E-submodule L' of L is O_E-free. We have by construction

$$L = L' + p_E L.$$

The O_E-modules L'^K, L^K are free of finite rank and $L^K = L'^K + p_E L^K$. The theory of invariants for free modules of finite rank over a principal ring implies that $L'^K = L^K$. As $L = \cup_K L^K,\ L' = \cup_K L'^K$, we deduce $L = L'$. Thus Theorem 1 is proved.

II. Integral structures in induced representations (criteria A and B). The framework of this section is very general, R is a commutative ring and G is a locally profinite group that contains an open compact subgroup C of pro-order invertible in R, such that G/C is countable. The criteria A and B are given in (II.6) and (II.7). The proofs are given at the end of the section.

II.1. We fix the notations:

Mod_R is the category of R-modules;

$\mathrm{Mod}_R\, G$ is the category of smooth representations of G on R-modules;

$\mathrm{Irr}_R\, G$ is the subset of irreducible representations;

H is a closed subgroup of G;

O_E is a principal ring with quotient field E;

$(\Omega, W) \in \mathrm{Mod}_E\, H$ of countable dimension;

$\mathrm{Ind}_H^G(\Omega, W) \in \mathrm{Mod}_E\, G$ is the space of functions $f : G \to W$ right invariant by some open compact subgroup K_f, with functional equation $f(hg) = \Omega(h) f(g)$ for $h, g \in H, G$, with the action of G by right translations;

$\mathrm{ind}_H^G(\Omega, W) \in \mathrm{Mod}_E\, G$ is the compactly induced representation, subrepresentation of $\mathrm{Ind}_H^G(\Omega, W)$ on the functions f with compact support modulo H.

We often forget the module V or the action π in the notation (π, V) of a representation.

The induced representation $\mathrm{Ind}_H^G \Omega$ and the compactly induced representation $\mathrm{ind}_H^G \Omega$ can be equal even when G is not compact modulo H. There are two typical examples with $\mathrm{ind}_H^G \Omega = \mathrm{Ind}_H^G \Omega$:

– a metaplectic representation [MVW I.3, I.6]: G is a p-adic Heisenberg group of center Z, H is a maximal commutative subgroup of G, $\ell \neq p$ a prime number, E is the field generated over $\overline{\mathbf{Q}}_\ell$ by the roots of 1 of order any power of p (the ring of integers O_E is principal), Ω is an E-character of H nontrivial on Z.

– a cuspidal representation [V5]: G is a p-adic connected reductive group, H is the normalizer in G of a maximal parahoric subgroup K, R is an algebraically closed field of characteristic $\neq p$, $\Omega \in \mathrm{Irr}_R H$ such that $\Omega|_K$ contains the inflation of a cuspidal irreducible representation of the quotient K/K_p (a finite connected reductive group).

A representation $(\pi, V) \in \mathrm{Mod}_E G$ is called O_E-integral when it contains an O_E-integral structure L, i.e., a G-stable O_E-free submodule L that contains an E-basis of V.

II.2. Let L be an O_E-integral structure of a representation $(\pi, V) \in \mathrm{Mod}_E G$ and let (π', V') be a subrepresentation of (π, V). Then $L' := L \cap V$ is an O_E-integral structure of (π', V').

This is a basic fact with an easy proof: clearly L' is G-stable; as O_E is principal and the O_E-module L is free, the O_E-submodule $L' \subset L$ is free; if $(v_i)_{i\in I}$ is a basis of V' then for each $i \in I$ there exists $a_i \in O_E$ such that $v_i a_i \in L$ hence $v_i a_i \in L'$. Therefore L' is an O_E-integral structure of V'.

In contrast with (II.2): *a quotient of an integral representation is not always integral*. A counter-example is given after (II.3).

We suppose in this section that $(\Omega, W) \in \mathrm{Irr}_E H$ is *O_E-integral with an O_E-integral structure L_W*. The proofs are given at the end of the section. Are the induced representations $\mathrm{Ind}_H^G \Omega$ and $\mathrm{ind}_H^G \Omega$ integral? In general, the induced representation without condition on the support is not integral by (II.2) because $\mathrm{Ind}_H^G \Omega$ may contain a nonintegral irreducible representation. This contrasts with the compactly induced representation $\mathrm{ind}_H^G \Omega$, which is integral.

Proposition II.3. $\mathrm{ind}_H^G L_W$ *is an O_E-integral structure of* $\mathrm{ind}_H^G(\Omega, W)$.

The integral representation $\mathrm{ind}_H^G \Omega$ may have nonintegral quotients: $\mathrm{ind}_1^{\mathbf{Q}_p^*} 1_E$ is integral but there are characters of $\mathbf{Q}_p^*$ with values not contained in O_E^*.

The O_E-module $\mathrm{Ind}_H^G L_W$ is clearly G-stable. But when $\mathrm{ind}_H^G \Omega \neq \mathrm{Ind}_H^G \Omega$, the O_E-module $\mathrm{Ind}_H^G L_W$ is not free and does not contain a basis of $\mathrm{Ind}_H^G W$. Hence the following property is particularly nice:

Proposition II.4. *For any admissible subrepresentation* (π, V) *of* $\mathrm{Ind}_H^G (\Omega, W)$, *the O_E-module $V \cap \mathrm{Ind}_H^G L_W$ is free or zero.*

Clearly $V \cap \mathrm{Ind}_H^G L_W$ is G-stable, hence $V \cap \mathrm{Ind}_H^G L_W$ is an O_E-integral structure of (π, V) if and only if any element of V has a nonzero multiple in $\mathrm{Ind}_H^G L_W$:

Denominators. Let $(\pi, V) \subset \mathrm{Ind}_H^G(\Omega, W)$. We say that $v \in V$ has a denominator if there exists $a \in O_E$ nonzero with $av \in \mathrm{Ind}_H^G L_W$. We say that V has a bounded denominator if there exists $a \in O_E$ nonzero and an E-basis of V with $av \in \mathrm{Ind}_H^G L_W$ for all v in the basis.

Two O_E-integral structures L_W, L'_W of $(\Omega, W) \in \mathrm{Irr}_E H$ are commensurable:

$$aL_W \subset L'_W \subset bL_W, \quad \text{for some } a, b \in O_E$$

and the definition of a denominator or of a bounded denominator does not depend on the choice of L_W. Any element of V has a denominator iff every element in a set of generators of V has a denominator. If (π, V) is finitely generated, any element of V has a denominator iff V has a bounded denominator; this is false if (π, V) is not finitely generated. From (II.4) we deduce:

COROLLARY II.5. *Let* $(\pi, V) \in \mathrm{Mod}_E\, G$ *admissible contained in* $\mathrm{Ind}_H^G(\Omega, W)$. *Then any element of* V *has a denominator iff* $V \cap \mathrm{Ind}_H^G L_W$ *is an* O_E*-integral structure of* (π, V).

We give two criteria A in (II.6), B in (II.7) for this property.

II.6. Criterion A uses the H-equivariant projection

$$p_\Omega : V \to V_\Omega$$

on the Ω-coinvariants V_Ω of $(\pi, V) \in \mathrm{Mod}_E\, G$; by definition V_Ω is the maximal semi-simple Ω-isotypic quotient of the restriction of (π, V) to H.

CRITERION A. *Let* $(\pi, V) \in \mathrm{Mod}_E\, G$ *contained in* $\mathrm{Ind}_H^G(\Omega, W)$. *If* (π, V) *contains an* O_E*-integral model* L *such that the* $O_E H$*-module* $p_\Omega L$ *is finitely generated, then* V *has a bounded denominator.*

Criterion A is equivalent to: V_Ω is isomorphic to a finite sum $\oplus^{m(\pi)}\Omega$ and $p_\Omega L$ is an O_E-structure of V_Ω. This is clear except may be the O_E-freeness of $p_\Omega L$ that results from the fact that O_E is principal and that a multiple of $p_\Omega L$ is contained in the O_E-integral structure of V_Ω defined by L_W. By adjunction

$$m(\pi) = \dim_E \mathrm{Hom}_{EG}\left(\pi, \mathrm{Ind}_H^G \Omega\right).$$

Criterion A is an integral version of the finite multiplicity of π in $\mathrm{Ind}_H^G \Omega$.

In the section III, for (F, G, ℓ) as in the introduction under the restriction on F given in the Theorem 2, we will prove that any highest Whittaker model of

$(\pi, V) \in \mathrm{Irr}_{\overline{\mathbf{Q}}_\ell} G$ satisfies the criterion A. As (π, V) is admissible, it follows that (1) implies (2) in Theorem 2.

II.7. Criterion B uses the Hecke algebras. One denotes by $\simeq_R$ an isomorphism of R-modules. For any open compact subgroup K of G, one defines the Hecke R-algebra of (G, K),

$$\mathcal{H}_R(G, K) := \mathrm{End}_{RG}\, R[K\backslash G] \simeq_R R[K\backslash G/K].$$

For $g \in G$, the RG-endomorphism of $R[K\backslash G]$ sending the characteristic function of K to the characteristic function of KgK identifies with the natural image $[KgK]$ of KgK in $R[K\backslash G/K]$. The set V^K of K-invariants of $(\pi, V) \in \mathrm{Mod}_R G$, has a natural structure of right $\mathcal{H}_R(G, K)$-module, which satisfies for any $v \in V^K$, $g \in G$:

$$v * [KgK] = \sum_h \pi(h)^{-1} v, \tag{1}$$

where $KgK = \cup_h Kh$ (disjoint union).

Criterion B. *We suppose that the* $\mathcal{H}_{O_E}(G, K)$*-module* $(\mathrm{ind}_H^G L_W)^K$ *is finitely generated for all* K *in a separated decreasing sequence of open compact subgroups of* G *of pro-order invertible in* O_E*. Let* $(\pi, V) \in \mathrm{Mod}_E G$ *be a quotient of* $\mathrm{ind}_H^G(\Omega, W)$*. Then* (π, V) *is* O_E*-integral iff the image of* $\mathrm{ind}_H^G L_W$ *in* V *is an* O_E*-integral structure of* (π, V).

Criterion B does not depend on the choice of L_W. There is no restriction on (π, V). Its application to the integral structures of subrepresentations of Ind_H^G (Ω, W) is obtained by using the contragredient (II.8.3); for the contragredient, we need to restrict to admissible representations.

Criterion B implies that the $\mathcal{H}_E(G, K)$-modules $(\mathrm{ind}_H^G \Omega)^K$ are finitely generated. This implies that for any admissible representation$(\pi, V) \in \mathrm{Mod}_E G$,

$$m_K(\pi) := \dim_E \mathrm{Hom}_{\mathcal{H}_E(G,K)} \left(\left(\mathrm{ind}_H^G \Omega\right)^K, \pi^K \right) < \infty.$$

The converse is false in general, the finite multiplicity $m_K(\pi) < \infty$ for all $(\pi, V) \in \mathrm{Mod}_E G$ does not implies that the $\mathcal{H}_E(G, K)$-modules $(\mathrm{ind}_H^G \Omega)^K$ are finitely generated.

For (F, G, ℓ) as in the introduction, we will prove in (IV.2.1) that any generic compact Whittaker $\overline{\mathbf{Q}}_\ell$-representation of G satisfies the Criterion B. Therefore (1) implies (2) in the Theorem 2 for any generic admissible representation, without restriction on F.

II.8. We recall some general properties of the contragredient. The contragredient $(\tilde\pi, \tilde V) \in \mathrm{Mod}_R G$ of an R-representation $(\pi, V) \in \mathrm{Mod}_R G$ of G is given the natural action of G on the smooth linear forms of V [V2, I.4.12]. The representation (π, V) is called reflexive when (π, V) is the contragredient of $(\tilde{\pi}, \tilde{V})$.

The contragredient $\tilde{}$: $\mathrm{Mod}_E\, G \to \mathrm{Mod}_E\, G$ is exact [V I.4.18] and relates the induced representation to the compactly induced representation

$$\left(\mathrm{ind}_H^G\, \Omega\right)\tilde{} \simeq \mathrm{Ind}_H^G(\tilde{\Omega} \otimes \delta_H),$$

where δ_H is the module of H [V2, I.5.11].

Admissible representations of $\mathrm{Mod}_E\, G$ are reflexive and conversely [V2, I.4.18]. Note that the induced representations $\mathrm{Ind}_H^G\, \Omega$, $\mathrm{ind}_H^G\, \Omega$ are not admissible in general. To apply the Criterion B to a subrepresentation (π, V) of $\mathrm{Ind}_H^G(\Omega, W)$, we suppose (π, V) *and* (Ω, W) *admissible so* that:

$$\left(\mathrm{ind}_H^G\, \tilde{\Omega} \otimes \delta_H^{-1}\right)\tilde{} \simeq \mathrm{Ind}_H^G\, \Omega$$

and $(\tilde{\pi}, \tilde{V})$ *is a quotient of* $\mathrm{ind}_H^G(\tilde{\Omega} \otimes \delta_H^{-1}, \tilde{W})$.

The assertion on the quotient results from a property (II.8.1) of the following isomorphism [V2, I.4.13]: Let $V_1, V_2 \in \mathrm{Mod}_R\, G$, then there is an isomorphism

$$\mathrm{Hom}_{RG}(V_1, \tilde{V}_2) \simeq \mathrm{Hom}_{RG}(V_2, \tilde{V}_1)$$

sending $f \in \mathrm{Hom}_{RG}(V_1, \tilde{V}_2)$ to $\phi \in \mathrm{Hom}_{RG}(V_2, \tilde{V}_1)$ such that

$$< f(v_1), v_2 >=< v_1, \phi(v_2) > \quad \text{for all } v_1 \in V_1, v_2 \in V_2$$

(for $\tilde{v} \in \tilde{V}$, $v \in V$, one denotes $\tilde{v}(v)$ by $< \tilde{v}, v >$ or by $< v, \tilde{v} >$).

II.8.1. Suppose that R is a field. If ϕ is surjective then f is injective; if V_1 is admissible then the converse is true.

An integral O_E-structure L (II.1) of an admissible representation $(\pi, V) \in \mathrm{Mod}_E\, G$ is an admissible integral O_E-structure in the sense of [V2, I.9.1-2] and conversely. The contragredient $\tilde{L}$ of L in $\mathrm{Mod}_{O_E}\, G$ is an O_E-structure of $(\tilde{\pi}, \tilde{V})$ [V2, I.9.7], and L is reflexive in $\mathrm{Mod}_{O_E}\, G$, i.e., the contragredient of $\tilde{L}$ is equal to L. These results are false without the admissibility.

The values of the module δ_H are units in O_E hence $\tilde{L}_W \subset \tilde{W}$ is stable by the action of $\tilde{\Omega} \otimes \delta_H^{-1}$. The O_E-module $\tilde{L}_W$ is an O_E-integral structure of $(\tilde{\Omega} \otimes \delta_H^{-1}, \tilde{W})$. The space of $\mathrm{ind}_H^G(\tilde{\Omega} \otimes \delta_H^{-1}, \tilde{L}_W) \in \mathrm{Mod}_{O_E}\, G$ is the O_E-module of functions $f \in \mathrm{ind}_H^G(\tilde{\Omega} \otimes \delta_H^{-1}, \tilde{W})$ with values in $\tilde{L}_W$.

$\mathrm{ind}_H^G(\tilde{\Omega} \otimes \delta_H^{-1}, \tilde{L}_W)$ *is an* O_E*-integral structure of* $\mathrm{ind}_H^G(\tilde{\Omega} \otimes \delta_H^{-1}, \tilde{W})$ by (II.3).

$(\mathrm{Ind}_H^G\, \Omega, \mathrm{Ind}_H^G\, L_W)$ *is the contragredient of* $\mathrm{ind}_H^G(\tilde{\Omega} \otimes \delta_H^{-1}, \tilde{L}_W)$ by [V2, I.5.11].

But $\mathrm{Ind}_H^G\, L_W$ is not an O_E-integral structure of $\mathrm{Ind}_H^G(\Omega, W)$ in general.

We deduce:

Let $(\pi, V) \in \mathrm{Mod}_E\, G$ admissible, O_E-integral, and contained in $\mathrm{Ind}_H^G(\Omega, W)$. Then $(\tilde{\pi}, \tilde{V}) \in \mathrm{Mod}_E\, G$ is admissible, O_E-integral, and a quotient of $\mathrm{ind}_H^G(\tilde{\Omega} \otimes \delta_H^{-1}, \tilde{W})$.

The image L' of $\mathrm{ind}_H^G(\tilde{\Omega} \otimes \delta_H^{-1}, \tilde{L}_W)$ in $\tilde{V}$ is always nonzero. When L' is an O_E-integral structure of $(\tilde{\pi}, \tilde{V})$, then $\tilde{L}'$ is an O_E-integral structure of (π, V).

PROPOSITION II.8.2. *Let* $(\Omega, W) \in \mathrm{Irr}_E\, H$ *admissible and let* $(\pi, V) \in \mathrm{Mod}_E\, G$ *admissible contained in* $\mathrm{Ind}_H^G(\Omega, W)$ *and* O_E*-integral. The following properties are equivalent:*

– $L := V \cap \mathrm{Ind}_H^G L_W$ *contains an* E*-basis of* V,

– *the image* L' *of* $\mathrm{ind}_H^G(\tilde{\Omega} \otimes \delta_H^{-1}, \tilde{L}_W)$ *in* $\tilde{V}$ *is* O_E*-free.*

– L, L' are O_E*-integral structures of* $(\pi, V), (\tilde{\pi}, \tilde{V})$, *contragredient of each other*.

Remarks: (i) When π is irreducible, the first property is equivalent to $L \neq 0$.

(ii) When L' is $O_E G$-finitely generated, the second property is satisfied because a multiple of L' is contained in an O_E-integral structure of $(\tilde{\pi}, \tilde{V})$ and O_E is principal.

With Criterion B (II.7) we deduce:

COROLLARY II.8.3. *Suppose that the* $\mathcal{H}_{O_E}(G, K)$*-module* $\mathrm{ind}_H^G(\tilde{\Omega} \otimes \delta_H^{-1}, \tilde{L}_W)^K$ *is finitely generated for all* K *as in (II.7). Let* $(\pi, V) \subset \mathrm{Ind}_H^G(\Omega, W)$ *admissible. Then* (π, V) *is* O_E*-integral iff* $V \cap \mathrm{Ind}_H^G L_W$ *is an* O_E*-integral structure of* (π, V).

Proofs of II.3, II.4, II.6, II.7, II.8.

Proof of II.3. Let K be an arbitrary open compact subgroup of G of pro-order invertible in O_E. We have the Mackey relations [V2, I.5.5]:

$$(\mathrm{II}.3.1) \qquad \left(\mathrm{ind}_H^G W\right)^K = \oplus_{HgK}\, \mathrm{ind}_H^{HgK} W, \quad \mathrm{ind}_H^{HgK} W \simeq_R W^{H \cap gKg^{-1}}.$$

The hypotheses on G, H, W insure that the dimension of $\mathrm{ind}_H^G W = \cup_K (\mathrm{ind}_H^G W)^K$ is countable. The relations (II.3.1) are valid for any O_E-representation of H. We apply them to $L_W \in \mathrm{Mod}_{O_E} H$. As O_E is principal and L_W is an O_E-free module that generates W, the O_E-module $L_W^{H \cap gKg^{-1}}$ is free and generates $W^{H \cap gKg^{-1}}$. We deduce that the O_E-module $(\mathrm{ind}_H^G L_W)^K$ is free and contains a basis of $(\mathrm{ind}_H^G W)^K$.

As K is arbitrary, this implies that $\mathrm{ind}_H^G L_W$ contains a basis of the vector space $\mathrm{ind}_H^G W$ and is free as an O_E-module, by the characterization of free modules on a principal commutative ring [V2, I.9.2 or I.C.4]. $\square$

Proof of II.4. Let $(e_i)_{i\in I}$ be an O_E-basis of $(\mathrm{ind}_H^G L_W)^K$. We have

$$\left(\mathrm{Ind}_H^G W\right)^K = \prod_{i\in I} Ee_i, \quad \left(\mathrm{Ind}_H^G L_W\right)^K = \prod_{i\in I} O_E e_i.$$

We suppose, as we may, $\mathrm{Ind}_H^G W \neq \mathrm{ind}_H^G W$; the set I is infinite and countable. The E-dimension N of V^K if finite because V is admissible. Let $(v_j)_{1\leq j\leq N}$ be an E-basis of V^K. We write $v_j = \sum_{i\in I} x_{j,i}e_i$ with $x_{j,i} \in E$, and the support of the map $i \to x_{j,i}$ is finite. We can extract a square submatrix $A = (x_{j,i})$ for $i = i_1, \dots, i_N$ and $1 \leq j \leq N$ of nonzero determinant; the projection $p : V^K \to \oplus_{1\leq k\leq N} Ee_{i_k}$ is an isomorphism. The projection p restricts to an injective O_E-homomorphism

$$V^K \cap \left(\mathrm{Ind}_H^G L_W\right)^K = \left(V \cap \mathrm{Ind}_H^G L_W\right)^K \to \oplus_{1\leq k\leq N} O_E e_{i_k}.$$

As O_E is principal, the O_E-submodule $p(V \cap \mathrm{Ind}_H^G L_W)^K$ of $\oplus_{1\leq k\leq N} O_E e_{i_k}$ is O_E-free or zero. This is true for all K and we deduce that $V \cap \mathrm{Ind}_H^G L_W$ is O_E-free or zero as in the proof of II.3. □

Proof of II.6. The value at 1 defines an H-equivariant nonzero linear form $V \to W$, and hence factorizes through $p_\Omega V$. There exists an H-equivariant linear map $q : p_\Omega V \to W$ such that $v(1) = q \circ p_\Omega(v)$ for all $v \in V$. As V_Ω is semi-simple, q splits and we can suppose that q corresponds to the first projection $\oplus^{m(\pi)} W \to W$.

By hypothesis $p_\Omega(L)$ is $O_E H$-finitely generated, the same is true for its image by the H-equivariant linear map q, therefore there exists $a \in O_E$ such that $a(q \circ p_\Omega)L \subset L_W$. Let $(v, g) \in L \times G$ arbitrary. We have $v(g) = gv(1) = q \circ p_\Omega(gv)$ and $gv \in L$, hence $av(g) \in L_W$, that is, $aL \subset \mathrm{Ind}_H^G L_W$. As L contains an E-basis of V, V has a bounded denominator. □

Proof of II.7. We suppose that (π, V) is O_E-integral. We want to prove that the image L of $\mathrm{ind}_H^G L_W$ in V is an O_E-integral structure of (π, V). Clearly L is G-stable and generates the E-vector space V. The only property that needs some argument is the O_E-freeness of L. As in the proofs of (II.3) and of (II.6) it is equivalent to prove that L^K is contained in a O_E-free module for all K, as in the Criterion B with $V^K \neq 0$. This results from the fact that the right $\mathcal{H}_{O_E}(G, K)$-module L^K is finitely generated, being the quotient of $(\mathrm{ind}_H^G L_W)^K$ (as $p \neq \ell$, the K-invariant functor is exact), hence a multiple of L^K is contained in an O_E-structure of (π, V), and O_E is principal. □

Proof of II.8.1. f is not injective iff there exist $v_1 \in V_1$ nonzero such that $f(v_1) = 0$, i.e., $< \phi(v_2), v_1 >= 0$ for any $v_2 \in V_2$. Let K be an open compact subgroup of G of pro-order invertible in R such that $v_1 \in V_1^K$. Then $(\tilde{V}_1)^K$ is the linear dual of V_1^K, and as we supposed that R is a field, there exists a linear form of V_1^K that does not vanish on v_1. Hence ϕ is not surjective.

ϕ is not surjective iff there exists K, as above, such that $\phi(V_2)^K$ is not the linear dual of V_1^K. Suppose V_1 admissible. The vector spaces V_1^K are finite dimensional and $\phi(V_2)^K$ is not the linear dual of V_1^K iff there exists $v_1 \in V_1^K$ nonzero such that $\phi(V_2)^K$ vanish on v_1. Hence f is not injective. □

Proof of II.8.2. (a) By definition $L = V \cap \mathrm{Ind}_H^G L_W$ and L' is the image in $\tilde{V}$ of $\mathrm{ind}_H^G \tilde{L}_W$ (we supressed $\Omega, \tilde{\Omega} \otimes \delta_H^{-1}$ to simplify).

An element $v \in \mathrm{Ind}_H^G W$ belongs to $\mathrm{Ind}_H^G L_W$ iff $< v, \phi > \in O_E$ for all $\phi \in \mathrm{ind}_H^G(\tilde{L}_W)$, because $\mathrm{Ind}_H^G(\Omega, L_W)$ is the contragredient of $\mathrm{ind}_H^G(\tilde{\Omega} \otimes \delta_H^{-1}, \tilde{L}_W)$. An element $\phi \in \mathrm{ind}_H^G(\tilde{W})$ acts on V via the quotient map $\mathrm{ind}_H^G(\tilde{W}) \to \tilde{V}$.

We deduce that L is the set of $v \in V$ such that $< v, \phi > \in O_E$ for all $\phi \in \mathrm{ind}_H^G(\tilde{L}_W)$ and $< L', L > \subset O_E$.

(b) Suppose that L' is O_E-free. Then L' is an O_E-integral structure of $(\tilde{\pi}, \tilde{V})$. Its contragredient $\tilde{L}'$ is equal to L by the above description of L. Hence $L = \tilde{L}'$ is an O_E-integral structure of (π, V).

(c) Suppose that L contains an E-basis of V, that is, by (II.4), L is an O_E-integral structure of (π, V). Its contragredient $\tilde{L}$ is an O_E-integral structure of $\tilde{V}$. By the last formula in (a), $L' \subset \tilde{L}$ hence L' is O_E-free because O_E is principal. From (b) we deduce that L' is the O_E-integral structure of $(\tilde{\pi}, \tilde{V})$ contragredient to L. □

III. Integral highest Whittaker model. Let (F, G, ℓ) be as in the introduction with the restriction: the characteristic of F is zero and $p \neq 2$.

We define a Whittaker data and a Whittaker representation following [MW]. We choose:

(a) A continuous homomorphism $\phi : F \to \mathbf{C}^*$, trivial on O_F but not on $p_F^{-1} O_F$.

(b) A nondegenerate $\mathrm{Ad}\, G$-invariant bilinear form $B : \mathcal{G} \times \mathcal{G} \to F$ on the Lie algebra $\mathcal{G}$ of G.

(c) An exponential $\exp : \mathcal{V}(0) \to V(1)$, which is a bijective G-equivariant homeomorphism defined on an $\mathrm{Ad}\, G$-invariant open closed subset $\mathcal{V}(0)$ of $\mathcal{G}$ containing the nilpotent elements with image an G-invariant open closed subset $V(1)$ of G, with inverse a logarithm $\log : V(1) \to \mathcal{V}(0)$.

(d) A nilpotent element Y of $\mathcal{G}$ of orbit $\mathcal{O} = \mathrm{Ad}\, G.Y$.

(e) A cocharacter $\mu : F^* \to G$ of G defining via the adjoint action a grading of the Lie algebra $\mathcal{G} = \oplus_{i \in \mathbf{Z}} \mathcal{G}_i$,

$$\mathcal{G}_i := \{X \in \mathcal{G} \mid \mathrm{Ad}\, \mu(s).X = s^i X \ \text{ for all } s \in \mu(F^*)\}$$

such that $Y \in \mathcal{G}_{-2}$. Set $\mathcal{G}_{\geq ?} := \oplus_{i \geq ?} \mathcal{G}_i$ and $?_i := \mathcal{G}_i \cap ?$.

Clearly the grading is finite, $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$ and $B(\mathcal{G}_i, \mathcal{G}_j) = 0$ if $i + j \neq 0$. The centralizer $\mathcal{G}^Y := \{Z \in \mathcal{G} \mid [Y, Z] = 0\}$ of Y in $\mathcal{G}$ satisfies $B(Y, \mathcal{G}^Y) = 0$ [MW, p. 438]. There is a unique $\mu(F^*)$-invariant decomposition

$$\mathcal{G} = \mathcal{M} \oplus \mathcal{G}^Y$$

and $\mathcal{M} = \oplus_{i \in \mathbf{Z}} \mathcal{M}_i, \quad \mathcal{G}^Y = \oplus_{i \in \mathbf{Z}} \mathcal{G}_i^Y$. The skew bilinear form

$$B_Y(X, Z) := B(Y, [X, Z]) : \mathcal{G} \times \mathcal{G} \to F$$

has a radical $\{Z \in \mathcal{G} \mid B(Y, [X, Z]) = 0 \text{ for all } X \in \mathcal{G}\}$ equal to $\mathcal{G}^Y$. Therefore B_Y induces a duality between $\mathcal{M}_i$ and $\mathcal{M}_{i+2}$ for all $i \in \mathbf{Z}$ and a symplectic form on $\mathcal{M}_1$. The dimension of $\mathcal{M}_1$ is an even integer $2m_1$.

(f) An O_F-lattice $\mathcal{M}_1(O_F)$ of $\mathcal{M}_1$, which is self-dual for B:

$$\mathcal{M}_1(O_F) = \{m \in \mathcal{M}_1 \mid B_Y(m, \mathcal{M}_1(O_F)) \subset O_F\}.$$

The group $N := \exp \mathcal{G}_{\geq 1}$ is unipotent and depends only on the choice of μ and exp. We consider the open subgroup H of N and the character χ of H defined by:

$$H := \exp\left(\mathcal{M}_1(O_F) \oplus \mathcal{G}_1^Y \oplus \mathcal{G}_{i\geq 2}\right), \quad \chi(\exp X) := \phi(B(Y, X)).$$

Clearly $H = N$ iff $\mathcal{M}_1 = 0$, and $\chi(\exp X) = \chi(\exp X_2)$, where X_2 is the component of X in $\mathcal{M}_2$. The character χ does not change if (ϕ, B) is replaced by $(\phi_a, a^{-1}B)$, where $\phi_a(x) := \phi(ax)$ with $a \in O_F^*$.

Definition III.1. We call $(\phi, B, \exp, Y, \mathcal{O}, \mu, \mathcal{M}_1(O_F))$ a Whittaker data of G and

$$\mathrm{Ind}_H^G \chi = \mathrm{Ind}_N^G \Omega, \quad \text{where } \Omega := \mathrm{Ind}_H^N \chi$$

a Whittaker representation of G.

When $H \neq N$ the representation Ω is a metaplectic representation of the Heisenberg group $H/\operatorname{Ker}\chi$. The representation Ω is irreducible and admissible [MVW chapter 2, I.6 (3)] and its isomorphism class does not depend on the choice of $\mathcal{M}_1(O_F)$. The isomorphism class of the Whittaker representation depends only on (Y, μ) when $(\phi, B, \exp)$ are fixed, and does not change if (Y, μ) is replaced by a G-conjugate.

The complex field $\mathbf{C}$ appears only in the definition of the nontrivial additive character ϕ of F. The same definitions can be given over any field (or even a commutative ring) R, which contains roots of 1 of any p-power order.

We define the highest Whittaker models of $(\pi, V) \in \mathrm{Irr}_{\overline{\mathbf{Q}}_\ell} G$ as in the introduction. When $V \subset \mathrm{Ind}_H^G \chi$ is a highest Whittaker model of π, we want to show that the projection on the (H, χ)-coinvariant vectors

$$p_\chi : V \to V_\chi$$

behaves well with integral structures.

THEOREM III.2. *Let* $(\pi, V) \in \mathrm{Irr}_{\overline{\mathbf{Q}}_\ell} G$ *integral with* $V \subset \mathrm{Ind}_H^G \chi$ *a highest Whittaker model. Let L be a* $\overline{\mathbf{Z}}_\ell$*-integral structure of* (π, V)*. Then* $p_\chi L$ *is a* $\overline{\mathbf{Z}}_\ell$*-free module.*

As $p_\chi V$ is a finite dimensional $\overline{\mathbf{Q}}_\ell$-space by Moeglin and Waldspurger, and as $p_\chi L$ is a $\overline{\mathbf{Z}}_\ell$-integral structure (a lattice) of $p_\chi V$ by the Theorem III.2, (π, V) satisfies the criterion A, modulo the fact that $\overline{\mathbf{Z}}_\ell$ is not a principal ring. But we may replace $(\overline{\mathbf{Z}}_\ell, \overline{\mathbf{Q}}_\ell)$ by (O_E, E), where O_E is the ring of integers of a finite extension $E/\mathbf{Q}_\ell(\mu_{p^\infty})$ such that π is defined over E, where μ_{p^∞} is the group of roots of 1 of any order of p in $\overline{\mathbf{Q}}_\ell$. The extension $\mathbf{Q}_\ell(\mu_{p^\infty})/Q_\ell$ is infinite and unramified hence O_E is principal.

Therefore (III.2) implies the Theorem 2 of the introduction under the restrictions on (F, π). The theorem (III.2) results from (III.4.6) and the remark following (III.4.3). The rest of the section III is devoted to the proof of (III.2).

The fundamental idea due to Rodier is to approximate the character χ of H by characters χ_n of open compact subgroups K_n with the property that the projections e_n on the (K_n, χ_n)-invariant vectors approximate the projection p_χ on the (H, χ)-coinvariant vectors in the following sense: when n is big enough, p_χ restricts to an isomorphism $e_n V \to p_\chi V$. We want to prove the same thing for an integral structure L instead of V. There is not much to add to the original proof for V, only another technical computation (III.4.1), and this is the purpose of this section.

III.3. We recall the construction of the geometric approximation (K_n, χ_n) of (H, χ) following [MW I.2 (2), I.4 (1), I.9, I.13] (our χ_n is not the character χ_n of [MW]). Set $t := \mu(p_F)$. We choose a lattice $\mathcal{L}$ of $\mathcal{G}$ such that $[\mathcal{L}, \mathcal{L}] \subset \mathcal{L}$ and we complete $\mathcal{M}_1(O_F)$ to a self-dual lattice $\mathcal{M}(O_F) = \oplus_i \mathcal{M}_i(O_F)$ of $\mathcal{M}$. The O_F-module

$$\mathcal{L}' := \mathcal{M}(O_F) \oplus \oplus_{i \in \mathbf{Z}} \left(\mathcal{L} \cap \mathcal{G}_i^Y\right)$$

is an O_F-lattice of $\mathcal{G}$. For a big enough fixed integer A and a fixed integer $c \geq A$, we set for all $n \geq A$

$$G_n := \exp\left(p_F^n \mathcal{L}'\right), \quad A_n := \exp\left(p_F^{[n/2]+c} (\mathcal{L} \cap \mathcal{G}_1)^Y\right), \quad K_n := t^{-n}(G_n A_n) t^n$$

$$\xi_n(\exp X) = \chi_n(t^{-n} \exp(Z_1) \exp(X) t^n) := \phi\left(p_F^{-2n} B(Y, X)\right)$$

for all $X \in p_F^n \mathcal{L}'$, $Z_1 \in p_F^{[n/2]+c} (\mathcal{L} \cap \mathcal{G}_1)^Y$, where $[n/2]$ is the smallest integer $\leq n/2$. The particular form of K_n will be explained soon.

We set $N' := \exp(\mathcal{G}_1^Y \oplus \mathcal{G}_{i \geq 2})$. The Campbell-Hausdorff formula shows that N' is a normal subgroup of H. The closed subgroup C of H generated by $\exp(\mathcal{M}_1(O_F))$ is compact and $H = CN'$. The character χ of H is trivial on C. The sequence $(K_n, \chi_n)_{n \geq A}$ is an approximation of (H, χ) in the following sense:

$K_n = (K_n \cap P^-)(K_n \cap H) = (K_n \cap H)(K_n \cap P^-)$ [MW I.4] where P^- is the stabilizer in G of $\oplus_{i<0}\mathcal{G}_i$, the sequence of groups $K_n \cap P^-$ is decreasing with trivial intersection, the sequence of groups $K_n \cap H = C(K_n \cap N')$ is increasing with union H, the restriction of χ_n to $K_n \cap P^-$ is trivial and $\chi_n = \chi$ on $K_n \cap H$.

The sequence of open compact subgroups G_n of G is decreasing with trivial intersection, and ξ_n is a character of G_n. A basic property of (G_n, ξ_n) is [MW I.6]:

III.3.1. For any integers $A \le m \le n$, the group G_n is normal in G_m and the stabilizer of ξ_n in G_m is equal to $G_n \exp(p_F^m \mathcal{L}^Y)$.

We introduce now an admissible representation $(\pi, V) \in \mathrm{Mod}_{\overline{\mathbf{Q}}_\ell} G$. Let I_n be the projection of V on its (G_n, ξ_n)-invariant vectors. The dimension of the $\overline{\mathbf{Q}}_\ell$-vector space $I_n V$ is finite. The profinite group $\exp(p_F^{c+[n/2]}\mathcal{L}^Y)$ acts on $I_n V$ by (III.3.1). The action is trivial iff the trace $\mathrm{tr}_{I_n V}\, u$ of the action of any element $u \in \exp(p_F^{c+[n/2]}\mathcal{L}^Y)$ is equal to $\dim I_n V$.

Suppose that (π, V) is irreducible hence admissible. When n is big enough, $\mathrm{tr}_{I_n V}\, u$ can be computed using the expansion of the trace $\mathrm{tr}\,\pi$ of π around 1. The computation simplifies when the nilpotent orbit $\mathcal{O}$ is maximal among the nilpotent orbits with a nonzero coefficient. When $\mathcal{O}$ satisfies this property we say that $\mathcal{O}$ is maximal for $\mathrm{tr}\,\pi$. Then we have [MW I.13]:

III.3.2. Let $(\pi, V) \in \mathrm{Irr}_{\overline{\mathbf{Q}}_\ell} G$. When $\mathcal{O}$ is maximal for $\mathrm{tr}\,\pi$ and when n is big enough, the action of $\exp(p_F^{c+[n/2]}\mathcal{L}^Y)$ on $I_n V$ is trivial.

For two integers $n, m \ge A$, we denote by $I_{n,m} : I_n V \to I_m V$ the restriction to $I_n V$ of $I_m t^{m-n}$. In particular

$$I_{n+1,n} = I_n t^{-1} : I_{n+1} V \to I_n V,$$
$$I_{n,n+1} = I_{n+1} t : I_n V \to I_{n+1} V.$$

Au: Eq. ok as set?

The property [MW I.15]: "*Let* $(\pi, V) \in \mathrm{Irr}_{\overline{\mathbf{Q}}_\ell} G$. *When* $\mathcal{O}$ *is maximal for* $\mathrm{tr}\,\pi$ *and when n is big enough,* $I_{n+1,n} I_{n,n+1} I_n$ *is a nonzero multiple of* I_n" is used to prove that the nilpotent orbits maximal for tr_π are those maximal for the Whittaker models [MW I.16] and that the dimension of the (H, χ)-coinvariants of (V, π) is equal to the coefficient attached to $\mathcal{O}$ in the expansion of tr_π [MW I.17].

III.4. We give variants of this property that will be the key to prove (III.2).

LEMMA III.4.1. *Let* $(\pi, V) \in \mathrm{Mod}_{\overline{\mathbf{Q}}_\ell} G$ *such that the action of* $\exp(p_F^{c+[n/2]}\mathcal{L}^Y)$ *on* $I_n V$ *is trivial when n is big enough. Then, when* $n \ge n_o$ *is big enough, there exist integers* $b(n), b'(n) \ge 0$ *such that*

$$I_{n+1,n} I_{n,n+1} I_n = p^{b(n)} I_n, \quad I_{n,n+1} I_{n+1,n} I_{n+1} = p^{b'(n)} I_{n+1}.$$

We will prove in (III.4.2) that $b(n) = b'(n)$.

Proof of III.4.1. To simplify we set $gv := \pi(g)v$ for $g \in G, v \in V$.

(a) It is proved in [MW I.15] under the hypothesis that π is irreducible but without using this property, that for any $w_n \in I_n V$, $I_{n+1,n} I_{n,n+1} w_n$ is the product of a power of p and of a sum

$$\sum_h \xi_{n+1}(h^{-1}) t^{-1} h t w_n,$$

where $h \in G_{n+1}/(G_{n+1} \cap t G_n t^{-1})$ and $t^{-1}ht$ stabilizes ξ_n. The number of terms of the sum is a power of p. It is claimed in [MW I.15] that each term of the sum is equal to w_n when n is big enough; we deduce that there exists an integer $b(n)$ such that $I_{n+1,n} I_{n,n+1} I_n = p^{b(n)} I_n$. We give a proof of the claim because the same method is used for the second equality of the lemma. Let $h \in G_{n+1}$ such that $t^{-1}ht$ stabilizes ξ_n. The definition of G_n shows that if n is big enough, the group $t^{-1} G_{n+1} t$ is contained in G_{n+1-a} for some integer a such that $c + [n/2] \leq n + 1 - a$. There exists $g \in G_n$ and $y \in \exp p_F^{n+1-a} \mathcal{L}^Y$ such that $t^{-1}ht = gy$ by (III.3.1). By (III.3.2) y acts trivially on $I_n V$ hence $t^{-1}htw_n = gw_n = \xi_n(g) w_n$. Denote by X_2 the component of $\log g$ in $\mathcal{M}_2$. Then

$$\xi_n(g) = \phi\big(p_F^{-2n} B(Y, X_2)\big) = \phi\big(p_F^{-2n-2} B(Y, \operatorname{Ad} t . X_2)\big) = \xi_{n+1}(h).$$

Hence each term in the sum is equal to w_n.

(b) We prove the second equality with the same method. For all $n \geq A$, we choose on G_n the Haar measure normalized by $\operatorname{vol} G_n = 1$. By definition, $I_{n,n+1} I_{n+1,n} I_{n+1} = I_{n+1} t I_n t^{-1} I_{n+1}$ is equal to

$$\int_{G_{n+1}} \int_{G_n} \int_{G_{n+1}} \xi_{n+1}(g')^{-1} \xi_n(h)^{-1} \xi_{n+1}(g)^{-1} g' t h t^{-1} g \, dg' \, dh \, dg.$$

When $h \in G_n \cap t^{-1} G_{n+1} t$, the action of $\xi_n(h)^{-1} t h t^{-1}$ on $I_{n+1} V$ is trivial because $\xi_n(h) = \xi_{n+1}(tht^{-1})$ as in (a). The volume of $G_n \cap t^{-1} G_{n+1} t$ is a power of p. The triple integral is the product of this volume and of:

$$\sum_{h \in G_n/(G_n \cap t^{-1} G_{n+1} t)} \xi_n(h)^{-1} \int_{G_{n+1}} \int_{G_{n+1}} \xi_{n+1}(g')^{-1} \xi_{n+1}(g)^{-1} g' t h t^{-1} g \, dg' \, dg.$$

The group $t G_n t^{-1}$ normalizes G_{n+1}, because $t G_n t^{-1}$ is contained in G_{n-a} and $n - a \geq A$ when n is big enough. After the change of variables $y = (tht^{-1})^{-1} g' t h t^{-1}$ and $x = yg$ in G_{n+1} we get

$$\sum_{h \in G_n/(G_n \cap t^{-1} G_{n+1} t)} \xi_n(h)^{-1} t h t^{-1} \int_{G_{n+1}} \int_{G_{n+1}} \xi_{n+1}(tht^{-1} y (tht^{-1})^{-1} y^{-1} x)^{-1} x \, dx \, dy,$$

which is equal to the product of a power of p and of

$$J := \sum_h \xi_n(h)^{-1} t h t^{-1} \int_{G_{n+1}} \xi_{n+1}(x)^{-1} x \, dx = \sum_h \xi_n(h)^{-1} t h t^{-1} I_{n+1},$$

where $h \in G_n/(G_n \cap t^{-1}G_{n+1}t)$ and tht^{-1} stabilizes ξ_{n+1}. The number of h is a power of p. Let $w_{n+1} \in I_{n+1}V$. We have

$$Jw_{n+1} = \sum_h \xi_n(h)^{-1}tht^{-1}w_{n+1}$$

for h as above. As in (a), one shows that each term of the sum is equal to w_{n+1}. Let $h \in G_n$ such that tht^{-1} stabilizes ξ_{n+1}. As in (a), $tht^{-1} \in G_{n-a}$ and the stabilizer of ξ_{n+1} in G_{n-a} is $G_{n+1} \exp p_F^{n-a}\mathcal{L}^Y$ with $n - a > c + [(n+1)/2]$ when n is big enough. Hence $tht^{-1} = gy$ for some $g \in G_{n+1}$ and the action of y on $I_{n+1}V$ is trivial. Hence $tht^{-1}w_{n+1} = gw_{n+1} = \xi_{n+1}(g)w_{n+1}$. Denote by X_2 the component of $\log g$ in $\mathcal{G}_2$. Then

$$\xi_{n+1}(g) = \phi\big(p_F^{-2n-2}B(Y, X_2)\big) = \phi\big(p_F^{-2n}B(Y, \mathrm{Ad}\, t^{-1}.X_2)\big) = \xi_n(h).$$

Hence each term in the sum is equal to w_{n+1}. We deduce that there exists an integer $b'(n)$ such that $I_{n,n+1}I_{n+1,n}I_{n+1} = p^{b'(n)}I_{n+1}$. The lemma is proved. □

For the application that we have in mind, we replace the projection I_n on the (G_n, ξ_n)-invariant vectors by the projection e_n on the (K_n, χ_n)-invariant vectors in the Lemma III.4.1, and we prove $b(n) = b'(n)$.

STABILIZATION LEMMA III.4.2. *Let $(\pi, V) \in \mathrm{Mod}_{\overline{\mathbf{Q}}_\ell}\, G$ such that the action of $\exp(p_F^{c+[n/2]}\mathcal{L}^Y)$ on I_nV is trivial when n is big enough. Then, when $n \geq n_o$ is big enough, there exists an integer $b(n) \geq 0$ such that*

$$e_ne_{n+1}e_n = p^{b(n)}e_n, \quad e_{n+1}e_ne_{n+1} = p^{b(n)}e_{n+1}.$$

In particular, e_{n+1} induces an isomorphism $e_nV \simeq e_{n+1}V$ of inverse $p^{-b(n)}e_n$ restricted to $e_{n+1}V$.

Proof of III.4.2. Suppose that n is big enough. By (III.3) $K_n = t^{-n}A_nt^n$ $t^{-n}G_nt^n$, as $t^{-n}A_nt^n$ acts trivially on $t^{-n}I_nV$ and as $\chi_n(t^{-n}gt^n) = \xi_n(g)$ for all $g \in G_n$, we have

$$I_n = t^ne_nt^{-n}.$$

The action of t on V is invertible hence $I_nV = t^ne_nV$. We have

$$\begin{aligned}
I_{n+1,n} &= I_nt^{-1} = t^ne_nt^{-n-1} : t^{n+1}e_{n+1}V \to t^ne_nV,\\
I_{n,n+1} &= I_{n+1}t = t^{n+1}e_{n+1}t^{-n} : t^ne_nV \to t^{n+1}e_{n+1}V,\\
I_{n+1,n}I_{n,n+1} &= t^ne_ne_{n+1}t^{-n} : t^ne_nV \to t^ne_nV,\\
I_{n,n+1}I_{n+1,n} &= t^{n+1}e_{n+1}e_nt^{-n-1} : t^{n+1}e_{n+1}V \to t^{n+1}e_{n+1}V,\\
I_{n+1,n}I_{n,n+1}I_n &= t^ne_ne_{n+1}e_nt^{-n} : V \to t^ne_nV,\\
I_{n,n+1}I_{n+1,n}I_{n+1} &= t^{n+1}e_{n+1}e_ne_{n+1}t^{-n-1} : V \to t^{n+1}e_{n+1}V.
\end{aligned}$$

The equalities in III.4.1 are equivalent to

$$e_n e_{n+1} e_n = p^{b(n)} e_n, \quad e_{n+1} e_n e_{n+1} = p^{b'(n)} e_{n+1}.$$

We compute $e_n e_{n+1} e_n e_{n+1}$ in two different ways using the two equalities. We get $p^{b(n)} e_n e_{n+1} = p^{b'(n)} e_n e_{n+1}$. The first equality implies $e_n e_{n+1} \neq 0$, hence $b(n) = b'(n)$. The equalities in (III.4.2) are proved.

Let $v_n \in e_n V$. The first equality gives $e_n e_{n+1} v_n = p^{b(n)} v_n$. In particular e_{n+1} is injective on $e_n V$. For $v_{n+1} \in e_{n+1} V$ the second equality gives $e_{n+1} e_n v_{n+1} = p^{b(n)} v_{n+1}$. In particular $e_{n+1} e_n V = e_{n+1} V$. Hence e_{n+1} induces an isomorphism $e_n V \to e_{n+1} V$. By the first equality $p^{-b(n)} e_n e_{n+1} v_n = v_n$, by the second equality $e_{n+1} p^{-b(n)} e_n v_{n+1} = v_{n+1}$. Hence $p^{-b(n)} e_n$ induces the inverse isomorphism $e_{n+1} V \to e_n V$. □

STABILIZATION PROPERTY III.4.3. *We say that the stabilization property holds for* (H, χ) *in* $(\pi, V) \in \mathrm{Mod}_{\overline{\mathbf{Q}}_\ell} G$ *when: for all big enough integers* $n \geq n_o$, *there exists an integer* $b(n)$ *such that* e_{n+1} *restricted to* $e_n V$ *is an isomorphism* $e_n V \simeq e_{n+1} V$ *of inverse* $p^{-b(n)} e_n$ *restricted to* $e_{n+1} V$.

Remark. When $(\pi, V) \in \mathrm{Irr}_{\overline{\mathbf{Q}}_\ell} G$ and $V \subset \mathrm{Ind}_H^G \chi$ is a highest Whittaker model, then the stabilization property III.4.3 holds for (H, χ) in (π, V) by (III.3.2) and (III.4.2).

We consider finally the projections ε_n on the $(K_n \cap H, \chi|_{K_n \cap H})$-invariant vectors.

LEMMA III.4.4. *The stabilization property for* (H, χ) *in* $(\pi, V) \in \mathrm{Mod}_{\overline{\mathbf{Q}}_\ell} G$ *implies for any big enough integers* $n \geq m \geq n_o$:

(a) $\varepsilon_n = e_n$ *on* $e_m V$ *and* ε_n *restricted to* $e_m V$ *is an isomorphism* $e_m V \to e_n V$,

(b) *if* (π, V) *has an integral structure* L, *we can replace* V *by* L *in (a).*

Proof of III.4.4. $\varepsilon_n v = e_n v$ for any $v \in V$, which is invariant by $K_n \cap P^-$ because $K_n = (K_n \cap P^-)(K_n \cap H)$ and χ_n is trivial on $K_n \cap P^-$ and equal to χ on $K_n \cap H$. In particular $\varepsilon_n v_m = e_n v_m$ for any $v_m \in e_m V$ because the sequence of groups $K_n \cap P^-$ is decreasing and χ_m is trivial on $K_m \cap P^-$. The stabilization property implies that ε_{m+1} restricted to $e_m V$ is an isomorphism $e_m V \simeq e_{m+1} V$. By induction, $\varepsilon_n \circ \ldots \circ \varepsilon_{m+1}$ restricted to $e_m V$ is an isomorphism $e_m V \simeq e_n V$. The open compact groups $K_n \cap H$ form an increasing sequence, hence for any $n \geq m$ and m big enough, $\varepsilon_n = \varepsilon_n \circ \ldots \circ \varepsilon_m$. We proved (a).

If (π, V) has an integral structure L, e_{n+1} and $p^{-b(n)} e_n$ give by restriction isomorphisms $e_n L \simeq e_{n+1} L$, which are inverse of each other, because the K_n are pro-p-groups, $p \neq \ell$, and $e_n L = L \cap e_n V$. The arguments given in the proof (a) are valid when V is replaced by L. $\square$

As the open compact groups $K_n \cap H$ form an increasing sequence of union H, the projections ε_n on the $(K_n \cap H, \chi|_{K_n \cap H})$-invariant vectors approximate the projection p_χ on the (H, χ)-invariants in the following sense:

$$p_\chi \varepsilon_n = p_\chi, \quad \operatorname{Ker} p_\chi = \cup_{n \geq m} \operatorname{Ker} \varepsilon_n \tag{III.4.5}$$

for any integer m.

PROPOSITION III.4.6. *The stabilization property (III.4.3) for (H, χ) in $(\pi, V) \in \operatorname{Mod}_{\overline{\mathbf{Q}}_\ell} G$ implies for a big enough integer $m \geq n_o$:*

(1) *p_χ restricted to $e_m V$ is an isomorphism $e_m V \simeq p_\chi V$,*

(2) *if (π, V) is integral with integral structure L, $p_\chi e_m L \simeq p_\chi L$ is a lattice of $p_\chi V$.*

The property (1) is a reformulation of [MW I.14] when $(\pi, V) \in \operatorname{Irr}_{\overline{\mathbf{Q}}_\ell} G$ and $V \subset \operatorname{Ind}_H^G \chi$ is a highest Whittaker model.

Proof of III.4.6. (a) Injectivity of p_χ restricted to $e_m V$. Apply (III.4.5), (III.4.4), and the injectivity of ε_n restricted to $e_m V$ for all $n \geq m \geq n_o$.

(b) Surjectivity of p_χ restricted to $e_m V$. We have $V = \cup_{n \geq m} V^{K_n \cap P^-}$ and by (III.4.4), and its proof:

$$p_\chi(V^{K_n \cap P^-}) = p_\chi \varepsilon_n(V^{K_n \cap P^-}) = p_\chi e_n(V^{K_n \cap P^-}) \subset p_\chi e_n V = p_\chi \varepsilon_n e_m V = p_\chi e_m V.$$

Hence $p_\chi V = p_\chi e_m V$.

(c) $p_\chi e_m L = p_\chi L$. The arguments of (b) apply to L instead of V.

(d) $e_m L$ is a lattice of $e_m V$; this remains true when one applies the isomorphism p_χ. $\square$

IV. Integral generic compact Whittaker representation

Notation IV.1. Let (F, G) be as in the introduction and let R be a commutative ring that contains roots of the unity of any power of p. The characteristic of R is automatically different from p. We choose in G a maximal split F-torus T (the group of rational points a maximal split F-torus) and a minimal parabolic F-group $B = TU$ that contains T and of unipotent radical U. We denote by Z the centralizer of T in G (not the center of G), and by $\overline{B} = T\overline{U}$ the opposite of B in G. We denote by Φ, Φ^{red}, Δ, Φ^+, $\Phi^{+,red}$ the set of roots of (G, T) in $\operatorname{Lie} U$, of reduced roots, of simple positive roots, of positive roots, of positive reduced roots with respect to B. Let $U_{(\alpha)}$ be the unipotent subgroup of U normalized by Z with Lie algebra $\mathcal{U}_\alpha + \mathcal{U}_{2\alpha}$ for any root $\alpha \in \Phi$ (when 2α is not a root, $\mathcal{U}_{2\alpha} = 0$ and $U_{(2\alpha)} = \{1\}$).

Definition IV.1.1. A character $\phi : U \to R^*$ is nondegenerate if the restrictions $\phi_{(\alpha)}$ of ϕ to $U_{(\alpha)}$ satisfy the two following properties (1) and (2):

1. $\phi_{(\alpha)}$ is trivial for any $\alpha \in \Phi^+ - \Delta$.

The character ϕ satisfying (1) identifies to a character of the direct product

$$\prod_{\alpha\in\Delta} U_{(\alpha)}/U_{(2\alpha)} \to R^*.$$

2. The kernel $\operatorname{Ker}\phi_{(\alpha)}$ of $\phi_{(\alpha)}$ is an open compact subgroup of $U_{(\alpha)}$ for all $\alpha \in \Delta$.

In particular (2) implies that $\phi_{(\alpha)}$ is nontrivial for all $\alpha \in \Delta$.

Remarks IV.1.2. (1) When G is anisotropic, $U = \{1\}$ is the trivial group, the regular representation of G on the R-module $C_c^\infty(G; R)$ of locally constant functions $f : G \to R$ with compact support is the compact generic Whittaker R-representation of G.

(2) When G is split, the property (1) of (IV.1.1) is true except in some exceptional cases [Borel Tits *Ann. Math.* **97** (1973), 449–571, see page 519 4.3], and the property (2) of (IV.1.1) is equivalent to: $\phi_{(\alpha)}$ is nontrivial for all $\alpha \in \Delta$.

(3) The set of nondegenerate characters of U is stable by the natural action of Z, because Z normalizes $U_{(\alpha)}$ for all roots $\alpha \in \Phi$.

IV.2. We choose an open compact subgroup K_o of G such that

$$G = BK_o$$

and a normal subgroup K of K_o of finite index, normalized by

$$T^+ := \{t \in T \mid |\alpha(t)| \leq 1 \text{ for all } \alpha \in \Delta\},$$

with an Iwahori decomposition

$$K = (K \cap \overline{U})(K \cap Z)(K \cap U) = (K \cap U)(K \cap Z)(K \cap \overline{U}).$$

$$K \cap U = \prod_{\alpha\in\Phi^{+,red}} K \cap U_{(\alpha)}, \quad K \cap \overline{U} = \prod_{\alpha\in\Phi^{+,red}} K \cap U_{(-\alpha)}.$$

The theory of Bruhat-Tits gives a subgroup K_o of G and a decreasing separated sequence of subgroups K of G satisfying these properties.

THEOREM IV.2.1. *The right $\mathcal{H}_R(G, K)$-module $(\operatorname{ind}_U^G \phi)^K$ is finitely generated for any nondegenerate character $\phi : U \to R^*$.*

This implies that a generic compact Whittaker representation satisfies the Criterion B of (II.7). The rest of this section is devoted to the proof of the theorem.

When $R = \mathbf{C}$ is the field of complex numbers, this is a theorem of Bushnell and Henniart [BH 7.1].

The theorem follows from a geometric property (IV.2.2) and a computation (IV.2.3). This proof is valid over any R and does not use the theorem over $\mathbf{C}$, and is a variant of the proof of [BH].

The support of $(\mathrm{ind}_U^G \phi)^K$ is

$$(1) \qquad G(U, \phi, K) = \{g \in G \mid gKg^{-1} \cap U \subset \mathrm{Ker}\, \phi\}$$

by the Mackey decomposition of $(\mathrm{ind}_U^G \phi)^K$ (proof of (II.3)). This means the following :

- for $g \in G(U, \phi, K)$ there exists a function $\phi_{UgK} \in (\mathrm{ind}_U^G \phi)^K$ with support UgK and value 1 at g,
- the functions ϕ_{UgK} for the (U, K)-cosets UgK of $G(U, \phi, K)$ form a basis of the R-module $(\mathrm{ind}_U^G \phi)^K$ over R.

The support $G(U, \phi, K)$ of $(\mathrm{ind}_U^G \phi)^K$ satisfies the geometric property:

IV.2.2. $G(U, \phi, K)$ is a finite union of UzT_+K_o with $z \in Z \cap G(U, \phi, K)$.
We consider now the right action of the Hecke algebra $\mathcal{H}_R(G, K)$ on $(\mathrm{ind}_U^G \phi)^K$.

IV.2.3. (a) We have for $x \in G(U, \phi, K)$ and $k_o \in K_o$:

$$\phi_{UxK} * [Kk_oK] = \phi_{Uxk_oK}$$

(b) We have for $z \in Z \cap G(U, \phi, K)$ and $t_+ \in T_+$:

$$\phi_{UzK} * [Kt_+K] = \phi_{Uzt_+K}.$$

Clearly, the Theorem IV.2.1 follows from the claims (IV.2.2) and (IV.2.3).

IV.3. The geometric property (IV.2.2) results from a known fact: when $X_{(\alpha)}$ is a group in the Bruhat-Tits filtration of $U_{(\alpha)}$ for $\alpha \in \Delta$ [T 1.4.2], we have the equality of semi-groups (deduced from [T 1.2 (1), 1.4.2]):

$$(\mathrm{IV.3.1}) \qquad T(X, X) = T_+,$$

where $T(X, X) := \{t \in T \mid tX_{(\alpha)}t^{-1} \subset X_{(\alpha)} \text{ for all } \alpha \in \Delta\}$.

For (IV.2.2) it is enough to know that for any $\alpha \in \Delta$ there exists an open compact subgroup $X_{(\alpha)}$ of $U_{(\alpha)}$ such that (IV.3.1) is true. We give a variant of (IV.3.1) when T is replaced by Z and the $X_{(\alpha)}$ are replaced by pairs $(K_{(\alpha)}, C_{(\alpha)})$ of open compact subgroups of $U_{(\alpha)}$ with $K_{(\alpha)}$ normalized by T_+ for any $\alpha \in \Delta$, and $T(X, X)$ is replaced by

$$Z(K, C) := \{z \in Z \mid zK_{(\alpha)}z^{-1} \subset C_{(\alpha)} \text{ for all } \alpha \in \Delta\}.$$

IV.3.2. $Z(K, C) = Z_o T_+$ for some compact subset Z_o of $Z(K, C)$.

The proof of the variant (IV.3.2) uses the particular case (IV.3.1) and the fact that T_+ contains the maximal compact open subgroup T^o of T with semi-group quotient $T/T^o \simeq \mathbf{N}^d$ where $\mathbf{N}$ is the set of natural integers and $d > 0$ an integer. One reduces (IV.3.2) to the combinatorial finiteness property:

IV.3.3. Any non-empty subset Y of $\mathbf{N}^d$ saturated under addition by $\mathbf{N}^d$ is a finite union of $y + \mathbf{N}^d$ for $y \in Y$.

The proof is elementary. When $d = 1$, we choose the minimum element y of Y. Then $Y = y + \mathbf{N}$. By induction on d, we suppose that the property is true for $d - 1$. Let us call "minimal" an element y of Y such that $z + \mathbf{N}^d \subset y + \mathbf{N}^d$ and $z \in Y$ implies $z = y$. Then Y is the union of $y + \mathbf{N}^d$ for $y \in Y$ minimal. The property is equivalent to the finiteness of minimum elements. If $y, z \in Y$ are minimum and distinct, then some component of z is strictly smaller than some component of y. We are reduced to prove that the set $M(i, m)$ of minimum elements of Y with a given i-th component $m \in \mathbf{N}$ is finite, for any $1 \leq i \leq d$ and any $m \in \mathbf{N}$. Suppose that $M(i, m)$ is not empty and let $Y_{i,m}$ be the union of $y + \mathbf{N}^d$ for $y \in M(i, m)$. Via the components different from i, the set of elements of $Y_{i,m}$ with i-th component m, identifies with a non-empty subset of $Y(i, m) \subset \mathbf{N}^{d-1}$ saturated under under addition by $\mathbf{N}^{d-1}$. Under this identification $M(i, m)$ becomes the set of minimum elements of $Y(i, m)$. By induction hypothesis, the set $M(i, m)$ is finite.

IV.3.4. We explain how (IV.3.1) and (IV.3.3) imply (IV.3.2).

(1) We replace Z by T. There exists an open compact subgroup Z_o of Z that normalizes $K_{(\alpha)}$ for any $\alpha \in \Delta$. There exists a finite set of $z_k \in Z$ such that

$$Z = \cup_k z_k T Z_o$$

because the quotient Z/T is compact. The subset $C_{k,(\alpha)} := z_k^{-1} C_{(\alpha)} z_k$ of $U_{(\alpha)}$ is open and compact. Let $t \in T$, $z_o \in Z_o$. Then $z_k t z_o \in Z(K, C)$ iff t belongs to $T(K, C_k)$ where

$$T(K, C) := \{t \in T \mid t K_{(\alpha)} t^{-1} \subset C_{(\alpha)} \text{ for all } \alpha \in \Delta\}.$$

Hence $Z(T, C) = \cup_k z_k T(K, C_k) Z_o$. The set $T(K, C)$ is stable by multiplication by T_+ because the $K_{(\alpha)}$ are normalized by T_+. Hence the property (IV.3.2) is true if for any (K, C) iff $T(K, C) = T_o T_+$ for some compact $T_o \subset T(K, C)$ for any (K, C). When $T(K, C)$ satisfies this property we say simply that $T(K, C)$ is compact modulo T_+.

(2) Change of (K, C) by (K', C'). The conjugation by $t \in T$ respects the property of being an open compact subgroup of T or of being an open compact subgroup of T normalized by T_+. Let $t_1, t_2 \in T$. Then $(t_1^{-1} K t_1, t_2 C t_2^{-1})$ satisfies the same hypotheses than (K, C). An element $t \in T$ satisfies $t t_1^{-1} K_{(\alpha)} t_1 t^{-1} \subset t_2 C_{(\alpha)} t_2^{-1}$

iff $x := t(t_1t_2)^{-1}$ satisfies $xK_{(\alpha)}x^{-1} \subset C_{(\alpha)}$. In other terms,

$$T(K,C) = T(t_1^{-1}Kt_1, t_2Ct_2^{-1})(t_1t_2)^{-1}. \tag{2a}$$

We deduce that $T(t_1^{-1}Kt_1, t_2Ct_2^{-1})$ is compact modulo T_+ iff the same is true for $T(K,C)$.

Let (K', C') satisfying the same hypotheses than (K, C). For $Y = K, C$ and $\alpha \in \Delta$, there exists $t_+ \in T^+$ such that

$$t_+Y_{(\alpha)}t_+^{-1} \subset Y'_{(\alpha)} \subset t_+^{-1}Y_{(\alpha)}t_+.$$

We can choose t_+ independent of the finite set of $\alpha \in \Delta$. The inclusions $K'_{(\alpha)} \subset t_+^{-1}K_{(\alpha)}t_+,\ t_+C_{(\alpha)}t_+^{-1} \subset C'_{(\alpha)}$ imply $T(t_+^{-1}Kt_+, t_+Ct_+^{-1}) \subset T(K', C')$. By symmetry and by (2a), we obtain:

$$T(K', C')t_+^2 \subset T(K,C) \subset T(K', C')t_+^{-2}. \tag{2b}$$

(3) Choosing $(K', C') = (X, X)$ and applying (IV.3.1) we deduce from (2a) and (2b) that there exists $t_+ \in T_+$ such that $T_+t_+^4 \subset T(t_+^{-1}Kt_+, t_+Ct_+^{-1}) \subset T_+$. Using the remark following (2a) and that $t_+^4 \in T_+$, we deduced that $T(K,C)$ is compact modulo T_+ for all (K,C) iff this is true when

$$T_+t_+ \subset T(K,C) \subset T_+$$

for some $t_+ \in T_+$. The image of these inclusions under the natural projection $T \to T/T^o$ followed by an isomorphism $T/T^o \simeq \mathbf{N}^d$ is

$$a + \mathbf{N}^d \subset Y \subset \mathbf{N}^d,$$

where (Y, a) is the image of $(T(K,C), t_+)$ in $\mathbf{N}^d$. We have $Y + \mathbf{N}^d \subset Y$ because $T(K,C)$ is stable by multiplication by T^+. By (IV.3.3), Y is a finite union of $y + \mathbf{N}^d$ with $y \in Y$. We deduce that $T(K,C) = T_oT_+$ is compact modulo T^+.

The claim (IV.3.2) is proved.

IV.3.5. We explain how the geometric property (IV.2.2) can be deduced from (IV.3.2). We start from the decomposition $G = UZK_o$. As K is normal in K_o, the support $G(U, \phi, K)$ of $\mathrm{ind}_U^G \phi$ described in (IV.2.1) (1) is a union of double (U, K_o)-cosets. Hence $G(U, \phi, K) = U(Z \cap G(U, \phi, K))K_o$. We have

$$Z \cap G(U, \phi, K) = \{z \in Z \mid z(K \cap U)z^{-1} \subset \operatorname{Ker}\phi\}.$$

because $zKz^{-1} \cap U = z(K \cap U)z^{-1}$ as $z \in Z$ normalizes U. As $\phi_{(\alpha)}$ is trivial for all positive non simple roots $\alpha \in \Phi^+ - \Delta$ by hypothesis (IV.1.1), and as $z \in Z$ normalizes $U_{(\alpha)}$ for all roots $\alpha \in \Phi$, the decomposition of $K \cap U$ implies that

$$Z \cap G(U, \phi, K) = \{z \in Z \mid z(K \cap U_{(\alpha)})z^{-1} \subset \operatorname{Ker}\phi_{(\alpha)} \ \text{ for all } \alpha \in \Delta\}.$$

By hypothesis (IV.1.1), $\operatorname{Ker}\phi_{(\alpha)}$ is an open compact subgroup of $U_{(\alpha)}$ for all $\alpha \in \Delta$. The open compact subgroups $K \cap U_{(\alpha)}$ of $U_{(\alpha)}$ are normalized by T_+. Hence by

(IV.3.2) $Z \cap G(U, \phi, K)$ is compact modulo T^+. Therefore $G(U, \phi, K)$ is a finite union of UzK_o with $z \in Z$. The geometric property (IV.2.2) is proved.

IV.3.6. We check the computations of (IV.2.3). The first one (a) follows from the formula (II.7) (1) and from the fact that K is normal in K_o hence $Kk_oK = k_oK = Kk_o$ and $UxKk_oK = Uxk_oK$ for any $k_o \in K_o$, $x \in G$. We check now the second one (b). Any element $t_+ \in T_+$ satisfies the relations

$$t_+(K \cap U)t_+^{-1} \subset K \cap U, \quad t_+(K \cap Z)t_+^{-1} = K \cap Z, \quad t_+^{-1}(K \cap \overline{U})t_+ \subset K \cap \overline{U}.$$

These relations and the Iwahori decomposition of K imply

(a) $t_+K = (K \cap Z\overline{U})t_+K$,

(b) $Kt_+ = Kt_+(K \cap ZU)$,

(c) $Kt_+K = \cup_{u^-} Kt_+u^-$(disjoint) with $K \cap U^- = \cup_{u^-} t_+^{-1}(K \cap U^-)t_+u^-$ (disjoint),

(d) $UzKt_+K = Uz(K \cap Z\overline{U})t_+K = Uzt_+K$ for any $z \in Z$ (z normalizes $U \cap K$).

By (d) the support of $f := \phi_{UzK} * [Kt_+K]$ is contained in Uzt_+K. Hence $f = f(zt_+)\phi_{Uzt_+K}$. We want to prove $f(zt^+) = 1$. We have using (c):

$$f(zt_+) = \sum_{u^-} \phi_{UzK}\left(zt_+(t_+u^-)^{-1}\right) = \sum_{u^-} \phi_{UzK}\left(zt_+u^{-^{-1}}t_+^{-1}\right)$$

for u^- as in (c). Only the u^- with $zt_+u^{-^{-1}}t_+^{-1} \in UzK$ give a nonzero contribution. As z normalises U, we can forget it and the condition is $u^{-^{-1}} \in t_+^{-1}UKt_+$ which means $u^- \in t_+^{-1}(K \cap U^-)t_+$ because $UK \subset B(K \cap U^-)$. With (c), only one term contributes and $f(zt^+) = 1$.

Appendix. Let (F, G, ℓ) be as in the introduction and let R be any algebraically closed field of characteristic ℓ. The aim of this appendix is to compare three properties of a representation $(\rho, V) \in \mathrm{Mod}_R\, G$:

(i) The $\mathcal{H}_R(G, K)$-module V^K is finitely generated for all K in a separated decreasing sequence of open compact pro-p-subgroups of G.

(ii) (ρ, V) is finitely generated in each block of $\mathrm{Mod}_R\, G$.

(iii) For any irreducible R-representation π, the quotient multiplicity $\dim_R \mathrm{Hom}_{RG}(\rho, \pi)$ is finite.

Example. $G = GL(2, F)$, H is a maximal torus (split or not split), $\Omega : H \to R^*$ a character. The representation $\rho = \mathrm{ind}_H^G\, \Omega$ was originally considered by Waldspurger in his work on modular forms of half integral weight leading to a proof of nonvanishing of values of L functions of automorphic cuspidal representations for $GL(2)$ at the center of the critical strip. We call it a *Waldspurger representation*.

THEOREM.
- *(i) is equivalent to (ii).*
- *(ii) implies (iii).*
- *(iii) implies (ii) for a complex Waldspurger representation.*

Remarks. (1) The finite quotient multiplicity of $\rho \in \mathrm{Mod}_R\, G$ is equivalent to the finite multiplicity of the contragredient $\tilde{\rho}$: for all $\pi \in \mathrm{Irr}_R\, G$, the multiplicity $\dim_R \mathrm{Hom}_{RG}(\pi, \tilde{\rho})$ is finite. To prove this, one uses that the contragredient is an involution on $\mathrm{Irr}_R\, G$ and the isomorphism (see II.8): $\mathrm{Hom}_{RG}(\pi, \tilde{\rho}) \simeq \mathrm{Hom}_{RG}(\rho, \tilde{\pi})$.

(2) When G is noncompact, their are infinitely many irreducible representations in a block, their direct sum is not finitely generated but satisfies the finite quotient multiplicity.

(3) When R is the field of complex numbers, the equivalence between (i) and (ii) is proved in [BH].

(4) The category $\mathrm{Mod}_R\, G$ is a product of blocks. Each block has a level $r \in \mathbf{Q}$ and there are finitely many blocks of a given level [V, II.5.8, II.5.9] and [V3, III.6]

(5) By the theory of Bernstein, in the complex case, the cuspidal blocks are well understood and the blocks are related with the cuspidal blocks of the Levi subgroups M of the parabolic subgroups of G. The groups M are the F-points of a reductive connected group, just as G, always with a noncompact center when $M \neq G$.

Proof (i) $\Leftrightarrow$ (ii). We need some preliminaries on the theory of Moy-Prasad minimal unrefined R-types. There are finitely many blocks of a given level $r \in \mathbf{Q}$. We denote by $\mathrm{Mod}_R\, G(r)$ their sum. The Moy-Prasad minimal unrefined types of level r contained in $V \in \mathrm{Mod}_{\mathbf{C}}\, G$ generate the component $V(r)$ of V in $\mathrm{Mod}_R\, G(r)$. There are only finitely many Moy-Prasad minimal unrefined types of a given level r, modulo G-conjugation [V, II.5.5]. *For each level r, there exists $K(r)$ such that $V(r)$ is generated by $V(r)^{K(r)}$, this is also true for a smaller $K \subset K(r)$.* Note that V is generated by V^K for some K iff V has only finitely many non zero components in the blocks of G. The letter K or $K(r)$ always stands for an open compact pro-p-subgroup of G. The properties (i), (ii) are respectively equivalent to: *For any level $r \in \mathbf{Q}$,*

(i) *the $\mathcal{H}_R(G, K)$-module $V(r)^K$ is finitely generated for some $K \subset K(r)$.*

(ii) *$V(r)$ finitely generated.*

We prove that (i)′ and (ii)′ are equivalent. We have $V^K = e_K V$ where $e_K \in \mathcal{H}_R(G)$ is an idempotent such that the Hecke algebra $\mathcal{H}_R(G, K)$ identifies to the subalgebra $e_K\mathcal{H}_R(G)e_K$ of the global Hecke algebra $\mathcal{H}_R(G)$, using that K is a pro-p-group [V, I.3.2]. Let $(v_i)_{i\in I}$ be elements of V^K. The two relations

$$V^K = \sum_{i\in I} \mathcal{H}_R(G, K)v_i, \quad \mathcal{H}_R(G)V^K = \sum_{i\in I} \mathcal{H}_R(G)v_i$$

are equivalent. Take $V = V(r)$ then $\mathcal{H}_R(G)V(r)^K = V(r)$ for any $K \subset K(r)$; we deduce from this the equivalence of (i)$'$ and (ii)$'$. $\square$

Comparaison between (ii) and (iii). It is clear that the finite generation in each block implies the finite quotient multiplicity because each irreducible representation is admissible. The converse is not true in general. We will describe certain properties which imply that the converse is true for complex representations.

We consider first a cuspidal block $\mathcal{B} \subset \mathrm{Mod}_{\mathbf{C}}\, G$. We recall some known facts [BDK]. As for a torus (IV.3), the compact subgroups of G generate a normal subgroup G^o with quotient isomorphic to $\mathbf{Z}^d$ where d is the rank of the maximal central split torus T of G. The unipotent subgroups of G are contained in G^o. If Z is the center of G (and not the centralizer of T as in the chapter IV), the quotient G/G^oZ is finite. Let $\pi \in \mathcal{B}$ irreducible. The restriction

$$\pi|_{G^o} = \oplus\sigma_i, \quad \sigma_i \in \mathrm{Irr}_{\mathbf{C}}\, G^o,$$

of π to G^o is semi-simple of finite length, and the irreducible representations in $\mathcal{B}$ are the representations of G with the same restriction to G^o. Each σ_i is the unique irreducible representation in a block of $\mathrm{Mod}_{\mathbf{C}}\, G^o$. We denote by $\mathcal{B}^o$ the sum of the blocks containing the σ_i. For $V \in \mathrm{Mod}_{\mathbf{C}}\, G$, the restriction of V to G^o belongs to $\mathcal{B}^o$ iff V belongs to $\mathcal{B}$. There are infinitely many irreducible non isomorphic cuspidal representations in $\mathcal{B}$ iff $d > 0$. The abelian subcategory $\mathcal{B}_\omega$ of representations in $\mathcal{B}$ with a central character ω contains only finitely many irreducible representations modulo isomorphism.

The categories $\mathcal{B}^o$ and $\mathcal{B}_\omega$ are semi-simple. In these categories, the properties finitely generated, finite length, finite multiplicity, finite quotient multiplicity are trivially equivalent.

For any representation $V = \mathrm{ind}_{G^o}^G W \in \mathcal{B}$ compactly induced from $W \in \mathcal{B}^o$, the property: V has finite quotient multiplicity is equivalent to the same property for W using that the functor $\mathrm{ind}_{G^o}^G$ is the left adjoint of the restriction from G to G^o. It implies that W is finitely generated hence V is finitely generated. By transitivity of the compact induction, this is also true for any $V \in \mathcal{B}$ compactly induced from a closed subgroup H of G^o. Any complex irreducible representation of a closed subgroup H of G has a central character because the cardinal of $\mathbf{C}$ is strictly bigger than the cardinal of G, hence $V = \mathrm{ind}_H^G W$ has a central character when $Z \subset H$. We summarize:

Let H be a closed subgroup of G with $H \subset G^o$ or $Z \subset H$ and let $\Omega \in \mathrm{Irr}_{\mathbf{C}}\, H$. Then the cuspidal irreducible quotients of $\mathrm{ind}_H^G \Omega$ have finite multiplicity if and only if $\mathrm{ind}_H^G \Omega$ is finitely generated in any cuspidal block.

Remarks. (1) This applies to all the representations used to give models of irreducible representations in the theory of automorphic forms related with L-functions, that I am aware of. For the Whittaker representations, H is nilpotent hence $H \subset G^o$. For the Waldspurger representations, H contains the center Z of G.

(2) There are of course other properties of (H, Ω) implying the same property for $\mathrm{ind}_H^G \Omega$. A variant that we will use for the component of a Waldspurger representation in a non cuspidal block is: $H = G^o Z'$ where Z' is a closed subgroup acting on $\Omega \in \mathrm{Mod}_{\mathbf{C}} H$ by a character.

Reduction to a cuspidal block. We consider now a noncuspidal block $\mathcal{B}$ of $\mathrm{Mod}_{\mathbf{C}} G$. There exists a pair $(P, \mathcal{B}_M)$ where $P = MN$ is a parabolic subgroup of G with unipotent radical N and Levi subgroup M and $\mathcal{B}_M$ is a cuspidal block of M, unique modulo association, such that the normalized functor of N-coinvariants, called the Jacquet functor, $r_P^G : \mathcal{B} \to \sum \mathcal{B}_M$ restricted to $\mathcal{B}$ is exact and faithful [R] Corollary 2.4 of image contained in the finite sum $\sum \mathcal{B}_M$ of the blocks of $\mathrm{Mod}_{\mathbf{C}} M$ conjugate to $\mathcal{B}_M$ by the normalizer of M in G. We need all of them, at the level of blocks $r_P^G(\mathcal{B}) = \sum \mathcal{B}_M$. Let $(\pi, V) \in \mathcal{B}$. We claim:

(π, V) is finitely generated iff $r_P^G(\pi, V)$ is finitely generated.

(π, V) has finite quotient multiplicity iff $r_P^G(\pi, V)$ has finite quotient multiplicity.

$r_P^G(\pi, V)$ is finitely generated iff $r_P^G(\pi, V)$ is finitely generated in each cuspidal block because the sum $\sum \mathcal{B}_M$ is finite. The computation of the Jacquet functors of the representations used for models in the theory of automorphic forms is a well known and basic question, originally considered by Rodier, Casselman, and Shalika for the generic Whittaker representation.

The proof of the claim is easy. Finitely generated: if because of exactness and faithfulness of r_P^G, any subset (v_i) of V which lifts a set of generators of $r_P^G(\pi, V)$ generates (π, V). Iff because G/P is compact, a finite set (v_i) of generators of (π, V) is fixed by an open compact subgroup K, $G = \cup_j P k_j K$ (finite union), the finite set $(k_j v_i)$ generates $r_P^G(\pi, V)$.

Finite quotient multiplicity: r_P^G is the left adjoint of the normalized parabolic induction i_P^G, so $\mathrm{Hom}_{\mathbf{C}G}(\pi, i_P^G \tau) \simeq \mathrm{Hom}_{\mathbf{C}M}(r_P^G \pi, \tau)$ for all $\tau \in \mathrm{Irr}_{\mathbf{C}} M$. As $i_P^G \tau$ has finite length, the finite quotient multiplicity for π implies the finite quotient multiplicity for $r_P^G \pi$ (one does not need to suppose $\pi \in \mathcal{B}$).

Conversely, the faithfulness of r_P^G on $\mathcal{B}$ implies that $r_P^G \rho \neq 0$ for any irreducible representation ρ which is a quotient of $\pi \in \mathcal{B}$; as $r_P^G \rho$ has finite length it has an irreducible quotient τ; by adjunction ρ is contained in $i_P^G \tau$ and $\dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{C}G}(\pi, \rho) \leq \dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{C}G}(r_P^G \pi, \tau)$. Hence the finite quotient multiplicity for $r_P^G \pi$ implies the finite quotient multiplicity for π.

Example. Let $G = GL(2, F)$ and $B = TN$ is the upper triangular subgroup with unipotent radical N and T the diagonal subgroup. Let $V \in \mathrm{Mod}_{\mathbf{C}} G$. Then

(iii) implies (ii) for the noncuspidal part of V iff (iii) implies (ii) for the N-coinvariants V_N. We need to analyze V_N. We take the example of a complex Waldspurger representation $\mathrm{ind}_H^G \Omega$ defined at the beginning of the appendix.

First case: $H = T$. We have $G = B \cup BsN$ where $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and a $\mathbf{C}N$-equivariant exact sequence:

(1) $$0 \to \mathrm{ind}_T^{BsN} \Omega \to \mathrm{ind}_T^G \Omega \to \mathrm{ind}_T^B \Omega \to 0.$$

The functor of N-coinvariants is exact and $(\mathrm{ind}_T^G \Omega)_N$ can be computed using (2) and (3) below. We have

(2) $$(\mathrm{ind}_T^B \Omega)_N \simeq \Omega$$

by the linear form $f \to \int_N f(n)\, dn$ for $f \in \mathrm{ind}_T^B \Omega$ and a Haar measure dn on N. We can neglect the character Ω for the properties (ii) and (iii). We compute $(\mathrm{ind}_T^{BsN} \Omega)_N$. The linear map $f(bsn) \to \phi(b) := \int_N f(bsn) dn$ for $b \in B$, followed by the restriction to N identifies $(\mathrm{ind}_T^{BsN} \Omega)_N$ with the space $C_c^\infty(N; \mathbf{C})$. The action of $t \in T$ on $\phi \in C_c^\infty(N; \mathbf{C})$ is

$$(t * \phi)(n') = \int_N f(n'snt)\, dn = \Omega(sts) \int_N f(n''s\, t^{-1}nt) dn = \Omega\delta_B(sts)\phi(n'')$$

where δ_B is the module of B and $n'' := (sts)^{-1}n'sts$ for $n' \in N$. We have

(3) $$\left(\mathrm{ind}_T^{BsN} \Omega\right)_N \simeq (\Omega\delta_B \otimes \rho) \circ s,$$

where ρ is the natural action of T on $C_c^\infty(N; \mathbf{C})$ by $(t.\phi)(n) = \phi(t^{-1}nt)$. For the properties (ii) and (iii) we can neglect the character $\Omega\delta_B$ and s. As T has two orbits in N, the trivial element of stabilizer T and the nontrivial elements of stabilizer the center Z of G, we have a T-equivariant exact sequence

(4) $$0 \to \mathrm{ind}_Z^T 1 \to \rho \to 1 \to 0.$$

For (ii) and (iii) we can neglect the trivial character, and we are reduced to examine $\mathrm{ind}_Z^T 1$. The blocks of $\mathrm{Mod}_{\mathbf{C}} T$ are parametrized by the characters χ^o of the maximal compact subgroup T^o of T, and the component of $\mathrm{ind}_Z^T 1$ in the block parametrized by χ^o is the cyclic representation $\mathrm{ind}_{ZT^o}^T \chi_o$ if χ^o is trivial on $Z \cap T^o$ and 0 otherwise. We deduce that the Waldspurger representation $\mathrm{ind}_T^G \Omega$ is finitely generated in the non cuspidal blocks of G.

Second case: H nonsplit. Modulo conjugation, H is contained in one of the two maximal, compact modulo the center Z, subgroups of G

$$C_1 := KZ, \quad C_2 := ZI \cup ZIt,$$

where $K = GL(2, O_F)$, I is the standard Iwahori subgroup normalized by $t := \begin{pmatrix} 0 & 1 \\ p_F & 0 \end{pmatrix}$. We suppose $H \subset C$ where $C = C_1$ or C_2. Using $G = CTN$ and the transitivity of the compact induction, we compute:

$$\left(\operatorname{ind}_H^G \Omega\right)_N \simeq \operatorname{ind}_{C\cap T}^T(\tau_{C\cap N}), \tag{5}$$

with $\tau_{C\cap N}$ equal to the $C\cap N$-coinvariants of $\tau = \operatorname{ind}_H^C \Omega$. As $C\cap T = T^o Z$ and Z acts on $\tau_{C\cap N}$ by multiplication by a character. We deduce from the cuspidal case seen above, that the Waldspurger representation $\operatorname{ind}_H^G \Omega$ are finitely generated in the non cuspidal blocks if and only if the noncuspidal quotients have finite multiplicity.

REFERENCES

[BDK] Bernstein, Deligne, and Kazhdan, *Le Centre de Bernstein, Vignéras Representations des groups reductifs sur un corps local,* Hemann, paris, 1984.

[BH] Colin Bushnell and Guy Henniart, Generalized Whittaker models and the Bernstein centre, preprint, May 2001.

[MW] Colette Moeglin Jean-Loup Waldspurger, Modéles de Whittake dégénérés pour des groupes p-adiques. *Math. Z.* **196,** no. 3 (1987), 427–452.

[MVW] Colette Moeglin, Marie-France Vignéras, and Jean-Loup Waldspurger and *Correspondances de Howe sur un corps p-adique. Lecture Notes in Mathematics,* vol. 1291, Springer-Verlag, Berlin, 1987.

[R] Alain Roche, Parabolic induction and the Bernstein decomposition.. **Au: Please provide the rest of the pub. info.**

[V1] Marie-France Vignéras, ℓ-principe de Brauer pour un groupe de Lie p-adique, $p \neq \ell$, *Math. Nachr.* **159** (1992), 37–45.

[V2] ———, Représentations l-modulaires d'un groupe réductif p-adique avec $\ell \neq p$, *Progress in Mathematics,* vol. 137, Birkhäuser, Boston, 1996.

[V3] ———, Vignéras induced representations of p-adic reductive groups, *Sel. Math.,* new series 4 (1998), 549–623.

[V4] ———, Integral Kirillov model, *C. R. Acad. Sci. Paris S? r.1 Math.* **326**, no.4 (1998), 411–416.

[V5] ———, Irreducible modular representations of a reductive p-adic group and simple modules for Hecke algebras, ECM3, Barcelone, 2000.

[V6] ———, Congruence modulo ℓ between ϵ factors for cuspidal representations of GL(2), *Journal de Theorie des Nombres de Bordeaux* **12** (2000), 571–580.