Automorphy for some l-adic lifts of automorphic mod l Galois representations.

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Introduction

In this paper we discuss the extension of the methods of Wiles [W] and Taylor-Wiles [TW] from GL_2 to unitary groups of any rank.

The method of [TW] does not extend to GL_n as the basic numerical coincidence on which the method depends (see corollary 2.43 and theorem 4.49 of [DDT]) breaks down. For the Taylor-Wiles method to work when considering a representation

$$r : \mathrm{Gal}\,(\overline{F}/F) \hookrightarrow G(\overline{\mathbb{Q}}_l)$$

one needs

$$[F : \mathbb{Q}](\dim G - \dim B) = \sum_{v|\infty} H^0(\mathrm{Gal}\,(\overline{F}_v/F_v), \mathrm{ad}^{\,0}\overline{r})$$

where B denotes a Borel subgroup of a (not necessarilly connected) reductive group G and $\mathrm{ad}^{\,0}$ denotes the kernel of $\mathrm{ad} \rightarrow \mathrm{ad}_{G}$. This is an 'oddness' condition, which can only hold if F is totally real (or $\mathrm{ad}^{\,0} = (0)$) and $\overline{r}$ satisfies some sort of self-duality. For instance one can expect positive results if $G = GSp_{2n}$ or $G = GO(n)$, but not if $G = GL_n$ for $n > 2$.

In this paper we work with a disconnected group $\mathcal{G}_n$ which we define to be the semidirect product of $GL_n \times GL_1$ by the two element group $\{1, \jmath\}$ with

$$\jmath(g, \mu)\jmath^{-1} = (\mu\, {}^t g^{-1}, \mu).$$

The advantage of this group is its close connection to GL_n and the fact that Galois representations valued in the l-adic points of this group should be connected to automorphic forms on unitary groups, which are already quite well understood. This choice can give us information about Galois representations

$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

where F is a CM field and where there is a symmetric pairing $\langle\ ,\ \rangle$ on $\overline{\mathbb{Q}}_l^n$ satisfying

$$\langle \sigma x, c\sigma c^{-1} y\rangle = \chi(\sigma)\langle x, y\rangle$$

for all $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$ and with c denoting complex conjugation. By a simple twisting argument this also gives us information about Galois representations

$$r : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

where F^+ is a totally real field and

$$r^c \cong \chi r^\vee$$

with χ a totally odd character.

In this setting the Taylor-Wiles argument carries over well, and we are able to prove $R = \mathbb{T}$ theorems in the 'minimal' case. Here, as usual, R denotes a universal deformation ring for certain Galois representations and $\mathbb{T}$ denotes a Hecke algebra for a definite unitary group. By 'minimal' case, we mean that we consider deformation problems where the lifts on the inertia groups away from l are completely prescribed. (This is often achieved by making them as unramified as possible, hence the word 'minimal'.) That this is possible may come as no surprise to experts. The key insights that allow this to work are aleady in the literature:

1. The discovery by Diamond [Dia] and Fujiwara that Mazur's 'multiplicity one principle' (or better 'freeness principle' - it states that a certain natural module for a Hecke algebra is free) was not needed for the Taylor-Wiles argument. In fact they show how the Taylor-Wiles argument can be improved to give a new proof of this principle.

2. The discovery by Skinner and Wiles [SW] of a beautiful trick using base change to avoid the use of Ribet's 'lowering the level' results.

3. The proof of the local Langlands conjecture for GL_n and its compatibility with the instances of the global correspondence studied by Kottwitz and Clozel. (See [HT].)

Indeed a preliminary version of this manuscript has been available for many years. One of us (R.T.) apologises for the delay in producing the final version.

We have not, however, been able to resolve the non-minimal case. We will explain that there is just one missing ingredient, the analogue of Ihara's lemma for the unitary groups we consider. One purpose of this paper is to convince the reader of the importance of attacking this problem.

To describe this conjecture we need some notation. Let F^+ be a totally real field and let G/F^+ be a unitary group with $G(F^+_\infty)$ compact. Then G becomes an inner form of GL_n over some totally imaginary quadratic extension F/F^+. Let v be a place of F^+ with $G(F^+_v) \cong GL_n(F^+_v)$ and consider an open compact subgroup $U = \prod_w U_w \subset G(\mathbb{A}^{\infty,v}_{F^+})$. Let l be a prime not divisible by v. Then we will consider the space $\mathcal{A}(U, \overline{\mathbb{F}}_l)$ of functions

$$G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U \longrightarrow \overline{\mathbb{F}}_l.$$

It is naturally an admissible representation of $GL_n(F^+_v)$ and of the commutative Hecke algebra

$$\mathbb{T} = \mathrm{Im}\,({\bigotimes_w}' \overline{\mathbb{F}}_l[U_w\backslash G(F^+_w)/U_w] \longrightarrow \mathrm{End}\,(\mathcal{A}(U, \overline{\mathbb{F}}_l)),$$

with the restricted tensor product taken over places for which $U_w \cong GL_n(\mathcal{O}_{F^+,w})$ (compatibly with $G(F_w^+) \cong GL_n(F_w^+)$). Subject to some minor restrictions on G we can define what it means for a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ in the support of $\mathcal{A}(U, \overline{\mathbb{F}}_l)$ to be *Eisenstein* - the associated mod$l$ Galois representation of $\mathrm{Gal}\,(\overline{F}/F)$ should be reducible. (See section 2.4 for details.) Then we conjecture the following.

Conjecture A *For any F^+, G, U, v and l as above, and for any irreducible $G(F_v^+)$-submodule*

$$\pi \subset \mathcal{A}(U, \overline{\mathbb{F}}_l)$$

either π is generic or it has an Eisenstein prime of $\mathbb{T}$ in its support.

In fact a slightly weaker statement would suffice for our purposes. See section 2.5 for details. For rank 2 unitray groups this conjecture follows from the strong approximation theorem. There is another argument which uses the geometry of quotients of the Drinfeld upper half plane. An analogous statement for $GL_2/\mathbb{Q}$ is equivalent to Ihara's lemma (lemma 3.2 of [I]). This can be proved in two ways. Ihara deduced it from the congruence subgroup property for $SL_2(\mathbb{Z}[1/v])$. Diamond and Taylor [DT] found an arithmetic algebraic geometry argument. The case of GL_2 seems to be unusually easy as non-generic irreducible representations of $GL_2(F_v^+)$ are one dimensional. We have some partial results when $n = 3$, to which we hope to return in a future paper. We stress the word 'submodule' in the conjecture. The conjecture is not true for 'subquotients'. The corresponding conjecture is often known to be true in characteristic 0, where one can use trace formula arguments to compare with GL_n. (See section 2.5 for more details.)

We will now state a sample of the sort of theorem we prove. (See corollary 4.5.4.)

Theorem B *Let $n \in \mathbb{Z}_{\geq 1}$ be even and let $l > \max\{3, n\}$ be a prime. Let*

$$r : \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GSp_n(\mathbb{Z}_l)$$

be a continuous irreducible representation with the following properties.

1. *r ramifies at only finitely many primes.*
2. *$r|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)}$ is crystalline.*
3. *$\dim_{\mathbb{Q}_l} \mathrm{gr}^i(r \otimes_{\mathbb{Q}_l} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)} = 0$ unless $i \in \{0, 1, ..., n-1\}$ in which case it has dimension 1.*

4. *There is a prime $q \neq l$ such that $q^i \not\equiv 1 \bmod l$ for $i = 1, ..., n$ and $r|^{\rm ss}_{G_{\mathbb{Q}_q}}$ is unramified and $r|^{\rm ss}_{G_{\mathbb{Q}_q}}(\mathrm{Frob}_q)$ has eigenvalues $\{\alpha q^i : \ i = 0, 1, ..., n-1\}$ for some α.*

5. *The image of $r \bmod l$ contains $Sp_n(\mathbb{F}_l)$.*

6. *$r \bmod l$ arises from a cuspidal automorphic representation π_0 of $GL_n(\mathbb{A})$ for which $\pi_{0,\infty}$ has trivial infinitessimal character and $\pi_{0,q}$ is an unramified twist of the Steinberg representation.*

Assume further that conjecture A is true.

Then r arises from a cuspidal automorphic representation π of $GL_n(\mathbb{A})$ for which π_∞ has trivial infinitessimal character and π_q is an unramified twist of the Steinberg representation.

We remark that to prove this theorem we need conjecture A not just for unitary groups defined over $\mathbb{Q}$, but also over other totally real fields.

We also remark that we actually prove a more general theorem which among other things allows one to work over any totally real field, and with any weight which is small compared to l, and with $\overline{r}$ with quite general image. (See theorems 4.3.4 and 4.5.3.) We go to considerable length to prove a similar theorem where instead of assuming that $\overline{r}$ is automorphic one can assume that it is induced from a character. (See theorems 4.4.4 and 4.5.5.) Along the way to the proof of these latter theorems we prove an analogue of Ramakrishna's lifting theorem [Ra] for $\mathcal{G}_n$. (See theorem 1.4.6 and, for a simple special case which may be easier to appreciate, corollary 1.4.7.)

As mentioned above we also obtain some unconditional theorems in the 'minimal case' (see for example theorem 3.1.1), but we have not emphasised this, as we believe they will not be so useful. It should not be hard however to extract such results from our paper, if they had an application.

One of the problems in writing this paper has been to decide exactly what generality to work in. We could certainly have worked in greater generality, but in the interests of clarity we have usually worked in the minimal generality which we believe will be useful. In particular we have restricted ourselves to the 'crystalline' case. It would be useful, and not very difficult, to include also the ordinary case. It would also be useful to clarify the more general results that are available in the case $n = 2$.

In the first section of this paper we discuss deformation theory and Galois theory. We set up the Galois theoretic machinary needed for the Taylor-Wiles method (see proposition 1.4.5) and also take the opportunity to give an analogue (see theorem 1.4.6 and corollary 1.4.7) of Ramakrishna's lifting theorem

[Ra] for $\mathcal{G}_n$. In the second section we discuss automorphic forms on definite unitary groups, their associated Hecke algebras, their associated Galois representations and results about congruences between such automorphic forms. In the third section we put these results together to prove two $R = T$ theorems. Theorem 3.1.1 is for the 'minimal' case and is unconditional. Theorem 3.1.2 is for the general case, but is conditional on the truth of Ihara's lemma (conjecture I or conjecture A). In the final section we combine these theorems with base change arguments to obtain various modularity theorems (theorems 4.3.4 and 4.5.3), along the lines of theorem B above.

Some of the results (those in the non-minimal case) in this paper depend on previously unpublished work of Marie-France Vignéras and of Russ Mann. Marie-France has kindly written up her results in an appendix to this paper. She has kindly written up these results in an appendix. Russ has left academia and as it seems unlikely ever fully write up his results (see [M2]) we have included an account of his work in another appendix.

Since this paper was written one of us (R.T.) has found a way to avoid Ihara's lemma in dealing with non-minimal lifts (see [Tay]). This still depends on much of the theory developed here, but not on Ihara's lemma or on the results of the appendices.

1 Galois deformation rings.

1.1 Some algebra

For n a positive integer let $\mathcal{G}_n$ denote the group scheme over $\mathbb{Z}$ which is the semi-direct product of $GL_n \times GL_1$ by the group $\{1, \jmath\}$ acting on $GL_n \times GL_1$ by

$$\jmath(g,\mu)\jmath^{-1} = (\mu^t g^{-1}, \mu).$$

There is a homomorphism $\nu : \mathcal{G}_n \to GL_1$ which sends (g,μ) to μ and $\jmath$ to -1. Let $\mathcal{G}_n^0$ denote the connected component of $\mathcal{G}_n$. Let $\mathfrak{g}_n$ demote $\mathrm{Lie}\, GL_n \subset \mathrm{Lie}\, \mathcal{G}_n$ and ad the adjoint action of $\mathcal{G}_n$ on $\mathfrak{g}_n$. Over $\mathbb{Z}[1/2]$ we have

$$\mathfrak{g}_n^{\mathcal{G}_n} = (0).$$

Let $\mathfrak{g}_n^0$ denote the trace zero subspace of $\mathfrak{g}_n$.

Suppose that Γ is a group, that Δ is a subgroup of index 2, and that $c \in \Gamma - \Delta$ satisfies $c^2 = 1$. Whenever we endow Γ with a topology we will assume that Δ is closed.

Lemma 1.1.1 *Suppose that R is a ring. Then there is a natural bijection between the following two sets.*

1. *Homomorphisms $r : \Gamma \to \mathcal{G}_n(R)$ that induce isomorphisms $\Gamma/\Delta \xrightarrow{\sim} \mathcal{G}_n/\mathcal{G}_n^0$.*
2. *Pairs $(\rho, \langle\ ,\ \rangle)$, where $\rho : \Delta \to GL_n(R)$ is a homomorphism and*

$$\langle\ ,\ \rangle : R^n \times R^n \longrightarrow R$$

is a perfect R linear pairing such that for all $x, y \in R^n$ and all $\delta \in \Delta$ we have

- *$\langle x, y\rangle = -\mu(c)\langle y, x\rangle$ for some $\mu(c) \in R$, and*
- *$\mu(\delta)\langle \delta^{-1}x, y\rangle = \langle x, c\delta c y\rangle$ for some $\mu(\delta) \in R$.*

Under this correspondence $\mu(\gamma) = (\nu \circ r)(\gamma)$ for all $\gamma \in \Gamma$. If Γ and R have topologies then under this correspondence continuous r's correspond to continuous ρ's.

Proof: The proof is elementary. A homomorphism r corresponds to $r|_\Delta$ with the pairing

$$\langle x, y\rangle = {}^t x A^{-1} y$$

where $r(c) = (A, -(\nu \circ r)(c))\jmath$. $\Box$

Lemma 1.1.2 *Suppose that k is a field of characteristic $\neq 2$ and that $r : \Gamma \to \mathcal{G}_n(k)$ such that $\Delta = r^{-1}(GL_n \times GL_1)(k)$. Then*

$$\dim_k \mathfrak{g}_n^{c=\delta} = n(n + \delta(\nu \circ r)(c))/2$$

for $\delta = 1$ or -1.

Proof: We have $r(c) = (A, -(\nu \circ r)(c), \jmath)$ where ${}^tA = -(\nu \circ r)(c)A$. Then

$$\mathfrak{g}_n^{c=\delta} = \{g \in M_n(k) : \; gA - \delta(\nu \circ r)(c)^t(gA) = 0\}.$$

The lemma follows. □

Lemma 1.1.3 *Suppose k is a field, that $\chi : \Gamma \to k^\times$ is a homomorphism and that*

$$\rho : \Delta \longrightarrow GL_n(k)$$

is absolutely irreducible and satisfies $\chi\rho^\vee \cong \rho^c$. Then there exists a homomorphism

$$r : \Gamma \longrightarrow \mathcal{G}_n(k)$$

such that $r|_\Delta = \rho$, $\nu \circ r|_\Delta = \chi|_\Delta$ and $r(c) \in \mathcal{G}_n(k) - GL_n(k)$.

If $\alpha \in k^\times$ define

$$r_\alpha : \Gamma \longrightarrow \mathcal{G}_n(k)$$

by $r_\alpha|_\Delta = \rho$ and, if $\gamma \in \Gamma - \Delta$ and $r(\gamma) = (A, \mu, \jmath)$, then

$$r_\alpha(\gamma) = (\alpha A, \mu, \jmath).$$

This sets up a bijection between $GL_n(k)$-conjugacy classes of extensions of ρ to $\Gamma \to \mathcal{G}_n(k)$ and $k^\times/(k^\times)^2$.

Note that $\nu \circ r_\alpha = \nu \circ r$. Also note that, if k is algebraically closed then r is unique up to $GL_n(k)$-conjugacy.

If Γ and R have topologies and ρ is continuous then so is r.

Proof: There exists a perfect pairing

$$\langle \;,\; \rangle : k^n \times k^n \longrightarrow k$$

such that $\chi(\delta)\langle \delta^{-1}x, y\rangle = \langle x, c\delta c y\rangle$ for all $\delta \in \Delta$ and all $x, y \in k^n$. The absolute irreducibility of ρ implies that $\langle \;,\; \rangle$ is unique up to $k^\times$-multiples. If we set

$$\langle x, y\rangle' = \langle y, x\rangle$$

then $\chi(\delta)\langle\delta^{-1}x, y\rangle' = \langle x, c\delta cy\rangle'$ for all $\delta \in \Delta$ and all $x, y \in k^n$. Thus

$$\langle\ ,\ \rangle' = \varepsilon\langle\ ,\ \rangle$$

for some $\varepsilon \in k^\times$. As

$$\langle\ ,\ \rangle'' = \langle\ ,\ \rangle$$

we see that $\varepsilon^2 = 1$. The first assertion now follows from lemma 1.1.1. For the second assertion note that conjugation by $\alpha \in k^\times \subset GL_n(k)$ leaves ρ unchanged and replaces $\langle\ ,\ \rangle$ by $\alpha^2\langle\ ,\ \rangle$. $\Box$

Suppose that k is a field and $r : \Gamma \to \mathcal{G}_n(k)$ is a homomorphism with $\Delta = r^{-1}(GL_n \times GL_1)(k)$. We will call r *Schur* if all irreducible Δ-subquotients of k^n are absolutely irreducible and if for all Δ-invariant subspaces $k^n \supset W_1 \supset W_2$ with k^n/W_1 and W_2 irreducible, we have

$$W_2^\vee(\nu \circ r) \not\cong (k^n/W_1)^c.$$

This is certainly satisfied if k^n is an absolutely irreducible Δ-module. Note that if k'/k is a field extension then $r : \Gamma \to \mathcal{G}_n(k)$ is Schur if and only if $r : \Gamma \to \mathcal{G}_n(k')$ is.

Lemma 1.1.4 *Suppose that k is a field and $r : \Gamma \to \mathcal{G}_n(k)$ is a homomorphism with $\Delta = r^{-1}(GL_n \times GL_1)(k)$. If r is Schur then the following assertions hold.*

1. *$r|_\Delta$ is semisimple.*

2. *If $r' : \Gamma \to \mathcal{G}_n(k)$ is another representation with $\Delta = (r')^{-1}GL_n(k)$ and $\operatorname{tr} r|_\Delta = \operatorname{tr} r'|_\Delta$, then r' is $GL_n(k^{ac})$-conjugate to r.*

3. *If k does not have characteristic 2 then $\mathfrak{g}_n^\Gamma = (0)$.*

Proof: We may suppose that k is algebraically closed.

Suppose that r corresponds to $(r|_\Delta, \langle\ ,\ \rangle)$ as in lemma 1.1.1, and let $V \subset k^n$ be an irreducible Δ-submodule. Then $(k^n/V^\perp)^c \cong V^\vee(\nu \circ r)$ and so we can not have $V \subset V^\perp$. Thus $k^n \cong V \oplus V^\perp$ as Δ-modules. Arguing recursively we see that we have a decomposition

$$k^n \cong V_1 \oplus \ldots \oplus V_r$$

and

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle_1 \perp \ldots \perp \langle\ ,\ \rangle_r,$$

where each V_i is an irreducible $k[\Delta]$-module and each $\langle\ ,\ \rangle_i$ is a perfect pairing on V_i. The first part of the lemma follows. Note also that for $i \neq j$ we have $V_i \not\cong V_j$ as $k[\Delta]$-modules and $V_i^c \cong V_i^\vee(\nu \circ r)$.

Note that if ρ and τ are representations $\Delta \to GL_n(k)$ with ρ semi-simple and multiplicity free and with $\operatorname{tr}\rho = \operatorname{tr}\tau$, then the semisimplification of τ is equivalent to ρ. Thus $r'|_\Delta$ has the same Jordan-Holder factors as $r|_\Delta$ (with multiplicity). Thus r' satisfies the same hypothesis as r and so by part one $r'|_\Delta$ is also semisimple. Hence $r'|_\Delta \cong r|_\Delta$, and we may suppose that in fact $r'|_\Delta = r|_\Delta$. Then corresponding to our decomposition

$$k^n \cong V_1 \oplus ... \oplus V_r$$

we see that r corresponds to

$$(r|_\Delta, \langle\ ,\ \rangle_1 \perp ... \perp \langle\ ,\ \rangle_r)$$

while r' corresponds to

$$(r|_\Delta, \mu_1\langle\ ,\ \rangle_1 \perp ... \perp \mu_r\langle\ ,\ \rangle_r)$$

for some $\mu_i \in k^\times$. Conjugation by the element of $GL_n(k)$ which acts on V_i by $\sqrt{\mu_i}$ takes r to r'.

For the third part note that

$$\mathfrak{g}_n^\Delta = \operatorname{End}_{k[\Delta]}(V_1) \oplus ... \oplus \operatorname{End}_{k[\Delta]}(V_r) = k^r.$$

Then c sends $(\alpha_1, ..., \alpha_r)$ to $(-\alpha_1^{*_1}, ..., -\alpha_r^{*_r}) = (-\alpha_1, ..., -\alpha_r)$, where $*_i$ denotes the adjoint with respect to $\langle\ ,\ \rangle_i$. Thus $\mathfrak{g}_n^\Gamma = (0)$. $\square$

Lemma 1.1.5 *Let R be a complete local noetherian ring with maximal ideal $\mathfrak{m}_R$ and residue field $k = R/\mathfrak{m}_R$ of characteristic $l > 2$. Let Γ be a group and let $r : \Gamma \to \mathcal{G}_n(R)$ be a homomorphism such that $\Delta = r^{-1}(GL_n \times GL_1)(R)$ has index 2 in Γ. Suppose moreover that $r \bmod \mathfrak{m}_R$ is Schur. Then the centraliser of r in $1 + M_n(\mathfrak{m}_R)$ is $\{1\}$.*

Proof: This lemma is easily reduced to the case that R is Artinian. In this case we argue by induction on the length of R, the case of length 1 (i.e. $R = k$) being immediate. In general we may choose an ideal I of R such that I has length 1. By the inductive hypothesis any element of the centraliser in $1 + M_n(\mathfrak{m}_R)$ of the image of r lies in $1 + M_n(I)$. It follows from lemma 1.1.4 that this centraliser is $\{1\}$. $\square$

Lemma 1.1.6 *Suppose that Γ is profinite and that*

$$r : \Gamma \longrightarrow \mathcal{G}_n(\mathbb{Q}_l^{ac})$$

is a continuous representation with $\Delta = r^{-1}(GL_n \times GL_1)(\mathbb{Q}_l^{ac})$. Then there exists a finite extension $K/\mathbb{Q}_l$ and a continuous representation

$$r' : \Gamma \longrightarrow \mathcal{G}_n(\mathcal{O}_K)$$

which is $GL_n(\mathbb{Q}_l^{ac})$-conjugate to r.

Proof: It follows from the Baire category theorem that $r(\Delta) \subset GL_n(K)$ for some finite extension $K/\mathbb{Q}_l$. The existence of a bilinear form $\langle\ ,\ \rangle$ as in lemma 1.1.1 over $\mathbb{Q}_l^{ac}$ implies the existence of one over K. (It can be thought of as an $n \times n$-matrix with non-zero determinant satisfying certain K-linear constraints on its coefficients.) Thus $r(\Gamma) \subset \mathcal{G}_n(K)$. A standard argument using the compactness of Δ shows that there is a Δ-invariant $\mathcal{O}_K$-lattice $\Lambda \subset K^n$. (Choose any lattice and add it to all its translates by elements of Δ.) We may further suppose that the $\langle\ ,\ \rangle$-dual lattice Λ^* contains Λ. (If not replace Λ by a suitable scalar multiple.) Choose a maximal Δ-invariant $\mathcal{O}_K$-lattice $\Lambda^* \supset M \supset \Lambda$ such that $M^* \supset M$, and replace Λ by M. Then if $\Lambda^* \supset N \supset \Lambda$ is any Δ-invariant $\mathcal{O}_K$-lattice with N/Λ simple, we must have $N^* \cap N = \Lambda$. We conclude that Λ^*/Λ must be a direct sum of simple $\mathcal{O}_K[\Delta]$-modules. Replacing K by a ramified quadratic extension and repeating this procedure we get a Δ-invariant $\mathcal{O}_K$-lattice Λ with $\Lambda^* = \Lambda$. The lemma now follows from lemma 1.1.1. $\Box$

The next two lemmas are standard.

Lemma 1.1.7 *Let R be a noetherian complete local ring. Let Δ be a profinite group and $\rho : \Delta \longrightarrow GL_n(R)$ a continuous representation. Suppose that $\rho \bmod \mathfrak{m}_R$ is absolutely irreducible. Then the centraliser in $GL_n(R)$ of the image of ρ is $R^\times$.*

Proof: It suffices to consider the case that R is Artinian. We can then argue by induction on the length of R. The case R is a field is well known. So suppose that I is a non-zero ideal of R with $\mathfrak{m}_R I = (0)$. If $z \in Z_{GL_n(R)}(\operatorname{Im}\rho)$ then we see by the inductive hypothesis that $z \in R^\times(1 + M_n(I))$. With out loss of generality we can suppose $z = 1 + y \in 1 + M_n(I)$. Thus $y \in (\mathrm{ad}\,(\rho \bmod \mathfrak{m}_R))^\Delta \otimes_{R/\mathfrak{m}_R} I = I$, and the lemma is proved. $\Box$

Lemma 1.1.8 *Let $R \supset S$ be noetherian complete local rings with $\mathfrak{m}_R \cap S = \mathfrak{m}_S$ and common residue field. Let Δ be a profinite group and let $\rho, \rho' : \Delta \longrightarrow GL_n(S)$ be continuous representations with $\rho \bmod \mathfrak{m}_S$ absolutely irreducible. Suppose that for all ideals $I \subset J$ of R we have*

$$Z_{1+M_n(\mathfrak{m}_R/I)}(\mathrm{Im}\,(\rho \bmod I)) \twoheadrightarrow Z_{1+M_n(\mathfrak{m}_R/J)}(\mathrm{Im}\,(\rho \bmod J)).$$

If ρ and ρ' are conjugate in $GL_n(R)$ then they are conjugate in $GL_n(S)$.

Proof: It suffices to consider the case that R is Artinian (because $S = \lim_{\leftarrow} S/I \cap S$ as I runs over open ideals of R). Again we argue by induction on the length of R. If R is a field there is nothing to do. So suppose that I is an ideal of R and $\mathfrak{m}_R I = (0)$. By the inuctive hypothesis we may suppose that $\rho \bmod I \cap S = \rho' \bmod I \cap S$. Thus $\rho' = (1+\phi)\rho$ where $\phi \in Z^1(\Delta, \mathrm{ad}\,(\rho \bmod \mathfrak{m}_S)) \otimes (I \cap S)$. As ρ and ρ' are conjugate in R, our assumption (on surjections of centralisers) tells us that they are conjugate by an element of $1+M_n(I)$. Hence $[\phi] = 0$ in $H^1(\Delta, \mathrm{ad}\,(\rho \bmod \mathfrak{m}_S)) \otimes I$. Thus $[\phi] = 0$ in $H^1(\Delta, \mathrm{ad}\,(\rho \bmod \mathfrak{m}_S)) \otimes (I \cap S)$, so that ρ and ρ' are conjugate by an element of $1 + M_n(I \cap S)$. $\Box$

The next lemma is essentially due to Carayol [Ca], but he makes various unnecessary hypotheses, so we reproduce some of the proof here.

Lemma 1.1.9 *Let $R \supset S$ be noetherian complete local rings with $\mathfrak{m}_R \cap S = \mathfrak{m}_S$ and common residue field. Let Δ be a profinite group and $\rho : \Delta \longrightarrow GL_n(R)$ a continuous representation. Suppose that $\rho \bmod \mathfrak{m}_R$ is absolutely irreducible and that $\mathrm{tr}\,\rho\Delta \subset S$. If I is an ideal of R such that $\rho \bmod I$ has image in $S/I \cap S$, then there is a $1_n + M_n(I)$-conjugate ρ' of ρ such that the image of ρ' is contained in $GL_n(S)$. In particular there is always a $1_n + M_n(\mathfrak{m}_R)$-conjugate ρ' of ρ such that the image of ρ' is contained in $GL_n(S)$.*

Proof: A simple recursion allows one to reduce to the case that $\mathfrak{m}_R I = (0)$ and $\dim_{R/\mathfrak{m}_R} I = 1$. Replacing R by the set of elements in R which are congruent mod I to an element of S we may further assume that $S/I \cap S \xrightarrow{\sim} R/I$. If $I \subset S$ then $R = S$ and there is nothing to prove. Otherwise $R = S \oplus I$ with multiplication

$$(s,i)(s',i') = (ss', s'i + si').$$

In particular $\mathfrak{m}_S$ is an ideal of R and $R/\mathfrak{m}_S \cong (S/\mathfrak{m}_S)[\epsilon]/(\epsilon^2)$. If we know the result for $S/\mathfrak{m}_S \subset R/\mathfrak{m}_S$ then the result follows for $S \subset R$ (because then we can find $A \in M_n(I)$ such that

$$(1_n - A)\rho(1_n + A) \bmod \mathfrak{m}_S$$

is valued in $GL_n(S/\mathfrak{m}_S)$ so that

$$(1_n - A)\rho(1_n + A)$$

is valued in S.)

Thus we are reduced to the case $S = k$ is a field, $R = k[\epsilon]/(\epsilon^2)$ and $I = (\epsilon)$. Replacing Δ by its image we may assume that $\Delta \subset GL_n(R)$. If $\delta \in \Delta$ we will write $\overline{\delta}$ for its projection to $GL_n(k)$. If $\gamma \in \Delta \cap (1_n + M_n(I))$ then for all $\delta_1, \delta_2 \in \Delta$ we have

$$\operatorname{tr} \overline{\delta}_1((\gamma - 1_n)/\epsilon)\overline{\delta}_2 = 0.$$

As $\overline{\rho}$ is absolutely irreducible we deduce that

$$\operatorname{tr} A((\gamma - 1_n)/\epsilon) = 0$$

for all $A \in M_n(k)$ and hence that $\gamma = 1_n$. Thus we may consider $\Delta \subset GL_n(k)$, when we have

$$\rho(\delta) = (1_n + c(\delta)\epsilon)\delta$$

for all $\delta \in \Delta$. We see that

- $c(\delta_1\delta_2) = c(\delta_1) + \delta_1 c(\delta_2)\delta_1^{-1}$ for all $\delta_1, \delta_2 \in \Delta$,
- and $\operatorname{tr} c(\delta)\delta = 0$ for all $\delta \in \Delta$.

As $\overline{\rho}$ is absolutely irreducible, it follows from lemma 1 of [Ca] that there exists $A \in M_n(k)$ such that

$$c(\delta) = \delta A \delta^{-1} - A$$

for all $\delta \in \Delta$. (Although Carayol makes the running assumption at the start of section 1 of [Ca] that k is perfect, this assumption is not used in the proof of lemma 1.) Then we see that

$$(1_n - A\epsilon)\rho(\delta)(1_n + A\epsilon) = \delta$$

for all $\delta \in \Delta$. The lemma follows. $\Box$

Lemma 1.1.10 *Suppose that $R \supset S$ are complete local noetherian rings with $\mathfrak{m}_R \cap S = \mathfrak{m}_S$ and common residue field k of characteristic $l > 2$. Suppose that Γ is a profinite group and that $r : \Gamma \to \mathcal{G}_n(R)$ is a continuous representation with $\Delta = r^{-1}(GL_n \times GL_1)(R)$. Suppose moreover that $r|_\Delta \bmod \mathfrak{m}_R$ is absolutely irreducible and that $\operatorname{tr} r(\Delta) \subset S$. Then r is $GL_n(R)$-conjugate to a homomorphism $r' : \Gamma \to \mathcal{G}_n(S)$.*

Proof: By lemma 1.1.9 we may suppose that $r(\Delta) \subset (GL_n \times GL_1)(S)$. Because $r|_\Delta^c$ and $r|_\Delta^\vee(\nu \circ r)$ are $GL_n(R)$-conjugate, it follows from lemma 1.1.8 that they are $GL_n(S)$-conjugate. Suppose that

$$r|_\Delta^c = Ar|_\Delta^\vee(\nu \circ r)A^{-1}$$

with $A \in GL_n(S)$. Then $A^{-1t}A$ commutes with the image of $r|_\Delta^\vee$. Hence by lemma 1.1.7, ${}^tA = \mu A$ for some $\mu \in R$ with $\mu^2 = 1$, i.e. with $\mu = \pm 1$. The lemma now follows from lemmas 1.1.1 and 1.1.3. $\Box$

Finally in this section we consider induction in this setting. Suppose that Γ' is a finite index subgroup of Γ containing c and set $\Delta' = \Delta \cap \Gamma'$. Suppose also that $\chi : \Gamma \to R^\times$ is a homomorphism. Let $r' : \Gamma' \to \mathcal{G}_n(R)$ be a homomorphism with $\nu \circ r' = \chi|_{\Gamma'}$ and suppose r' corresponds to a pair $(\rho', \langle\ ,\ \rangle')$ as in lemma 1.1.1. We define

$$\mathrm{Ind}\,_{\Gamma',\Delta'}^{\Gamma,\Delta,\chi} r' : \Gamma \to \mathcal{G}_{n[\Gamma:\Gamma']}(R)$$

to be the homomorphism corresponding to the pair $(\rho, \langle\ ,\ \rangle)$ where ρ acts by right translation on the R-module of functions $f : \Delta \to R^n$ such that

$$f(\delta'\delta) = \rho(\delta')f(\delta)$$

for all $\delta' \in \Delta'$ and $\delta \in \Delta$. We set

$$\langle f, f' \rangle = \sum_{\delta \in \Delta' \backslash \Delta} \chi(\delta)^{-1} \langle f(\delta), f'(c\delta c^{-1}) \rangle'.$$

We have $\nu \circ (\mathrm{Ind}\,_{\Gamma',\Delta'}^{\Gamma,\Delta,\chi} r') = \chi$. If Γ and R carry topologies, if Γ' is open in Γ and if r' is continuous then we consider only continuous functions f. We will sometimes write $\mathrm{Ind}\,_{\Gamma'}^{\Gamma,\chi}$ for $\mathrm{Ind}\,_{\Gamma',\Delta'}^{\Gamma,\Delta,\chi}$, although it depends essentially on Δ' and Δ as well as Γ', Γ and χ.

1.2 Deformation theory

Next we will turn to deformation theory. We will follow the approach of Dickinson [Dic1]. Let l be an odd prime. Let k denote an algebraic extension of the finite field with l elements, let $\mathcal{O}$ denote the ring of integers of a finite totally ramified extension K of the fraction field of the Witt vectors $W(k)$, let λ denote the maximal ideal of $\mathcal{O}$, let $\mathcal{C}_\mathcal{O}^f$ denote the category of Artinian local $\mathcal{O}$-algebras for which the structure map $\mathcal{O} \to R$ induces an isomorphism on residue fields, and let $\mathcal{C}_\mathcal{O}$ denote the full subcategory of the category of topological $\mathcal{O}$-algebras whose objects are inverse limits of objects of $\mathcal{C}_\mathcal{O}^f$. Also fix a profinite group Γ together with a closed subgroup $\Delta \subset \Gamma$ such that there is an element $c \in \Gamma - \Delta$ with $c^2 = 1$. Also fix a continuous Schur homomorphism

$$\overline{r} : \Gamma \longrightarrow \mathcal{G}_n(k)$$

and a homomorphism $\chi : \Gamma \to \mathcal{O}^\times$, such that $\Delta = \overline{r}^{-1}(GL_n \times GL_1)(k)$ and $\nu \circ \overline{r} = \chi$. Let $S \supset S_0$ be finite index sets. For $q \in S$ let Δ_q be a profinite group provided with a continuous homomorphism $\Delta_q \to \Delta$. For $q \in S_0$ fix a decreasing filtration of k^n by Δ_q-invariant subspaces $\overline{\mathrm{Fil}}^i_q$, such that $\overline{\mathrm{Fil}}^i_q$ is k^n for i sufficiently small and $\overline{\mathrm{Fil}}^i_q = (0)$ for i sufficiently large.

By a *lifting* of $(\overline{r}, \{\overline{\mathrm{Fil}}^i_q\}_{q \in S_0, i})$ to an object R of $\mathcal{C}_\mathcal{O}$ we shall mean a pair $(r, \{\mathrm{Fil}^i_q\}_{q\in S_0,i})$, where $r : \Gamma \to \mathcal{G}_n(R)$ is a continuous homomorphism with $r \bmod \mathfrak{m}_R = \overline{r}$ and $\nu \circ r = \chi$, and where Fil^i_q is a decreasing filtration of R^n by Δ_q-invariant subspaces such that the natural maps $\mathrm{Fil}^i_q \otimes_R k \to k^n$ give isomorphisms $\mathrm{Fil}^i_q \otimes_R k \xrightarrow{\sim} \overline{\mathrm{Fil}}^i_q$. By a *lifting* of $(\overline{r}|_{\Delta_q}, \{\overline{\mathrm{Fil}}^i_q\}_i)$ (resp. $\overline{r}|_{\Delta_q}$) to an object R of $\mathcal{C}_\mathcal{O}$ we shall mean a pair $(r, \{\mathrm{Fil}^i_q\}_i)$ (resp. r), where $r : \Delta_q \to GL_n(R)$ is a continuous homomorphism with $r \bmod \mathfrak{m}_R = \overline{r}$ and $\nu \circ r = \chi$, and where Fil^i_q is a decreasing filtration of R^n by Δ_q-invariant subspaces such that the natural maps $\mathrm{Fil}^i_q \otimes_R k \to k^n$ give isomorphisms $\mathrm{Fil}^i_q \otimes_R k \xrightarrow{\sim} \overline{\mathrm{Fil}}^i_q$. We will call two liftings *equivalent* if they are conjugate by an element of $1 + M_n(\mathfrak{m}_R) \subset GL_n(R)$. By a *deformation* of $(\overline{r}, \{\overline{\mathrm{Fil}}^i_q\})$ we shall mean an equivalence class of liftings.

For $q \in S_0$ define a filtration Fil_q on $\mathrm{ad}\,\overline{r}$ by setting

$$\mathrm{Fil}^i_q \mathrm{ad}\,\overline{r} = \{a \in \mathrm{ad}\,\overline{r} : \ a\mathrm{Fil}^j_q \subset \mathrm{Fil}^{j+i} q \ \ \forall j\}.$$

To simplify notation set $\mathrm{Fil}^0_q \mathrm{ad}\,\overline{r} = \mathrm{ad}\,\overline{r}$ and $\mathrm{Fil}^1_q \mathrm{ad}\,\overline{r} = (0)$ if $q \notin S_0$. We will write $Z^1(\Delta_q, \{\overline{\mathrm{Fil}}^i_q\}_i, \mathrm{ad}\,\overline{r})$ for the set of pairs (ϕ, A) with $\phi \in Z^1(\Delta_q, \mathrm{ad}\,\overline{r})$ and $A \in \mathrm{ad}\,\overline{r}/\mathrm{Fil}^0_q \mathrm{ad}\,\overline{r}$ satisfying

$$\phi + (\mathrm{ad}\,\overline{r} - 1)A = 0 \in Z^1(\Delta_q, \mathrm{ad}\,\overline{r}/\mathrm{Fil}^0_q \mathrm{ad}\,\overline{r}).$$

There is a natural map

$$\begin{array}{rcl} \mathrm{ad}\,\overline{r} & \longrightarrow & Z^1(\Delta_q, \{\overline{\mathrm{Fil}}^i_q\}_i, \mathrm{ad}\,\overline{r}) \\ A & \longmapsto & ((1 - \mathrm{ad}\,\overline{r})A, A). \end{array}$$

This gives rise to an exact sequence

$$\begin{array}{l} (0) \to H^0(\Delta_q, \mathrm{Fil}^0_q \mathrm{ad}\,\overline{r}) \to \mathrm{ad}\,\overline{r} \to \\ \to Z^1(\Delta_q, \{\overline{\mathrm{Fil}}^i_q\}_i, \mathrm{ad}\,\overline{r}) \to H^1(\Delta_q, \mathrm{Fil}^0_q \mathrm{ad}\,\overline{r}) \to (0), \end{array}$$

where the penultimate map sends (ϕ, A) to $[\phi + (\operatorname{ad}\overline{r} - 1)\widetilde{A}]$ for any lifting $\widetilde{A}$ of A to $\operatorname{ad}\overline{r}$.

For $q \in S$ there is a universal lifting (*not* deformation) of $(\overline{r}, \{\overline{\mathrm{Fil}}^i_q\}_i)$ over a an object R^{loc}_q of $\mathcal{C}_{\mathcal{O}}$. Note that R^{loc}_q has a natural action of $1_n + M_n(\mathfrak{m}_{R^{\mathrm{loc}}_q})$. There are natural isomorphisms

$$\operatorname{Hom}_k(\mathfrak{m}_{R^{\mathrm{loc}}_q}/(\mathfrak{m}^2_{R^{\mathrm{loc}}_q}, \lambda), k) \cong \operatorname{Hom}_{\mathcal{C}_{\mathcal{O}}}(R^{\mathrm{loc}}_q, k[\epsilon]/(\epsilon^2)) \cong Z^1(\Delta_q, \{\overline{\mathrm{Fil}}^i_q\}_i, \operatorname{ad}\overline{r}).$$

The first is standard. Under the second a pair (ϕ, A) corresponds to the homomorphism arising from the lifting

$$((1+\phi\epsilon)\overline{r}|_{\Delta_q}, \{(1+\epsilon A)\overline{\mathrm{Fil}}^i_q + \epsilon\overline{\mathrm{Fil}}^i_q\}_i)$$

of $(\overline{r}|_{\Delta_q}, \{\overline{\mathrm{Fil}}^i_q\}_i)$. The action of $M_2(\mathfrak{m}_{R^{\mathrm{loc}}_q}/(\mathfrak{m}^2_{R^{\mathrm{loc}}_q}, \lambda))$ on $R^{\mathrm{loc}}_q/(\mathfrak{m}^2_{R^{\mathrm{loc}}_q}, \lambda)$ gives an action on $Z^1(\Delta_q, \{\overline{\mathrm{Fil}}^i_q\}_i, \operatorname{ad}\overline{r})$ which can be described as follows. If $\psi \in \operatorname{Hom}_k(\mathfrak{m}_{R^{\mathrm{loc}}_q}/(\mathfrak{m}^2_{R^{\mathrm{loc}}_q}, \lambda), k)$ corresponds to $z \in Z^1(\Delta_q, \{\overline{\mathrm{Fil}}^i_q\}_i, \operatorname{ad}\overline{r})$, then $B \in M_2(\mathfrak{m}_{R^{\mathrm{loc}}_q}/(\mathfrak{m}^2_{R^{\mathrm{loc}}_q}, \lambda))$ takes z to z plus the image of $\psi(B) \in \operatorname{ad}\overline{r}$. In particular there is a bijection between $M_2(\mathfrak{m}_{R^{\mathrm{loc}}_q}/(\mathfrak{m}^2_{R^{\mathrm{loc}}_q}, \lambda))$ invariant subspaces of $Z^1(\Delta_q, \{\overline{\mathrm{Fil}}^i_q\}_i, \operatorname{ad}\overline{r})$ and subspaces of $H^1(\Delta_q, \mathrm{Fil}^0_q\operatorname{ad}\overline{r})$.

Let R be an object of $\mathcal{C}_{\mathcal{O}}$ and I be a closed ideal of R with $\mathfrak{m}_R I = (0)$. Suppose that $(r_1, \{\mathrm{Fil}^i_{q,1}\})$ and $(r_2, \{\mathrm{Fil}^i_{q,2}\})$ are two liftings of $(\overline{r}|_{\Delta_q}, \{\overline{\mathrm{Fil}}^i_q\})$ with the same reduction mod I. Choose $A \in M_n(I)$ such that $(1_n + A)\mathrm{Fil}^i_{q,1} = \mathrm{Fil}^i_{q,2}$ for all i. Then

$$\gamma \longmapsto r_2(\gamma)r_1(\gamma)^{-1} - 1 + (\operatorname{ad}\overline{r}(\gamma) - 1)A$$

defines an element of $H^1(\Delta_q, \mathrm{Fil}^0_q\operatorname{ad}\overline{r}) \otimes_k I$ which is independent of the choice of A and which we shall denote $[(r_2, \{\mathrm{Fil}^i_{q,2}\}) - (r_1, \{\mathrm{Fil}^i_{q,1}\})]$. In fact this sets up a bijection between $H^1(\Delta_q, \mathrm{Fil}^0_q\operatorname{ad}\overline{r}) \otimes_k I$ and $(1+M_n(I))$-conjugacy classes of lifts which agree with $(r_1, \{\mathrm{Fil}^i_{q,1}\})$ modulo I. Now suppose that $(r, \{\mathrm{Fil}^i_q\})$ is a lift of $(\overline{r}|_{\Delta_q}, \{\overline{\mathrm{Fil}}^i_q\})$ to R/I. Choose a lifting $\{\widetilde{\mathrm{Fil}}^i_q\}$ to R of $\{\mathrm{Fil}^i_q\}$ and for each $\gamma \in \Delta_q$ choose a lifting $\widetilde{r(\gamma)}$ to $GL_n(R)$ of $r(\gamma)$ such that $\widetilde{r(\gamma)}\widetilde{\mathrm{Fil}}^i_q \subset \widetilde{\mathrm{Fil}}^i_q$ for all i. Then

$$(\gamma, \delta) \longmapsto \widetilde{r(\gamma\delta)}\,\widetilde{r(\delta)}^{-1}\,\widetilde{r(\gamma)}^{-1}$$

defines a class $\mathrm{obs}_{R,I}(r, \{\mathrm{Fil}^i_q\}) \in H^2(\Delta_q, \mathrm{Fil}^0\operatorname{ad}\overline{r}) \otimes_k I$ which is independent of the choices made and vanishes if and only if $(r, \{\mathrm{Fil}^i_q\})$ lifts to R.

Now suppose that $(r_q, \{\mathrm{Fil}\,^i_q\}_i)$ is a lifting of $(r|_{\Delta_q}, \{\overline{\mathrm{Fil}}\,^i_q\}_i)$ to $\mathcal{O}$ corresponding to a homomorphism $\alpha : R_q^{\mathrm{loc}} \to \mathcal{O}$. Write $Z^1(\Delta_q, \{\mathrm{Fil}\,^i_q\}_i, \mathrm{ad}\, r_q \otimes K/\mathcal{O})$ for the set of pairs (ϕ, A) with $\phi \in Z^1(\Delta_q, \mathrm{ad}\, r_q \otimes K/\mathcal{O})$ and $A \in (\mathrm{ad}\, r_q/\mathrm{Fil}\,^0_q \mathrm{ad}\, r_q) \otimes K/\mathcal{O}$ satisfying

$$\phi + (\mathrm{ad}\, r_q - 1)A = 0 \in Z^1(\Delta_q, (\mathrm{ad}\, r_q/\mathrm{Fil}\,^0_q \mathrm{ad}\, r_q)).$$

As above, the map

$$\begin{array}{rcl} \mathrm{ad}\, r_q \otimes K/\mathcal{O} & \longrightarrow & Z^1(\Delta_q, \{\mathrm{Fil}\,^i_q\}_i, \mathrm{ad}\, r_q \otimes K/\mathcal{O}) \\ A & \longmapsto & ((1 - \mathrm{ad}\, r_q)A, A) \end{array}$$

has kernel $H^0(\Delta_q, \mathrm{Fil}\,^0_q \mathrm{ad}\, r_q \otimes K/\mathcal{O})$ and cokernel $H^1(\Delta_q, \mathrm{Fil}\,^0_q \mathrm{ad}\, r_q \otimes K/\mathcal{O})$ (via the map

$$\begin{array}{rcl} Z^1(\Delta_q, \{\mathrm{Fil}\,^i_q\}_i, \mathrm{ad}\, r_q \otimes K/\mathcal{O}) & \longrightarrow & H^1(\Delta_q, \mathrm{Fil}\,^0_q \mathrm{ad}\, r_q \otimes K/\mathcal{O}) \\ (\phi, A) & \longmapsto & [\phi + (\mathrm{ad}\, r_q - 1)\widetilde{A}], \end{array}$$

where $\widetilde{A}$ is any lifting of A to $\mathrm{ad}\, r_q \otimes K/\mathcal{O}$). There is also a natural identification

$$\mathrm{Hom}\,_{\mathcal{O}}(\ker \alpha/(\ker \alpha)^2, K/\mathcal{O}) \cong Z^1(\Delta_q, \{\mathrm{Fil}\,^i_q\}_i, \mathrm{ad}\, r_q \otimes K/\mathcal{O}).$$

This may be described as follows. Consider the topological $\mathcal{O}$-algebra $\mathcal{O} \oplus K/\mathcal{O}\epsilon$ where $\epsilon^2 = 0$. Although $\mathcal{O} \oplus K/\mathcal{O}\epsilon$ is not and object of $\mathcal{C}_{\mathcal{O}}$, it still makes sense to talk about liftings of $(r_q, \{\mathrm{Fil}\,^i_q\}_i)$ to $\mathcal{O} \oplus K/\mathcal{O}\epsilon$. One can then check that such liftings are parametrised by $Z^1(\Delta_q, \{\mathrm{Fil}\,^i_q\}_i, \mathrm{ad}\, r_q \otimes K/\mathcal{O})$. (Any such lifting arises by extension of scalars from a lifting to some $\mathcal{O} \oplus \lambda^{-N}/\mathcal{O}\epsilon$.) On the other hand such liftings correspond to homomorphisms $R_q^{\mathrm{loc}} \to \mathcal{O} \oplus K/\mathcal{O}\epsilon$ lifting α and such liftings correspond to $\mathrm{Hom}\,_{\mathcal{O}}(\ker \alpha/(\ker \alpha)^2, K/\mathcal{O})$.

If $q \in S$ then by a *local deformation problem* at q we mean a collection $\mathcal{D}_q$ of liftings of $(\overline{r}|_{\Delta_q}, \{\overline{\mathrm{Fil}}\,^i_q\})$ (or simply of $\overline{r}|_{\Delta_q}$ if $q \in S - S_0$) to objects of $\mathcal{C}_{\mathcal{O}}$ satisfying the following conditions.

1. $(k, \overline{r}|_{\Delta_q}, \{\overline{\mathrm{Fil}}\,^i_q\}) \in \mathcal{D}_q$.

2. If $(R, r, \{\mathrm{Fil}\,^i_q\}) \in \mathcal{D}_q$ and if $f : R \to S$ is a morphism in $\mathcal{C}_{\mathcal{O}}$ then $(S, f \circ r, \{f\mathrm{Fil}\,^i_q\}) \in \mathcal{D}_q$.

3. Suppose that $(R_1, r_1, \{\mathrm{Fil}\,^i_{q,1}\})$ and $(R_2, r_2, \{\mathrm{Fil}\,^i_{q,2}\}) \in \mathcal{D}_q$, that I_1 (resp. I_2) is a closed ideal of R_1 (resp. R_2) and that $f : R_1/I_1 \xrightarrow{\sim} R_2/I_2$ is an isomorphism in $\mathcal{C}_{\mathcal{O}}$ such that $f((r_1, \{\mathrm{Fil}\,^i_{q,1}\}) \bmod I_1) = ((r_2, \{\mathrm{Fil}\,^i_{q,2}\}) \bmod I_2)$. Let R_3 denote the subring of $R_1 \oplus R_2$ consisting of pairs with the same image in $R_1/I_1 \xrightarrow{\sim} R_2/I_2$. Then $(R_3, r_1 \oplus r_2, \{\mathrm{Fil}\,^i_{q,1} \oplus \mathrm{Fil}\,^i_{q,2}\}) \in \mathcal{D}_q$.

4. If $(R_j, r_j, \{\mathrm{Fil}\,^i_{q,j}\})$ is an inverse system of elements of $\mathcal{D}_q$ then $(\lim_{\leftarrow} R_j, \lim_{\leftarrow} r_j, \{\lim_{\leftarrow} \mathrm{Fil}\,^i_{q,j}\}) \in \mathcal{D}_q$.

5. $\mathcal{D}_q$ is closed under equivalence.

It is equivalent to give a $1_n + M_n(\mathfrak{m}_{R_q^{\mathrm{loc}}})$ invariant ideal $\mathcal{I}_q$ of R_q^{loc}. The collection $\mathcal{D}_q$ is simply the collection of all liftings $(r, \{\mathrm{Fil}\,^i_q\}_i)$ over rings R such that the kernel of the induced map $R_q^{\mathrm{loc}} \to R$ contains $\mathcal{I}_q$. We will write $L_q = L_q(\mathcal{D}_q)$ for the image in $H^1(\Delta_q, \mathrm{Fil}\,^0_q \mathrm{ad}\,\overline{r})$ of the annihilator L_q^1 in $Z^1(\Delta_q, \{\overline{\mathrm{Fil}}\,^i_q\}_i, \mathrm{ad}\,\overline{r})$ of $\mathcal{I}_q/(\mathcal{I}_q \cap (\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda)) \subset \mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda)$. Because $\mathcal{I}_q$ is $1_n + M_n(\mathfrak{m}_{R_q^{\mathrm{loc}}})$ invariant we see that the annihilator in $Z^1(\Delta_q, \{\overline{\mathrm{Fil}}\,^i_q\}_i, \mathrm{ad}\,\overline{r})$ of $\mathcal{I}_q/(\mathcal{I}_q \cap (\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda))$ equals the preimage of L_q.

Lemma 1.2.1 *Keep the above notation and assumptions. Suppose that R is an object of $\mathcal{C}_{\mathcal{O}}$ and I is a closed ideal of R with $\mathfrak{m}_R I = (0)$. Suppose also that $(r_1, \{\mathrm{Fil}\,^i_{q,1}\})$ and $(r_2, \{\mathrm{Fil}\,^i_{q,2}\})$ are two liftings of $(\overline{r}|_{\Delta_q}, \{\overline{\mathrm{Fil}}\,^i_q\}_i)$ with the same reduction* mod$I$. *Suppose finally that $(r_1, \{\mathrm{Fil}\,^i_{q,1}\})$ is in $\mathcal{D}_q$. Then $(r_2, \{\mathrm{Fil}\,^i_{q,2}\})$ is in $\mathcal{D}_q$ if and only if $[(r_2, \{\mathrm{Fil}\,^i_{q,2}\}) - (r_1, \{\mathrm{Fil}\,^i_{q,1}\})] \in L_q$.*

Proof: Suppose that $(r_j, \{\mathrm{Fil}\,^i_{q,j}\}_i)$ corresponds to $\alpha_j : R_q^{\mathrm{loc}} \to R$. Then $\alpha_2 = \alpha_1 + \beta$ where

$$\beta : R_q^{\mathrm{loc}} \longrightarrow I$$

satisfies

- $\beta(x+y) = \beta(x) + \beta(y)$;
- $\beta(xy) = \beta(x)\alpha_1(y) + \alpha_1(x)\beta(y) + \beta(x)\beta(y)$;
- and $\beta|_{\mathcal{O}} = 0$.

Thus β is determined by $\beta|_{\mathfrak{m}_{R_q^{\mathrm{loc}}}}$ and β is trivial on $(\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda)$. Hence β gives rise to and is determined by an $\mathcal{O}$-linear map:

$$\beta : \mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda) \longrightarrow I.$$

A straightforward calculation shows that

$$[(r_2, \{\mathrm{Fil}\,^i_{q,2}\}) - (r_1, \{\mathrm{Fil}\,^i_{q,1}\})] \in H^1(\Delta_q, \mathrm{Fil}\,^0_q \mathrm{ad}\,\overline{r})$$

is the image of

$$\beta \in \mathrm{Hom}\,(\mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda), I) \cong Z^1(\Delta_q, \{\overline{\mathrm{Fil}}\,^i_q\}_i, \mathrm{ad}\,\overline{r}) \otimes_k I.$$

The homomorphism α_1 vanishes on $\mathcal{I}_q$. Thus we must show that β vanishes on $\mathcal{I}_q$ if and only if β maps to $L_q \otimes_k I$, i.e. if and only if

$$\beta \in \mathrm{Hom}\,(\mathfrak{m}_{R_q^{\mathrm{loc}}}/(\mathfrak{m}^2_{R_q^{\mathrm{loc}}}, \lambda, \mathcal{I}_q), k) \otimes_k I.$$

This is tautological. □

Again let $(r_q, \{\mathrm{Fil}\,_q^i\})$ be a lift of $(\overline{r}|_{\Delta_q}, \{\overline{\mathrm{Fil}}\,_q^i\})$ to $\mathcal{O}$ corresponding to a homomorphism $\alpha : R_q^{\mathrm{loc}} \to \mathcal{O}$. Suppose that $(r_q, \{\mathrm{Fil}\,_q^i\}_i\})$ is in $\mathcal{D}_q$. We will call a lift of $(r_q, \{\mathrm{Fil}\,_q^i\}_i)$ to $\mathcal{O} \oplus K/\mathcal{O}\epsilon$ *of type* $\mathcal{D}_q$ if it arises by extension of scalars from a lift to some $\mathcal{O} \oplus \lambda^{-N}/\mathcal{O}\epsilon$ which is in $\mathcal{D}_q$. Such liftings correspond to homomorphisms $R_q^{\mathrm{loc}}/\mathcal{I}_q \to \mathcal{O} \oplus K/\mathcal{O}\epsilon$ which lift α. Because $\mathcal{I}_q$ is $1_n + M_n(\mathfrak{m}_{R_q^{\mathrm{loc}}})$ invariant, the subspace of $Z^1(\Delta_q, \{\overline{\mathrm{Fil}}\,_q^i\}_i, \mathrm{ad}\, r_q \otimes K/\mathcal{O})$ corresponding to

$$\mathrm{Hom}\,_{\mathcal{O}}(\ker\alpha/((\ker\alpha)^2, \mathcal{I}_q), K/\mathcal{O}) \subset \mathrm{Hom}\,_{\mathcal{O}}(\ker\alpha/(\ker\alpha)^2, K/\mathcal{O})$$

is the inverse image of a sub-$\mathcal{O}$-module

$$L(r_q) \subset H^1(\Delta_q, \mathrm{Fil}\,_q^0 \mathrm{ad}\, r_q \otimes K/\mathcal{O}).$$

Thus a lift of $(r_q, \{\mathrm{Fil}\,_q^i\}_i)$ to $\mathcal{O} \oplus K/\mathcal{O}\epsilon$ is of type $\mathcal{D}_q$ if and only if its class in $Z^1(\Delta_q, \{\overline{\mathrm{Fil}}\,_q^i\}_i, \mathrm{ad}\, r_q \otimes K/\mathcal{O})$ maps to an element of $L_q(r_q)$.

We will call $\mathcal{D}_q$ *liftable* if the following condition is satisfied:

- for each object R of $\mathcal{C}_{\mathcal{O}}$, for each ideal I of R with $\mathfrak{m}_R I = (0)$ and for each lifting $(r, \{\mathrm{Fil}\,_q^i\})$ to R/I in $\mathcal{D}_q$ there is a lifting of $(r, \{\mathrm{Fil}\,_q^i\})$ to R.

This is equivalent to $R_q^{\mathrm{loc}}/\mathcal{I}_q$ being a power series ring over $\mathcal{O}$. We will call L_q *minimal* if

$$\dim_k L_q = \dim_k H^0(\Delta_q, \mathrm{Fil}\,^0 \mathrm{ad}\, \overline{r}).$$

This is equivalent to the preimage of L_q in $Z^1(\Delta_q, \{\overline{\mathrm{Fil}}\,_q^i\}_i, \mathrm{ad}\, \overline{r})$ having dimension n^2.

Let $\mathcal{S}$ be a collection of deformation problems $\mathcal{D}_q$ for each $q \in S$. We call a lifting $(R, r, \{\mathrm{Fil}\,_q^i\}_{q \in S_0, i})$ of $(\overline{r}, \{\overline{\mathrm{Fil}}\,_q^i\})$ *of type* $\mathcal{S}$ if for all $q \in S$ the restriction $(R, r|_{\Delta_q}, \{\mathrm{Fil}\,_q^i\}_i) \in \mathcal{D}_q$. If $(R, r, \{\mathrm{Fil}\,_q^i\})$ is of type $\mathcal{S}$, so is any equivalent lifting. We say that a deformation $[(R, r, \{\mathrm{Fil}\,_q^i\})]$ is of type $\mathcal{S}$ if some (or equivalently, every) element $(R, r, \{\mathrm{Fil}\,_q^i\})$ of $[(R, r, \{\mathrm{Fil}\,_q^i)]$ is of type $\mathcal{S}$. We let $\mathrm{Def}_{\mathcal{S}}$ denote the functor from $\mathcal{C}_{\mathcal{O}}$ to sets which sends R to the set of deformations $[(R, r, \{\mathrm{Fil}\,_q^i\})]$ of type $\mathcal{S}$.

We need to introduce a variant of the cohomology group $H^i(\Gamma, \mathrm{ad}\,\overline{r})$. More specifically we will denote by $H^i(\Gamma, \{\overline{\mathrm{Fil}}^i_q\}, \mathrm{ad}\,\overline{r})$ the homology in degree i of the complex

$$C^i(\Gamma, \{\overline{\mathrm{Fil}}^i_q\}, \mathrm{ad}\,\overline{r}) = C^i(\Gamma, \mathrm{ad}\,\overline{r}) \oplus \bigoplus_{q \in S_0} C^{i-1}(\Delta_q, \mathrm{ad}\,\overline{r}/\mathrm{Fil}^0_q \mathrm{ad}\,\overline{r}),$$

where the boundary map sends

$$\begin{array}{rcl} C^i(\Gamma, \{\overline{\mathrm{Fil}}^i_q\}, \mathrm{ad}\,\overline{r}) & \longrightarrow & C^{i+1}(\Gamma, \{\overline{\mathrm{Fil}}^i_q\}, \mathrm{ad}\,\overline{r}) \\ (\phi, (\psi_q)) & \longmapsto & (\partial\phi, (\phi|_{\Delta_q} - \partial\psi_q)). \end{array}$$

Note that we have long exact sequences with a morphism between them:

$$\begin{array}{ccc} \downarrow & & \downarrow \\ H^i(\Gamma, \{\overline{\mathrm{Fil}}^i_q\}, \mathrm{ad}\,\overline{r}) & \longrightarrow & \bigoplus_{q \in S_0} H^i(\Delta_q, \mathrm{Fil}^0_q \mathrm{ad}\,\overline{r}) \\ \downarrow & & \downarrow \\ H^i(\Gamma, \mathrm{ad}\,\overline{r}) & \longrightarrow & \bigoplus_{q \in S_0} H^i(\Delta_q, \mathrm{ad}\,\overline{r}) \\ \downarrow & & \downarrow \\ \bigoplus_{q \in S_0} H^i(\Delta_q, \mathrm{ad}\,\overline{r}/\mathrm{Fil}^0_q \mathrm{ad}\,\overline{r}) & = & \bigoplus_{q \in S_0} H^i(\Delta_q, \mathrm{ad}\,\overline{r}/\mathrm{Fil}^0_q \mathrm{ad}\,\overline{r}) \\ \downarrow & & \downarrow . \end{array}$$

Similarly if $(r, \{\mathrm{Fil}^i_q\})$ is a lifting of $(\overline{r}, \{\overline{\mathrm{Fil}}^i_q\})$ to $\mathcal{O}$ then we will denote by $H^i(\Gamma, \{\mathrm{Fil}^i_q\}, \mathrm{ad}\,r \otimes K/\mathcal{O})$ the homology in degree i of the complex $C^i(\Gamma, \{\mathrm{Fil}^i_q\}, \mathrm{ad}\,r \otimes K/\mathcal{O})$ defined as

$$C^i(\Gamma, \mathrm{ad}\,r \otimes K/\mathcal{O}) \oplus \bigoplus_{q \in S_0} C^{i-1}(\Delta_q, \mathrm{ad}\,r/\mathrm{Fil}^0_q \mathrm{ad}\,r \otimes K/\mathcal{O}),$$

where the boundary map sends

$$\begin{array}{rcl} C^i(\Gamma, \{\mathrm{Fil}^i_q\}, \mathrm{ad}\,r \otimes K/\mathcal{O}) & \longrightarrow & C^{i+1}(\Gamma, \{\mathrm{Fil}^i_q\}, \mathrm{ad}\,r \otimes K/\mathcal{O}) \\ (\phi, (\psi_q)) & \longmapsto & (\partial\phi, (\phi|_{\Delta_q} - \partial\psi_q)). \end{array}$$

Note that we have a morphism:

$$H^i(\Gamma, \{\mathrm{Fil}^i_q\}, \mathrm{ad}\,r \otimes K/\mathcal{O}) \longrightarrow \bigoplus_{q \in S_0} H^i(\Delta_q, \mathrm{Fil}^0_q \mathrm{ad}\,r \otimes K/\mathcal{O}).$$

We will also denote by $H^i_{\mathcal{S}}(\Gamma, \mathrm{ad}\,\overline{r})$ the cohomology of the complex

$$C^i_{\mathcal{S}}(\Gamma, \mathrm{ad}\,\overline{r}) = C^i(\Gamma, \mathrm{ad}\,\overline{r}) \oplus \bigoplus_{q \in S} C^{i-1}(\Delta_q, \mathrm{ad}\,\overline{r})/L^{i-1}_q,$$

where $L_q^i = (0)$ for $i > 1$,

$$L_q^0 = C^0(\Delta_q, \mathrm{Fil}^0 \mathrm{ad}\, \overline{r})$$

and L_q^1 denotes the preimage of L_q in $C^1(\Delta_q, \mathrm{Fil}^0 \mathrm{ad}\, \overline{r})$. The boundary map sends

$$\begin{array}{rcl} C^i_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r}) & \longrightarrow & C^{i+1}_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r}) \\ (\phi, (\psi_q)) & \longmapsto & (\partial\phi, (\phi|_{\Delta_q} - \partial\psi_q)). \end{array}$$

We have long exact sequences

$$\begin{array}{ccccccc} & & & & & (0) & \to \\ \to & H^0_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r}) & \to & H^0(\Gamma, \mathrm{ad}\, \overline{r}) & \to & \bigoplus_q \ker(H^0(\Delta_q, \mathrm{ad}\, \overline{r}/\mathrm{Fil}^0 \mathrm{ad}\, \overline{r}) \to & \\ & & & & & \to H^1(\Delta_q, \mathrm{Fil}^0 \mathrm{ad}\, \overline{r})/L_q) & \to \\ \to & H^1_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r}) & \to & H^1(\Gamma, \mathrm{ad}\, \overline{r}) & \to & \bigoplus_q H^1(\Delta_q, \mathrm{ad}\, \overline{r})/L_q & \to \\ \to & H^2_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r}) & \to & H^2(\Gamma, \mathrm{ad}\, \overline{r}) & \to & \bigoplus_q H^2(\Delta_q, \mathrm{ad}\, \overline{r}) & \to \\ \to & H^3_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r}) & \to & H^3(\Gamma, \mathrm{ad}\, \overline{r}) & \to & \ldots & \end{array}$$

and

$$\begin{array}{ccccccc} & & & & & (0) & \to \\ \to & H^0_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r}) & \to & H^0(\Gamma, \{\overline{\mathrm{Fil}}^i_q\}, \mathrm{ad}\, \overline{r}) & \to & (0) & \to \\ \to & H^1_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r}) & \to & H^1(\Gamma, \{\overline{\mathrm{Fil}}^i_q\}, \mathrm{ad}\, \overline{r}) & \to & \bigoplus_q H^1(\Delta_q, \mathrm{Fil}^0 \mathrm{ad}\, \overline{r})/L_q & \to \\ \to & H^2_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r}) & \to & H^2(\Gamma, \{\overline{\mathrm{Fil}}^i_q\}, \mathrm{ad}\, \overline{r}) & \to & \bigoplus_q H^2(\Delta_q, \mathrm{Fil}^0 \mathrm{ad}\, \overline{r}) & \to \\ \to & H^3_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r}) & \to & H^3(\Gamma, \{\overline{\mathrm{Fil}}^i_q\}, \mathrm{ad}\, \overline{r}) & \to & \ldots & \end{array}$$

Lemma 1.2.2 *Suppose that all the groups $H^i(\Gamma, \mathrm{ad}\, \overline{r})$ and $H^i(\Delta_q, \mathrm{ad}\, \overline{r})$ are finite and that they all vanish for i sufficiently large. Set*

$$\chi(\Gamma, \mathrm{ad}\, \overline{r}) = \prod_i \# H^i(\Gamma, \mathrm{ad}\, \overline{r})^{(-1)^i},$$

and

$$\chi(\Delta_q, \mathrm{ad}\, \overline{r}) = \prod_i \# H^i(\Delta_q, \mathrm{ad}\, \overline{r})^{(-1)^i},$$

and

$$\chi_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r}) = \prod_i \# H^i_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r})^{(-1)^i}.$$

Then

$$\chi_{\mathcal{S}}(\Gamma, \mathrm{ad}\, \overline{r}) = \chi(\Gamma, \mathrm{ad}\, \overline{r}) \prod_q (\chi(\Delta_q, \mathrm{ad}\, \overline{r})^{-1} \# H^0(\Delta_q, \mathrm{Fil}^0 \mathrm{ad}\, \overline{r})/\# L_q).$$

The next result is a variant of well known results for GL_n without filtrations. Filtrations were introduced into the picture by Dickinson [Dic2]. Our proof follows his.

Proposition 1.2.3 *Keep the above notation and assumptions. Then* $\mathrm{Def}_{\mathcal{S}}$ *is represented by an object* $R_{\mathcal{S}}^{\mathrm{univ}}$ *of* $\mathcal{C}_{\mathcal{O}}$*. We will let* $r_{\mathcal{S}}^{\mathrm{univ}}$ *denote the universal deformation over* $R_{\mathcal{S}}^{\mathrm{univ}}$*. There is a canonical isomorphism*

$$\mathrm{Hom}_{\,\mathrm{cts}}(\mathfrak{m}_{R_{\mathcal{S}}^{\mathrm{univ}}}/(\mathfrak{m}^2_{R_{\mathcal{S}}^{\mathrm{univ}}},\lambda),k)\cong H^1_{\mathcal{S}}(\Gamma,\mathrm{ad}\,\overline{r}).$$

If $H^1_{\mathcal{S}}(\Gamma,\mathrm{ad}\,\overline{r})$ *is finite dimensional then* $R_{\mathcal{S}}^{\mathrm{univ}}$ *is a complete local noetherian* $\mathcal{O}$*-algebra.*

Proof: First we consider representability. By properties 1, 2, 3 and 4 of $\mathcal{D}_q$ we see that the functor sending R to the set of all lifts of $(\overline{r},\{\overline{\mathrm{Fil}}^{\,i}_q\})$ to R of type $\mathcal{S}$ is representable. By property 5 we see that $\mathrm{Def}_{\mathcal{S}}$ is the quotient of this functor by the smooth group valued functor $R\mapsto\ker(GL_n(R)\to GL_n(k))$ acting by conjugation. Thus by [Dic1] it suffices to check that if $\phi:R\twoheadrightarrow S$ in $\mathcal{C}_{\mathcal{O}}$, if $(r,\{\mathrm{Fil}^{\,i}_q\})$ is a lift of $(\overline{r},\{\overline{\mathrm{Fil}}^{\,i}_q\})$ to R, and if $g\in 1+M_n(\mathfrak{m}_S)$ conjugates $\phi(r,\{\mathrm{Fil}^{\,i}_q\})$ to itself, then there is a lift $\tilde{g}$ of g in $1+M_n(\mathfrak{m}_R)$ which conjugates $(r,\{\mathrm{Fil}^{\,i}_q\})$ to itself. This is clear from lemma 1.1.5.

Recall that

$$\mathrm{Hom}_{\,\mathrm{cts}}(\mathfrak{m}_{R_{\mathcal{S}}^{\mathrm{univ}}}/(\mathfrak{m}^2_{R_{\mathcal{S}}^{\mathrm{univ}}},\lambda),k)\cong\mathrm{Hom}\,(R_{\mathcal{S}}^{\mathrm{univ}},k[\epsilon]/(\epsilon^2))\cong\mathrm{Def}_{\mathcal{S}}(k[\epsilon]/(\epsilon^2)).$$

For $q\in S_0$ define a filtration $\mathrm{Fil}_{\,q}$ on $\mathrm{ad}\,\overline{r}$ by setting

$$\mathrm{Fil}^{\,i}_q\mathrm{ad}\,\overline{r}=\{a\in\mathrm{ad}\,\overline{r}:\ \ a\mathrm{Fil}^{\,j}_q\subset\mathrm{Fil}^{\,j+i}q\ \ \forall j\}.$$

Any lifting $(r,\{\mathrm{Fil}^{\,i}_q\})$ of $(\overline{r},\{\overline{\mathrm{Fil}}^{\,j}_q\})$ to $k[\epsilon]/(\epsilon^2)$ is of the form

- $r=(1+\phi\epsilon)\overline{r}$,
- $\mathrm{Fil}^{\,i}_q=(1+a_q\epsilon)\overline{\mathrm{Fil}}^{\,i}_q+\epsilon\overline{\mathrm{Fil}}^{\,i}_q$,

where

- $\phi\in Z^1(\Gamma,\mathrm{ad}\,\overline{r})$,
- $a_q\in\mathrm{ad}\,\overline{r}/\mathrm{Fil}^{\,0}_q\mathrm{ad}\,\overline{r}$, and
- $\phi=(1-\mathrm{ad}\,\overline{r})a_q$ in $Z^1(\Delta_q,\mathrm{ad}\,\overline{r}/\mathrm{Fil}^{\,0}_q\mathrm{ad}\,\overline{r})$.

This establishes a bijection between lifts of $(\overline{r}, \{\overline{\mathrm{Fil}}^j_q\})$ to $k[\epsilon]/(\epsilon^2)$ and collections of data $(\phi, \{a_q\})$ satisfying these conditions. Two collections of data $(\phi, \{a_q\})$ and $(\phi', \{a'_q\})$ correspond to equivalent lifts if there is an $A \in \mathrm{ad}\,\overline{r}$ such that

- $\phi' = \phi + (1 - \mathrm{ad}\,\overline{r})A$ and
- $a'_q = a_q + A$.

It is straightforward to complete the proof of the proposition. $\Box$

Lemma 1.2.4 *Suppose that R is an object of $\mathcal{C}_{\mathcal{O}}$ and that I is an ideal of R with $\mathfrak{m}_R I = (0)$. Suppose that $(r, \{\mathrm{Fil}^i_q\})$ is a lifting of $(\overline{r}, \{\overline{\mathrm{Fil}}^i_q\})$ to R/I of type $\mathcal{S}$. Suppose moreover that for each $q \in S$ the restriction $(r|_{\Delta_q}, \{\mathrm{Fil}^i_q\})$ has a lift to R in $\mathcal{D}_q$. Pick such a lifting $(\widehat{r}_q, \{\widehat{\mathrm{Fil}}^i_q\})$ and for each $\gamma \in \Gamma$ pick a lifting $\widetilde{r(\gamma)}$ of $r(\gamma)$ to $\mathcal{G}_n(R)$. Set*

$$\phi(\gamma, \delta) = \widetilde{r(\gamma\delta)}\,\widetilde{r(\delta)}^{-1}\,\widetilde{r(\gamma)}^{-1} - 1$$

and, for $\delta \in \Delta_q$, set

$$\psi_q(\delta) = \widetilde{r(\delta)}\,\widehat{r}(\delta)^{-1} - 1.$$

Then $(\phi, (\psi_q))$ defines a class $\mathrm{obs}_{\mathcal{S},R,I}(r, \{\mathrm{Fil}^i_q\}) \in H^2_{\mathcal{S}}(\Gamma, \mathrm{ad}\,\overline{r}) \otimes I$ which is independent of the various choices and vanishes if and only if $(r, \{\mathrm{Fil}^i_q\})$ has a lifting to R of type $\mathcal{S}$.

Proof: We leave the proof to the reader. $\Box$

Corollary 1.2.5 *Suppose that each $\mathcal{D}_q$ is liftable and that $H^2_{\mathcal{S}}(\Gamma, \mathrm{ad}\,\overline{r}) = (0)$. Then $R^{\mathrm{univ}}_{\mathcal{S}}$ is a power series ring in $\dim H^1_{\mathcal{S}}(\Gamma, \mathrm{ad}\,\overline{r})$ variables over $\mathcal{O}$.*

Corollary 1.2.6 *Suppose that for each $q \in S$ the ring R^{loc}_q is a complete intersection. Then $R^{\mathrm{univ}}_{\mathcal{S}}$ is the quotient of a power series ring in $\dim H^1_{\mathcal{S}}(\Gamma, \mathrm{ad}\,\overline{r})$ variables by*

$$\dim H^2_{\mathcal{S}}(\Gamma, \mathrm{ad}\,\overline{r}) + \sum_{q \in \mathcal{S}} (n^2 + 1 + \dim L_q - \dim R^{\mathrm{loc}}_q/\mathcal{I}_q - \dim H^0(\Delta_q, \mathrm{Fil}^0_q \mathrm{ad}\,\overline{r}))$$

relations. Thus $R^{\mathrm{univ}}_{\mathcal{S}}$ has Krull dimension at least

$$\begin{gathered} 1 + \dim H^1_{\mathcal{S}}(\Gamma, \mathrm{ad}\,\overline{r}) - \dim H^2_{\mathcal{S}}(\Gamma, \mathrm{ad}\,\overline{r}) + \\ + \textstyle\sum_{q \in \mathcal{S}} (\dim R^{\mathrm{loc}}_q/\mathcal{I}_q - n^2 - 1 + \dim H^0(\Delta_q, \mathrm{Fil}^0_q \mathrm{ad}\,\overline{r}) - \dim L_q). \end{gathered}$$

Proof: $R_q^{\rm loc}$ is topologically generated by $\dim L_q + n^2 - \dim H^0(\Delta_q, {\rm Fil}\,_q^0 {\rm ad}\,\overline{r})$ elements. $\Box$

Now suppose that $\alpha : R_{\mathcal{S}}^{\rm univ} \twoheadrightarrow \mathcal{O}$ and let $(r, \{{\rm Fil}\,_q^i\})$ be a corresponding lift of $(\overline{r}, \{\overline{{\rm Fil}}\,_q^i\})$. Let $H^1_{\mathcal{S}}(\Gamma, {\rm ad}\, r \otimes K/\mathcal{O})$ denote the kernel of

$$H^1(\Gamma, \{{\rm Fil}\,_q^i\}, {\rm ad}\, r \otimes K/\mathcal{O}) \longrightarrow \bigoplus_{q \in S} H^1(\Delta_q, {\rm Fil}\,_q^0 {\rm ad}\, r \otimes K/\mathcal{O})/L_q(r_q).$$

The next lemma is now immediate.

Lemma 1.2.7 *Keep the notation and assumptions of the previous paragraph. Then there is a natural isomorphism*

$$\operatorname{Hom}_{\mathcal{O}}(\ker \alpha/(\ker \alpha)^2, K/\mathcal{O}) \cong H^1_{\mathcal{S}}(\Gamma, {\rm ad}\, r \otimes_{\mathcal{O}} K/\mathcal{O}).$$

1.3 Deformations of Galois representations

Fix an odd prime l. Also fix an imaginary quadratic field E in which l splits and a totally real field F^+. Set $F = F^+E$. Fix an algebraic extension k of $\mathbb{F}_l$ and a finite totally ramified extension K of the fraction field of $W(k)$. Let $\mathcal{O}$ denote the ring of integers of K and λ the maximal ideal of $\mathcal{O}$. We will suppose that K contains the image of each embedding of F into $\widehat{\overline{\mathbb{Q}}}_l$. Fix a character $\chi : G_F \to \mathcal{O}^\times$. Let S_∞ (resp. S_l, resp. I_l) denote the places of F^+ above ∞ (resp. places of F^+ above l, resp. embeddings $F^+ \hookrightarrow K$). For $v \in S_\infty$, write c_v for the notrivial element of G_{F_v}. There is a natural surjection $I_l \twoheadrightarrow S_l$. Choose a prime of E above l and let $\widetilde{S}_l$ denote the set of primes of F above this prime and $\widetilde{I}_l$ the set of embeddings of F into $\overline{\mathbb{Q}}_l$ above $\widetilde{S}_l$. Thus S_l and $\widetilde{S}_l$ (resp. I_l and $\widetilde{I}_l$) are in natural bijection. If $v \in S_l$ (resp. $\tau \in I_l$) we write $\widetilde{v}$ (resp. $\widetilde{\tau}$) for its lifting to $\widetilde{S}_l$ (resp. $\widetilde{I}_l$). Let ϵ denote the l-adic cyclotomic character. We will write $M(a)$ for $M \otimes_{\mathbb{Z}_l} \mathbb{Z}_l(\epsilon^a)$.

Fix a finite set of primes S of F^+ which split in F and such that $S \supset S_l$. Also choose a set $\widetilde{S} \supset \widetilde{S}_l$ consisting of the choice of one prime of F above each prime in S. Let $S_0 \subset S$ contain all ramified elements of S_l. Set $S_{l,0} = S_l \cap S_0$, set $\widetilde{S}_0$ equal to the preimage of S_0 in $\widetilde{S}$ and set $\widetilde{S}_{l,0}$ equal to the preimage of $S_{l,0}$ in $\widetilde{S}$. Let $F(S)/F$ denote the maximal extension unramified outside S and set $G_{F^+,S} = {\rm Gal}\,(F(S)/F^+)$ (resp. $G_{F,S} = {\rm Gal}\,(F(S)/F)$). Let $n < l$ be a positive integer and let $\overline{r} : G_{F^+,S} \to \mathcal{G}_n(k)$ be a continuous Schur homomorphism such that $G_{F,S} = \overline{r}^{-1}(GL_n(k))$ and $\nu \circ \overline{r} = \chi \bmod \lambda$.

We suppose that for each $\widetilde{v} \in \widetilde{S}_{l,0}$ there are n characters

$$\overline{\chi}_{\widetilde{v},0}, ..., \overline{\chi}_{\widetilde{v},n-1} : G_{F_{\widetilde{v}}} \longrightarrow k^\times,$$

and a $G_{F_{\widetilde{v}}}$-invariant decreasing filtration $\overline{\mathrm{Fil}}^i_{\widetilde{v}}$ on k^n such that

- $\overline{\mathrm{Fil}}^i_{\widetilde{v}} = (0)$ for $i \geq n$;
- $\overline{\mathrm{Fil}}^i_{\widetilde{v}} = k^n$ for $i \leq 0$;
- if $i = 0, ..., n-1$ then $\dim \overline{\mathrm{gr}}^i_{\widetilde{v}} k^n = 1$ and $G_{F_{\widetilde{v}}}$ acts on $\overline{\mathrm{gr}}^i_{\widetilde{v}}$ by $\overline{\chi}_{\widetilde{v},i}$;
- if $i > j+1$ then $\overline{\chi}_{\widetilde{v},i} \neq \overline{\chi}_{\widetilde{v},j}\epsilon$.

We need to impose one more assumption at primes $\widetilde{v} \in \widetilde{S}_{l,0}$, for which we will require some preliminaries. Suppose that R is an object of $\mathcal{C}_{\mathcal{O}}$ and that M is a free rank two R module M with a continuous action of $G_{F_{\widetilde{v}}}$ and a $G_{F_{\widetilde{v}}}$-invariant submodule Fil with M/Fil free of rank one over R. Suppose moreover that, if $G_{F_{\widetilde{v}}}$ acts on Fil (resp. M/Fil) by χ_0 (resp. χ_1), then $\chi_1\epsilon\chi_0^{-1}$ is unramified. Then we will define an invariant $\mathrm{val}\,(M, \mathrm{Fil}) \in R$ as follows. Suppose first that R is Artinian. Choose a finite unramified extension $F'/F_{\widetilde{v}}$ such that $\chi_0 = \chi_1\epsilon$ on $G_{F'}$. Thus, as a $G_{F'}$-module, (M, Fil) is an extension of $R(\chi_1)$ by $R(\chi_1\epsilon)$ and so gives rise to a class in

$$H^1(G_{F'}, R(1)) \cong (F')^\times \otimes R.$$

(By $(F')^\times \otimes R$ we mean $(F')^\times/((F')^\times)^{l^a} \otimes_{\mathbb{Z}/l^a\mathbb{Z}} R$ for any sufficiently large a.) The invariant $\mathrm{val}\,(M, \mathrm{Fil})$ is just the image of this class in $\mathbb{Z}\otimes R = R$ under the valuation map. Note that this does not depend on the choice of F'. Also note that if $R \to S$ in $\mathcal{C}_{\mathcal{O}}$ then $\mathrm{val}\,(M \otimes_R S, \mathrm{Fil} \otimes_R S)$ is the image of $\mathrm{val}\,(M, \mathrm{Fil})$ in S. We extend the definition to the case that R is any object of $\mathcal{C}_{\mathcal{O}}$ by using inverse limits. This preserves the invariance of val under pushforwards by morphisms in $\mathcal{C}_{\mathcal{O}}$. Our additional assumption is that if $\overline{\chi}_{\widetilde{v},i+1} = \overline{\chi}_{\widetilde{v},i}\epsilon$ then $\mathrm{val}\,(\mathrm{Fil}^i_{\widetilde{v}}/\mathrm{Fil}^{i+2}_{\widetilde{v}}, \mathrm{Fil}^{i+1}_{\widetilde{v}}/\mathrm{Fil}^{i+2}_{\widetilde{v}}) = 0$.

Suppose that $\widetilde{v} \in \widetilde{S}_l - \widetilde{S}_{l,0}$. Let $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$ denote the category of finite $\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}$-modules M together with

- a decreasing filtration $\mathrm{Fil}^i M$ by $\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}$-submodules which are $\mathcal{O}_{F,\widetilde{v}}$ direct summands with $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{l-1} M = (0)$;
- and $\mathrm{Fr} \otimes 1$-linear maps $\Phi^i : \mathrm{Fil}^i M \to M$ with $\Phi^i|_{\mathrm{Fil}^{i+1} M} = l\Phi^{i+1}$ and $\sum_i \Phi^i \mathrm{Fil}^i M = M$.

Let $\mathcal{MF}_{k,\widetilde{v}}$ denote the full subcategory of objects killed by λ. Fontaine and Lafaille (see [FL]) define an exact, fully faithful functor of $\mathcal{O}$-linear categories $\mathbb{G}_{\widetilde{v}}$ from $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$ to the category of finite $\mathcal{O}$-modules with a continuous action of $G_{F_{\widetilde{v}}}$. They show that the image of $\mathbb{G}_{\widetilde{v}}$ is closed under taking sub-objects and quotients and that $[\mathcal{O}_F/\widetilde{v}:\mathbb{F}_l]$ times the length of $\mathbb{G}_{\widetilde{v}}(M)$ as an $\mathcal{O}$-module is the length of M as a $\mathcal{O}$-module. (In fact in [FL] a slight variant U_S of $\mathbb{G}_{\widetilde{v}}$ is defined. We define $\mathbb{G}_{\widetilde{v}}(M) = U_S(\mathrm{Hom}\,(M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}\{l-2\})))(2-l)$. Here $\mathrm{Hom}\,(M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}\{l-2\})) \in \mathcal{MF}_{\mathcal{O},\widetilde{v}}$ is defined as follows.

- The underlying $\mathcal{O}$-module is $\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}})$.
- $\mathrm{Fil}^a\mathrm{Hom}\,(M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}\{l-2\})) = \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M/\mathrm{Fil}^{l-1-a}M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}})$.
- If $f \in \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M/\mathrm{Fil}^{l-1-a}M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}})$ and if $m \in \Phi^b\mathrm{Fil}^b M$ set
$$\Phi^a(f)(m) = l^{l-2-a-b}\mathrm{Fr}f(\Phi^b)^{-1}(m)).$$

To check that $\Phi^a f$ is well defined one uses the exact sequence

$$\begin{array}{ccccccc} (0) & \to & \bigoplus_{i=1}^{l-2}\mathrm{Fil}^i M & \to & \bigoplus_{i=0}^{l-2}\mathrm{Fil}^i M & \to & M & \to & (0) \\ & & (m_i) & \mapsto & (lm_i - m_{i+1})_i & & & & \\ & & & & (m_i) & \mapsto & \sum \Phi^i m_i. & & \end{array}$$

To check that

$$\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}) = \sum_a \Phi^a\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M/\mathrm{Fil}^{l-1-a}M, F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}})$$

it suffices to check that

$$\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M[l], F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}) = \sum_a \Phi^a\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M[l]/\mathrm{Fil}^{l-1-a}M[l], F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}).$$

But $M[l] = \bigoplus_i \Phi^i\mathrm{gr}^i M[l]$ and $\Phi^a\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(M[l]/\mathrm{Fil}^{l-1-a}M[l], F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}}) = \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}}}(\Phi^{l-2-a}\mathrm{gr}^{l-2-a}M[l], F_{\widetilde{v}}/\mathcal{O}_{F,\widetilde{v}})$.) For any objects M and N of $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$ (resp. $\mathcal{MF}_{k,\widetilde{v}}$), the map

$$\mathrm{Ext}^1_{\mathcal{MF}_{\mathcal{O},\widetilde{v}}}(M,N) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}[G_{F_{\widetilde{v}}}]}(\mathbb{G}_{\widetilde{v}}(M), \mathbb{G}_{\widetilde{v}}(N))$$

(resp.

$$\begin{array}{rcl} \mathrm{Ext}^1_{\mathcal{MF}_{k,\widetilde{v}}}(M,N) & \longrightarrow & \mathrm{Ext}^1_{k[G_{F_{\widetilde{v}}}]}(\mathbb{G}_{\widetilde{v}}(M), \mathbb{G}_{\widetilde{v}}(N)) \\ & \cong & H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_k(\mathbb{G}_{\widetilde{v}}(M), \mathbb{G}_{\widetilde{v}}(N)))) \end{array}$$

is an injection. Moreover

$$\mathrm{Hom}_{\mathcal{MF}_{\mathcal{O},\widetilde{v}}}(M,N) \xrightarrow{\sim} H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_{\mathcal{O}}(\mathbb{G}_{\widetilde{v}}(M), \mathbb{G}_{\widetilde{v}}(N))).$$

For $\widetilde{v} \in \widetilde{S}_l - \widetilde{S}_{l,0}$, we will assume that $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is in the image of $\mathbb{G}_{\widetilde{v}}$ and that for each i and each $\widetilde{\tau} \in \widetilde{I}_l$ above $\widetilde{v}$ we have

$$\dim_k(\operatorname{gr}^i \mathbb{G}_{\widetilde{v}}^{-1}(\overline{r}|_{G_{F_{\widetilde{v}}}})) \otimes_{\mathcal{O}_{F_{\widetilde{v}}},\widetilde{\tau}} \mathcal{O} \leq 1.$$

For $\widetilde{\tau} \in \widetilde{I}_l$ above $\widetilde{v} \in \widetilde{S}_l - \widetilde{S}_{l,0}$, we will denote by $m_{\widetilde{\tau},0} \leq ... \leq m_{\widetilde{\tau},n-1}$ the integers $l-2 \geq m \geq 0$ such that

$$\dim_k(\operatorname{gr}^m \mathbb{G}_{\widetilde{v}}^{-1}(\overline{r}|_{G_{F_{\widetilde{v}}}})) \otimes_{\mathcal{O}_{F_{\widetilde{v}}},\widetilde{\tau}} \mathcal{O} = 1.$$

For $\widetilde{v} \in \widetilde{S}_0 - \widetilde{S}_{l,0}$ fix a $G_{F_{\widetilde{v}}}$-invariant filtration $\{\overline{\mathrm{Fil}}^i_{\widetilde{v}}\}$ of k^n such that

- $\overline{\mathrm{Fil}}^i_{\widetilde{v}} = (0)$ for $i >> 0$, and
- $\overline{\mathrm{Fil}}^i_{\widetilde{v}} = k^n$ for $i << 0$.

We are going to define a deformation problem

$$(G_{F^+,S} \supset G_{F,S}, S \supset S_0, \{G_{F_{\widetilde{v}}}\}_{v \in S}, \mathcal{O}, \overline{r}, \chi, \{\overline{\mathrm{Fil}}^i_{\widetilde{v}}\}, \{\mathcal{D}_{\widetilde{v}}\}, \{L_{\widetilde{v}}\})$$

as in the last section. It remains to describe the $\mathcal{D}_{\widetilde{v}}$ and the $L_{\widetilde{v}}$.

1.3.1 Ordinary deformations

The following discussion is a bit ad hoc. We do not feel that we have found the right degree of generality here.

First of all we will discuss $\widetilde{v} \in \widetilde{S}_{l,0}$. For $i = 0, \dots, n-1$ choose characters $\chi_{\widetilde{v},i} : G_{F_{\widetilde{v}}} \to \mathcal{O}^\times$ lifting $\overline{\chi}_{\widetilde{v},i}$ with the following properties.

1. For each $\widetilde{\tau} \in \widetilde{I}_{l,0}$ above $\widetilde{v}$ there are integers $m_{\widetilde{\tau},0} \leq \ldots \leq m_{\widetilde{\tau},n-1}$ such that if we consider $\chi_{\widetilde{v},i}|_{I_{\widetilde{v}}}$ as a character of $\mathcal{O}_{F,\widetilde{v}}^\times$ (by class field theory) then
$$\chi_{\widetilde{v},i}(x) = \prod_{\widetilde{\tau}} (\widetilde{\tau} x)^{m_{\widetilde{\tau},i}}.$$
2. If $\overline{\chi}_{\widetilde{v},i+1} = \overline{\chi}_{\widetilde{v},i}\epsilon$ then $\chi_{\widetilde{v},i+1} = \chi_{\widetilde{v},i}\epsilon$.

We will take $\mathcal{D}_{\widetilde{v}}$ to be the set of all lifts $(r, \{\mathrm{Fil}^i_{\widetilde{v}}\})$ of $(\overline{r}|_{G_{F_{\widetilde{v}}}}, \{\overline{\mathrm{Fil}}^i_{\widetilde{v}}\})$ to objects R of $\mathcal{C}_{\mathcal{O}}$ such that $I_{F_{\widetilde{v}}}$ acts on $\operatorname{gr}^i_{\widetilde{v}} R^n$ by $\chi_{\widetilde{v},i}$ and such that, if $\overline{\chi}_{\widetilde{v},i+1} = \overline{\chi}_{\widetilde{v},i}\epsilon$ then $\operatorname{val}(\mathrm{Fil}^i_{\widetilde{v}} R^n / \mathrm{Fil}^{i+2}_{\widetilde{v}} R^n) = 0$. It is easy to see that $\mathcal{D}_{\widetilde{v}}$ is a local deformation problem.

Set

$$L'_{\widetilde{v}} = \ker(H^1(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^0_{\widetilde{v}}\overline{r}) \longrightarrow H^1(I_{F_{\widetilde{v}}}, \mathrm{gr}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})).$$

Then there is a natural map

$$L'_{\widetilde{v}} \to H^1(I_{F_{\widetilde{v}}}, \mathrm{Fil}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r})/\partial H^0(I_{F_{\widetilde{v}}}, \mathrm{gr}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \to \bigoplus_{i=1}^{n-1} H^1(I_{F_{\widetilde{v}}}, k(\chi_{\widetilde{v},i-1}/\chi_{\widetilde{v},i}))/\langle c_i\rangle,$$

where c_i is the class defined by the extension $\overline{\mathrm{Fil}}\,^{i-1}_{\widetilde{v}}/\overline{\mathrm{Fil}}\,^{i+1}_{\widetilde{v}}$. Then it is not hard to see that $L_{\widetilde{v}} = L_{\widetilde{v}}(\mathcal{D}_{\widetilde{v}})$ is the kernel of the composite

$$L'_{\widetilde{v}} \longrightarrow \bigoplus_j H^1(I_{F_{\widetilde{v}}}, k(\chi_{\widetilde{v},j-1}/\chi_{\widetilde{v},j}))/\langle c_j\rangle \stackrel{\mathrm{val}}{\longrightarrow} \bigoplus_j k,$$

where j runs over indices such that $\overline{\chi}_{j-1} = \overline{\chi}_j\epsilon$.

Lemma 1.3.1 *For $\widetilde{v} \in \widetilde{S}_{l,0}$ the set $\mathcal{D}_{\widetilde{v}}$ is liftable.*

Proof: Suppose that R is an object of $\mathcal{C}_{\mathcal{O}}$ and I is a closed ideal of R with $\mathfrak{m}_R I = (0)$. Suppose also that $(r, \{\mathrm{Fil}\,^i_{\widetilde{v}}\})$ is a deformation in $\mathcal{D}_{\widetilde{v}}$ of $(\overline{r}|_{G_{F_{\widetilde{v}}}}, \{\overline{\mathrm{Fil}}\,^i_{\widetilde{v}}\})$ to R/I. We will show by reverse induction on i that we can find a lifting $\mathrm{Fil}\,^i$ of $\mathrm{Fil}\,^i_{\widetilde{v}}r$ to R so that for $j \geq i$, $I_{F_{\widetilde{v}}}$ acts on $\mathrm{gr}\,^j_{\widetilde{v}}\mathrm{Fil}\,^i$ by $\chi_{\widetilde{v},j}$ and, if $\overline{\chi}_{\widetilde{v},j+1} = \overline{\chi}_{\widetilde{v},j}\epsilon$, then $\mathrm{val}\,(\mathrm{Fil}\,^j_{\widetilde{v}}\mathrm{Fil}\,^i/\mathrm{Fil}\,^{j+2}_{\widetilde{v}}\mathrm{Fil}\,^i) = 0$.

The case $i = n-1$ is trivial. Suppose that $\mathrm{Fil}\,^{i+1}$ is such a lifting. Also choose a lifting $\mathrm{gr}\,^i$ of $\mathrm{gr}\,^i_{\widetilde{v}}r$ such that $I_{F_{\widetilde{v}}}$ acts by $\chi_{\widetilde{v},i}$. We will choose $\mathrm{Fil}\,^i$ to be an extension of $\mathrm{gr}\,^i$ by $\mathrm{Fil}\,^{i-1}$ which lifts $\mathrm{Fil}\,^i_{\widetilde{v}}r$. Such extensions are parametrised by $H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i, \mathrm{Fil}\,^{i+1}))$.

We have a commutative diagram with its first two columns exact:

$$\begin{array}{ccccc}
H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}}\overline{r}, I\mathrm{Fil}\,^{i+1})) & \to & H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}}\overline{r}, I\mathrm{gr}\,^{i+1})) & \stackrel{\mathrm{val}}{\to} & (\mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}}\overline{r}, I\mathrm{gr}\,^{i+1})(\epsilon^{-1})_{I_{F_{\widetilde{v}}}})^{G_{F_{\widetilde{v}}}} \\
\downarrow & & \downarrow & & \downarrow \\
H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i, \mathrm{Fil}\,^{i+1})) & \to & H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i, \mathrm{gr}\,^{i+1})) & \stackrel{\mathrm{val}}{\to} & (\mathrm{Hom}\,(\mathrm{gr}\,^i, \mathrm{gr}\,^{i+1})(\epsilon^{-1})_{I_{F_{\widetilde{v}}}})^{G_{F_{\widetilde{v}}}} \\
\downarrow & & \downarrow & & \downarrow \\
H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}}r, \mathrm{Fil}\,^{i+1}_{\widetilde{v}}r)) & \to & H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}}r, \mathrm{gr}\,^{i+1}_{\widetilde{v}}r)) & \stackrel{\mathrm{val}}{\to} & (\mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}}r, \mathrm{gr}\,^{i+1}_{\widetilde{v}}r)(\epsilon^{-1})_{I_{F_{\widetilde{v}}}})^{G_{F_{\widetilde{v}}}} \\
\downarrow & & \downarrow & & \\
H^2(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}}\overline{r}, I\mathrm{Fil}\,^{i+1})) & \to & H^2(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}}\overline{r}, I\mathrm{gr}\,^{i+1})). & &
\end{array}$$

The last column is also exact, as either each term is zero or $I_{F_{\widetilde{v}}}$ acts trivially on each of the modules so that we can suppress the $I_{F_{\widetilde{v}}}$-coinvariants.

It suffices to check that the kernel of the map

$$H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}}r, \mathrm{Fil}\,^{i+1}_{\widetilde{v}}r)) \to H^2(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}}\overline{r}, I\mathrm{Fil}\,^{i+1}))$$

contains the kernel of the map

$$H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} r, \mathrm{Fil}\,^{i+1}_{\widetilde{v}} r)) \to (\mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} r, \mathrm{gr}\,^{i+1}_{\widetilde{v}} r)(\epsilon^{-1})_{I_{F_{\widetilde{v}}}})^{G_{F_{\widetilde{v}}}},$$

and that, in the case $\overline{\chi}_{\widetilde{v},i+1} = \overline{\chi}_{\widetilde{v},i}\epsilon$, the map

$$H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} \overline{r}, I\mathrm{Fil}\,^{i+1})) \to (\mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} \overline{r}, I\mathrm{gr}\,^{i+1})(\epsilon^{-1})_{I_{F_{\widetilde{v}}}})^{G_{F_{\widetilde{v}}}}$$

is surjective.

For the first property we dualise. If M is a $\mathbb{Z}_l[G_{F_{\widetilde{v}}}]$-module we set $M^* = \mathrm{Hom}\,(M, \mathbb{Q}_l/\mathbb{Z}_l(\epsilon))$. We have the commutative diagram

$$\begin{array}{ccc} & & H^1(G_{F_{\widetilde{v}}}/I_{\widetilde{v}}, (\mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} r, \mathrm{gr}\,^{i+1}_{\widetilde{v}} r)^*)^{I_{F_{\widetilde{v}}}}) \\ & & \downarrow \\ H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} \overline{r}, I\mathrm{gr}\,^{i+1})^*) & \to & H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} r, \mathrm{gr}\,^{i+1}_{\widetilde{v}} r)^*) \\ \downarrow & & \downarrow \\ H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} \overline{r}, I\mathrm{Fil}\,^{i+1})^*) & \to & H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} r, \mathrm{Fil}\,^{i+1}_{\widetilde{v}} r)^*) \end{array}$$

and we need to check that the image of

$$H^1(G_{F_{\widetilde{v}}}/I_{\widetilde{v}}, (\mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} r, \mathrm{gr}\,^{i+1}_{\widetilde{v}} r)^*)^{I_{F_{\widetilde{v}}}}) \to H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} r, \mathrm{Fil}\,^{i+1}_{\widetilde{v}} r)^*)$$

contains the image of

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} \overline{r}, I\mathrm{Fil}\,^{i+1})^*) \to H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} r, \mathrm{Fil}\,^{i+1}_{\widetilde{v}} r)^*).$$

The map in the last row equals

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{Fil}\,^{i+1}_{\widetilde{v}} \overline{r}, \mathrm{gr}\,^i_{\widetilde{v}} \overline{r})(\epsilon)) \otimes I^* \leftarrow H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^{i+1}_{\widetilde{v}} \overline{r}, \mathrm{gr}\,^i_{\widetilde{v}} \overline{r})(\epsilon)) \otimes I^*.$$

It is surjective because, by our assumptions,

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^j_{\widetilde{v}} \overline{r}, \mathrm{gr}\,^i_{\widetilde{v}} \overline{r})(\epsilon)) = (0)$$

for $j > i+1$. Thus we need only show that the image of

$$H^1(G_{F_{\widetilde{v}}}/I_{\widetilde{v}}, (\mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} r, \mathrm{gr}\,^{i+1}_{\widetilde{v}} r)^*)^{I_{F_{\widetilde{v}}}}) \to H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} r, \mathrm{gr}\,^{i+1}_{\widetilde{v}} r)^*)$$

contains the image of

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} \overline{r}, I\mathrm{gr}\,^{i+1})^*) \to H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,_t v^i r, \mathrm{gr}\,^{i+1}_{\widetilde{v}} r)^*),$$

i.e. that

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} \overline{r}, I\mathrm{gr}\,^{i+1})^*) \to H^1(I_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^i_{\widetilde{v}} r, \mathrm{gr}\,^{i+1}_{\widetilde{v}} r)^*)$$

is zero. If $\overline{\chi}_{\widetilde{v},i+1} \neq \overline{\chi}_{\widetilde{v},i}\epsilon$ then the domain is trivial so there is nothing to prove. Otherwise $I_{F_{\widetilde{v}}}$ acts trivially $\mathrm{Hom}\,(\mathrm{gr}^{\,i}, \mathrm{gr}^{\,i+1})^*$ and again we see this map is zero.

For the second property we suppose that $\overline{\chi}_{\widetilde{v},i+1} = \overline{\chi}_{\widetilde{v},i}\epsilon$. It suffices to check that

$$H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}^{\,i}_{\widetilde{v}}\overline{r}, I\mathrm{Fil}^{\,i+1})) \to H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}^{\,i}_{\widetilde{v}}\overline{r}, I\mathrm{gr}^{\,i+1}))$$

is surjective, or even that

$$H^2(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}^{\,i}_{\widetilde{v}}\overline{r}, I\mathrm{Fil}^{\,i+2})) = H^2(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}^{\,i}_{\widetilde{v}}\overline{r}, \mathrm{Fil}^{\,i+2}_{\widetilde{v}}\overline{r})) \otimes I = (0).$$

Dually it suffices to check that

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{Fil}^{\,i+2}_{\widetilde{v}}\overline{r}, \mathrm{gr}^{\,i}_{\widetilde{v}}\overline{r})(\epsilon)) = (0),$$

which follows from our assumption that $\overline{\chi}_{\widetilde{v},i}/\overline{\chi}_{\widetilde{v},j} \neq \epsilon$ for $j > i+1$. $\Box$

Lemma 1.3.2

$$\begin{aligned}&\mathrm{lg}_{\mathcal{O}}\, L'_{\widetilde{v}} - \mathrm{lg}_{\mathcal{O}}\, H^0(G_{F_{\widetilde{v}}}, \mathrm{Fil}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r}) = \\ &n(n-1)[F_{\widetilde{v}} : \mathbb{Q}_l]/2 + \mathrm{lg}_{\mathcal{O}}\ker(H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r}/\mathrm{Fil}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon)) \to \mathrm{gr}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r}),\end{aligned}$$

where the last map is the composite of

$$H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r}/\mathrm{Fil}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon)) \stackrel{\partial}{\longrightarrow} H^1(G_{F_{\widetilde{v}}}, (\mathrm{gr}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon))$$

and

$$H^1(G_{F_{\widetilde{v}}}, (\mathrm{gr}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon)) = (\mathrm{gr}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \otimes F_{\widetilde{v}}^{\times} \stackrel{\mathrm{val}}{\longrightarrow} \mathrm{gr}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r}.$$

Proof: Looking at the diagram

$$\begin{array}{ccccc} & & (0) & & \\ & & \downarrow & & \\ & & H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{gr}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r}) & & \\ & & \downarrow & & \\ H^1(G_{F_{\widetilde{v}}}, \mathrm{Fil}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r}) & \longrightarrow & H^1(G_{F_{\widetilde{v}}}, \mathrm{gr}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r}) & \longrightarrow & H^2(G_{F_{\widetilde{v}}}, \mathrm{Fil}^{\,1}_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \\ & & \downarrow & & \\ & & H^1(I_{F_{\widetilde{v}}}, \mathrm{gr}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r}) & & \end{array}$$

with an exact row and column, we see that

$$\begin{aligned}\mathrm{lg}_{\mathcal{O}}\, L'_{\widetilde{v}} = {} & \mathrm{lg}_{\mathcal{O}}\ker(H^1(G_{F_{\widetilde{v}}}, \mathrm{Fil}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \to H^1(G, \mathrm{gr}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r})) + \\ & \mathrm{lg}_{\mathcal{O}}\ker(H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{gr}^{\,0}_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \to H^2(G_{F_{\widetilde{v}}}, \mathrm{Fil}^{\,1}_{\widetilde{v}}\mathrm{ad}\,\overline{r})).\end{aligned}$$

The long exact sequence corresponding to the short exact sequence

$$(0) \longrightarrow \mathrm{Fil}\,^1_{\tilde v}\mathrm{ad}\,\overline{r} \longrightarrow \mathrm{Fil}\,^0_{\tilde v}\mathrm{ad}\,\overline{r} \longrightarrow \mathrm{gr}\,^0_{\tilde v}\mathrm{ad}\,\overline{r} \longrightarrow (0)$$

tells us that

$$\begin{array}{l} \lg_{\mathcal{O}} \ker(H^1(G_{F_{\tilde v}}, \mathrm{Fil}\,^0_{\tilde v}\mathrm{ad}\,\overline{r}) \to H^1(G, \mathrm{gr}\,^0_{\tilde v}\mathrm{ad}\,\overline{r})) = \\ \lg_{\mathcal{O}} H^1(G_{F_{\tilde v}}, \mathrm{Fil}\,^1_{\tilde v}\mathrm{ad}\,\overline{r}) - \lg_{\mathcal{O}} H^0(G_{F_{\tilde v}}, \mathrm{gr}\,^0_{\tilde v}\mathrm{ad}\,\overline{r}) + \\ + \lg_{\mathcal{O}} H^0(G_{F_{\tilde v}}, \mathrm{Fil}\,^0_{\tilde v}\mathrm{ad}\,\overline{r}) - \lg_{\mathcal{O}} H^0(G_{F_{\tilde v}}, \mathrm{Fil}\,^1_{\tilde v}\mathrm{ad}\,\overline{r}). \end{array}$$

The local Euler characteristic formula in turn, tells us that this is

$$[F_{\tilde v}:\mathbb{Q}_l]n(n-1)/2 + \lg_{\mathcal{O}} H^0(G_{F_{\tilde v}}, \mathrm{Fil}\,^0_{\tilde v}\mathrm{ad}\,\overline{r}) - \lg_{\mathcal{O}} \mathrm{gr}\,^0_{\tilde v}\mathrm{ad}\,\overline{r} + \lg_{\mathcal{O}} H^2(G_{F_{\tilde v}}, \mathrm{Fil}\,^1_{\tilde v}\mathrm{ad}\,\overline{r}).$$

On the other hand local duality tells us that

$$\begin{array}{l} \lg_{\mathcal{O}} \ker(H^1(G_{F_{\tilde v}}/I_{F_{\tilde v}}, \mathrm{gr}\,^0_{\tilde v}\mathrm{ad}\,\overline{r}) \to H^2(G_{F_{\tilde v}}, \mathrm{Fil}\,^1_{\tilde v}\mathrm{ad}\,\overline{r})) = \\ \lg_{\mathcal{O}} \mathrm{coker}\,(H^0(G_{F_{\tilde v}}, (\mathrm{ad}\,\overline{r}/\mathrm{Fil}\,^0_{\tilde v}\mathrm{ad}\,\overline{r})(\epsilon)) \to \mathrm{gr}\,^0_{\tilde v}\mathrm{ad}\,\overline{r}), \end{array}$$

where the second map is the one described in the statement of the lemma. Local duality also tells us that

$$\lg_{\mathcal{O}} H^2(G_{F_{\tilde v}}, \mathrm{Fil}\,^1_{\tilde v}\mathrm{ad}\,\overline{r}) = \lg_{\mathcal{O}} H^0(G_{F_{\tilde v}}, (\mathrm{ad}\,\overline{r}/\mathrm{Fil}\,^0_{\tilde v}\mathrm{ad}\,\overline{r})(\epsilon)).$$

Thus

$$\begin{array}{l} \lg_{\mathcal{O}} L'_{\tilde v} - \lg_{\mathcal{O}} H^0(G_{F_{\tilde v}}, \mathrm{Fil}\,^0_{\tilde v}\mathrm{ad}\,\overline{r}) = \\ {[F_{\tilde v}:\mathbb{Q}_l]}n(n-1)/2 - \lg_{\mathcal{O}} \mathrm{gr}\,^0_{\tilde v}\mathrm{ad}\,\overline{r} + \lg_{\mathcal{O}} H^0(G_{F_{\tilde v}}, (\mathrm{ad}\,\overline{r}/\mathrm{Fil}\,^0_{\tilde v}\mathrm{ad}\,\overline{r})(\epsilon)) + \\ \lg_{\mathcal{O}} \mathrm{coker}\,(H^0(G_{F_{\tilde v}}, (\mathrm{ad}\,\overline{r}/\mathrm{Fil}\,^0_{\tilde v}\mathrm{ad}\,\overline{r})(\epsilon)) \to \mathrm{gr}\,^0_{\tilde v}\mathrm{ad}\,\overline{r}), \end{array}$$

and the lemma follows. □

Corollary 1.3.3

$$\begin{array}{r} \lg_{\mathcal{O}} L'_{\tilde v} - \lg_{\mathcal{O}} H^0(G_{F_{\tilde v}}, \mathrm{Fil}\,^0_{\tilde v}\mathrm{ad}\,\overline{r}) = n(n-1)[F_{\tilde v}:\mathbb{Q}_l]/2 + \\ + \lg_{\mathcal{O}} H^0(G_{F_{\tilde v}}, (\mathrm{gr}\,^{-1}_{\tilde v}\mathrm{ad}\,\overline{r})(\epsilon)). \end{array}$$

Proof: The natural map

$$(\mathrm{gr}\,^{-1}_{\tilde v}\mathrm{ad}\,\overline{r})(\epsilon)^{G_{F_{\tilde v}}} \longrightarrow (\mathrm{ad}\,\overline{r}/\mathrm{Fil}\,^0_{\tilde v}\mathrm{ad}\,\overline{r})(\epsilon)^{G_{F_{\tilde v}}}$$

is an isomorphism. The map

$$(\mathrm{gr}\,^{-1}_{\tilde v}\mathrm{ad}\,\overline{r})(\epsilon)^{G_{F_{\tilde v}}} \stackrel{\mathrm{val}\circ\partial}{\longrightarrow} \mathrm{gr}\,^0_{\tilde v}\mathrm{ad}\,\overline{r}$$

is zero, because for all j we have $\mathrm{val}\,(\overline{\mathrm{Fil}}\,^{j-1}_{\tilde v}/\overline{\mathrm{Fil}}\,^{j+1}_{\tilde v}) = 0$. Thus

$$(\mathrm{gr}\,^{-1}_{\tilde v}\mathrm{ad}\,\overline{r})(\epsilon)^{G_{F_{\tilde v}}} \stackrel{\sim}{\longrightarrow} \ker(H^0(G_{F_{\tilde v}}, (\mathrm{ad}\,\overline{r}/\mathrm{Fil}\,^0_{\tilde v}\mathrm{ad}\,\overline{r})(\epsilon)) \to \mathrm{gr}\,^0_{\tilde v}\mathrm{ad}\,\overline{r}).$$

□

Lemma 1.3.4 *The composite*

$$H^1(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \longrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{gr}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \stackrel{\mathrm{val}}{\longrightarrow} ((\mathrm{gr}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon^{-1})_{I_{F_{\widetilde{v}}}})^{G_{F_{\widetilde{v}}}}$$

is surjective.

Proof: We must show that

$$\ker(H^1(G_{F_{\widetilde{v}}}, \mathrm{gr}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \longrightarrow H^2(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^2_{\widetilde{v}}\mathrm{ad}\,\overline{r})) \stackrel{\mathrm{val}}{\twoheadrightarrow} ((\mathrm{gr}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon^{-1})_{I_{F_{\widetilde{v}}}})^{G_{F_{\widetilde{v}}}}.$$

Dually it suffices to show that

$$H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\mathrm{gr}\,^{-1}_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon)^{I_{F_{\widetilde{v}}}})$$

injects into the cokernel of the map

$$H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r}/\mathrm{Fil}\,^{-1}_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon)) \longrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{gr}\,^{-1}_{\widetilde{v}}\mathrm{ad}\,\overline{r}(\epsilon)).$$

Equivalently we must check that

$$H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\mathrm{gr}\,^{-1}_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon)^{I_{F_{\widetilde{v}}}}) \longrightarrow H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r}/\mathrm{Fil}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon)^{I_{F_{\widetilde{v}}}})$$

is injective. This follows because

$$((\mathrm{ad}\,\overline{r}/\mathrm{Fil}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon)^{I_{F_{\widetilde{v}}}}/(\mathrm{gr}\,^{-1}_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon)^{I_{F_{\widetilde{v}}}})^{G_{F_{\widetilde{v}}}} = (0).$$

$\Box$

Corollary 1.3.5

$$\mathrm{lg}_{\mathcal{O}}\, L_{\widetilde{v}} - \mathrm{lg}_{\mathcal{O}}\, H^0(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}) = [F_{\widetilde{v}} : \mathbb{Q}_l]n(n-1)/2.$$

Proof: The lemma tells us that

$$\mathrm{lg}_{\mathcal{O}}\, L'_{\widetilde{v}} - \mathrm{lg}_{\mathcal{O}}\, L_{\widetilde{v}} = \mathrm{lg}_{\mathcal{O}}((\mathrm{gr}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon^{-1})_{I_{F_{\widetilde{v}}}})^{G_{F_{\widetilde{v}}}} = \mathrm{lg}_{\mathcal{O}}(\mathrm{gr}\,^{-1}_{\widetilde{v}}\mathrm{ad}\,\overline{r})(\epsilon)^{G_{F_{\widetilde{v}}}}.$$

$\Box$

1.3.2 Crystalline deformations

Secondly we will second discuss the case $\widetilde{v} \in \widetilde{S}_l - \widetilde{S}_{l,0}$. In this case we will let $\mathcal{D}_{\widetilde{v}}$ consist of all lifts $r : G_{F_{\widetilde{v}}} \to GL_n(R)$ of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ such that, for each Artinian quotient R' of R, $r \otimes_R R'$ is in the essential image of $\mathbb{G}_{\widetilde{v}}$. It is easy to verify that this is a local deformation problem and that

$$L_{\widetilde{v}} = L_{\widetilde{v}}(\mathcal{D}_{\widetilde{v}}) = \mathrm{Ext}\,^1_{\mathcal{MF}_{k,\widetilde{v}}}(\mathbb{G}_{\widetilde{v}}^{-1}(\overline{r}), \mathbb{G}_{\widetilde{v}}^{-1}(\overline{r})) \hookrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}).$$

Lemma 1.3.6 *For $\widetilde{v} \in \widetilde{S}_l - \widetilde{S}_{l,0}$ the set $\mathcal{D}_{\widetilde{v}}$ is liftable.*

Proof: Suppose that R is an Artinian object of $\mathcal{C}_{\mathcal{O}}$ and I is an ideal of R with $\mathfrak{m}_R I = (0)$. Suppose also that r is a deformation in $\mathcal{D}_{\widetilde{v}}$ of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ to R/I. Write $M = \mathbb{G}_{\widetilde{v}}^{-1}(r)$ and for $\widetilde{\tau} : F_{\widetilde{v}} \hookrightarrow K$ write $M_{\widetilde{\tau}} = M \otimes_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}, \widetilde{\tau} \otimes 1} \mathcal{O}$. Then $\mathrm{Fil}\,^i M = \bigoplus_{\widetilde{\tau}} \mathrm{Fil}\,^i M_{\widetilde{\tau}}$ for all i. As $M/\mathfrak{m}_R M = \mathbb{G}_{\widetilde{v}}^{-1}(\overline{r})$ we see that we can find a surjection $(R/I)^n \twoheadrightarrow M_{\widetilde{\tau}}$ such that $(R/I)^i \twoheadrightarrow \mathrm{Fil}\,^{m_{\widetilde{\tau},n-i}} M_{\widetilde{\tau}}$ for all i (where $(R/I)^i \subset (R/I)^n$ consists of vectors whose last $n-i$ entries are zero). Counting orders we see that $(R/I)^n \xrightarrow{\sim} M_{\widetilde{\tau}}$, and hence $(R/I)^i \xrightarrow{\sim} \mathrm{Fil}\,^{m_{\widetilde{\tau},n-i}} M_{\widetilde{\tau}}$ for all i. Define an object $N = \bigoplus_{\widetilde{\tau}} N_{\widetilde{\tau}}$ of $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$ with an action of R as follows. We take $N_{\widetilde{\tau}} = R^n$ with an $\mathcal{O}_{F,\widetilde{v}}$-action via $\widetilde{\tau}$. We set $\mathrm{Fil}\,^j N_{\widetilde{\tau}} = R^i$ where $m_{\widetilde{\tau},n-i} \geq j > m_{\widetilde{\tau},n-1-i}$ (and where we set $m_{\widetilde{\tau},n} = \infty$ and $m_{\widetilde{\tau},-1} = -\infty$). Then $N/I \cong M$ as filtered $\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} R$-modules. Finally we define $\Phi^{m_{\widetilde{\tau},i}} : \mathrm{Fil}\,^{m_{\widetilde{\tau},i}} N_{\widetilde{\tau}} \to N_{\widetilde{\tau} \circ \mathrm{Frob}_l}$ by reverse recursion on i. For $i = n-1$ we take any lift of $\Phi^{m_{\widetilde{\tau},n-1}} : \mathrm{Fil}\,^{m_{\widetilde{\tau},n-1}} M_{\widetilde{\tau}} \to M_{\widetilde{\tau} \circ \mathrm{Frob}_l}$. In general we choose any lift of $\Phi^{m_{\widetilde{\tau},i}} : \mathrm{Fil}\,^{m_{\widetilde{\tau},i}} M_{\widetilde{\tau}} \to M_{\widetilde{\tau} \circ \mathrm{Frob}_l}$ which restricts to $l^{m_{\widetilde{\tau},i+1} - m_{\widetilde{\tau},i}} \Phi^{m_{\widetilde{\tau},i+1}}$ on $\mathrm{Fil}\,^{m_{\widetilde{\tau},i+1}} N_{\widetilde{\tau}}$. This is possible as $\mathrm{Fil}\,^{m_{\widetilde{\tau},i+1}} M_{\widetilde{\tau}}$ is a direct summand of $\mathrm{Fil}\,^{m_{\widetilde{\tau},i}} M_{\widetilde{\tau}}$. Nakayama's lemma tells us that $\sum_i \Phi^{m_{\widetilde{\tau},i}} \mathrm{Fil}\,^{m_{\widetilde{\tau},i}} N_{\widetilde{\tau}} = N_{\widetilde{\tau} \circ \mathrm{Frob}_l}$, so that N is an object of $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$. As our lifting of r we take $\mathbb{G}_{\widetilde{v}}(N)$. $\Box$

We will need to calculate $\mathrm{lg}_{\mathcal{O}} L_{\widetilde{v}}$. To this end we have the following lemma.

Lemma 1.3.7 *Suppose that M and N are objects of $\mathcal{MF}_{k,\widetilde{v}}$. Then there is an exact sequence*

$$\begin{aligned}&(0) \to \mathrm{Hom}\,_{\mathcal{MF}_{k,\widetilde{v}}}(M,N) \to \mathrm{Fil}\,^0 \mathrm{Hom}\,_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}}(M,N) \to \\ &\to \mathrm{Hom}\,_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}, \mathrm{Fr} \otimes 1}(\mathrm{gr}\, M, N) \to \mathrm{Ext}\,^1_{\mathcal{MF}_{k,\widetilde{v}}}(M,N) \to (0),\end{aligned}$$

where $\mathrm{Fil}\,^i \mathrm{Hom}\,_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}}(M,N)$ *denotes the subset of* $\mathrm{Hom}\,_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}}(M,N)$ *consisting of elements which take* $\mathrm{Fil}\,^j M$ *to* $\mathrm{Fil}\,^{i+j} N$ *for all j and where* $\mathrm{gr}\, M = \bigoplus_i \mathrm{gr}\,^i M$. *The central map sends β to $(\beta \Phi^i_M - \Phi^i_N \beta)$.*

Proof: Any extension

$$(0) \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow (0)$$

in $\mathcal{MF}_{k,\widetilde{v}}$ can be written $E = N \oplus M$ such that $\mathrm{Fil}^i E = \mathrm{Fil}^i N \oplus \mathrm{Fil}^i M$ (and such that $N \to E$ is the natural inclusion and $E \to M$ is the natural projection). Then

$$\Phi^i_E = \begin{pmatrix} \Phi^i_N & \alpha_i \\ 0 & \Phi^i_M \end{pmatrix}$$

with $\alpha_i \in \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}, \mathrm{Fr} \otimes 1}(\mathrm{gr}^i M, N)$. Conversly, any

$$\alpha = (\alpha_i) \in \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}, \mathrm{Fr} \otimes 1}(\mathrm{gr}\, M, N)$$

gives rise to such an extension. Two such extensions corresponding to α and α' are isomorphic if there is a $\beta \in \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}}(M, N)$ which preserves the filtrations and such that for all i

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi^i_N & \alpha_i \\ 0 & \Phi^i_M \end{pmatrix} = \begin{pmatrix} \Phi^i_N & \alpha'_i \\ 0 & \Phi^i_M \end{pmatrix} \begin{pmatrix} 1 & \beta|_{\mathrm{gr}^i M} \\ 0 & 1 \end{pmatrix}.$$

The lemma now follows easily. □

Corollary 1.3.8 *Keep the above notation. We have*

$$\mathrm{lg}_{\mathcal{O}}\, L_{\widetilde{v}} - \mathrm{lg}_{\mathcal{O}}\, H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\, \overline{r}) = [F_{\widetilde{v}} : \mathbb{Q}_l] n(n-1)/2.$$

Proof: If M is an object of $\mathcal{MF}_{\mathcal{O},\widetilde{v}}$ and if $\widetilde{\tau} : F_{\widetilde{v}} \hookrightarrow K$ set

$$M_{\widetilde{\tau}} = M \otimes_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}, \tau \otimes 1} \mathcal{O}.$$

Thus $\mathrm{Fil}^i M = \bigoplus_{\widetilde{\tau}} \mathrm{Fil}^i M_{\widetilde{\tau}}$ and $\Phi^i : \mathrm{Fil}^i M_{\widetilde{\tau}} \to M_{\widetilde{\tau} \circ \mathrm{Fr}^{-1}}$. We have

$$\mathrm{Fil}^0 \mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}}(M, N) \cong \bigoplus_{\widetilde{\tau}} \mathrm{Fil}^0 \mathrm{Hom}_{\mathcal{O}}(M_{\widetilde{\tau}}, N_{\widetilde{\tau}})$$

and

$$\mathrm{Hom}_{\mathcal{O}_{F,\widetilde{v}} \otimes_{\mathbb{Z}_l} \mathcal{O}, \mathrm{Fr} \otimes 1}(\mathrm{gr}\, M, N) \cong \bigoplus_{\widetilde{\tau}} \mathrm{Hom}_{\mathcal{O}}(\mathrm{gr}\, M_{\widetilde{\tau}}, N_{\widetilde{\tau} \circ \mathrm{Fr}^{-1}}).$$

Note that $\dim_k \mathrm{Fil}^0 \mathrm{Hom}_k(\mathbb{G}^{-1}_{\widetilde{v}}(\overline{r})_{\widetilde{\tau}}, \mathbb{G}^{-1}_{\widetilde{v}}(\overline{r})_{\widetilde{\tau}}) = n(n+1)/2$ and that $\dim_k \mathrm{Hom}_k(\mathrm{gr}\, \mathbb{G}^{-1}_{\widetilde{v}}(\overline{r})_{\widetilde{\tau}}, \mathbb{G}^{-1}_{\widetilde{v}}(\overline{r})_{\widetilde{\tau} \circ \mathrm{Fr}^{-1}}) = n^2$. The corollary follows. □

Corollary 1.3.9 *If $n = 1$ then*

$$L_{\widetilde{v}} = H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \operatorname{ad}\overline{r}).$$

Proof: One checks that $L_{\widetilde{v}} \supset H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \operatorname{ad}\overline{r})$ and then uses the equality of dimensions. $\Box$

The next lemma is clear.

Lemma 1.3.10 *If $\overline{r}|_{G_{F_{\widetilde{v}}}} = \oplus_i \overline{s}_i$ then*

$$H^1(G_{F_{\widetilde{v}}}, \operatorname{ad}\overline{r}) = \oplus_{i,j} H^1(G_{F_{\widetilde{v}}}, \operatorname{Hom}(\overline{s}_i, \overline{s}_j))$$

and $L_{\widetilde{v}} = \oplus_{i,j}(L_{\widetilde{v}})_{i,j}$, where $(L_{\widetilde{v}})_{i,j}$ denotes the image of

$$\operatorname{Ext}^1_{\mathcal{MF}_{k,\widetilde{v}}}(\mathbb{G}_{\widetilde{v}}^{-1}(\overline{s}_i), \mathbb{G}_{\widetilde{v}}^{-1}(\overline{s}_j)) \longrightarrow H^1(G_{F_{\widetilde{v}}}, \operatorname{Hom}(\overline{s}_i, \overline{s}_j)).$$

1.3.3 Unrestricted deformations

Suppose $\widetilde{v} \in \widetilde{S} - \widetilde{S}_l$. We can take $\mathcal{D}_{\widetilde{v}}$ to consist of all lifts of $(\overline{r}|_{G_{F_{\widetilde{v}}}}, \{\overline{\operatorname{Fil}}^i_{\widetilde{v}}\})$ and $L_{\widetilde{v}} = L_{\widetilde{v}}(\mathcal{D}_{\widetilde{v}}) = H^1(G_{F_{\widetilde{v}}}, \operatorname{Fil}^0_{\widetilde{v}}\operatorname{ad}\overline{r})$. In this case

$$\lg_{\mathcal{O}} L_{\widetilde{v}} - \lg_{\mathcal{O}} H^0(G_{F_{\widetilde{v}}}, \operatorname{Fil}^0_{\widetilde{v}}\operatorname{ad}\overline{r}) = \lg_{\mathcal{O}} H^0(G_{F_{\widetilde{v}}}, (\operatorname{ad}\overline{r}/\operatorname{Fil}^1_{\widetilde{v}})(1)).$$

If $H^0(G_{F_{\widetilde{v}}}, (\operatorname{ad}\overline{r}/\operatorname{Fil}^1_{\widetilde{v}})(1)) = (0)$ then $L_{\widetilde{v}}$ is minimal and (using local duality, we see that) $\mathcal{D}_{\widetilde{v}}$ is liftable.

1.3.4 Minimal deformations

Suppose $\widetilde{v} \in \widetilde{S} - \widetilde{S}_l$. Suppose moreover that $\overline{\operatorname{Fil}}^0_{\widetilde{v}}\overline{r} = (0)$ and that for $i < 0$, $\overline{\operatorname{Fil}}^i_{\widetilde{v}}\overline{r}/\overline{\operatorname{Fil}}^{i+1}_{\widetilde{v}}\overline{r}$ is the maximal submodule of $\overline{r}/\overline{\operatorname{Fil}}^{i+1}_{\widetilde{v}}\overline{r}$ on which $I_{F_{\widetilde{v}}}$ acts semisimply. (Thus if $r|_{I_{F_{\widetilde{v}}}}$ is semisimple then $\widetilde{v} \notin \widetilde{S}_0$, while otherwise $\widetilde{v} \in \widetilde{S}_0$.) For every $\widetilde{v} \not| l$, there is a unique such filtration on $\overline{r}$.

Before describing $\mathcal{D}_{\widetilde{v}}$ we first give a description of all lifts of $(\overline{r}|_{G_{F_{\widetilde{v}}}}, \{\overline{\operatorname{Fil}}^i_{\widetilde{v}}\}_i)$. Let $P_{F_{\widetilde{v}}}$ denote the kernel of any surjection $I_{F_{\widetilde{v}}} \twoheadrightarrow \mathbb{Z}_l$. (Hence it is the kernel of every such surjection.) Then $P_{F_{\widetilde{v}}}$ has (pro-)order prime to l and $I_{F_{\widetilde{v}}}$ is the semidirect product of $P_{F_{\widetilde{v}}}$ by $\mathbb{Z}_l$. Let σ denote a topological generator of this $\mathbb{Z}_l$. Moreover $G_{F_{\widetilde{v}}}$ is the semidirect product of $I_{F_{\widetilde{v}}}$ by $\widehat{\mathbb{Z}}$. Let ϕ be a topological generator of this $\widehat{\mathbb{Z}}$ which lifts $\operatorname{Frob}_{\widetilde{v}}^{-1}$. Then $\phi\mathbb{Z}_l\phi^{-1}$ is a Sylow pro-l-subgroup of $I_{F_{\widetilde{v}}}$ and so $\phi\mathbb{Z}_l\phi^{-1} = \tau\mathbb{Z}_l\tau^{-1}$ for some $\tau \in I_{F_{\widetilde{v}}}$. Replacing ϕ by $\tau^{-1}\phi$ we may assume that $\phi\sigma\phi^{-1} = \sigma^{\# k(\widetilde{v})}$. Set $T_{F_{\widetilde{v}}} = G_{F_{\widetilde{v}}}/P_{F_{\widetilde{v}}}$. Thus we have written $G_{F_{\widetilde{v}}}$ as the semidirect product of $P_{F_{\widetilde{v}}}$ by $T_{F_{\widetilde{v}}}$.

Suppose that τ is an irreducible representation of $P_{F_{\tilde{v}}}$ over k. Then τ admits a unique lift $\widetilde{\tau}$ to $\mathcal{O}$. Let G_τ denote the set of $\sigma \in G_{F_{\tilde{v}}}$ with $\tau^\sigma \sim \tau$. Write I_τ for $G_\tau \cap I_{F_{\tilde{v}}}$ and T_τ for $G_\tau \cap T_{F_{\tilde{v}}}$. Then I_τ is the semidirect product of P_{F_τ} by $\langle \sigma^{l^{a_\tau}} \rangle$ for some $a_\tau \in \mathbb{Z}_{\geq 0}$. Write $\sigma_\tau = \sigma^{l^{a_\tau}}$. Moreover we can find $\phi_\tau \in T_{F_{\tilde{v}}}$ such that T_τ is the semidirect product of $I_\tau \cap T_{F_{\tilde{v}}}$ by the copy of $\widehat{\mathbb{Z}}$ with topological generator ϕ_τ. Note that $l \not| \dim \tau$, as $P_{F_{\tilde{v}}}$ has pro-order prime to l.

The representation τ has a unique (up to equivalence) extension to I_τ. (Suppose $\tau(\sigma_\tau g \sigma_\tau^{-1}) = A\tau(g)A^{-1}$ for all $g \in P_{F_{\tilde{v}}}$. Suppose also that $\sigma_\tau^{l^b}$ centralises $\tau P_{F_{\tilde{v}}}$. Then we see that $z = A^{l^b}$ lies in the centraliser Z_τ of the image of τ. As τ is irreducible we see that Z_τ is the multiplicative group of a finite extension of k and so is a torsion abelian group with orders prime to l. Moreover $\mathbb{Z}/l^b\mathbb{Z}$ acts on Z_τ by letting 1 act by conjugation by A. As $H^2(\mathbb{Z}/l^b\mathbb{Z}, Z_\tau) = (0)$ we see that there is $w \in Z_\tau$ with $z^{-1} = w(AwA^{-1})(A^2wA^{-2})...(A^{l^b-1}wA^{1-l^b}) = (wA)^{l^b}A^{-l^b}$. We can extend τ to I_τ by sending σ_τ to wA. Now write A for wA. Any other extension sends σ_τ to uA for some $u \in Z_\tau$ with $u(AuA^{-1})...(A^{l^b-1}uA^{1-l^b})$ equalling an element of Z_τ of l-power order, i.e. equalling 1. As $H^1(\mathbb{Z}/l^b\mathbb{Z}, Z_\tau) = 1$ we see that $u = v^{-1}AvA^{-1}$ for some $v \in Z_\tau$. Hence our second extension of $\tau|_{P_{F_{\tilde{v}}}}$ is $v^{-1}\tau v$, i.e. our extension is unique up to equivalence.) Similarly the lifting $\widetilde{\tau}$ has a unique extension to I_τ with determinant of order prime to l. (Argue as before but choose A with $\det A$ having order prime to l, which is possible as for $z \in \mathcal{O}^\times$ we have $\det(zA) = z^{\dim \tau}\det(A)$. Then take Z_τ to be the set of elements of the centraliser of $\tau(P_{F_{\tilde{v}}})$ with order prime to l. The same argument shows the existence of one extension with determinant of order prime to l and also its uniqueness.)

By the uniqueness of the extension, $\widetilde{\tau}$ and $\widetilde{\tau}^{\phi_\tau}$ are equivalent as representations of $G_\tau \cap I_{F_{\tilde{v}}}$. Hence $\widetilde{\tau}$ extends to a representation of G_τ. Pick one such extension and let τ denote its reduction modulo λ.

Suppose that R is an object of $\mathcal{C}_\mathcal{O}$ and that M is a finite R-module with a continuous action of $G_{F_{\tilde{v}}}$. Then we can write

$$M = \bigoplus_\tau M_\tau$$

where τ runs over irreducible $k[P_{F_{\tilde{v}}}]$-modules and where M_τ is the biggest $R[P_{F_{\tilde{v}}}]$-submodule all whose irreducible subquotients are isomorphic to τ. Then M_τ is in fact a $R[G_\tau]$-module. Moreover $M'_\tau = \mathrm{Hom}_{\mathcal{O}[P_{F_{\tilde{v}}}]}(\widetilde{\tau}, M)$ is naturally an $R[T_\tau]$-module and

$$M_\tau = \widetilde{\tau} \otimes_\mathcal{O} M'_\tau.$$

Moreover we see that

$$M = \bigoplus_{[\tau]} \mathrm{Ind}\,_{G_\tau}^{G_{F_{\widetilde{v}}}} (\widetilde{\tau} \otimes_{\mathcal{O}} M'_\tau),$$

where $[\tau]$ runs over $G_{F_{\widetilde{v}}}$-conjugacy classes of irreducible $k[P_{F_{\widetilde{v}}}]$-modules. In fact the category of finite R-modules with continuous $G_{F_{\widetilde{v}}}$-action is naturally equivalent to the direct sum over $[\tau]$ of the categories of finite R-modules with a continuous action of T_τ. This equivalence sends M to $(M'_\tau)_{[\tau]}$. We will say that M *lacks unipotent ramification* if each M'_τ is unramified, i.e. the action of T_τ restricts trivially to $(I_{F_{\widetilde{v}}} \cap G_\tau)/P_{F_{\widetilde{v}}}$. In this case

$$M|_{I_{F_{\widetilde{v}}}} \cong \bigoplus_{[\tau]} (\mathrm{Ind}\,_{I_\tau}^{I_{F_{\widetilde{v}}}} \widetilde{\tau}) \otimes_{\mathcal{O}} (M'_\tau)^{[G_{F_{\widetilde{v}}} : I_{F_{\widetilde{v}}} G_\tau]}.$$

Conversely if

$$M|_{I_{F_{\widetilde{v}}}} \cong \bigoplus_{[\tau]} (\mathrm{Ind}\,_{I_\tau}^{I_{F_{\widetilde{v}}}} \widetilde{\tau}) \otimes_{\mathcal{O}} M''_\tau$$

for some trivial $I_{F_{\widetilde{v}}}$-modules M''_τ then M lacks unipotent monodromy.

We now return to lifts of $(r, \{\mathrm{Fil}\,^i_{\widetilde{v}}\})$ of $(\overline{r}|_{G_{F_{\widetilde{v}}}}, \{\overline{\mathrm{Fil}}\,^i_{\widetilde{v}}\})$ over an object R of $\mathcal{C}_{\mathcal{O}}$. We will say that such a lift is *minimally ramified* if each $\mathrm{gr}\,^i_{\widetilde{v}} r$ lacks unipotent ramification. Let $\mathcal{D}_{\widetilde{v}}$ denote the set of minimally ramified lifts. Using the above equivalence of categories it is easy to see that $\mathcal{D}_{\widetilde{v}}$ is a local deformation problem. Moreover

$$L_{\widetilde{v}} = L_{\widetilde{v}}(\mathcal{D}_{\widetilde{v}}) = \ker(H^1(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^0_{\widetilde{v}} \mathrm{ad}\,\overline{r}) \longrightarrow H^1(I_{F_{\widetilde{v}}}, \mathrm{gr}\,^0_{\widetilde{v}} \mathrm{ad}\,\overline{r})).$$

Note that

$$\overline{\mathrm{Fil}}\,^j_{\widetilde{v}} \overline{r} \cap \overline{r}'_\tau = \overline{\mathrm{Fil}}\,^j_{\widetilde{v}} \overline{r}'_\tau = \ker(\sigma_\tau - 1)^{-j}|_{\overline{r}'_\tau}.$$

Lemma 1.3.11 *$L_{\widetilde{v}}$ is minimal.*

Proof:

$$\begin{aligned}
&\dim_k L_{\widetilde{v}} - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \\
= \;& \dim_k \mathrm{Im}\,(H^1(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \to H^1(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}))+ \\
& + \dim_k \ker(H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\mathrm{gr}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})^{I_{F_{\widetilde{v}}}}) \to H^2(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r}))- \\
& - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \\
= \;& \dim_k H^1(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r}) - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{gr}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})- \\
& - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r}) - \dim_k H^2(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r})+ \\
& + \dim_k H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\mathrm{gr}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})^{I_{F_{\widetilde{v}}}})+ \\
& + \dim_k \mathrm{coker}\,(H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\mathrm{gr}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})^{I_{F_{\widetilde{v}}}}) \to H^2(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^1_{\widetilde{v}}\mathrm{ad}\,\overline{r})) \\
= \;& \dim_k \ker(H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r}/\mathrm{Fil}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})(1)) \to H^1(I_{F_{\widetilde{v}}}, (\mathrm{gr}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})(1))).
\end{aligned}$$

Thus to prove the lemma it suffices to show that

$$H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r}/\mathrm{Fil}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})(1)) \longrightarrow H^1(I_{F_{\widetilde{v}}}, (\mathrm{gr}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})(1))$$

is injective.

We have

$$\mathrm{ad}\,\overline{r} = \bigoplus_{[\tau],[\tau']} \mathrm{Hom}\,_k(\mathrm{Ind}\,^{G_{F_{\widetilde{v}}}}_{G_{\tau'}}(\widetilde{\tau}' \otimes \overline{r}'_{\tau'}), \mathrm{Ind}\,^{G_{F_{\widetilde{v}}}}_{G_\tau}(\widetilde{\tau} \otimes \overline{r}'_\tau)).$$

Hence

$$\begin{aligned}
(\mathrm{ad}\,\overline{r})^{P_{F_{\widetilde{v}}}} &= \bigoplus\nolimits_{[\tau],[\tau']} \mathrm{Ind}\,^{T_{F_{\widetilde{v}}}}_{T_\tau} \mathrm{Hom}\,_{k[P_{F_{\widetilde{v}}}]}(\mathrm{Ind}\,^{G_{F_{\widetilde{v}}}}_{G_{\tau'}}(\widetilde{\tau}' \otimes \overline{r}'_{\tau'}), (\widetilde{\tau} \otimes \overline{r}'_\tau)) \\
&= \bigoplus\nolimits_{[\tau]} (\mathrm{End}\,_{k[P_{F_{\widetilde{v}}}]}(\tau) \otimes \mathrm{Ind}\,^{T_{F_{\widetilde{v}}}}_{T_\tau} \mathrm{ad}\,\overline{r}'_\tau.
\end{aligned}$$

Thus we must show that for each $[\tau]$ the map

$$H^0(T_\tau, (\mathrm{ad}\,\overline{r}'_\tau/\mathrm{Fil}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}'_\tau)(1)) \longrightarrow H^1(I_\tau/P_{F_{\widetilde{v}}}, (\mathrm{gr}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}'_\tau)(1))$$

is injective. In fact it suffices to show that

$$H^0(I_\tau/P_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r}'_\tau/\mathrm{Fil}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}'_\tau)) \longrightarrow H^1(I_\tau/P_{F_{\widetilde{v}}}, (\mathrm{gr}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}'_\tau)) = \mathrm{gr}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}'_\tau$$

is injective. (Note that $I_\tau/P_{F_{\widetilde{v}}}$ acts trivially on each $\mathrm{gr}\,^j_{\widetilde{v}}\overline{r}'_\tau$.) In concrete terms, it suffices to show that if $\alpha \in \mathrm{ad}\,\overline{r}'_\tau$ and $\sigma_\tau \alpha \sigma_\tau^{-1} - \alpha$ sends $\overline{\mathrm{Fil}}\,^j_{\widetilde{v}}\overline{r}'_\tau$ to $\overline{\mathrm{Fil}}\,^{j+1}_{\widetilde{v}}\overline{r}'_\tau$ for all j then α sends $\overline{\mathrm{Fil}}\,^j_{\widetilde{v}}\overline{r}'_\tau$ to $\overline{\mathrm{Fil}}\,^j_{\widetilde{v}}\overline{r}'_\tau$ for all j.

We prove this last assertion by reverse induction on j. It is vacuously true for $j \geq 0$. Now consider $j < 0$. Our assumption tells us that

$$((\sigma_\tau - 1)\alpha - \alpha(\sigma_\tau - 1))\overline{\mathrm{Fil}}^j_{\widetilde{v}}\overline{r}'_\tau \subset \overline{\mathrm{Fil}}^{1+j}_{\widetilde{v}}\overline{r}'_\tau.$$

The inductive hypothesis and the fact that $(\sigma_\tau - 1)\overline{\mathrm{Fil}}^j_{\widetilde{v}}\overline{r}'_\tau \subset \overline{\mathrm{Fil}}^{1+j}_{\widetilde{v}}\overline{r}'_\tau$ implies that

$$(\sigma_\tau - 1)\alpha\overline{\mathrm{Fil}}^j_{\widetilde{v}}\overline{r}'_\tau \subset \overline{\mathrm{Fil}}^{1+j}_{\widetilde{v}}\overline{r}'_\tau.$$

Hence $\alpha\overline{\mathrm{Fil}}^j_{\widetilde{v}}\overline{r}'_\tau \subset \overline{\mathrm{Fil}}^j_{\widetilde{v}}\overline{r}'_\tau$, as desired. □

Lemma 1.3.12 *$\mathcal{D}_{\widetilde{v}}$ is liftable.*

Proof: Because of the equivalence of categories discussed above it suffices to prove the corresponding result for representations of T_τ. More precisely suppose that $\overline{r}'_\tau$ is a representation of T_τ over k. Define a filtration $\overline{\mathrm{Fil}}^j$ on $\overline{r}'_\tau$ by setting $\overline{\mathrm{Fil}}^j\overline{r}'_\tau = (0)$ for $j \geq 0$ and $= \ker(\overline{r}(\sigma_\tau) - 1)^{-j}$ for $j \leq 0$. Let $\mathcal{D}$ denote the set of liftings $(r, \{\mathrm{Fil}^j\})$ of $(\overline{r}'_\tau, \{\overline{\mathrm{Fil}}^j\})$ such that σ_τ acts trivially on $\mathrm{gr}^j r$ for all j. We need to show that $\mathcal{D}$ is liftable.

Let R be an object of $\mathcal{C}_\mathcal{O}$ and let I be an ideal of R with $\mathfrak{m}_R I = (0)$. Let $(r, \{\mathrm{Fil}^j\})$ be a lifting of $(\overline{r}'_\tau, \{\overline{\mathrm{Fil}}^j\})$ to R/I. We will show by induction on i that $(r/\mathrm{Fil}^i, \{\mathrm{Fil}^j/\mathrm{Fil}^i\}_j)$ can be lifted to R in such a way that σ_τ acts trivially on each graded piece. For i sufficiently negative this is vacuous. Assume we have done this for $i-1$ and we will show it for i. Choose bases compatible with the filtration. Write

$$r(\sigma_\tau) = \begin{pmatrix} 1 & X_0 \\ 0 & V_0 \end{pmatrix} \qquad\qquad r(\phi_\tau) = \begin{pmatrix} A_0 & B_0 \\ 0 & D_0 \end{pmatrix}.$$

Let $m = (\#k(\widetilde{v}))^{[G_{F_{\widetilde{v}}}:I_{F_{\widetilde{v}}}G_\tau]}$. Then V_0 is unipotent; A_0 and D_0 are invertible; $D_0V_0 = V_0^r D_0$ and

$$A_0X_0 + B_0V_0 = B_0 + X_0(1 + V_0 + ... + V_0^{m-1})D_0.$$

Moreover we are assuming that we are given lifts D of D_0 and V of V_0 to R such that V is unipotent and $DV = V^r D$. We wish to show we can find lifts A of A_0, B of B_0 and X of X_0 such that

$$AX + BV = B + X(1 + V + ... + V^{m-1})D.$$

Choose any lifts A_1, B_1 and X_1 and set $A = A_1 + A_2$, $B = B_1 + B_2$ and $X = X_1$ where A_2 and B_2 are matrices with entries in I. We need to find A_2 and B_2 such that

$$A_2 X + B_2(V-1) = B_1(1-V) + X(1+V+...+V^{m-1})D - A_1 X.$$

By assumption the right hand side is a matrix with entries in I. Thus it suffices to show that after reduction modulo $\mathfrak{m}_R$ the rows of X and $V-1$ taken together span $k^{\dim \overline{r}/\overline{\mathrm{Fil}}^{i-1}\overline{r}}$, i.e. that $\sigma_\tau - 1$ acting on $\overline{r}/\overline{\mathrm{Fil}}^{i}\overline{r}$ has rank $\dim(\overline{r}/\overline{\mathrm{Fil}}^{i-1}\overline{r})$. This follows because $\overline{\mathrm{Fil}}^{i-1}\overline{r}/\overline{\mathrm{Fil}}^{i}\overline{r}$ is the kernel of $\sigma_\tau - 1$ acting on $\overline{r}/\overline{\mathrm{Fil}}^{i}\overline{r}$. $\Box$

Lemma 1.3.13 $H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}/\mathrm{Fil}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \hookrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{Fil}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})/L_{\widetilde{v}}$.

Proof: It suffices to show that

$$H^0(I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}/\mathrm{Fil}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \hookrightarrow H^1(I_{F_{\widetilde{v}}}, \mathrm{gr}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})$$

or equivalently that

$$H^0(I_{F_{\widetilde{v}}}, \mathrm{gr}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \twoheadrightarrow H^0(I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}/\mathrm{Fil}^1_{\widetilde{v}}\mathrm{ad}\,\overline{r}).$$

In fact we will show by induction on j that

$$H^0(I_{F_{\widetilde{v}}}, \mathrm{gr}^0_{\widetilde{v}}\mathrm{ad}\,(\overline{r}/\overline{\mathrm{Fil}}^j_{\widetilde{v}}\overline{r})) \twoheadrightarrow H^0(I_{F_{\widetilde{v}}}, \mathrm{ad}\,(\overline{r}/\overline{\mathrm{Fil}}^j_{\widetilde{v}}\overline{r})/\mathrm{Fil}^1_{\widetilde{v}}\mathrm{ad}\,(\overline{r}/\overline{\mathrm{Fil}}^j_{\widetilde{v}}\overline{r})).$$

To establish the claim for $j+1$ consider the commutative diagram with exact columns

$$\begin{array}{ccc}
(0) & & (0) \\
\downarrow & & \downarrow \\
H^0(I_{F_{\widetilde{v}}}, \mathrm{ad}\,\mathrm{gr}^j_{\widetilde{v}}\overline{r}) & \stackrel{\sim}{\longrightarrow} & H^0(I_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}^j_{\widetilde{v}}\overline{r}, \overline{r}/\overline{\mathrm{Fil}}^{j+1}_{\widetilde{v}}\overline{r})) \\
\downarrow & & \downarrow \\
H^0(I_{F_{\widetilde{v}}}, \mathrm{gr}^0_{\widetilde{v}}\mathrm{ad}\,(\overline{r}/\overline{\mathrm{Fil}}^{j+1}_{\widetilde{v}}\overline{r})) & \longrightarrow & H^0(I_{F_{\widetilde{v}}}, \mathrm{ad}\,(\overline{r}/\overline{\mathrm{Fil}}^{j+1}_{\widetilde{v}}\overline{r})/\mathrm{Fil}^1_{\widetilde{v}}\mathrm{ad}\,(\overline{r}/\overline{\mathrm{Fil}}^{j+1}_{\widetilde{v}}\overline{r})) \\
\downarrow & & \downarrow \\
H^0(I_{F_{\widetilde{v}}}, \mathrm{gr}^0_{\widetilde{v}}\mathrm{ad}\,(\overline{r}/\overline{\mathrm{Fil}}^{j}_{\widetilde{v}}\overline{r})) & \stackrel{\sim}{\longrightarrow} & H^0(I_{F_{\widetilde{v}}}, \mathrm{ad}\,(\overline{r}/\overline{\mathrm{Fil}}^{j}_{\widetilde{v}}\overline{r})/\mathrm{Fil}^1_{\widetilde{v}}\mathrm{ad}\,(\overline{r}/\overline{\mathrm{Fil}}^{j}_{\widetilde{v}}\overline{r})) \\
\downarrow & & \\
(0). & &
\end{array}$$

(The top horizontal arrow is an isomorphism by the definition of $\overline{\mathrm{Fil}}^j_{\widetilde{v}}$, and the bottom horizontal arrow is an isomorphism by the inductive hypothesis.) $\Box$

For example if $\overline{r}$ is unramified at $\widetilde{v}$ then $\mathcal{D}_{\widetilde{v}}$ consists of all unramified lifts.

1.3.5 Discrete series deformations

Suppose that $m|n$, that there is a representation $\widetilde{r}_{\widetilde{v}} : G_{F_{\widetilde{v}}} \to GL_{n/m}(\mathcal{O})$ with $\widetilde{r}_{\widetilde{v}} \otimes k$ absolutely irreducible. Note that if $R \twoheadrightarrow S$ is a surjection in $\mathcal{C}_{\mathcal{O}}$ then

$$Z_{1+M_{n/m}(\mathfrak{m}_R)}(\widetilde{r}_{\widetilde{v}}(G_{F_{\widetilde{v}}})) = 1 + \mathfrak{m}_R \twoheadrightarrow 1 + \mathfrak{m}_S = Z_{1+M_{n/m}(\mathfrak{m}_S)}(\widetilde{r}_{\widetilde{v}}(G_{F_{\widetilde{v}}})).$$

Lemma 1.3.14 *If $R \twoheadrightarrow S$ is a surjection in $\mathcal{C}_{\mathcal{O}}$ then*

$$Z_{1+M_{n/m}(\mathfrak{m}_R)}(\widetilde{r}_{\widetilde{v}}(I_{F_{\widetilde{v}}})) \twoheadrightarrow Z_{1+M_{n/m}(\mathfrak{m}_S)}(\widetilde{r}_{\widetilde{v}}(I_{F_{\widetilde{v}}})),$$

and both groups are abelian.

Proof: It suffices to prove that

$$Z_{M_{n/m}(R)}(\widetilde{r}_{\widetilde{v}}(I_{F_{\widetilde{v}}})) \twoheadrightarrow Z_{M_{n/m}(S)}(\widetilde{r}_{\widetilde{v}}(I_{F_{\widetilde{v}}})).$$

As these modules are defined linearly and because $W(\overline{k})$ is free over $W(k)$ it suffices to prove this after tensoring with $W(\overline{k})$. Thus we may assume that k is algebraicly closed.

Let $\overline{r}_1$ be an irreducible constituent of $\widetilde{r}_{\widetilde{v}}|_{I_{F_{\widetilde{v}}}} \otimes k$ and let $\overline{r}_1'$ denote the $\overline{r}_1$-isotypic component of $\widetilde{r}_{\widetilde{v}}|_{I_{F_{\widetilde{v}}}} \otimes k$. Let $H \subset G_{F_{\widetilde{v}}}$ denote the group of $\sigma \in G_{F_{\widetilde{v}}}$ such that $\overline{r}_1^\sigma \cong \overline{r}_1$. Thus $\overline{r}_1'$ is an H-module and $\mathrm{Ind}_H^{G_{F_{\widetilde{v}}}} \overline{r}_1' \hookrightarrow \widetilde{r}_{\widetilde{v}}|_{I_{F_{\widetilde{v}}}} \otimes k$. Because $H/I_{F_{\widetilde{v}}}$ is pro-cyclic we can extend $\overline{r}_1$ to a representation of H and we get

$$\overline{r}_1' = \overline{r}_1 \otimes \mathrm{Hom}_{I_{F_{\widetilde{v}}}}(\overline{r}_1, \overline{r}_1')$$

as H-modules. Because $\widetilde{r}_{\widetilde{v}} \otimes k$ is an irreducible $G_{F_{\widetilde{v}}}$-module we see that $\mathrm{Hom}_{I_{F_{\widetilde{v}}}}(\overline{r}_1, \overline{r}_1')$ must be an irreducible $H/I_{F_{\widetilde{v}}}$-module and hence one dimensional. Twisting $\overline{r}_1$ by a character of $H/I_{F_{\widetilde{v}}}$ we may assume that

$$\widetilde{r}_{\widetilde{v}} \otimes k = \mathrm{Ind}_H^{G_{F_{\widetilde{v}}}} \overline{r}_1$$

where $\overline{r}_1|_{I_{F_{\widetilde{v}}}}$ is irreducible. Thus

$$\widetilde{r}_{\widetilde{v}}|_{I_{F_{\widetilde{v}}}} \otimes k = \overline{r}_1 \oplus ... \oplus \overline{r}_s$$

where each $\overline{r}_i$ is irreducible and where $\overline{r}_i \not\cong \overline{r}_j$ if $i \neq j$.

We claim that $\widetilde{r}_{\widetilde{v}}|_{I_{F_{\widetilde{v}}}} = r_1 \oplus ... \oplus r_s$ where r_i is a lifting of $\overline{r}_i$. We prove this modulo λ^t by induction on t, the case $t = 1$ being immediate. So suppose this is true modulo λ^t. As $I_{F_{\widetilde{v}}}$ has cohomological dimension 1 we see that we may lift r_i to a continuous representation $r_i' : I_{F_{\widetilde{v}}} \to GL_{\dim \overline{r}_1}(\mathcal{O}/\lambda^{t+1})$. Then $\widetilde{r}_{\widetilde{v}}|_{I_{F_{\widetilde{v}}}} \bmod \lambda^{t+1}$ differs from $r_1' \oplus ... \oplus r_s'$ by an element of

$$H^1(I_{F_{\widetilde{v}}}, \mathrm{ad}\, \widetilde{r}_{\widetilde{v}} \otimes k) = \bigoplus_{i,j} H^1(I_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{r}_i, \overline{r}_j)).$$

For $i \neq j$ we have $\mathrm{Hom}\,(\overline{r}_i, \overline{r}_j)_{I_{F_{\widetilde{v}}}} = (0)$ so

$$H^1(I_{F_{\widetilde{v}}}, \mathrm{ad}\, \widetilde{r}_{\widetilde{v}} \otimes k) = \bigoplus_{i} H^1(I_{F_{\widetilde{v}}}, \mathrm{ad}\, \overline{r}_i).$$

Hence $\widetilde{r}_{\widetilde{v}}|_{I_{F_{\widetilde{v}}}} \bmod \lambda^{t+1} = r_1 \oplus ... \oplus r_s$, as desired.

We deduce (from lemma 1.1.7) that

$$Z_{M_{n/m}(R)}(\widetilde{r}_{\widetilde{v}}(I_{F_{\widetilde{v}}})) = R^s$$

and the lemma follows. $\Box$

Note that $(1+M_{n/m}(\mathfrak{m}_R))$-conjugacy classes of lifts to R of $\widetilde{r}_{\widetilde{v}} \otimes k$ correspond to

$$Z_{1+M_{n/m}(\mathfrak{m}_R)}(\widetilde{r}_{\widetilde{v}}(I_{F_{\widetilde{v}}})))/\sim$$

where $z \sim z'$ if and only if

$$z' = \widetilde{r}_{\widetilde{v}}(\mathrm{Frob}_{\widetilde{v}})^{-1} w \widetilde{r}_{\widetilde{v}}(\mathrm{Frob}_{\widetilde{v}}) z' w^{-1}$$

for some $w \in Z_{1+M_{n/m}(\mathfrak{m}_R)}(\widetilde{r}_{\widetilde{v}}(I_{F_{\widetilde{v}}}))$. This correspondence sends z to the lift

$$g \longmapsto \widetilde{r}_{\widetilde{v}}(g) z^{\mathrm{val}_l g},$$

where $\mathrm{val}_l : G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}} \twoheadrightarrow \mathbb{Z}_l$ sends $\mathrm{Frob}_{\widetilde{v}}$ to 1.

Suppose that there is a filtration $\overline{\mathrm{Fil}}^i_{\widetilde{v}}$ of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ and an isomorphism

$$\kappa_{\widetilde{v}} : (\widetilde{r}_{\widetilde{v}} \otimes k)|_{I_{F_{\widetilde{v}}}} \xrightarrow{\sim} (\overline{\mathrm{gr}}^0_{\widetilde{v}} \overline{r})|_{I_{F_{\widetilde{v}}}}$$

such that

$$\overline{\mathrm{gr}}^i_{\widetilde{v}} \overline{r} \cong (\overline{\mathrm{gr}}^0_{\widetilde{v}} \overline{r})(\epsilon^i)$$

for $i = 1, ..., m-1$. Suppose moreover that for $j = 0, ..., m-2$ we have

$$\mathrm{Hom}_{k[G_{F_{\widetilde{v}}}]}(\overline{\mathrm{Fil}}^j_{\widetilde{v}} \overline{r}, \overline{\mathrm{gr}}^j_{\widetilde{v}} \overline{r}) = k.$$

(This will be true if for instance

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\, \widetilde{r}_{\widetilde{v}} \otimes k(\epsilon^i)) = (0)$$

for $i = 1-m, ..., -1$.) In this case an easy induction on j shows that $\overline{\mathrm{Fil}}^j_{\widetilde{v}}$ is determined uniquely by the isomorphism class of $\overline{\mathrm{gr}}^0_{\widetilde{v}}\overline{r}$ as a $k[G_{F_{\widetilde{v}}}]$-module.

Let $\mathcal{D}_{\widetilde{v}}$ be the set of all liftings r of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ such that r has a filtration $\mathrm{Fil}^i_{\widetilde{v}}$ by direct summands which lifts $\overline{\mathrm{Fil}}^i_{\widetilde{v}}$ and satisfies

- $\kappa_{\widetilde{v}}$ lifts to an isomorphism

$$(\widetilde{r}_{\widetilde{v}} \otimes R)|_{I_{F_{\widetilde{v}}}} \xrightarrow{\sim} (\mathrm{gr}^0_{\widetilde{v}} r)|_{I_{F_{\widetilde{v}}}}$$

- and,

$$\mathrm{gr}^i_{\widetilde{v}} r \cong (\mathrm{gr}^0_{\widetilde{v}} r)(\epsilon^i)$$

  for $i = 1, ..., m-1$.

If such a filtration $\mathrm{Fil}^j_{\widetilde{v}}$ exists it is unique. (To see this one can reduce to the case that R is Artinian and then argue by induction on the length of R. Thus suppose that R is Artinian and I is an ideal of R of length 1. Suppose the filtration on $r \otimes_R R/I$ is unique. Any other such filtration is of the form $(1_n + h)\mathrm{Fil}^j_{\widetilde{v}}$ where $h \in M_n(I) \cong \mathrm{ad}\,\overline{r}$ has image in $\mathrm{ad}\,\overline{r}/\overline{\mathrm{Fil}}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}$ fixed by $G_{F_{\widetilde{v}}}$. Thus h has an image in

$$H^1(G_{F_{\widetilde{v}}}, \overline{\mathrm{Fil}}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \longrightarrow H^1(G_{F_{\widetilde{v}}}, \overline{\mathrm{gr}}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}) = \bigoplus_{i=0}^{m-1} H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{\mathrm{gr}}^i_{\widetilde{v}}\overline{r}.$$

Note that $H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{\mathrm{gr}}^0_{\widetilde{v}}\overline{r})^{I_{F_{\widetilde{v}}}}) = H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k1_{n/m})$. Thus we require the image of h in $H^1(G_{F_{\widetilde{v}}}, \overline{\mathrm{gr}}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})$ to lie in $H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k1_n)$. Altering h by an element of $\mathrm{Fil}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}$ does not change $(1_n + h)\mathrm{Fil}^j$. Thus possible filtrations are parametrised by elements of

$$\ker(H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}/\overline{\mathrm{Fil}}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \longrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{gr}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})/H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k1_n)).$$

As

$$H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k1_n) \hookrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}/\overline{\mathrm{Fil}}^1_{\widetilde{v}}\mathrm{ad}\,\overline{r})$$

we see that possible filtrations are actually parametrised by elements of

$$\ker(H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}/\overline{\mathrm{Fil}}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \longrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{gr}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})).$$

We prove by reverse induction on i that

$$\ker\left(H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{r}, \overline{\mathrm{Fil}}^i\overline{r})/\overline{\mathrm{Fil}}^0_{\widetilde{v}}\mathrm{Hom}\,(\overline{r}, \overline{\mathrm{Fil}}^i\overline{r})) \longrightarrow \bigoplus_{j=i}^{m-1} H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\mathrm{gr}^j_{\widetilde{v}}\overline{r})\right)$$

is trivial. For this consider the commutative diagram with left column exact at the centre:

$$\begin{array}{ccc} H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{r}, \overline{\mathrm{Fil}}^i\overline{r})/\overline{\mathrm{Fil}}^0_{\widetilde{v}}\mathrm{Hom}\,(\overline{r}, \overline{\mathrm{Fil}}^i\overline{r})) & \longrightarrow & \bigoplus_{j=i}^{m-1} H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\mathrm{gr}^{\,j}_{\widetilde{v}}\overline{r}) \\ \downarrow & & \downarrow \\ H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{r}, \overline{\mathrm{Fil}}^{i-1}\overline{r})/\overline{\mathrm{Fil}}^0_{\widetilde{v}}\mathrm{Hom}\,(\overline{r}, \overline{\mathrm{Fil}}^{i-1}\overline{r})) & \longrightarrow & \bigoplus_{j=i-1}^{m-1} H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\mathrm{gr}^{\,j}_{\widetilde{v}}\overline{r}) \\ \downarrow & & \downarrow \\ H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{\mathrm{Fil}}^i\overline{r}, \overline{\mathrm{gr}}^{i-1}\overline{r})) & \longrightarrow & H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\mathrm{gr}^{\,i}_{\widetilde{v}}\overline{r}). \end{array}$$

The injectivity of the last horizontal arrow follows from our assumption that $\mathrm{Hom}_{\,k[G_{F_{\widetilde{v}}}]}(\overline{\mathrm{Fil}}^{i-1}_{\widetilde{v}}\overline{r}, \overline{\mathrm{gr}}^{i-1}_{\widetilde{v}}\overline{r}) = k$.) It follows from this and from lemma 1.3.14 that $\mathcal{D}_{\widetilde{v}}$ is a local deformation problem.

Note that

$$\# H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{\mathrm{gr}}^0_{\widetilde{v}}\overline{r})^{I_{F_{\widetilde{v}}}}) = \# H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{\mathrm{gr}}^0_{\widetilde{v}}\overline{r}) = \# k$$

and so

$$H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{\mathrm{gr}}^0_{\widetilde{v}}\overline{r})^{I_{F_{\widetilde{v}}}}) = H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k1_{n/m}).$$

Let

$$\alpha_i : \overline{\mathrm{gr}}^i_{\widetilde{v}}\overline{r} \stackrel{\sim}{\longrightarrow} (\overline{\mathrm{gr}}^0_{\widetilde{v}}\overline{r})(\epsilon^i).$$

This map is unique up to scalars. We see that

$$\left(\sum_{i=0}^{m-1} \mathrm{ad}\,(\alpha_i) \right) H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{\mathrm{gr}}^0_{\widetilde{v}}\overline{r})^{I_{F_{\widetilde{v}}}}) = H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k1_n).$$

From this it is not hard to see that we may take $L_{\widetilde{v}} = L_{\widetilde{v}}(\mathcal{D}_{\widetilde{v}})$ to be the kernel of the map

$$H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) \longrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}/\overline{\mathrm{Fil}}^1_{\widetilde{v}}\mathrm{ad}\,\overline{r})/H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k1_n).$$

Lemma 1.3.15 *Recall our assumption that*

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{\mathrm{Fil}}^j_{\widetilde{v}}\overline{r}, \overline{\mathrm{gr}}^j_{\widetilde{v}}\overline{r})) = k$$

for $j = 0, ..., m-2$. *Then* $L_{\widetilde{v}}$ *is minimal and* $\mathcal{D}_{\widetilde{v}}$ *is liftable.*

Proof: We see that

$$\dim L_{\widetilde{v}} = 1 + \dim\ker(H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) \longrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}/\overline{\mathrm{Fil}}^1_{\widetilde{v}}\mathrm{ad}\,\overline{r})).$$

(Because $H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k1_n)$ is a subspace of $H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})$ of dimension 1.) Hence

$$\dim L_{\widetilde{v}} - \dim H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) = 1 + \dim H^1(G_{F_{\widetilde{v}}}, \overline{\mathrm{Fil}}^1_{\widetilde{v}}\mathrm{ad}\,\overline{r}) - \\ - \dim H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r})/\overline{\mathrm{Fil}}^1_{\widetilde{v}}\mathrm{ad}\,\overline{r}) - \dim H^0(G_{F_{\widetilde{v}}}, \overline{\mathrm{Fil}}^1_{\widetilde{v}}\mathrm{ad}\,\overline{r}).$$

Applying the local Euler characteristic formula this becomes

$$\dim L_{\widetilde{v}} - \dim H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) =$$
$$1 + \dim H^0(G_{F_{\widetilde{v}}}, ((\mathrm{ad}\,\overline{r})/(\overline{\mathrm{Fil}}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}))(1)) - \dim H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r})/(\overline{\mathrm{Fil}}^1_{\widetilde{v}}\mathrm{ad}\,\overline{r})).$$

From the exact sequence

$$(0) \to ((\mathrm{ad}\,\overline{r})/(\overline{\mathrm{Fil}}^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}))(1) \to (\mathrm{ad}\,\overline{r})/(\overline{\mathrm{Fil}}^1_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \to \mathrm{Hom}\,(\overline{r}, \overline{\mathrm{gr}}^0_{\widetilde{v}}\overline{r}) \to (0)$$

we see that $\dim L_{\widetilde{v}} - \dim H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})$ equals the dimension of the cokernel

$$H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r})/(\overline{\mathrm{Fil}}^1_{\widetilde{v}}\mathrm{ad}\,\overline{r})) \longrightarrow H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{r}, \overline{\mathrm{gr}}^0_{\widetilde{v}}\overline{r})).$$

By assumption the latter group is k and is in the image of

$$H^0(G_{F_{\widetilde{v}}}, k1_n) \subset H^0(G_{F_{\widetilde{v}}}, (\mathrm{ad}\,\overline{r})/(\overline{\mathrm{Fil}}^1_{\widetilde{v}}\mathrm{ad}\,\overline{r})).$$

Thus the cokernel is trivial and $L_{\widetilde{v}}$ is minimal.

We finally turn to the liftability of $\mathcal{D}_{\widetilde{v}}$. Suppose that R is an object of $\mathcal{C}_{\mathcal{O}}$ and that I is a closed ideal of R with $\mathfrak{m}_R I = (0)$. Suppose also that r is a lifting of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ to R/I in $\mathcal{D}_{\widetilde{v}}$. Let $\{\mathrm{Fil}^i\}$ be the corresponding filtration of $(R/I)^n$. Choose a lifting gr^0 of $\mathrm{gr}^0 r$ to R such that the isomorphism of $R/I[I_{F_{\widetilde{v}}}]$-modules

$$\widetilde{r}_{\widetilde{v}} \otimes R/I \stackrel{\sim}{\longrightarrow} \mathrm{gr}^0 r$$

lifts to an isomorphism of $R[I_{F_{\widetilde{v}}}]$-modules

$$\widetilde{r}_{\widetilde{v}} \otimes R \stackrel{\sim}{\to} \mathrm{gr}^0.$$

We will show by reverse induction on i that $\mathrm{Fil}^i r$ is liftable to a free R module Fil^i with a filtration and $G_{F_{\widetilde{v}}}$-action such that for $j = i, ..., m-1$ there is an isomorphism $\mathrm{gr}^0(\epsilon^j) \stackrel{\sim}{\to} \mathrm{gr}^j \mathrm{Fil}^i$. This is certainly possible for $i = m-1$. Suppose it is true for $i+1$. It suffices to show that

$$H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R(\mathrm{gr}^0(\epsilon^i), \mathrm{Fil}^{i+1})) \twoheadrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R((\mathrm{gr}^0 r)(\epsilon^i), \mathrm{Fil}^{i+1} r)),$$

or equivalently that

$$H^2(G_{F_{\widetilde{v}}}, I\mathrm{Hom}_R(\mathrm{gr}^0(\epsilon^i), \mathrm{Fil}^{i+1})) \hookrightarrow H^2(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R(\mathrm{gr}^0(\epsilon^i), \mathrm{Fil}^{i+1})).$$

Dualising, this is equivalent to the surjectivity of the map

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R(\mathrm{Fil}^{i+1}, \mathrm{gr}^0(\epsilon^{i+1})) \otimes_R R^\vee) \to H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}_R(\mathrm{Fil}^{i+1}, \mathrm{gr}^0(\epsilon^{i+1})) \otimes_R I^\vee),$$

where $M^\vee$ denotes $\mathrm{Hom}\,(M, \mathbb{Q}_l/\mathbb{Z}_l)$. However

$$\begin{array}{rl} & H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,_R(\mathrm{Fil}\,^{i+1}, \mathrm{gr}\,^0(\epsilon^{i+1})) \otimes_R I^\vee) \\ = & H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,_k(\overline{\mathrm{Fil}}\,_{\widetilde{v}}^{i+1}\overline{r}, \mathrm{gr}\,_{\widetilde{v}}^0\overline{r}(\epsilon^{i+1}))) \otimes_k I^\vee \\ = & H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,_k(\mathrm{gr}\,_{\widetilde{v}}^{i+1}\overline{r}, \mathrm{gr}\,_{\widetilde{v}}^{i+1}\overline{r})) \otimes_k I^\vee \\ = & I^\vee. \end{array}$$

As the composite

$$\begin{array}{rcl} R^\vee & = & H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\mathrm{gr}\,^{i+1}, \mathrm{gr}\,^{i+1}) \otimes_R R^\vee \\ & \to & H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,_R(\mathrm{Fil}\,^{i+1}, \mathrm{gr}\,^0(\epsilon^{i+1})) \otimes_R R^\vee) \\ & \to & H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,_R(\mathrm{Fil}\,^{i+1}, \mathrm{gr}\,^0(\epsilon^{i+1})) \otimes_R I^\vee) \\ & = & I^\vee \end{array}$$

is surjective, it follows that the map

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,_R(\mathrm{Fil}\,^{i+1}, \mathrm{gr}\,^0(\epsilon^{i+1})) \otimes_R R^\vee) \to H^0(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,_R(\mathrm{Fil}\,^{i+1}, \mathrm{gr}\,^0(\epsilon^{i+1})) \otimes_R I^\vee),$$

is surjective and hence $\mathcal{D}_{\widetilde{v}}$ is liftable. □

1.3.6 Taylor-Wiles deformations

Suppose that $\mathbf{N}\widetilde{v} \equiv 1 \bmod l$, that $\overline{r}$ is unramified at $\widetilde{v}$ and that $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{\psi} \oplus \overline{s}$ where $\dim_k \overline{\psi} = 1$ and $\overline{s}$ does not contain $\overline{\psi}$ as a sub-quotient. Take $\mathcal{D}_{\widetilde{v}}$ to consist of all lifts of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ which are $(1 + M_n(\mathfrak{m}_R))$-conjugate to one of the form $\psi \oplus s$ where ψ lifts $\overline{\psi}$, and where s lifts $\overline{s}$ and is unramified. Then $\mathcal{D}_{\widetilde{v}}$ is a local deformation problem and

$$L_{\widetilde{v}} = L_{\widetilde{v}}(\mathcal{D}_{\widetilde{v}}) = H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{s}) \oplus H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{\psi}).$$

Note that in this case

$$\mathrm{lg}_{\mathcal{O}}\, L_{\widetilde{v}} - \mathrm{lg}_{\mathcal{O}}\, H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) = \mathrm{lg}_{\mathcal{O}}\, H^1(I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{\psi})^{G_{F_{\widetilde{v}}}} = 1.$$

We will write $\Delta_{\widetilde{v}}$ for the maximal l-power quotient of the inertia subgroup of $G_{F_{\widetilde{v}}}^{\mathrm{ab}}$. It is cyclic of order the maximal power of l dividing $\mathbf{N}\widetilde{v} - 1$. If r is any deformation of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ in $\mathcal{D}_{\widetilde{v}}$ over a ring R then $\det r : \Delta_{\widetilde{v}} \to R^\times$ and so R becomes an $\mathcal{O}[\Delta_{\widetilde{v}}]$-module. If $\mathfrak{a}_{\widetilde{v}}$ denotes the augmentation ideal of $\mathcal{O}[\Delta_{\widetilde{v}}]$ then $R/\mathfrak{a}_{\widetilde{v}}R$ is the maximal quotient of R over which r becomes unramified at $\widetilde{v}$.

1.3.7 Ramakrishna deformations

Suppose that $(\mathbf{N}\widetilde{v}) \not\equiv 1 \bmod l$ and that $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{\psi}\epsilon \oplus \overline{\psi} \oplus \overline{s}$, where $\overline{\psi}$ and $\overline{s}$ are unramified and $\overline{s}$ contains neither $\overline{\psi}$ nor $\overline{\psi}\epsilon$ as a sub-quotient. Take $\mathcal{D}_{\widetilde{v}}$ to consist of the set of lifts of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ which are $(1 + M_n(\mathfrak{m}_R))$-conjugate to a lift of the form

$$\begin{pmatrix} \psi\epsilon & * & 0 \\ 0 & \psi & 0 \\ 0 & 0 & s \end{pmatrix}$$

with ψ an unramified lift of $\overline{\psi}$ and s an unramified lift of $\overline{s}$. Then $\mathcal{D}_{\widetilde{v}}$ is a local deformation problem and $L_{\widetilde{v}} = L_{\widetilde{v}}(\mathcal{D}_{\widetilde{v}})$ is

$$H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k \begin{pmatrix} 1_2 & 0 \\ 0 & 0 \end{pmatrix}) \oplus H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{\psi}, \overline{\psi}\epsilon)) \oplus H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{s}).$$

Then $\dim_k L_{\widetilde{v}} = 2 + \dim_k H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{s}) = 2 + \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{s}) = \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})$. Thus $L_{\widetilde{v}}$ is minimal. Moreover $\mathcal{D}_{\widetilde{v}}$ is liftable. (Because if R is an object of $\mathcal{C}_{\mathcal{O}}$ and if I is a closed ideal of R then

$$H^1(G_{F_{\widetilde{v}}}, R(\epsilon)) \twoheadrightarrow H^1(G_{F_{\widetilde{v}}}, (R/I)(\epsilon)).)$$

1.3.8 One more deformation

Suppose again that $(\mathbf{N}\widetilde{v}) \not\equiv 1 \bmod l$ and that $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{\psi}\epsilon \oplus \overline{\psi} \oplus \overline{s}$, where $\overline{\psi}$ and $\overline{s}$ are unramified and $\overline{s}$ contains neither $\overline{\psi}$ nor $\overline{\psi}\epsilon$ as a sub-quotient. Take $\mathcal{D}_{\widetilde{v}}$ to consist of the set of lifts of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ which are $(1 + M_n(\mathfrak{m}_R))$-conjugate to a lift of the form

$$\begin{pmatrix} \psi_1 & * & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & s \end{pmatrix}$$

with ψ_1 (resp. ψ_2) an unramified lift of $\overline{\psi}\epsilon$ (resp. $\overline{\psi}$) and s an unramified lift of $\overline{s}$. Note that $\mathcal{D}_{\widetilde{v}}$ includes all unramified lifts and all Ramakrishna lifts (see section 1.3.7). It is a local deformation problem and $L_{\widetilde{v}} = L_{\widetilde{v}}(\mathcal{D}_{\widetilde{v}})$ is

$$H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{\psi}\epsilon, \overline{\psi}\epsilon) \oplus \mathrm{Hom}\,(\overline{\psi}, \overline{\psi})) \oplus H^1(G_{F_{\widetilde{v}}}, \mathrm{Hom}\,(\overline{\psi}, \overline{\psi}\epsilon)) \oplus \\ \oplus H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{s}).$$

Then $\dim_k L_{\widetilde{v}} = 3 + \dim_k H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{s}) = 3 + \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{s}) = 1 + \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})$.

The next lemma is immediate.

Lemma 1.3.16 *Suppose that*

$$\mathcal{S} = (G_{F^+,S} \supset G_{F,S}, S \supset S_0, \{G_{F_{\tilde{v}}}\}_{v\in S}, \mathcal{O}, \overline{r}, \chi, \{\overline{\mathrm{Fil}}^i_{\tilde{v}}\}, \{\mathcal{D}_{\tilde{v}}\}, \{L_{\tilde{v}}\})$$

is a deformation problem as above. Suppose that $S' \supset S$ is a finite set of primes of F^+ which split in F and choose a set $\widetilde{S}' \supset \widetilde{S}$ consisting of one prime of F above each prime of S'. Define a deformation problem

$$\mathcal{S}' = (G_{F^+,S'} \supset G_{F,S'}, S' \supset S_0, \{G_{F_{\tilde{v}}}\}_{v\in S'}, \mathcal{O}, \overline{r}, \chi, \{\overline{\mathrm{Fil}}^i_{\tilde{v}}\}, \{\mathcal{D}_{\tilde{v}}\}, \{L_{\tilde{v}}\}),$$

where, for $v \in S$ the $\{\overline{\mathrm{Fil}}^i_{\tilde{v}}\}$, $\mathcal{D}_{\tilde{v}}$ and $L_{\tilde{v}}$ are as in $\mathcal{S}$, and for $v \notin S$ the set $\mathcal{D}_{\tilde{v}}$ consists of all unramified deformations and $L_{\tilde{v}} = H^1(G_{F_{\tilde{v}}}/I_{F_{\tilde{v}}}, \mathrm{ad}\,\overline{r})$. Then $\mathrm{Def}_{\mathcal{S}}$ *is naturally isomorphic to* $\mathrm{Def}_{\mathcal{S}'}$ *and in particular $R^{\mathrm{univ}}_{\mathcal{S}} = R^{\mathrm{univ}}_{\mathcal{S}'}$.*

Lemma 1.3.17 *Suppose that*

$$\mathcal{S} = (G_{F^+,S} \supset G_{F,S}, S \supset S_0, \{G_{F_{\tilde{v}}}\}_{v\in S}, \mathcal{O}, \overline{r}, \chi, \{\overline{\mathrm{Fil}}^i_{\tilde{v}}\}, \{\mathcal{D}_{\tilde{v}}\}, \{L_{\tilde{v}}\})$$

is a deformation problem as above. Suppose that $R \subset S-(S_0\cup S_l)$ only contains primes v for which $\overline{r}$ is unramified at v and $\mathcal{D}_{\tilde{v}}$ consists of all unramified lifts of $\overline{r}|_{G_{F_{\tilde{v}}}}$. Define a new deformation problem

$$\mathcal{S}' = (G_{F^+,S} \supset G_{F,S}, S \supset S_0, \{G_{F_{\tilde{v}}}\}_{v\in S}, \mathcal{O}, \overline{r}, \chi, \{\overline{\mathrm{Fil}}^i_{\tilde{v}}\}, \{\mathcal{D}'_{\tilde{v}}\}, \{L'_{\tilde{v}}\}),$$

where

- *for $v \in S - R$, $\mathcal{D}'_{\tilde{v}} = \mathcal{D}_{\tilde{v}}$ and $L'_{\tilde{v}} = L_{\tilde{v}}$, and*
- *for $v \in R$, $\mathcal{D}'_{\tilde{v}}$ consists of all deformations of $\overline{r}|_{G_{F_{\tilde{v}}}}$ and $L'_{\tilde{v}} = H^1(G_{F_{\tilde{v}}}, \mathrm{ad}\,\overline{r})$.*

Suppose that $\phi : R^{\mathrm{univ}}_{\mathcal{S}} \to \mathcal{O}$ and let ϕ_R denote the composite of ϕ with the natural map $R^{\mathrm{univ}}_{\mathcal{S}'} \twoheadrightarrow R^{\mathrm{univ}}_{\mathcal{S}}$. Also let r_ϕ denote $\phi(r^{\mathrm{univ}}_{\mathcal{S}})$. Then

$$\mathrm{lg}_{\mathcal{O}} \ker \phi_R/(\ker \phi_R)^2 \leq \mathrm{lg}_{\mathcal{O}} \ker \phi/(\ker \phi)^2 + \sum_{v\in R} \mathrm{lg}_{\mathcal{O}}\, H^0(G_{F_{\tilde{v}}}, (\mathrm{ad}\, r_\phi)(\epsilon^{-1})).$$

Proof: As described at the end of section 1.2 a class $[\psi] \in H^1_{\mathcal{S}'}(G_{F^+,S}, \mathrm{ad}\, r_\phi \otimes \lambda^{-N}/\mathcal{O})$ corresponds to a deformation $(1+\psi\epsilon)r_\phi$ of r_ϕ mod λ^N. This deformation corresponds to an element of $H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\, r_\phi \otimes \lambda^{-N}/\mathcal{O})$ if and only if $(1+\psi\epsilon)r_\phi$ is unramified at all $v \in R$ if and only if $\psi(I_{F_{\tilde{v}}}) = 0$ for all $v \in R$. Note that, for $v \in R$, we have

$$\begin{aligned} H^1(I_{F_{\tilde{v}}}, \mathrm{ad}\, r_\phi \otimes_{\mathcal{O}} \lambda^{-N}/\mathcal{O}) &= \mathrm{Hom}\,(I_{F_{\tilde{v}}}, \mathrm{ad}\, r_\phi \otimes_{\mathcal{O}} \lambda^{-N}/\mathcal{O}) \\ &= (\mathrm{ad}\, r_\phi) \otimes_{\mathcal{O}} \lambda^{-N}/\mathcal{O}(\epsilon^{-1}). \end{aligned}$$

Thus we have an exact sequence

$$\begin{aligned}&(0) \longrightarrow H^1_{\mathcal{S}}(G_{F^+,S}, \operatorname{ad} r_\phi \otimes \lambda^{-N}/\mathcal{O}) \longrightarrow H^1_{\mathcal{S}'}(G_{F^+,S}, \operatorname{ad} r_\phi \otimes \lambda^{-N}/\mathcal{O}) \longrightarrow \\ &\longrightarrow \bigoplus_{v\in R} H^0(G_{F_{\tilde{v}}}, (\operatorname{ad} r_\phi) \otimes_{\mathcal{O}} \lambda^{-N}/\mathcal{O}(\epsilon^{-1})).\end{aligned}$$

Taking a direct limit and applying lemma 1.2.7 we then get an exact sequence

$$\begin{aligned}&(0) \longrightarrow \operatorname{Hom}(\ker \phi/(\ker \phi)^2, K/\mathcal{O}) \longrightarrow \operatorname{Hom}(\ker \phi_R/(\ker \phi_R)^2, K/\mathcal{O}) \longrightarrow \\ &\longrightarrow \bigoplus_{v\in R} H^0(G_{F_{\tilde{v}}}, (\operatorname{ad} r_\phi) \otimes_{\mathcal{O}} K/\mathcal{O}(\epsilon^{-1}))\end{aligned}$$

and the lemma follows. □

1.4 Galois theory.

We will keep the notation and assumptions of the last section.

We will start with a lemma from algebraic number theory, which may be standard but for which we do not know a reference.

Lemma 1.4.1 *Let E/F be a Galois extension of number fields. Let S be a finite set of finite places of F and let $E(S)/E$ be the maximal extension unramified outside S. Thus $E(S)/F$ is Galois. Let M be a continuous* $\operatorname{Gal}(E(S)/F)$*-module of finite cardinality coprime to $[E : F]$. Suppose that S contains all finite places v such that $v|\#M$. Then*

$$\frac{\#H^1(\operatorname{Gal}(E(S)/F), M)}{\#H^0(\operatorname{Gal}(E(S)/F), M)\#H^2(\operatorname{Gal}(E(S)/F), M)} \prod_{v|\infty} \#H^0(\operatorname{Gal}(\overline{F}_v/F_v), M)$$

equals $(\#M)^{[F:\mathbb{Q}]}$.

Proof: This is proved in exactly the same way as the usual global Euler characteristic formula.

Firstly one shows that if there is a short exact sequence

$$(0) \to M_1 \to M_2 \to M_3 \to (0)$$

and the theorem is true for two of the terms, then it is also true for the third. To do this one considers the long exact sequences with $H^i(\operatorname{Gal}(E(S)/F),\ )$ and $H^i(\operatorname{Gal}(\overline{F}_v/F_v),\ )$. The key point is that

$$\operatorname{coker}(H^2(\operatorname{Gal}(E(S)/F), M_2) \longrightarrow H^2(\operatorname{Gal}(E(S)/F), M_3))$$

is isomorphic to

$$\operatorname{coker}\,(\bigoplus_{v|\infty} H^0(\mathrm{Gal}\,(\overline{F}_v/F_v), M_2) \longrightarrow \bigoplus_{v|\infty} H^0(\mathrm{Gal}\,(\overline{F}_v/F_v), M_3)).$$

This follows from the equalities

$$\begin{array}{rcl} H^3(\mathrm{Gal}\,(E(S)/F), M_i) & = & H^3(\mathrm{Gal}\,(E(S)/E), M_i)^{\mathrm{Gal}\,(E/F)} \\ & \cong & (\bigoplus_{w|\infty} H^1(\mathrm{Gal}\,(\overline{E}_w/E_w), M_i))^{\mathrm{Gal}\,(E/F)} \\ & = & \bigoplus_{v|\infty} H^1(\mathrm{Gal}\,(\overline{F}_v/F_v), M_i). \end{array}$$

Thus we are reduced to the case that M is an $\mathbb{F}_l$-module for some prime $l \not| [E : F]$.

Next choose a subfield L of $E(S)$ which contains $E(\zeta_l)$, which is totally imaginary and which is finite, Galois over F. Suppose that M is a $\mathrm{Gal}\,(L/F)$-module. Let $L \supset K \supset F$ and let $R_{\mathbb{F}_l}(\mathrm{Gal}\,(L/K))$ denote the representation ring for $\mathrm{Gal}\,(L/K)$ acting on finite dimensional $\mathbb{F}_l$-vector spaces. Define a homomorphism

$$\chi_K : R_{\mathbb{F}_l}(\mathrm{Gal}\,(L/K)) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}$$

by

$$\begin{array}{rcl} \chi[M] & = & \dim H^1(\mathrm{Gal}\,(E(S)/K), M) - \dim H^0(\mathrm{Gal}\,(E(S)/K), M) \\ & & - \dim H^2(\mathrm{Gal}\,(E(S)/K), M) + \sum_{v|\infty} H^0(\mathrm{Gal}\,(\overline{K}_v/K_v), M). \end{array}$$

This is well defined by the observation of the previous paragraph. We need to show that

$$\chi_F = [F : \mathbb{Q}] \dim .$$

It is easy to check that

$$\chi_F \circ \mathrm{Ind}_{\mathrm{Gal}\,(L/K)}^{\mathrm{Gal}\,(L/F)} = \chi_K.$$

As $R_{\mathbb{F}_l}(\mathrm{Gal}\,(L/F)) \otimes \mathbb{Q}$ is spanned by $\mathrm{Ind}_{\mathrm{Gal}\,(L/K)}^{\mathrm{Gal}\,(L/F)} R_{\mathbb{F}_l}(\mathrm{Gal}\,(L/K))$ as K runs over intermediate fields with L/K cyclic of degree prime to l, it suffices to prove that $\chi_K = [K : \mathbb{Q}] \dim$ when K is an intermediate field with L/K cyclic of degree prime to l.

Now assume that $L \supset K \supset F$ with L/K cyclic of degree prime to l. Define

$$\widetilde{\chi}_K : R_{\mathbb{F}_l}(\mathrm{Gal}\,(L/K)) \longrightarrow R_{\mathbb{F}_l}(\mathrm{Gal}\,(L/K))$$

by

$$\begin{array}{rcl} \widetilde{\chi}_K[M] & = & \sum_{v|\infty} [M \otimes \mathrm{Ind}_{\mathrm{Gal}\,(L_w/K_v)}^{\mathrm{Gal}\,(L/K)} \mathbb{F}_l] + [H^1(\mathrm{Gal}\,(E(S)/L), M)] \\ & & -[H^0(\mathrm{Gal}\,(E(S)/L), M)] - [H^2(\mathrm{Gal}\,(E(S)/L), M)], \end{array}$$

where w denotes a place of L above v. This is well defined because L totally imaginary implies $H^3(\mathrm{Gal}\,(E(S)/L), M) = (0)$. Note that $\widetilde{\chi}_K([M]) = [M(-1)] \otimes \widetilde{\chi}_K([\mu_l])$. Moreover as $l \not| [L:K]$ we see that

$$\chi_K = H^0(\mathrm{Gal}\,(L/K), \ \ ) \circ \widetilde{\chi}_K,$$

so that

$$\chi_k([M]) = H^0(\mathrm{Gal}\,(L/K), [M(-1)] \otimes \widetilde{\chi}_K([\mu_l])).$$

Thus it suffices to prove that

$$\widetilde{\chi}_K([\mu_l]) = [K:\mathbb{Q}][\mathrm{Ind}\,_{\{1\}}^{\mathrm{Gal}\,(L/K)}\mathbb{F}_l].$$

As $E(S)$ is the maximal extension of L unramified outside S one has the standard formulae

$$[H^0(\mathrm{Gal}\,(E(S)/L), \mu_l)] = [\mu_l]$$

and

$$[H^1(\mathrm{Gal}\,(E(S)/L), \mu_l)] = [\mathcal{O}_L[1/S]^\times \otimes \mathbb{F}_l] + [\mathrm{Cl}_S(L)[l]]$$

and

$$[H^2(\mathrm{Gal}\,(E(S)/L), \mu_l)] = [\mathrm{Cl}_S(L) \otimes \mathbb{F}_l] - [\mathbb{F}_l] + \sum_{v \in S}[\bigoplus_{w|v} \mathrm{Br}\,(L_w)[l]],$$

where $\mathcal{L}_S(L)$ denotes the S-class group of L (i.e. the quotient of the class group by classes of ideals supported over S) and $\mathrm{Br}\,(L_w)$ denotes the Brauer group of L_w. Using these formulae the proof is easily completed, just as in the case of the usual global Euler characteristic formula. $\Box$

We will write $L_{\widetilde{v}}^\perp$ for the annihilator in $H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}(1))$ of the image of $L_{\widetilde{v}}$ in $H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})$. We will also write $H^1_{\mathcal{L}^\perp}(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1))$ for the kernel of the map

$$H^1(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) \longrightarrow \bigoplus_{\widetilde{v} \in S} H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}(1))/L_{\widetilde{v}}^\perp.$$

Lemma 1.4.2 *1.* $H^i_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = (0)$ *unless* $i = 1, 2$ *or* 3.

2. $\dim_k H^3_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = \dim_k H^0(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1))$.

3. $\dim_k H^2_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = \dim_k H^1_{\mathcal{L}^\perp}(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1))$.

4.

$$\begin{array}{rl} & \dim_k H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) \\ = & \dim_k H^1_{\mathcal{L}^\perp}(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) - \dim_k H^0(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) \\ & -n\sum_{v|\infty}(\chi(c_v)+1)/2 + \sum_{\widetilde{v} \in \widetilde{S}-\widetilde{S}_l}(\dim_k L_{\widetilde{v}} - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})). \end{array}$$

Proof: For the first part we use the long exact sequences before lemma 1.2.2 and the vanishing of $H^i(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = H^i(G_{F,S}, \mathrm{ad}\,\overline{r})^{\mathrm{Gal}\,(F/F^+)}$ for $i \neq 1$ or 2.

For the second and third parts one compares the exact sequences

$$\begin{array}{ccccc} & & H^1(G_{F^+,S}, \mathrm{ad}\,\overline{r}) & \rightarrow & \bigoplus_{\widetilde{v}\in\widetilde{S}} H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})/L_{\widetilde{v}} \\ & & & & \downarrow \\ \bigoplus_{\widetilde{v}\in\widetilde{S}} H^2(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) & \leftarrow & H^2(G_{F^+,S}, \mathrm{ad}\,\overline{r}) & \leftarrow & H^2_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) \\ \downarrow & & & & \\ H^3_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) & \rightarrow & (0) & & \end{array}$$

and

$$\begin{array}{ccccc} & & H^1(G_{F^+,S}, \mathrm{ad}\,\overline{r}) & \rightarrow & \bigoplus_{\widetilde{v}\in\widetilde{S}} H^1(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})/L_{\widetilde{v}} \\ & & & & \downarrow \\ \bigoplus_{\widetilde{v}\in\widetilde{S}} H^2(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}) & \leftarrow & H^2(G_{F^+,S}, \mathrm{ad}\,\overline{r}) & \leftarrow & H^1_{\mathcal{L}^\perp}(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1))^\vee \\ \downarrow & & & & \\ H^0(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1))^\vee & \rightarrow & (0). & & \end{array}$$

(The latter exact sequence is a consequence of Poitou-Tate global duality and the identifications $H^i(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = H^i(G_{F,S}, \mathrm{ad}\,\overline{r})^{\mathrm{Gal}\,(F/F^+)}$ for $i = 1, 2$ and $H^i(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(1)) = H^i(G_{F,S}, (\mathrm{ad}\,\overline{r})(1))^{\mathrm{Gal}\,(F/F^+)}$ for $i = 0, 1$.)

For the fourth part we have the Euler characteristic formula

$$\begin{array}{cl} & \dim_k H^1(G_{F^+,S}, \mathrm{ad}\,\overline{r}) - \dim_k H^0(G_{F^+,S}, \mathrm{ad}\,\overline{r}) - \dim_k H^2(G_{F^+,S}, \mathrm{ad}\,\overline{r}) \\ = & n^2[F^+ : \mathbb{Q}] - \sum_{v|\infty} \dim_k H^0(G_{F_v^+}, \mathrm{ad}\,\overline{r}). \end{array}$$

(See lemma 1.4.1.) This, lemma 1.2.2, and the local Euler characteristic formulae tell us that

$$\begin{array}{l} \dim_k H^1_{\mathcal{S}}(G_{F^+}, \mathrm{ad}\,\overline{r}) - \dim_k H^2_{\mathcal{S}}(G_{F^+}, \mathrm{ad}\,\overline{r}) + \dim_k H^3_{\mathcal{S}}(G_{F^+}, \mathrm{ad}\,\overline{r}) = \\ \sum_{v|\infty}(n^2 - \dim_k H^0(G_{F_v^+}, \mathrm{ad}\,\overline{r})) - \sum_{\widetilde{v}\in\widetilde{S}_l} n^2[F_{\widetilde{v}} : \mathbb{Q}_l] \\ + \sum_{\widetilde{v}\in\widetilde{S}_l} n(n-1)[F_{\widetilde{v}} : \mathbb{Q}_l]/2 + \sum_{\widetilde{v}\in\widetilde{S}-\widetilde{S}_l}(\dim_k L_{\widetilde{v}} - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})). \end{array}$$

Lemma 1.1.2 tells us that for $v|\infty$

$$H^0(G_{F_v^+}, \mathrm{ad}\,\overline{r}) = n(n + (\nu \circ \overline{r})(c_v))/2 = n(n-1)/2 + n(1 + \chi(c_v))/2.$$

The fourth part of the lemma follows. □

Corollary 1.4.3 *Suppose that for $\widetilde{v} \in \widetilde{S}_l$ the deformation problem $\mathcal{D}_{\widetilde{v}}$ is as in section 1.3.1 or 1.3.2. Suppose that for all $\widetilde{v} \in \widetilde{S} - \widetilde{S}_l$ the set $\mathcal{D}_{\widetilde{v}}$ is liftable.*

Suppose also that $H^1_{\mathcal{L}^\perp}(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) = (0)$*. Then* $R^{\mathrm{univ}}_{\mathcal{S}}$ *is a power series ring over* $\mathcal{O}$ *in*

$$\begin{array}{l}\sum_{\widetilde{v}\in\widetilde{S}-\widetilde{S}_l}(\dim_k L_{\widetilde{v}} - \dim_k H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r})) \\ -\dim_k H^0(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) - n\sum_{v|\infty}(\chi(c_v)+1)/2\end{array}$$

variables.

Corollary 1.4.4 *Suppose that for* $\widetilde{v} \in \widetilde{S}_l$ *the deformation problem* $\mathcal{D}_{\widetilde{v}}$ *is as in section 1.3.1 or 1.3.2. Suppose that there is a subset* $Q \subset \widetilde{S} - \widetilde{S}_l$ *such that*

- *for* $\widetilde{v} \in \widetilde{S} - (\widetilde{S}_l \cup Q)$*,* $L_{\widetilde{v}}$ *is minimal, and*
- *for* $\widetilde{v} \in Q$ *the pair* $(\mathcal{D}_{\widetilde{v}}, L_{\widetilde{v}})$ *is as in example 1.3.6.*

Then

$$\begin{array}{l}\dim_k H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = \\ \#Q + \dim_k H^1_{\mathcal{L}^\perp}(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) - \dim_k H^0(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) \\ -n\sum_{v|\infty}(\chi(c_v)+1)/2.\end{array}$$

We will call a subgroup $H \subset \mathcal{G}_n(k)$ *big* if the following conditions are satisfied.

- $H^0(H, \mathfrak{g}_n(k)) = (0)$.
- $H^1(H, \mathfrak{g}_n(k)) = (0)$.
- For all irreducible $k[H]$-submodules W of $\mathfrak{g}_n(k)$ we can find $h \in H \cap \mathcal{G}^0_n(k)$ and $\alpha \in k$ with the following properties. The α generalised eigenspace $V_{h,\alpha}$ of h in k^n is one dimensional. Let $\pi_{h,\alpha} : k^n \to V_{h,\alpha}$ (resp. $i_{h,\alpha}$) denote the h-equivariant projection of k^n to $V_{h,\alpha}$ (resp. h-equivariant injection of $V_{h,\alpha}$ into k^n). Then $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq (0)$.

We note that the third property will also hold for any non-zero $\mathbb{F}_l[H]$-subspace W of $\mathfrak{g}_n(k)$. (Because it holds for W if and only if it holds for its k-linear span.) Also note that the first two properties are implied by

- $H^0(H \cap \mathcal{G}^0_n(k), \mathfrak{g}^0_n(k)) = (0)$, and
- $H^1(H \cap \mathcal{G}^0_n(k), \mathfrak{g}^0_n(k)) = (0)$.

The next proposition assures the existence of global 'Taylor-Wiles' type deformations.

Proposition 1.4.5 *Suppose that for $\widetilde{v} \in \widetilde{S}_l$ the deformation problem $\mathcal{D}_{\widetilde{v}}$ is as in section 1.3.1 or 1.3.2. Suppose that for all $\widetilde{v} \in \widetilde{S} - \widetilde{S}_l$ the space $L_{\widetilde{v}}$ is minimal. Suppose also that for all $m \in \mathbb{Z}_{\geq 1}$ the group $\overline{r}(G_{F^+(\zeta_{l^m})})$ is big. Then we can find an integer r with the following properties. If $N \in \mathbb{Z}_{\geq 1}$ then we can find a set Q of primes of F^+ which don't lie in S and which split in F, with the following properties.*

- $\#Q = r$.
- *If $v \in Q$ the* $\mathbf{N}v \equiv 1 \bmod l^N$.
- *For each prime $v \in Q$ we can choose a prime $\widetilde{v}$ of F above v and a set of deformations $\mathcal{D}_{\widetilde{v}}$ of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ as in example 1.3.6 such that, if $\mathcal{S}'$ denotes the extended deformation problem obtained by adding Q and these $\mathcal{D}_{\widetilde{v}}$ to $\mathcal{S}$, then*

$$\dim H^1_{\mathcal{S}'}(G_{F^+,S\cup Q}, \operatorname{ad} \overline{r}) = r - n \sum_{v|\infty} (\chi(c_v)+1)/2.$$

Proof: Suppose that Q is any finite set of primes of F^+ which don't lie in S, which split in F; and suppose that for $v \in Q$ there is a prime $\widetilde{v}$ of F above v and a pair $(\mathcal{D}_{\widetilde{v}}, L_{\widetilde{v}})$ as in example 1.3.6. Write $\widetilde{Q}$ for the set of $\widetilde{v}$ for $v \in S$. Also write $\mathcal{S}'$ for the deformation problem obtained from $\mathcal{S}$ by adjoining Q and the $\mathcal{D}_{\widetilde{v}}$ for $\widetilde{v} \in \widetilde{Q}$. If $\widetilde{v} \in \widetilde{Q}$ write $\overline{r} = \overline{\psi}_{\widetilde{v}} \oplus \overline{s}_{\widetilde{v}}$ as in example 1.3.6. Note that

$$\begin{aligned}(0) \longrightarrow H^1(G_{F^+,S}, (\operatorname{ad} \overline{r})(\epsilon)) \longrightarrow H^1(G_{F^+,S\cup Q}, (\operatorname{ad} \overline{r})(\epsilon)) \longrightarrow \\ \longrightarrow \textstyle\bigoplus_{\widetilde{v}\in\widetilde{Q}} H^1(I_{F_{\widetilde{v}}}, (\operatorname{ad} \overline{r})(\epsilon))^{G_{F_{\widetilde{v}}}}\end{aligned}$$

is left exact. As $\#H^1(I_{F_{\widetilde{v}}}, \operatorname{Hom}(\overline{\psi}_{\widetilde{v}}, \overline{s}_{\widetilde{v}})(\epsilon))^{G_{F_{\widetilde{v}}}} = \#\operatorname{Hom}(\overline{\psi}_{\widetilde{v}}, \overline{s}_{\widetilde{v}})_{G_{F_{\widetilde{v}}}} = 1$ and $\#H^1(I_{F_{\widetilde{v}}}, \operatorname{Hom}(\overline{s}_{\widetilde{v}}, \overline{\psi}_{\widetilde{v}})(\epsilon))^{G_{F_{\widetilde{v}}}} = \#\operatorname{Hom}(\overline{s}_{\widetilde{v}}, \overline{\psi}_{\widetilde{v}})_{G_{F_{\widetilde{v}}}} = 1$ we have a left exact sequence

$$\begin{aligned}(0) \longrightarrow H^1(G_{F^+,S}, (\operatorname{ad} \overline{r})(\epsilon)) \longrightarrow H^1(G_{F^+,S\cup Q}, (\operatorname{ad} \overline{r})(\epsilon)) \longrightarrow \\ \longrightarrow \textstyle\bigoplus_{\widetilde{v}\in\widetilde{Q}} (H^1(I_{F_{\widetilde{v}}}, (\operatorname{ad} \overline{s}_{\widetilde{v}})(\epsilon))^{G_{F_{\widetilde{v}}}} \oplus H^1(I_{F_{\widetilde{v}}}, (\operatorname{ad} \overline{\psi}_{\widetilde{v}})(\epsilon))^{G_{F_{\widetilde{v}}}}),\end{aligned}$$

and hence a left exact sequence

$$\begin{aligned}(0) \longrightarrow H^1_{(\mathcal{L}')^\perp}(G_{F^+,S\cup Q}, (\operatorname{ad} \overline{r})(\epsilon)) \longrightarrow H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\operatorname{ad} \overline{r})(\epsilon)) \longrightarrow \\ \longrightarrow \textstyle\bigoplus_{\widetilde{v}\in\widetilde{Q}} H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, (\operatorname{ad} \overline{\psi}_{\widetilde{v}})(\epsilon)) = \bigoplus_{\widetilde{v}\in\widetilde{Q}} k.\end{aligned}$$

The latter map sends the class of a cocycle $\phi \in Z^1(G_{F^+,S}, (\operatorname{ad} \overline{r})(\epsilon))$ to

$$(\pi_{\mathrm{Frob}_{\widetilde{v}}, \psi_{\widetilde{v}}(\mathrm{Frob}_{\widetilde{v}})} \circ \phi(\mathrm{Frob}_{\widetilde{v}}) \circ i_{\mathrm{Frob}_{\widetilde{v}}, \psi_{\widetilde{v}}(\mathrm{Frob}_{\widetilde{v}})})_{\widetilde{v}\in\widetilde{Q}}.$$

We take $r=\dim_k H^1_{\mathcal{L}^\perp}(G_{F^+,S},(\mathrm{ad}\,\overline{r})(\epsilon))$. By corollary 1.4.4 it suffices to find a set Q of primes of F^+ disjoint from S such that

- if $v\in Q$ then v splits completely in $F(\zeta_{l^N})$;
- if $v\in Q$ then $\overline{r}(\mathrm{Frob}_v)$ has an eigenvalue $\overline{\psi}_{\widetilde{v}}(\mathrm{Frob}_{\widetilde{v}})$ whose generalised eigenspace has dimension 1;
- $H^1_{\mathcal{L}^\perp}(G_{F^+,S},(\mathrm{ad}\,\overline{r})(\epsilon))\hookrightarrow\bigoplus_{\widetilde{v}\in\widetilde{Q}}H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}},(\mathrm{ad}\,\overline{\psi}_{\widetilde{v}})(\epsilon))$.

(If necessary we can then shrink Q to a set of cardinality r with the same properties.) By the Cebotarev density it suffices to show that if ϕ is an element of $Z^1(G_{F^+,S},(\mathrm{ad}\,\overline{r})(\epsilon))$ with non-zero image in $H^1(G_{F^+,S},(\mathrm{ad}\,\overline{r})(\epsilon))$, then we can find $\sigma\in G_{F(\zeta_{l^N})}$ such that

- $\overline{r}(\sigma)$ has an eigenvalue α whose generalised eigenspace has dimension 1;
- $\pi_{\sigma,\alpha}\circ\phi(\sigma)\circ i_{\sigma,\alpha}\neq 0$.

Let $L/F(\zeta_{l^N})$ be the extension cut out by $\mathrm{ad}\,\overline{r}$. If $\sigma'\in G_L$ then $\overline{r}(\sigma'\sigma)\in k^\times\overline{r}(\sigma)$ and $\phi(\sigma'\sigma)=\phi(\sigma')+\phi(\sigma)$. Thus it suffices to find $\sigma\in G_{F(\zeta_{l^N})}$ such that

- $\overline{r}(\sigma)$ has an eigenvalue α whose generalised eigenspace has dimension 1;
- $\pi_{\sigma,\alpha}\circ(\phi(G_L)+\phi(\sigma))\circ i_{\sigma,\alpha}\neq 0$.

It even suffices to find $\sigma\in\mathrm{Gal}\,(L/F(\zeta_{l^N}))$ such that

- $\overline{r}(\sigma)$ has an eigenvalue α whose generalised eigenspace has dimension 1;
- $\pi_{\sigma,\alpha}\circ\phi(G_L)\circ i_{\sigma,\alpha}\neq 0$.

As $H^1(\mathrm{Gal}\,(L/F(\zeta_{l^N})),\mathrm{ad}\,\overline{r})=(0)$ we see that $[\phi]\neq 0$ implies that $\phi(G_L)\neq(0)$. Then the existence of such a ϕ follows from our assumptions. $\Box$

Next we will prove a generalisation of Ramakrishna's lifting theorem [Ra]. The statement is rather complicated as we want to be able to apply it to certain representations $\overline{r}$ with small image, in particular $\overline{r}$ which are induced from a character.

Suppose that $\mathrm{ad}\,\overline{r}$ is a semisimple $k[G_{F^+}]$-module. If $W\subset\mathrm{ad}\,\overline{r}$ is a $k[G_{F^+}]$-submodule we will define

$$H^1_{\mathcal{L}}(G_{F^+,S},W)=\ker(H^1(G_{F^+,S},W)\longrightarrow\bigoplus_{\widetilde{v}\in\widetilde{S}}H^1(G_{F_{\widetilde{v}}},W)/(L_{\widetilde{v}}\cap H^1(G_{F_{\widetilde{v}}},W)))$$

and $H^1_{\mathcal{L}^\perp}(G_{F^+,S},W(1))$ to be the kernel of

$$H^1(G_{F^+,S},W(1))\longrightarrow\bigoplus_{\widetilde{v}\in\widetilde{S}}H^1(G_{F_{\widetilde{v}}},W(1))/(L^\perp_{\widetilde{v}}\cap H^1(G_{F_{\widetilde{v}}},W(1))).$$

Theorem 1.4.6 *Keep the notation and assumptions of section 1.3. In addition make the following assumptions.*

- *For all $\widetilde{v} \in \widetilde{S}_l$ the local deformation problem $\mathcal{D}_{\widetilde{v}}$ is as in section 1.3.1 or 1.3.2.*
- *For all $\widetilde{v} \in \widetilde{S} - \widetilde{S}_l$ the space $L_{\widetilde{v}}$ is minimal and the set $\mathcal{D}_{\widetilde{v}}$ is liftable. (This is true if $\mathcal{D}_{\widetilde{v}}$ is as in section 1.3.4 or section 1.3.5.)*
- *For $\widetilde{v} \in \widetilde{S}_0$*

$$H^0(G_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}/\mathrm{Fil}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r}) \hookrightarrow H^1(G_{F_{\widetilde{v}}}, \mathrm{Fil}\,^0_{\widetilde{v}}\mathrm{ad}\,\overline{r})/L_{\widetilde{v}}.$$

  (This is true if $\mathcal{D}_{\widetilde{v}}$ is as in section 1.3.4; or as in section 1.3.1 with $\overline{\chi}_i \neq \overline{\chi}_j$ for $i \neq j$.)
- *For each infinite place v of F^+ we have $\chi(c_v) = -1$.*
- $\mathrm{ad}\,\overline{r}$ *and* $(\mathrm{ad}\,\overline{r})(1)$ *are semisimple $k[G_{F^+}]$-modules and have no irreducible constituent in common.*
- *$H^i((\mathrm{ad}\,\overline{r})(G_{F^+(\zeta_l)}), \mathfrak{g}_n(k)) = (0)$ for $i = 0$ and 1.*
- *W_0 (resp. W_1) is a G_{F^+}-submodule of $\mathrm{ad}\,\overline{r}$ with $H^1_{\mathcal{L}}(G_{F^+,S}, W_0) = (0)$ (resp. $H^1_{\mathcal{L}^\perp}(G_{F^+,S}, W_1(1)) = (0)$).*

Suppose moreover that for all irreducible $k[G_{F^+,S}]$-submodules W and W' of $\mathfrak{g}_n(k)$ with $W' \not\subset W_0$ and $W \not\subset W_1$ we can find $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$ and $\alpha \in k$ with the following properties:

- $\epsilon(\sigma) \not\equiv 1 \bmod l$.
- *The α generalised eigenspace $V_{\sigma,\alpha}$ and the $\alpha\epsilon(\sigma)$ generalised eigenspace $V_{\sigma,\alpha\epsilon(\sigma)}$ of $\overline{r}(\sigma)$ are one dimensional. Let $i_{\sigma,\alpha}$ (resp. $i_{\sigma,\alpha\epsilon(\sigma)}$) denote the inclusions $V_{\sigma,\alpha} \hookrightarrow k^n$ (resp. $V_{\sigma,\alpha\epsilon(\sigma)} \hookrightarrow k^n$). Let $\pi_{\sigma,\alpha} : k^n \to V_{\sigma,\alpha}$ (resp. $\pi_{\sigma,\alpha\epsilon(\sigma)} : k^n \to V_{\sigma,\alpha\epsilon(\sigma)}$) denote the σ-equivariant projections.*
- $i_{\sigma,\alpha\epsilon(\sigma)}\pi_{\sigma\alpha} \not\in W_0$.
- $(i_{\sigma,\alpha\epsilon(\sigma)}\pi_{\sigma,\alpha\epsilon(\sigma)} - i_{\sigma,\alpha}\pi_{\sigma,\alpha}) \not\in W_1$.
- $\pi_{\sigma,\alpha} \circ W \circ i_{\sigma,\alpha\epsilon(\sigma)} \neq (0)$.
- *$\pi_{\sigma,\alpha} \circ w' \circ i_{\sigma,\alpha} \neq \pi_{\sigma,\alpha\epsilon(\sigma)} \circ w' \circ i_{\sigma,\alpha\epsilon(\sigma)}$ for some $w' \in W'$.*

(We note that this property will also hold for any non-zero $\mathbb{F}_l[G_{F^+,S}]$-subspaces W and W' of $\mathfrak{g}_n(k)$ with $W' \not\subset W_0$ and $W \not\subset W_1$. Because it holds for W and W' if and only if it holds for their k-linear spans.)

Then we can find a finite set Q of primes of F^+ which don't lie in S and which split in F with the following properties. Choose a set $\widetilde{Q}$ consisting of one prime of F above each element of Q.

- *If $v \in Q$ then $\mathbf{N}v \not\equiv 1 \bmod l$.*
- *If $\widetilde{v} \in \widetilde{Q}$ then $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{t}_{\widetilde{v}} \oplus \overline{s}_{\widetilde{v}}$ where $\overline{t}_{\widetilde{v}} = \overline{\psi}_{\widetilde{v}} \oplus \overline{\psi}_{\widetilde{v}}\epsilon$ and neither $\overline{\psi}_{\widetilde{v}}$ nor $\overline{\psi}_{\widetilde{v}}\epsilon$ is a subquotient of $\overline{s}_{\widetilde{v}}$. Let $\mathcal{D}'_{\widetilde{v}}$ and $L'_{\widetilde{v}}$ be chosen as in example 1.3.7.*
- *If $\mathcal{S}'$ denotes the problem obtained from $\mathcal{S}$ adding Q to S with the condition $\mathcal{D}_{\widetilde{v}}$ for $\widetilde{v} \in \widetilde{Q}$ then*
$$R^{\rm univ}_{\mathcal{S}'} = \mathcal{O}.$$

In particular there is a lifting $(r, \{\mathrm{Fil}^i_{\widetilde{v}}\})$ of $(\overline{r}, \{\overline{\mathrm{Fil}}^i_{\widetilde{v}}\})$ where $r : G_{F^+,S\cup Q} \to \mathcal{G}_n(\mathcal{O})$, where $\nu \circ r = \chi$, and where for all $\widetilde{v} \in \widetilde{S}$ the restriction $(r|_{G_{F_{\widetilde{v}}}}, \{\mathrm{Fil}^i_{\widetilde{v}}\})$ lies in $\mathcal{D}_{\widetilde{v}}$.

Proof: If $H^1_{\mathcal{L}^\perp}(G_{F^+,S}, \mathrm{ad}\,\overline{r}(1)) = (0)$ then the proposition follows at once from lemma 1.4.2 and corollary 1.4.3 (with $Q = \emptyset$). In the general case we need only show that we can find a prime $v \notin S$ of F^+ which splits as $\widetilde{v}^c\widetilde{v}$ in F such that

- $\mathbf{N}v \not\equiv 1 \bmod l$.
- $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{t}_{\widetilde{v}} \oplus \overline{s}_{\widetilde{v}}$ where $\overline{t}_{\widetilde{v}} = \overline{\psi}_{\widetilde{v}} \oplus \overline{\psi}_{\widetilde{v}}\epsilon$ and neither $\overline{\psi}_{\widetilde{v}}$ nor $\overline{\psi}_{\widetilde{v}}\epsilon$ is a subquotient of $\overline{s}_{\widetilde{v}}$. Let $\mathcal{D}'_{\widetilde{v}}$ and $L'_{\widetilde{v}}$ be chosen as in example 1.3.7.
- If $\mathcal{S}'$ denotes the problem obtained from $\mathcal{S}$ adding v to S with the condition $\mathcal{D}'_{\widetilde{v}}$ then
$$\dim H^1_{(\mathcal{L}')^\perp}(G_{F^+,S\cup\{v\}}, (\mathrm{ad}\,\overline{r})(1)) < \dim H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(1)).$$
- $H^1_{\mathcal{L}'}(G_{F^+,S\cup\{v\}}, W_0) = (0)$ and $H^1_{(\mathcal{L}')^\perp}(G_{F^+,S\cup\{v\}}, W_1(1)) = (0)$.

(Then one can add primes v as above to S recursively until
$$H^1_{(\mathcal{L}')^\perp}(G_{F^+,S\cup Q}, (\mathrm{ad}\,\overline{r})(1)) = (0).)$$

So let $v \notin S$ be a prime of F^+ which splits as $\widetilde{v}^c\widetilde{v}$ in F such that

- $\mathbf{N}v \not\equiv 1 \bmod l$.

- $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{t}_{\widetilde{v}} \oplus \overline{s}_{\widetilde{v}}$ where $\overline{t}_{\widetilde{v}} = \overline{\psi}_{\widetilde{v}} \oplus \overline{\psi}_{\widetilde{v}}\epsilon$ and neither $\overline{\psi}_{\widetilde{v}}$ nor $\overline{\psi}_{\widetilde{v}}\epsilon$ is a subquotient of $\overline{s}_{\widetilde{v}}$.

- Let $\pi_{\overline{\psi}_{\widetilde{v}}}$ (resp. $i_{\overline{\psi}_{\widetilde{v}}}$, resp. $\pi_{\overline{\psi}_{\widetilde{v}}\epsilon}$, resp. $i_{\overline{\psi}_{\widetilde{v}}\epsilon}$) denote the $G_{F_{\widetilde{v}}}$-equivariant projection $\overline{r} \twoheadrightarrow \overline{\psi}_{\widetilde{v}}$ (resp. inclusion $\overline{\psi}_{\widetilde{v}} \hookrightarrow \overline{r}$, resp. projection $\overline{r} \twoheadrightarrow \overline{\psi}_{\widetilde{v}}\epsilon$, resp. inclusion $\overline{\psi}_{\widetilde{v}}\epsilon \hookrightarrow \overline{r}$). Then $i_{\overline{\psi}_{\widetilde{v}}\epsilon}\pi_{\overline{\psi}_{\widetilde{v}}} \notin W_0$ and $i_{\overline{\psi}_{\widetilde{v}}\epsilon}\pi_{\overline{\psi}_{\widetilde{v}}\epsilon} - i_{\overline{\psi}_{\widetilde{v}}}\pi_{\overline{\psi}_{\widetilde{v}}} \notin W_1$.

Set $S' = S \cup \{v\}$ and consider three pairs $(\mathcal{D}_{\widetilde{v}}, L_{\widetilde{v}})$, $(\mathcal{D}'_{\widetilde{v}}, L'_{\widetilde{v}})$, $(\mathcal{D}''_{\widetilde{v}}, L''_{\widetilde{v}})$ defining three extensions $\mathcal{S}_v$, $\mathcal{S}'_v$ and $\mathcal{S}''_v$ of $\mathcal{S}$:

- $\mathcal{D}_{\widetilde{v}}$ consists of all unramified lifts of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ and $L_{\widetilde{v}} = H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \operatorname{ad}\overline{r}) = H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \operatorname{ad}\overline{t}) \oplus H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, \operatorname{ad}\overline{s})$;

- $\mathcal{D}'_{\widetilde{v}}$ and $L'_{\widetilde{v}}$ are as in example 1.3.7; and

- $\mathcal{D}''_{\widetilde{v}}$ and $L''_{\widetilde{v}}$ are as in example 1.3.8.

Note that

$$H^1_{\mathcal{S}_v}(G_{F^+,S'}, \operatorname{ad}\overline{r}) = H^1_{\mathcal{S}}(G_{F^+,S}, \operatorname{ad}\overline{r})$$

and

$$H^1_{\mathcal{L}_v^\perp}(G_{F^+,S'}, (\operatorname{ad}\overline{r})(1)) = H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\operatorname{ad}\overline{r})(1)).$$

Also note that there are left exact sequences

$$(0) \longrightarrow H^1_{\mathcal{S}_v}(G_{F^+,S'}, \operatorname{ad}\overline{r}) \longrightarrow H^1_{\mathcal{S}''_v}(G_{F^+,S'}, \operatorname{ad}\overline{r}) \longrightarrow H^1(I_{F_{\widetilde{v}}}, \operatorname{ad}\overline{r})$$

and (by our second assumption)

$$(0) \to H^1_{\mathcal{S}'_v}(G_{F^+,S'}, \operatorname{ad}\overline{r}) \to H^1_{\mathcal{S}''_v}(G_{F^+,S'}, \operatorname{ad}\overline{r}) \to H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k(i_{\overline{\psi}_{\widetilde{v}}\epsilon}\pi_{\overline{\psi}_{\widetilde{v}}\epsilon} - i_{\overline{\psi}_{\widetilde{v}}}\pi_{\overline{\psi}_{\widetilde{v}}}))$$

and

$$\begin{aligned}(0) \to H^1_{(\mathcal{L}''_v)^\perp}(G_{F^+,S'}, (\operatorname{ad}\overline{r})(1)) \to H^1_{(\mathcal{L}_v)^\perp}(G_{F^+,S'}, (\operatorname{ad}\overline{r})(1)) \to \\ \to H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, ((\operatorname{ad}\overline{t})/k(i_{\overline{\psi}_{\widetilde{v}}\epsilon}\pi_{\overline{\psi}_{\widetilde{v}}}))(1)).\end{aligned}$$

It follows from lemma 1.4.2 (and the discussions of sections 1.3.7 and 1.3.8) that

$$\begin{aligned} & \dim H^1_{(\mathcal{L}'_v)^\perp}(G_{F^+,S'}, (\operatorname{ad}\overline{r})(1)) - \dim H^1_{(\mathcal{L}''_v)^\perp}(G_{F^+,S'}, (\operatorname{ad}\overline{r})(1)) \\ = \; & \dim H^1_{\mathcal{S}'_v}(G_{F^+,S'}, \operatorname{ad}\overline{r}) - \dim H^1_{\mathcal{S}''_v}(G_{F^+,S'}, \operatorname{ad}\overline{r}) + \dim L''_{\widetilde{v}} - \dim L'_{\widetilde{v}} \\ = \; & \dim H^1_{\mathcal{S}'_v}(G_{F^+,S'}, \operatorname{ad}\overline{r}) - \dim H^1_{\mathcal{S}''_v}(G_{F^+,S'}, \operatorname{ad}\overline{r}) + 1. \end{aligned}$$

Moreover because $i_{\overline{\psi}_{\widetilde{v}}\epsilon}\pi_{\overline{\psi}_{\widetilde{v}}} \notin W_0$ we see that $H^1(G_{F_{\widetilde{v}}}, W_0) \cap L'_{\widetilde{v}}$ is contained in $H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, W_0)$ and so $H^1_{\mathcal{L}'_v}(G_{F^+,S'}, W_0) \subset H^1_{\mathcal{L}}(G_{F^+,S}, W_0) = (0)$. Similarly because $(i_{\overline{\psi}_{\widetilde{v}}\epsilon}\pi_{\overline{\psi}_{\widetilde{v}}\epsilon} - i_{\overline{\psi}_{\widetilde{v}}}\pi_{\overline{\psi}_{\widetilde{v}}}) \notin W_1$ we see that $H^1(G_{F_{\widetilde{v}}}, W_1(1)) \cap (L'_{\widetilde{v}})^\perp \subset H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, W_1(1))$ and so $H^1_{(\mathcal{L}'_v)^\perp}(G_{F^+,S'}, W_1(1)) \subset H^1_{\mathcal{L}^\perp}(G_{F^+,S}, W_1(1)) = (0)$.

Thus the prime v will have the desired properties if

$$H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(1)) \to H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, ((\mathrm{ad}\,\overline{t})/k(i_{\overline{\psi}_{\widetilde{v}}\epsilon}\pi_{\overline{\psi}_{\widetilde{v}}}))(1))$$

and

$$H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) \hookrightarrow H^1_{\mathcal{S}''_v}(G_{F^+,S'}, \mathrm{ad}\,\overline{r}) \to H^1(G_{F_{\widetilde{v}}}/I_{F_{\widetilde{v}}}, k(i_{\overline{\psi}_{\widetilde{v}}\epsilon}\pi_{\overline{\psi}_{\widetilde{v}}\epsilon} - i_{\overline{\psi}_{\widetilde{v}}}\pi_{\overline{\psi}_{\widetilde{v}}}))$$

are both non-trivial.

Suppose that $H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(1)) \neq (0)$. It follows from lemma 1.4.2 that

$$\dim H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r}) = \dim H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(1)) > 0.$$

Choose a non-zero classes $[\varphi] \in H^1_{\mathcal{L}^\perp}(G_{F^+,S}, (\mathrm{ad}\,\overline{r})(1))$ and a non-zero class $[\varphi''] \in H^1_{\mathcal{S}}(G_{F^+,S}, \mathrm{ad}\,\overline{r})$. By the Cebotarev density theorem it suffices to show that we can choose $\sigma \in G_F$ and $\alpha \in k$ with the following properties.

- $\sigma|_{F(\zeta_l)} \neq 1$.
- $\overline{r}(\sigma)$ has eigenvalues α and $\alpha\epsilon(\sigma)$ and the corresponding generalised eigenspaces U and U' have dimension 1. Let i (resp. i') denote the inclusion of U (resp. U') into k^n and let π (resp. π') denote the σ-equivariant projection of k^n onto U (resp. U').
- $i'\pi \notin W_0$.
- $i'\pi' - i\pi \notin W_1$.
- $\pi \circ \varphi(\sigma) \circ i' \neq 0$.
- $\pi \circ \varphi''(\sigma) \circ i \neq \pi' \circ \varphi''(\sigma) \circ i'$.

Let L denote the extension of $F(\zeta_l)$ cut out by $\mathrm{ad}\,\overline{r}$. Replacing σ by $\sigma'\sigma$ with $\sigma' \in G_L$ we need only show that we can find $\sigma \in G_F$ and $\alpha \in k$ with the following properties.

- $\sigma|_{F(\zeta_l)} \neq 1$.

- $\overline{r}(\sigma)$ has eigenvalues α and $\alpha\epsilon(\sigma)$ and the corresponding generalised eigenspaces U and U' have dimension 1. Let i (resp. i') denote the inclusion of U (resp. U') into k^n and let π (resp. π') denote the σ-equivariant projection of k^n onto U (resp. U').

- $i'\pi \notin W_0$.

- $i'\pi' - i\pi \notin W_1$.

- $\pi \circ \varphi(G_L) \circ i' \neq 0$.

- $\sigma' \mapsto \pi \circ \varphi''(\sigma') \circ i - \pi' \circ \varphi''(\sigma') \circ i'$ is not identically zero on G_L.

Note that $\varphi(G_L) \not\subset W_0$ and $\varphi''(G_L) \not\subset W_1$ (because $H^1_{\mathcal{L}}(G_{F^+,S}, W_0) = (0)$ and $H^1_{\mathcal{L}^\perp}(G_{F^+,S}, W_1(1)) = (0)$). Hence the existence of σ follows from the assumptions of the lemma. $\Box$

Because the hypotheses of this theorem are so complicated we give a concrete instance of the theorem. We will write $\mathrm{Cl}(F)$ for the class group of a number field F.

Corollary 1.4.7 *Suppose that $n > 1$ is an integer, that F^+ is a totally real field and that E is an imaginary quadratic field. Let $\mathrm{Cl}(EF^+)$ denote the class group of EF^+. Suppose that $l > n$ is a prime which is split in E, which is unramified in F^+ and which does not divide the order of the $\mathrm{Gal}\,(EF^+/F^+)$-coinvariants $\mathrm{Cl}(EF^+)_{\mathrm{Gal}\,(EF^+/F^+)}$. Suppose moreover that*

$$\overline{r} : G_{F^+} \longrightarrow \mathcal{G}_n(\mathbb{F}_l)$$

is a continuous, surjective homomorphism such that

- $\overline{r}^{-1}GL_n(\mathbb{F}_l) = G_{EF^+}$*;*

- $\overline{r}|_{G_{EF^+}}$ *only ramifies at primes which are split over F^+;*

- $\nu \circ \overline{r}(c) = -1$ *for any complex conjugation c;*

- *for any place w of EF^+ above l then $\overline{r}|_{G_{(EF^+)_w}}$ is in the image of $\mathbb{G}_w$ and for each $i = 0, ..., l-2$ we have*

$$\dim_{k(w)} \mathrm{gr}^{\,i} \mathbb{G}_w^{-1} \overline{r}|_{G_{(EF^+)_w}} \leq 1.$$

Then there is a finite extension $k/\mathbb{F}_l$ such that $\overline{r}$ lifts to a continuous homomorphism

$$r : G_{F^+} \longrightarrow \mathcal{G}_n(W(k))$$

which ramifies at only finitely many primes and which is crystalline at all primes of EF^+ above l.

Proof: We apply the theorem. We take $\mathcal{O} = W(k)$ for a suitably large finite extension $k/\mathbb{F}_l$. We take S to be the set of places above l or below a prime of $F = EF^+$ at which $\overline{r}|_{G_F}$ is ramified. For $\widetilde{v} \in \widetilde{S}_l$ we take $\mathcal{D}_{\widetilde{v}}$ as in section 1.3.2. For $\widetilde{v} \in \widetilde{S} - \widetilde{S}_l$ we take $\mathcal{D}_{\widetilde{v}}$ as in section 1.3.4. As $l > n$ we have $\mathrm{ad}\,\overline{r} = k1_n \oplus \mathrm{ad}^0\overline{r}$ and both summands are irreducible G_F-modules. As $F^+(\zeta_l)$ is linearly disjoint from F^+E over F^+ (look at ramification above l) we have that $H^0(\mathrm{ad}\,\overline{r}G_{F^+(\zeta_l)}, k1_n) = (0)$ and $H^1(\mathrm{ad}\,\overline{r}G_{F^+(\zeta_l)}, k1_n) = (0)$. Clearly $H^0(\mathrm{ad}\,\overline{r}G_{F^+(\zeta_l)}, \mathfrak{g}_n^0(k)) = (0)$. By [CPS] (see table (4.5)) we have $H^1(SL_n(\mathbb{F}_l), M_n(\mathbb{F}_l)^{\mathrm{tr}=0}) = (0)$, and so $H^1(\mathrm{ad}\,\overline{r}G_{F^+(\zeta_l)}, \mathfrak{g}_n^0(k)) = (0)$. We take $W_0 = k1_n$ and $W_1 = (k1_n)(1)$. Then

$$\begin{aligned} H^1_{\mathcal{L}}(G_{F^+,S}, W_0) &= \ker(H^1(G_{F^+}, k1_n) \longrightarrow \textstyle\bigoplus_{\widetilde{v}} H^1(I_{F_{\widetilde{v}}}, k1_n)) \\ &= \ker(H^1(G_{F^+}, k1_n) \longrightarrow \textstyle\bigoplus_{v} H^1(I_{F_v^+}, k1_n)) \\ &= \ker(H^1(G_F, k1_n) \longrightarrow \textstyle\bigoplus_{\widetilde{v}} H^1(I_{F_{\widetilde{v}}}, k1_n))^{\mathrm{Gal}\,(F/F^+)} \\ &= \mathrm{Hom}\,(\mathrm{Cl}(F)/(c-1)\mathrm{Cl}(F), k) = (0). \end{aligned}$$

(Note that if $\widetilde{v}$ is a prime of F ramified over F^+ then $H^1(I_{F^+_{\widetilde{v}}}, k1_n) \hookrightarrow H^1(I_{F_{\widetilde{v}}}, k1_n)$.) Also

$$H^1_{\mathcal{L}^\perp}(G_{F^+,S}, W_1) = \ker(H^1(G_{F^+}, (k1_n)(1)) \longrightarrow \bigoplus_v H^1(I_{F_v^+}, (k1_n)(1))).$$

(Note that if $\widetilde{v}$ is a prime of F ramified over F^+ then $H^1(I_{F^+_{\widetilde{v}}}, (k1_n)(1)) \hookrightarrow H^1(I_{F_{\widetilde{v}}}, (k1_n)(1))$.) By, for instance, theorem 2.19 of [DDT] we see that

$$H^1_{\mathcal{L}^\perp}(G_{F^+,S}, W_1) = (0).$$

The rest of the hypotheses of the theorem are easy to verify and the corollary follows. □

2 Hecke algebras.

2.1 GL_n over a local field: characteristic zero theory.

In this section let p be a rational prime and let F_w be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_{F_w}$ denote the maxinal order in F_w, let $\wp_w$ denote the maximal ideal in $\mathcal{O}_{F_w}$, let $k(w) = \mathcal{O}_{F_W}/\wp_w$ and let $q_w = \#k(w)$. We will use ϖ_w to denote a generator of $\wp_w$ in situations where the particular choice of generator does not matter. Also let $\overline{K}$ denote an algebraic closure of $\mathbb{Q}_l$. Also fix a positive integer n. We will write B_n for the Borel subgroup of GL_n consisting of upper triangular matrices.

We will use some, mostly standard, notation from [HT] without comment. For instance n-Ind , $\boxplus$, Sp_m, JL , rec and R_l. *On the other hand, if π is an irreducible smooth representation of $GL_n(F_w)$ over $\overline{K}$ we will use the notation $r_l(\pi)$ for the l-adic representation associated (as in [Tat]) to the Weil-Deligne representation*

$$\mathrm{rec}_l(\pi^\vee \otimes |\ |^{(1-n)/2}),$$

when it exists (i.e. when the eigenvalues of $\mathrm{rec}(\pi^\vee \otimes |\ |^{(1-n)/2})(\phi_w)$ are l-adic units for some lift ϕ_w of Frob_w). In [HT] we used $r_l(\pi)$ for the semisimplification of this representation.

For any integer $m \geq 0$ we will let $U_0(w^m)$ (resp. $U_1(w^m)$) denote the subgroup of $GL_n(\mathcal{O}_{F_w})$ consisting of matices with last row congruent to $(0, ..., 0, *)$ (resp. $(0, ..., 0, 1)$) modulo $\wp_w^m$. Thus $U_1(w^m)$ is a normal subgroup of $U_0(w^m)$ and we have a natural identification

$$U_0(w^m)/U_1(w^m) \cong (\mathcal{O}_{F_w}/\wp_w^m)^\times$$

by projection to the lower right entry of a matrix. We will also denote by $\mathrm{Iw}(w)$ the subgroup of $GL_n(\mathcal{O}_{F_w})$ consisting of matrices which are upper triangular modulo $\wp_w$ and by $\mathrm{Iw}_1(w)$ the subgroup of $\mathrm{Iw}(w)$ consisting of matrices whose diagonal entries are all congruent to one modulo $\wp_w$. Thus $\mathrm{Iw}_1(w)$ is a normal subgroup of $\mathrm{Iw}(w)$ and we have a natural identification

$$\mathrm{Iw}(w)/\mathrm{Iw}_1(w) \cong (k(w)^\times)^n.$$

We will let $\alpha_{w,j}$ denote the matrix

$$\begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix}.$$

For $j = 1, ..., n$ let $T_w^{(j)}$ denote the Hecke operator

$$[GL_n(\mathcal{O}_{F_w})\alpha_{w,j}GL_n(\mathcal{O}_{F_w})].$$

For $j=1,...,n-1$ and for $m>0$ let $U_w^{(j)}$ denote the Hecke operator

$$[U_0(w^m)\alpha_{w,j}U_0(w^m)]$$

or

$$[U_1(w^m)\alpha_{w,j}U_1(w^m)].$$

If W is a smooth representation of $GL_n(F_w)$ and if $m_1>m_2>0$ then the action of $U_w^{(j)}$ is compatible with the inclusions

$$W^{U_0(w^{m_2})}\subset W^{U_1(w^{m_2})}\subset W^{U_1(w^{m_1})}.$$

(This follows easilly from the coset decompositions given in [M1] for $U_1(w^m)\alpha_{w,j}U_1(w^m)/U_1(w^m)$.)

If $\alpha\in F_w^\times$ has non-negative valuation we will write V_α for the Hecke operators

$$[U_0(w)\begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha \end{pmatrix}U_0(w)]$$

and

$$[U_1(w)\begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha \end{pmatrix}U_1(w)].$$

If W is a smooth representation of $GL_n(F_w)$ then the action of V_α is compatible with the inclusion

$$W^{U_0(w)}\subset W^{U_1(w)}.$$

(This follows from the easily verified equalities

$$U_1(w)\left(U_0(w)\cap\begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha \end{pmatrix}U_0(w)\begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right)=U_0(w)$$

and

$$\begin{aligned} & U_1(w)\cap\begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha \end{pmatrix}U_0(w)\begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \\ = \; & U_1(w)\cap\begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha \end{pmatrix}U_1(w)\begin{pmatrix} 1_{n-1} & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.) \end{aligned}$$

It is well known that there is an isomorphism

$$\mathbb{Z}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]\cong\mathbb{Z}[T_1,T_2,...,T_n,T_n^{-1}],$$

under which T_j corresponds to $T_w^{(j)}$. (The latter ring is the polynomial algebra in the given variables.) Alternatively we have the Satake isomorphism

$$\mathbb{Z}[1/q_w][GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]\cong\mathbb{Z}[1/q_w][X_1^{\pm1},...,X_n^{\pm1}]^{S_n},$$

under which $T_w^{(j)}$ corresponds to $q_w^{j(1-j)/2} s_j(X_1, ..., X_n)$, where s_j is the j^{th} elementary symmetric function (i.e. the sum of all square free monomials of degree j). This is *not* the standard normalisation of the Satake isomorphism.

Lemma 2.1.1 *Suppose that $\chi_1, ..., \chi_n$ are unramified characters of $F_w^\times$. Then $(\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n))^{GL_n(\mathcal{O}_{F_w})}$ is one dimensional and $T_w^{(j)}$ acts on it by $q_w^{j(n-j)/2} s_j(\chi_1(\varpi_w), ..., \chi_n(\varpi_w))$, where s_j is the j^{th} elementary symmetric function (i.e. the sum of all square free monomials of degree j). If*

$$T \in \mathbb{Z}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$$

has Satake transform $P(X_1, ..., X_n)$ then the eigevalue of T on

$$(\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n))^{GL_n(\mathcal{O}_{F_w})}$$

is $P(q_w^{(n-1)/2}\chi_1(\varpi_w), ..., q_w^{(n-1)/2}\chi_n(\varpi_w))$.

Proof: The fixed space $(\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n))^{GL_n(\mathcal{O}_{F_w})}$ is spanned by the function φ_0 where

$$\varphi_0(bu) = \prod_{i=1}^n \chi_i(b_{ii})|b_{ii}|^{(n+1)/2-i}$$

for $b \in B_n(F_w)$ and $u \in GL_n(\mathcal{O}_{F_w})$. Then $(T_w^{(j)}\varphi_0)(1)$ equals the eigenvalue of $T_w^{(j)}$ on $(\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n))^{GL_n(\mathcal{O}_{F_w})}$. Let X denote a set of representatives for $k(w)$ in $\mathcal{O}_{F_w}$. Then

$$(T_w^{(j)}\varphi_0)(1) = \sum_I \sum_b \varphi_0(b)$$

where I runs over j element subsets of $\{1, ..., n\}$ and b runs over elements of $B_n(F_w)$ with

- $b_{ii} = \varpi_w$ if $i \in I$ and $b_{ii} = 1$ otherise;
- if $j > i$, $i \in I$ and $j \notin I$ then $b_{ij} \in X$;
- if $j > i$ and either $i \notin I$ or $j \in I$ then $b_{ij} = 0$.

Thus

$$\begin{array}{rcl} (T_w^{(j)}\varphi_0)(1) & = & \sum_I q_w^{\sum_{k=1}^j (n-j+k-i_k)} \prod_{i\in I} \chi_i(\varpi_w) q_w^{i-(n+1)/2} \\ & = & q_w^{j(n-j)/2} \sum_I \prod_{i\in I} \chi_i(\varpi_w), \end{array}$$

where $I = \{i_1 < ... < i_j\}$ runs over j element subsets of $\{1, ..., n\}$. The lemma follows. $\Box$

Corollary 2.1.2 *Suppose that π is an unramified irreducible admissible representation of $GL_n(F_w)$ over $\overline{K}$. Let $t_\pi^{(j)}$ denote the eigenvalue of $T_w^{(j)}$ on $\pi^{GL_n(\mathcal{O}_{F_w})}$. Then $r_l(\pi)^\vee(1-n)(\mathrm{Frob}_w)$ has characteristic polynomial*

$$X^n - t_\pi^{(1)} X^{n-1} + ... + (-1)^j q_w^{j(j-1)/2} t_\pi^{(j)} X^{n-j} + ... + (-1)^n q_w^{n(n-1)/2} t_\pi^{(n)}.$$

Proof: Suppose that $\pi = \chi_1 \boxplus ... \boxplus \chi_n$. Then

$$r_l(\pi)^\vee(1-n) = \bigoplus_i (\chi_i| \ |^{(1-n)/2}) \circ \mathrm{Art}^{-1},$$

so that $r_l(\pi)^\vee(1-n)(\mathrm{Frob}_w)$ has characteristic polynomial

$$(X - \chi_1(\varpi_w) q_w^{(n-1)/2}) ... (X - \chi_n(\varpi_w) q_w^{(n-1)/2}).$$

$\square$

Lemma 2.1.3 *Suppose that π is an unramified irreducible admissible representation of $GL_n(F_w)$ over $\overline{K}$. Let $t_\pi^{(j)}$ denote the eigenvalue of $T_w^{(j)}$ on $\pi^{GL_n(\mathcal{O}_{F_w})}$. Then $\pi^{U_0(w)} = \pi^{U_1(w)}$ and the characteristic polynomial of V_{ϖ_w} on $\pi^{U_0(w)}$ divides*

$$X^n - t_\pi^{(1)} X^{n-1} + ... + (-1)^j q_w^{j(j-1)/2} t_\pi^{(j)} X^{n-j} + ... + (-1)^n q_w^{n(n-1)/2} t_\pi^{(n)}.$$

Proof: The first assertion is immediate because the central character of π is unramified. Choose unramified characters $\chi_i : F_w^\times \to \overline{K}^\times$ for $i = 1, .., n$ such that the $q_w^{(n-1)/2} \chi_i(\varpi_w)$ are the roots of

$$X^n - t_\pi^{(1)} X^{n-1} + ... + (-1)^j q_w^{j(j-1)/2} t_\pi^{(j)} X^{n-j} + ... + (-1)^n q_w^{n(n-1)/2} t_\pi^{(n)}$$

with multiplicities. From the last lemma we see that π is a subquotient of $\text{n-Ind}_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n)$. Thus it suffices to show that the eigenvalues of V_ϖ on $\text{n-Ind}_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n)^{U_0(w)}$ are $\{q_w^{(n-1)/2} \chi_i(\varpi_w)\}$, with multiplicities (as roots of the characteristic polynomial).

Let w_m denote the $m \times m$-matrix with $(w_m)_{ij} = 1$ if $i + j = n + 1$ and $(w_m)_{ij} = 0$ otherwise. Let $w_{n,i}$ denote the matrix

$$\begin{pmatrix} 1_{i-1} & 0 \\ 0 & w_{n+1-i} \end{pmatrix}.$$

The space $\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n)^{U_0(w)}$ has a basis of functions φ_i for $i = 1, ..., n$ where the support of φ_i is contained in $B_n(F_w)w_{n,i}U_0(w)$ and $\varphi_i(w_{n,i}) = 1$. We have

$$V_{\varpi_w}\varphi_i = \sum_j (V_{\varpi_w}\varphi_i)(w_{n,j})\varphi_j.$$

Let X denote a set of representatives for $k(w)$ in $\mathcal{O}_{F_w}$ containing 0. Then

$$\begin{array}{rcl} (V_{\varpi_w}\varphi_i)(w_{n,j}) & = & \sum_{x \in X^{n-1}} \varphi_i \left(w_{n,j} \begin{pmatrix} 1_{n-1} & 0 \\ \varpi_w x & \varpi_x \end{pmatrix} \right) \\ & = & \sum_{x \in X^{j-1}} \sum_{y \in X^{n-j}} \varphi_i \begin{pmatrix} 1_{j-1} & 0 & 0 \\ \varpi_w x & \varpi_w y & \varpi_w \\ 0 & w_{n-j} & 0 \end{pmatrix} \\ & = & q_w^{(n-1)/2} \chi_j(\varpi_w) \sum_{x \in X^{j-1}} \varphi_i \begin{pmatrix} 1_{j-1} & 0 & 0 \\ x & 0 & 1 \\ 0 & w_{n-j} & 0 \end{pmatrix} \end{array}$$

A matrix $g \in GL_n(\mathcal{O}_{F_w})$ lies in $B_n(\mathcal{O}_{F_w})w_{n,i}U_0(w)$ if and only if i is the largest integer such that $(0, ..., 0, 1)$ lies in the $k(w)$ span of the reduction modulo $\wp_w$ of the last $n + 1 - i$ rows of g. Thus

$$(V_{\varpi_w}\varphi_i)(w_{n,j})$$

is

- 0 if $i > j$,
- $q_w^{(n-1)/2}\chi_j(\varpi_w)$ if $i = j$, and
- $(q_w - 1)q_w^{j-i-1}q_w^{(n-1)/2}\chi_j(\varpi_w)$ if $i < j$.

Thus the matrix of V_{ϖ_w} with respect to the basis $\{\varphi_i\}$ of the space $\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n)^{U_0(w)}$ is triangular with diagonal entries $q_w^{(n-1)/2}\chi_j(\varpi_w)$. The lemma follows. $\Box$

Lemma 2.1.4 *Suppose that we have a partition $n = n_1 + n_2$ and that π_1 (resp. π_2) is a smooth representation of $GL_{n_1}(F_w)$ (resp. $GL_{n_2}(F_w)$). Let $P \supset B_n$ denote the parabolic corresponding to the partition $n = n_1 + n_2$. Set $\pi = \text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)}(\pi_1 \otimes \pi_2)$. Then*

$$\pi^{U_1(w)} \cong (\pi_1^{GL_{n_1}(\mathcal{O}_{F_w})} \otimes \pi_2^{U_1(w)}) \oplus (\pi_1^{U_1(w)} \otimes \pi_2^{GL_{n_2}(\mathcal{O}_{F_w})}).$$

Moreover $U_w^{(j)}$ acts as

$$\begin{pmatrix} A & 0 \\ * & B \end{pmatrix}$$

where

$$A = \sum_{j_1+j_2=j} q_w^{(n_1j_2+n_2j_1)/2-j_1j_2}(T_w^{(j_1)} \otimes U_w^{(j_2)})$$

and

$$B = \sum_{j=j_1+j_2} q_w^{(n_1j_2+n_2j_1)/2-j_1j_2}(U_w^{(j_1)} \otimes T_w^{(j_2)})$$

and if $\alpha \in F_w^\times$ has positive valuation then V_α acts as

$$\begin{pmatrix} |\alpha|^{-n_1/2}(1 \otimes V_\alpha) & * \\ 0 & |\alpha|^{-n_2/2}(V_\alpha \otimes 1) \end{pmatrix}.$$

Proof: Let

$$\omega = \begin{pmatrix} 1_{n_1-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1_{n_2} & 0 \end{pmatrix}.$$

Then

$$GL_n(F_w) = P(F_w)U_1(w) \coprod P(F_w)\omega U_1(w)$$

so that

$$\begin{array}{rl} & (\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)}\pi_1 \otimes \pi_2)^{U_1(w)} \\ = & (\pi_1 \otimes \pi_2)^{P(F_w)\cap U_1(w)} \oplus (\pi_1 \otimes \pi_2)^{P(F_w)\cap \omega U_1(w)\omega^{-1}} \\ = & \pi_1^{GL_{n_1}(\mathcal{O}_{F_w})} \otimes \pi_2^{U_1(w)} \oplus \pi_1^{U_1(w)} \otimes \pi_2^{GL_{n_2}(\mathcal{O}_{F_w})}. \end{array}$$

Specifically $x \in \pi_1^{GL_{n_1}(\mathcal{O}_{F_w})} \otimes \pi_2^{U_1(w)}$ corresponds to a function φ_x supported on $P(F_w)U_1(w)$ with $\varphi_x(1) = x$, and $y \in \pi_1^{U_1(w)} \otimes \pi_2^{GL_{n_2}(\mathcal{O}_{F_w})}$ corresponds to a function φ'_y supported on $P(F_w)\omega U_1(w)$ with $\varphi'_y(\omega) = y$.

Choose a set X of representatives for $k(w)$ in $\mathcal{O}_{F_w}$, which contains 0. If $\varphi \in (\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)}\pi_1 \otimes \pi_2)^{U_1(w)}$ then

$$(U_w^{(j)}\varphi)(a) = \sum_I \sum_b \varphi_x(ab)$$

where I runs over j element subsets of $\{1, ..., n-1\}$ and where b runs over elements of $B_{n-1}(F_w)$ with

- $b_{ii} = \varpi_w$ if $i \in I$ and $= 1$ otherwise,
- $b_{ij} \in X$ if $j > i$, and $= 0$ unless $i \in I$ and $j \notin I$.

Thus
$$(U_w^{(j)}\varphi'_y)(1) = \sum_I \sum_b \varphi'_y(b) = 0$$
and
$$(U_w^{(j)}\varphi_x)(1) = \sum_{I_1,I_2} \sum_{a,b,c} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} x$$
where I_1 runs over subsets of $\{1,...,n_1\}$, I_2 runs over subsets of $\{1,...,n_2-1\}$, $a \in B_{n_1}(F_w)$, $b \in M_{n_1\times n_2}(F_w)$ and $c \in B_{n_2}(F_w)$ such that

- $\#I_1 + \#I_2 = j$,
- $a_{ii} = \varpi_w$ if $i \in I_1$ and $= 1$ otherwise,
- $c_{ii} = \varpi_w$ if $i \in I_2$ and $= 1$ otherwise,
- if $j > i$ then $a_{ij} \in X$ and $= 0$ unless $i \in I_1$ and $j \notin I_1$,
- if $j > i$ then $c_{ij} \in X$ and $= 0$ unless $i \in I_2$ and $j \notin I_2$,
- $b_{ij} \in X$ and $= 0$ unless $i \in I_1$ and $j \notin I_2$.

Equivalently
$$(U_w^{(j)}\varphi_x)(1) = \sum_{j_1+j_2=j} q_w^{(n_1j_2+n_2j_1)/2-j_1j_2}(T_w^{(j_1)} \otimes U_w^{(j_2)})x.$$

Similarly
$$(U_w^{(j)}\varphi'_y)(\omega) = \sum_{I_1,I_2} \sum_{a,b,c,d,e} \varphi'_y(\begin{pmatrix} a & c & b \\ 0 & 1 & 0 \\ 0 & e & d \end{pmatrix} \omega),$$
where $I_1 \subset \{1,...,n_1-1\}$, $I_2 \subset \{1,...,n_2\}$, $a \in B_{n_1-1}(F_w)$, $b \in M_{(n_1-1)\times n_2}(F_w)$, $c \in F_w^{n_1-1}$, $d \in B_{n_2}(F_w)$ and $e \in F_w^{n_2}$ with

- $\#I_1 + \#I_2 = j$,
- $a_{ii} = \varpi_w$ if $i \in I_1$ and $= 1$ otherwise,
- $d_{ii} = \varpi_w$ if $i \in I_2$ and $= 1$ otherwise,
- if $j > i$ then $a_{ij} \in X$ and $= 0$ unless $i \in I_1$ and $j \notin I_1$,
- if $j > i$ then $d_{ij} \in X$ and $= 0$ unless $i \in I_2$ and $j \notin I_2$,

- $b_{ij} \in X$ and $= 0$ unless $i \in I_1$ and $j \notin I_2$,
- $c_i \in X$ and $= 0$ unless $i \in I_1$,
- $e_i \in X$ and $= 0$ unless $i \in I_2$.

The matrix

$$\begin{pmatrix} a & c & b \\ 0 & 1 & 0 \\ 0 & e & d \end{pmatrix} \omega \in P(F_w)\omega U_1(w)$$

if and only if

$$\begin{pmatrix} a & c & b \\ 0 & 1 & 0 \\ 0 & d^{-1}e & 1_{n_2} \end{pmatrix} \in P(F_w)\omega U_1(w)\omega^{-1}$$

if and only if $e = 0$. Thus

$$(U_w^{(j)}\varphi'_y)(\omega) = \sum_{j=j_1+j_2} q_w^{(n_1j_2+n_2j_1)/2-j_1j_2}(U_w^{(j_1)} \otimes T_w^{(j_2)})y.$$

Now suppose $\alpha \in F_w^\times$ has non-negative valuation. If $\varphi \in (\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)}\pi_1 \otimes \pi_2)^{U_1(w)}$ then

$$(V_\alpha\varphi)(a) = \sum_{b \in (\mathcal{O}_{F_w}/(\alpha))^{n-1}} \varphi(a \begin{pmatrix} 1_{n-1} & 0 \\ \varpi_w b & \alpha \end{pmatrix}).$$

Thus

$$(V_\alpha\varphi_x)(1) = \sum_{b \in (\mathcal{O}_{F_w}/(\alpha))^{n_1}} \sum_{c \in (\mathcal{O}_{F_w}/(\alpha))^{n_2-1}} \varphi_x \begin{pmatrix} 1_{n_1} & 0 & 0 \\ 0 & 1_{n_2-1} & 0 \\ \varpi_w b & \varpi_w c & \alpha \end{pmatrix}.$$

However

$$\begin{pmatrix} 1_{n_1} & 0 & 0 \\ 0 & 1_{n_2-1} & 0 \\ \varpi_w b & \varpi_w c & \alpha \end{pmatrix} \in P(F_w)U_1(w)$$

if and only if

$$\begin{pmatrix} 1_{n_1} & 0 & 0 \\ 0 & 1_{n_2-1} & 0 \\ \alpha^{-1}\varpi_w b & 0 & 1 \end{pmatrix} \in P(F_w)U_1(w)$$

if and only if $b=0$. Hence

$$\begin{array}{rcl}(V_\alpha\varphi_x)(1) & = & \sum_{c\in(\mathcal{O}_{F_w}/(\alpha))^{n_2-1}}\varphi_x\begin{pmatrix}1_{n_1} & 0 & 0\\ 0 & 1_{n_2-1} & 0\\ 0 & \varpi_w c & \alpha\end{pmatrix}\\ & = & |\alpha|^{-n_1/2}(1\otimes V_\alpha)x.\end{array}$$

On the other hand

$$(V_\alpha\varphi_x)(\omega)=\sum_{b\in(\mathcal{O}_{F_w}/(\alpha))^{n_1-1}}\sum_{c\in(\mathcal{O}_{F_w}/(\alpha))^{n_2}}\varphi_x\left(\begin{pmatrix}1_{n_1-1} & 0 & 0\\ \varpi_w b & \alpha & \varpi_w c\\ 0 & 0 & 1_{n_2}\end{pmatrix}\omega\right)=0.$$

Similarly

$$\begin{array}{rl}(V_\alpha\varphi'_y)(\omega) & =\sum_{b\in(\mathcal{O}_{F_w}/(\alpha))^{n_1-1}}\sum_{c\in(\mathcal{O}_{F_w}/(\alpha))^{n_2}}\varphi_x\left(\begin{pmatrix}1_{n_1-1} & 0 & 0\\ \varpi_w b & \alpha & \varpi_w c\\ 0 & 0 & 1_{n_2}\end{pmatrix}\omega\right)\\ & =|\alpha|^{-n_2/2}(V_\alpha\otimes 1)y.\end{array}$$

The lemma follows. □

Lemma 2.1.5 *Suppose that π is an irreducible admissible representation of $GL_n(F_w)$ over $\overline{K}$ with a $U_1(w)$ fixed vector but no $GL_n(\mathcal{O}_{F_w})$-fixed vector. Then $\dim\pi^{U_1(w)}=1$ and there is a character with open kernel, $V_\pi:F_w^\times\to\overline{K}^\times$ such that $V_\pi(\alpha)$ is the eigenvalue of V_α on $\pi^{U_1(w)}$ for all $\alpha\in F_w^\times$ with non-negative valuation. For $j=1,...,n-1$, let $u_\pi^{(j)}$ denote the eigenvalue of $U_w^{(j)}$ on $\pi^{U_1(w)}$. Then there is an exact sequence*

$$(0)\to s\to r_l(\pi)^\vee(1-n)\to V_\pi\circ\mathrm{Art}_{F_w}^{-1}\to(0)$$

where s is unramified and $s(\mathrm{Frob}_w)$ has characteristic polynomial

$$X^{n-1}-u_\pi^{(1)}X^{n-2}+...+(-1)^j q_w^{j(j-1)/2}u_\pi^{(j)}X^{n-1-j}+...+(-1)^n q_w^{(n-1)(n-2)/2}u_\pi^{(n-1)}.$$

If $\pi^{U_0(w)}\neq(0)$ then $q_w^{-1}V_\pi(\varpi_w)$ is a root of

$$X^{n-1}-u_\pi^{(1)}X^{n-2}+...+(-1)^j q_w^{j(j-1)/2}u_\pi^{(j)}X^{n-1-j}+...+(-1)^n q_w^{(n-1)(n-2)/2}u_\pi^{(n-1)}.$$

If $\pi^{U_0(w)}=(0)$ then $r_l(\pi)^\vee(1-n)(\mathrm{Gal}(\overline{F}_w/F_w))$ is abelian.

Proof: If π is an irreducible, cuspidal, smooth representation of $GL_m(F_w)$ then the conductor of $\mathrm{rec}(\pi) \geq m$ unless $m = 1$ and π is unramified. If π is an irreducible, square integrable, smooth representation of $GL_m(F_w)$ then the conductor of $\mathrm{rec}(\pi) \geq m$ unless $\pi = \mathrm{Sp}_m(\chi)$ for some unramified character χ, in which case the conductor is $m - 1$. As any irreducible, square integrable, smooth representation π of $GL_m(F_w)$ is generic we see from [JPSS] that $\pi^{U_1(w)} \neq (0)$ if and only if either $m = 1$ and π has conductor ≤ 1, or $m = 2$ and $\pi = \mathrm{Sp}_2(\chi)$ for some unramified character χ of $F_w^\times$.

Now suppose that $n = n_1 + ... + n_r$ is a partition of n and let $P \supset B_n$ denote the corresponding parabolic. Let π_i be an irreducible, square integrable, smooth representation of $GL_{n_i}(F_w)$. If

$$(\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)} \pi_1 \otimes ... \otimes \pi_r)^{U_1(w)} \neq (0)$$

then by the last lemma there must exist an index i_0 such that:

- For $i \neq i_0$ we have $n_i = 1$ and π_i unramified.
- Either $n_{i_0} = 1$ and π_{i_0} has conductor ≤ 1 or $n_{i_0} = 2$ and $\pi_{i_0} = \mathrm{Sp}_2(\chi)$ for some unramified character χ of $F_w^\times$.

Thus if π is an irreducible smooth representation of $GL_n(F_w)$ with a $U_1(w)$ fixed vector but no $GL_n(\mathcal{O}_{F_w})$ fixed vector then

1. either $\pi = \chi_1 \boxplus ... \boxplus \chi_n$ with χ_i an unramified character of $F_w^\times$ for $i = 1, ..., n-1$ and with χ_n a character of $F_w^\times$ with conductor 1,
2. or $\pi = \chi_1 \boxplus ... \boxplus \chi_{n-2} \boxplus \mathrm{Sp}_2(\chi_{n-1})$ with χ_i an unramified character of $F_w^\times$ for $i = 1, ..., n-1$.

Consider first the first of these two cases. Let $\pi' = \chi_1 \boxplus ... \boxplus \chi_{n-1}$, an unramified representation of $GL_{n-1}(F_w)$. Also let $P \supset B_n$ denote the parabolic corresponding to the partition $n = (n-1) + 1$. As $(\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n))^{U_1(w)}$ and $(\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)} \pi' \otimes \chi_n)^{U_1(w)}$ are one dimensional we must have

$$\begin{aligned} \pi^{U_1(w)} &= (\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n))^{U_1(w)} \\ &= (\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)} \pi' \otimes \chi_n)^{U_1(w)} \\ &= (\pi')^{GL_{n-1}(\mathcal{O}_{F_w})} \otimes \chi_n. \end{aligned}$$

From the last lemma we see that $V_\pi = \chi_n |\ |^{(1-n)/2}$ and that $U_w^{(j)}$ acts as $q_w^{j/2} T_w^{(j)} \otimes 1$. In particular π has no $U_0(w)$ fixed vector. Because

$$r_l(\pi' \boxplus \chi_n)^\vee(1-n) = r_l(\pi')^\vee(2-n)|\mathrm{Art}\,_{F_w}^{-1}|^{-1/2} \oplus (V_\pi \circ \mathrm{Art}\,_{F_w}^{-1})$$

the lemma follows.

Consider now the second of our two cases. Let $\pi' = \chi_1 \boxplus ... \boxplus \chi_{n-2}$, an unramified representation of $GL_{n-2}(F_w)$. Also let $P \supset B_n$ (resp. $P' \supset B_n$) denote the parabolic corresponding to the partition $n = (n-2)+2$ (resp. $n = 1+...+1+2$). Because $\dim(\text{n-Ind}\,_{P'(F_w)}^{GL_n(F_w)} \chi_1 \otimes ... \otimes \chi_{n-2} \otimes \mathrm{Sp}\,_2(\chi_n))^{U_1(w)} = 1$ and $\dim(\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)} \pi' \otimes \mathrm{Sp}\,_2(\chi_n))^{U_1(w)} = 1$ we must have

$$\begin{aligned}\pi^{U_1(w)} &= (\text{n-Ind}\,_{P'(F_w)}^{GL_n(F_w)} \chi_1 \otimes ... \otimes \chi_{n-2} \otimes \mathrm{Sp}\,_2(\chi_n))^{U_1(w)} \\ &= (\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)} \pi' \otimes \mathrm{Sp}\,_2(\chi_n))^{U_1(w)} \\ &= (\pi')^{GL_{n-2}(\mathcal{O}_{F_w})} \otimes \mathrm{Sp}\,_2(\chi_n)^{U_1(w)}.\end{aligned}$$

Moreover V_α acts as $|\alpha|^{(2-n)/2}(1 \otimes V_\alpha)$ and $U_w^{(j)}$ acts as

$$q_w^j(T_w^{(j)} \otimes 1) + q_w^{n/2-1}(T_w^{(j-1)} \otimes U_w^{(1)}).$$

The representation $\text{n-Ind}\,_{B_2(F_w)}^{GL_2(F_w)}(\chi_n, \chi_n|\ \ |)$ has two irreducible constituents $(\chi_n|\ \ |^{1/2}) \circ \det$ and $\mathrm{Sp}\,_2(\chi_n)$. On $\mathrm{Sp}\,_2(\chi_n)^{U_1(w)}$ we have

$$V_\alpha = \begin{pmatrix} |\alpha|^{1/2}\chi_n(\alpha) & * \\ 0 & |\alpha|^{-1/2}\chi_n(\alpha) \end{pmatrix}$$

and

$$U_w^{(1)} = \begin{pmatrix} q_w^{1/2}\chi_n(\varpi_w) & 0 \\ * & q_w^{-1/2}\chi_n(\varpi_w) \end{pmatrix}.$$

On $(\chi_n|\ \ |^{1/2}) \circ \det$ we have

$$V_\alpha = |\alpha|^{1/2}\chi_n(\alpha)$$

and

$$U_w^{(1)} = q_w^{1/2}\chi_n(\varpi_w).$$

Thus on $\mathrm{Sp}\,_2(\chi_n)^{U_1(w)}$ we have

$$V_\alpha = |\alpha|^{-1/2}\chi_n(\alpha)$$

and

$$U_w^{(1)} = q_w^{-1/2}\chi_n(\varpi_w).$$

Hence on $\pi^{U_1(w)}$ we have

$$V_\alpha = |\alpha|^{(1-n)/2}\chi_n(\alpha)$$

and

$$U_w^{(j)} = q_w^j (T_w^{(j)} \otimes 1) + q_w^{(n-3)/2} (T_w^{(j-1)} \otimes \chi_n(\varpi_w)).$$

On the other hand

$$\begin{array}{l} (0) \to (r_l(\pi')^\vee(3-n)|\mathrm{Art}\,_{F_w}^{-1}|^{-1} \oplus (\chi_n|\ \ |^{(3-n)/2}) \circ \mathrm{Art}\,_{F_w}^{-1}) \to \\ \to r_l(\pi' \boxplus \mathrm{Sp}\,_2(\chi_n))^\vee(1-n) \to (\chi_n|\ \ |^{(1-n)/2}) \circ \mathrm{Art}\,_{F_w}^{-1} \to (0). \end{array}$$

This is a short exact sequence of the desired form and $s(\mathrm{Frob}_w)$ has characteristic polynomial $(X - q_w^{(n-3)/2}\chi_n(\varpi_w))$ times

$$X^{n-2} - q_w t^{(1)} X^{n-3} + ... + (-1)^j q_w^{j+j(j-1)} t^{(j)} X^{n-2-j} + ... + (-1)^n q_w^{(n-2)+(n-2)(n-3)} t^{(n-2)},$$

where $t^{(j)}$ is the eigenvalue of $T_w^{(j)}$ on $(\pi')^{GL_{n-2}(\mathcal{O}_{F_w})}$. The lemma follows. $\Box$

Lemma 2.1.6 *Let π be an irreducible smooth representation of $GL_n(F_w)$ over $\overline{K}$.*

1. *If $\pi^{\mathrm{Iw}_1(w)} \neq (0)$ then $r_l(\pi)^\vee(1-n)^{\mathrm{ss}}$ is a direct sum of one dimensional representations.*
2. *Suppose*
$$\chi = (\chi_1, ..., \chi_n) : (k(w)^\times)^n \to \overline{K}^\times$$
and $\chi_i \neq \chi_j$ whenever $i \neq j$. If $\pi^{\mathrm{Iw}_0(w),\chi} \neq (0)$ Then
$$r(\pi)^\vee(1-n)|_{I_{F_w}} = (\chi_1 \circ \mathrm{Art}\,_{F_w}^{-1}) \oplus ... \oplus (\chi_n \circ \mathrm{Art}\,_{F_w}^{-1}).$$
(Here we think of χ_i as a character of $\mathcal{O}_{F_w}^\times \twoheadrightarrow k(w)^\times$.)

Proof: The key point is that $\pi^{\mathrm{Iw}_1(w)} \neq (0)$ if and only if π is a subquotient of a principal series representation $\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1', ..., \chi_n')$ with each χ_i' tamely ramified. More precisely $\pi^{\mathrm{Iw}_0(w),\chi} \neq (0)$ if and only if π is a subquotient of a principal series representation $\text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1', ..., \chi_n')$ with each $\chi_i'|_{\mathcal{O}_{F_w}^\times} = \chi_i$. (See theorem 7.7 of [Ro]. In section 4 of that article some restrictions were placed on the characteristic of $\mathcal{O}_{F_w}/\wp_w$. However it is explained in remark 4.14 how these restrictions can be avoided in the case of GL_n. More precisely it is explained how to avoid these restrictions in the proof of theorem 6.3. The proof of theorem 7.7 relies only on lemma 3.6 and, via lemma 7.6, on lemma 6.2 and theorem 6.3. Lemmas 3.6 and 6.2 have no restrictions on the characteristic.) $\Box$

2.2 GL_n over a local field: finite characteristic theory.

We will keep the notation and assumptions of the last section. Let $l \not| q_w$ be a rational prime, K a finite extension of the field of fractions of the Witt vectors of an algebraic extension of $\mathbb{F}_l$, $\mathcal{O}$ the ring of integers of K, λ the maximal ideal of $\mathcal{O}$ and $k = \mathcal{O}/\lambda$. We will call l *quasi-banal* for $GL_n(F_w)$ if either $l \not| \# GL_n(k(w))$ (the *banal case*), or $l > n$ and $q_w \equiv 1 \bmod l$ (the *limit case*).

Lemma 2.2.1 *Suppose that $l > n$ and $l|(q_w - 1)$. Suppose also that π is an unramified irreducible smooth representation of $GL_n(F_w)$ over $\overline{\mathbb{F}}_l$. Then* $\dim \pi^{GL_n(\mathcal{O}_{F_w})} = 1$. *Let $t_\pi^{(j)}$ denote the eigenvalue of $T_w^{(j)}$ on $\pi^{GL_n(\mathcal{O}_{F_w})}$. Set*

$$P_\pi(X) = X^n - t_\pi^{(1)} X^{n-1} + ... + (-1)^j q_w^{j(j-1)/2} t_\pi^{(j)} X^{n-j} + ... + (-1)^n q_w^{n(n-1)/2} t_\pi^{(n)}.$$

Suppose that $P_\pi(X) = (X-a)^m Q(X)$ with $m > 0$ and $Q(a) \neq 0$. Then

$$Q(V_{\varpi_w})\pi^{GL_n(\mathcal{O}_{F_w})} \neq (0).$$

(Considered in $\pi^{U_0(w)}$.)

Proof: According to assertion VI.3 of [V2] we can find a partition $n = n_1 + ... + n_r$ corresponding to a parabolic $P \supset B_n$ and distinct, unramified characters $\chi_1, ..., \chi_r : F_w^\times \to \overline{\mathbb{F}}_l^\times$ such that $\pi = \text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)}(\chi_1 \circ \det, ..., \chi_r \circ \det)$. Then

$$P_\pi(X) = \prod_{i=1}^r (X - \chi_i(\varpi_w))^{n_i}.$$

Suppose without loss of generality that $a = \chi_1(\varpi_w)$.

For $i = 1, ..., r$ set $w_i' = w_{n, n_1 + ... + n_i - 1}$. Then $\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)}(\chi_1 \circ \det, ..., \chi_r \circ \det)^{U_0(w)}$ has a basis consisting of functions φ_i for $i = 1, ..., r$, where the support of φ_i is $P(F_w) w_i' U_0(w)$ and $\varphi_i(w_i') = 1$. Note that $\text{n-Ind}\,_{P(F_w)}^{GL_n(F_w)}(\chi_1 \circ \det, ..., \chi_r \circ \det)^{GL_n(\mathcal{O}_{F_w})}$ is spanned by $\varphi_1 + ... + \varphi_r$.

We have

$$V_{\varpi_w} \varphi_i = \sum_j (V_{\varpi_w} \varphi_i)(w_j') \varphi_j.$$

But, as in the proof of lemma 2.1.3, we also have

$$(V_{\varpi_w} \varphi_i)(w_j') = \chi_j(\varpi_w) \sum_{x \in X^{n_1 + ... + n_j - 1}} \varphi_i \begin{pmatrix} 1_{n_1 + ... + n_j - 1} & 0 & 0 \\ x & 0 & 1 \\ 0 & w_{n_{j+1} + ... + n_r} & 0 \end{pmatrix},$$

where X is a set of representatives for $k(w)$ in $\mathcal{O}_{F_w}$. A matrix $g \in GL_n(\mathcal{O}_{F_w})$ lies in $P(\mathcal{O}_{F_w})w_i'U_0(w)$ if and only if i is the largest integer such that $(0, ..., 0, 1)$ lies in the $k(w)$ span of the reduction modulo $\wp_w$ of the last $n_i + ... + n_r$ rows of g. Thus

$$(V_{\varpi_w}\varphi_i)(w_j')$$

is

- 0 if $i > j$,
- $q_w^{n_i-1}\chi_j(\varpi_w) = \chi_j(\varpi_w)$ if $i = j$, and
- $(q_w^{n_i} - 1)q_w^{n_{i+1}+...+n_j-1}\chi_j(\varpi_w) = 0$ if $i < j$.

Thus, for $i = 1, ..., r$, we have

$$V_{\varpi_w}\varphi_i = \chi_i(\varpi_w)\varphi_i$$

and

$$Q(V_{\varpi_w})(\varphi_1 + ... + \varphi_r) = Q(\chi_1(\varpi_w))\varphi_1$$

and the lemma follows. □

Lemma 2.2.2 *Suppose that $l > n$ and $l|(q_w - 1)$. Let R be a complete local $\mathcal{O}$-algebra. Let M be an R-module with a smooth action of $GL_n(F_w)$ such that for all open compact subgroups $U \subset GL_n(F_w)$ the module of invariants M^U is finite and free over $\mathcal{O}$. Suppose also that for $j = 1, .., n$ there are elements $t_j \in R$ with $T_w^{(j)} = t_j$ on $M^{GL_n(\mathcal{O}_{F_w})}$. Set*

$$P(X) = X^n + \sum_{j=1}^{n}(-1)^j q_w^{j(j-1)/2} t_j X^{n-j} \in R[X].$$

Suppose that in $R[X]$ we have a factorisation $P(X) = (X - a)Q(X)$ with $Q(a) \in R^\times$. Suppose finally that $M \otimes_{\mathcal{O}} \overline{K}$ is semi-simple over $R[GL_n(F_w)]$ and that, if π is an R-invariant irreducible $GL_n(F_w)$-constituent of $M \otimes_{\mathcal{O}} \overline{K}$ with a $U_0(w)$-fixed vector, then either π is unramified or

$$P(X) = (X - V_{\varpi_w})(X^{n-1} - U_w^{(1)}X^{n-2} + ... + (-1)^j q_w^{j(j-1)/2}U_w^{(j)}X^{n-1-j} + \\ + ... + (-1)^n q_w^{(n-1)(n-2)/2}U_w^{(n-1)})$$

on $\pi^{U_0(w)}$. Then $Q(V_\varpi)$ gives an isomorphism

$$Q(V_{\varpi_w}) : M^{GL_n(\mathcal{O}_{F_w})} \xrightarrow{\sim} M^{U_0(w), V_{\varpi_w}=a}.$$

Proof: Lemma 2.1.3 tells us that

$$Q(V_{\varpi_w}) : M^{GL_n(\mathcal{O}_{F_w})} \longrightarrow M^{U_0(w), V_{\varpi_w}=a}.$$

Let π be an R-invariant irreducible $GL_n(F_w)$-constituent of $M \otimes_{\mathcal{O}} \overline{K}$ with $\pi^{U_0(w), V_{\varpi_w}=a} \neq (0)$. If π is ramified then lemma 2.1.5 tells us that

$$\begin{aligned}(q_w^{-1}a)^{n-1} - U_w^{(1)}(q_w^{-1}a)^{n-2} ... + (-1)^j q_w^{j(j-1)/2} U_w^{(j)} (q_w^{-1}a)^{n-1-j} + \\ + ... + (-1)^n q_w^{(n-1)(n-2)/2} U_w^{(n-1)} = 0\end{aligned}$$

on $\pi^{U_0(w)}$. Thus $Q(a) \in \mathfrak{m}_R$, which contradicts our hypothesis. Thus π is unramified. Lemma 2.1.3 and the assumption that a is a simple root of $P(X)$, we see that $\dim \pi^{U_0(w), V_{\varpi_w}=a} \leq 1 = \dim \pi^{GL_n(\mathcal{O}_{F_w})}$. Thus

$$\dim(M \otimes_{\mathcal{O}} \overline{K})^{U_0(w), V_{\varpi_w}=a} \leq \dim(M \otimes_{\mathcal{O}} \overline{K})^{GL_n(\mathcal{O}_{F_w})}.$$

Hence it suffices to show that $Q(V_{\varphi_w}) \otimes \overline{k}$ is injective. Suppose not. Choose a non-zero vector $x \in \ker(Q(V_{\varphi_w}) \otimes \overline{k})$ such that $\mathfrak{m}_R x = (0)$. Let N' denote the $\overline{k}[GL_n(F_w)]$-submodule of $M \otimes_{\mathcal{O}} \overline{k}$ generated by x. Let N denote an irreducible quotient of N'. Then by lemma 2.2.1

$$Q(V_{\varpi_w}) N^{GL_n(\mathcal{O}_{F_w})} \neq (0),$$

a contradiction and the lemma is proved. $\Box$

Suppose that U is an open subgroup of $GL_n(\mathcal{O}_{F_w})$ and that

$$\phi : \overline{k}[GL_n(\mathcal{O}_{F_w}) \backslash GL_n(F_w) / GL_n(\mathcal{O}_{F_w})] \longrightarrow \overline{k}$$

is a $\overline{k}$-algebra homomorphism. Set

$$\begin{aligned} & \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi \\ = \; & \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \otimes_{\overline{k}[GL_n(\mathcal{O}_{F_w}) \backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})], \phi} \overline{k} \end{aligned}$$

and

$$\begin{aligned} & \overline{k}[U \backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi \\ = \; & \overline{k}[U \backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \otimes_{\overline{k}[GL_n(\mathcal{O}_{F_w}) \backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})], \phi} \overline{k} \end{aligned}$$

If V is any smooth $\overline{k}[GL_n(F_w)]$-module and if $v \in V^{GL_n(\mathcal{O}_{F_w})}$ satisfies $Tv = \phi(T)v$ for all $T \in \overline{k}[GL_n(\mathcal{O}_{F_w}) \backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$, then there is a unique map of $\overline{k}[GL_n(F_w)]$-modules

$$\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi \longrightarrow V$$

sending $[GL_n(\mathcal{O}_{F_w})]$ to v, and a unique map of $\overline{k}[U\backslash GL_n(F_w)/U]$-modules

$$\overline{k}[U\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi \longrightarrow V^U$$

sending $[GL_n(\mathcal{O}_{F_w})]$ to v. (These observations were previously used in a similar context by Lazarus [La].)

Fix an additive character $\psi : F_w \to \overline{k}$ with kernel $\mathcal{O}_{F_w}$. Let B_n denote the Borel subgroup of GL_n consisting of upper triangular matrices and let N_n denote its unipotent radical. Let P_n denote the subgroup of GL_n consisting of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a \in GL_{n-1}$. We will think of ψ as a character of $N_n(F_w)$ by

$$\psi : \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n-1} & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n-1} & a_{2n} \\ 0 & 0 & 1 & \dots & a_{3n-1} & a_{3n} \\ & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_{n-1n} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \longmapsto \psi(a_{12} + a_{23} + \dots + a_{n-1n}).$$

We will gen_n the compact induction $\text{c-Ind}\,_{N_n(F_w)}^{P_n(F_w)}\psi$ and by $\mathcal{W}_n$ the induction $\mathrm{Ind}\,_{N_n(F_w)}^{GL_n(F_w)}\psi$. We will use the theory of derivatives over $\overline{k}$ as it is developed in section III.1 of [V1]. Note that if π is a smooth $\overline{k}[GL_n(F_w)]$-module then

$$\mathrm{Hom}\,_{GL_n(F_w)}(\pi, \mathcal{W}_n) \cong \pi^\vee_{N_n(F_w),\psi} \cong \mathrm{Hom}\,_{P_n(F_w)}(\mathrm{gen}_n, \pi)^\vee,$$

where $\vee$ denote linear dual and $\pi_{N_n(F_w),\psi}$ denotes the maximal quotient of π on which $N_n(F_w)$ acts by ψ. If π is irreducible we will call it *generic* if these spaces are non-trivial.

The next lemma is proved exactly as in characteristic zero (see [Sh]).

Lemma 2.2.3 *Suppose that $\phi : \overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \to \overline{k}$ is a homomorphism. Then the ϕ eigenspace in $\mathcal{W}_n^{GL_n(\mathcal{O}_{F_w})}$ is one dimensional and spanned by a function W^0_ϕ with $W^0_\phi(1) = 1$.*

The next lemma is due to Vignéras, see parts 1 and 3 of theorem 1 of her appendix to this article.

Lemma 2.2.4 (Vignéras) *Suppose that l is quasi-banal for $GL_n(F_w)$. Then the functor $V \mapsto V^{\mathrm{Iw}(w)}$ is an equivalence of categories from the category of smooth $\overline{k}[GL_n(F_w)]$-modules generated by their $\mathrm{Iw}(w)$-fixed vectors to the category of $\overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/\mathrm{Iw}(w)]$-modules. Moreover the category of smooth $\overline{k}[GL_n(F_w)]$-modules generated by their $\mathrm{Iw}(w)$-fixed vectors is closed under passage to subquotients (in the category of smooth $\overline{k}[GL_n(F_w)]$-modules).*

Lemma 2.2.5 *Suppose that l is quasi-banal for $GL_n(F_w)$ and that*

$$\phi : \overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \longrightarrow \overline{k}$$

is a $\overline{k}$-algebra homomorphism. Then $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi$ has finite length (as a smooth $\overline{k}[GL_n(F_w)]$-module) and its Jordan-Holder constituents are the same as those of any unramified principal series representation π for which $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ acts on $\pi^{GL_n(\mathcal{O}_{F_w})}$ by ϕ. In particular the smooth representation $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi$ has exactly one generic irreducible subquotient.

Proof: In the banal case this is due to Lazarus [La].

By lemma 2.2.4 the $\mathrm{Iw}(w)$-invariants functor is exact on the category of subquotients of smooth $k[GL_n(F_w)]$-modules generated by their $\mathrm{Iw}(w)$-fixed vectors.

Let $T_1, ..., T_{n+1}$ generate $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ as a $\overline{k}$-algebra. Then we have an exact sequence

$$\begin{array}{l}(0) \rightarrow \sum_i \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})](T_i - \phi(T_i)) \rightarrow \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \rightarrow \\ \rightarrow \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi \rightarrow (0).\end{array}$$

Thus

$$\begin{array}{l}(0) \rightarrow (\sum_i \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})](T_i - \phi(T_i)))^{\mathrm{Iw}(w)} \rightarrow \\ \rightarrow \overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \rightarrow (\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi)^{\mathrm{Iw}(w)} \rightarrow (0)\end{array}$$

is exact. On the other hand if A and B are two $\overline{k}[GL_n(F_w)]$-submodules of $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ then the exact sequence

$$(0) \rightarrow A \rightarrow A + B \rightarrow B/(A \cap B) \rightarrow (0)$$

gives an exact sequence

$$(0) \rightarrow A^{\mathrm{Iw}(w)} \rightarrow (A + B)^{\mathrm{Iw}(w)} \rightarrow B^{\mathrm{Iw}(w)}/(A^{\mathrm{Iw}(w)} \cap B^{\mathrm{Iw}(w)}) \rightarrow (0).$$

Thus

$$(A + B)^{\mathrm{Iw}(w)} = A^{\mathrm{Iw}(w)} + B^{\mathrm{Iw}(w)}$$

and we get an exact sequence

$$(0) \to \sum_i (\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})](T_i - \phi(T_i)))^{\mathrm{Iw}(w)} \to$$
$$\to \overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \to (\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi)^{\mathrm{Iw}(w)} \to (0).$$

As

$$(T_i - \phi(T_i)) : \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \twoheadrightarrow \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})](T_i - \phi(T_i))$$

we see that $(T_i - \phi(T_i))$ maps

$$\overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \twoheadrightarrow (\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})](T_i - \phi(T_i)))^{\mathrm{Iw}(w)}.$$

Finally we get an exact sequence

$$(0) \to \sum_i \overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})](T_i - \phi(T_i)) \to$$
$$\to \overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \to (\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi)^{\mathrm{Iw}(w)} \to (0)$$

and we see that

$$(\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi)^{\mathrm{Iw}(w)} = \overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi.$$

Following Kato and Lazarus [La] we see that the Satake isomorphism extends to an isomorphism

$$\overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \cong \overline{k}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$$

as $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \cong \overline{k}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^{S_n}$-modules. We deduce immediately that

$$\dim_{\overline{k}} \overline{k}[\mathrm{Iw}(w)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi = n!$$

and hence (from lemma 2.2.4) that $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi$ has finite length. Moreover the argument of section 3.3 of [La] then shows that the Jordan-Holder constituents of $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi$ are the same as the Jordan-Holder constituents of any unramified principal series representation π for which $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ acts on $\pi^{GL_n(\mathcal{O}_{F_w})}$ by ϕ. The final assertion of the lemma then follows from the results of section III.1 of [V1]. □

We will now recall some results of Russ Mann [M1] and [M2]. See also appendix A of this article.

The first result follows at once from proposition 4.4 of [M1].

Lemma 2.2.6 (Mann) *Suppose that* $\chi_1, ..., \chi_n$ *are unramified characters* $F_w^\times \to \overline{K}^\times$ *and set* $\pi = \text{n-Ind}_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n)$. *The simultaneous eigenspaces of the operators* $U_w^{(j)}$ *(for* $j = 1, ..., n-1$*) on* $\pi^{U_1(w^n)}$ *are parametrised by subsets* $A \subset \{1, ..., n\}$ *of cardinality less than* n. *Let* $u_A^{(j)}$ *denote the eigenvalue of* $U_w^{(j)}$ *on the eigenspace corresponding to* A. *Then*

$$\begin{array}{l} X^n - q_w^{(1-n)/2} u_A^{(1)} X^{n-1} + ... + (-1)^j q_w^{j(j-n)/2} u_A^{(j)} X^{n-j} + \\ + ... + (-1)^{n-1} q_w^{(n-1)/2} u_A^{(n-1)} X = X^{n-\#A} \prod_{i \in A}(X - \chi_i(\varpi_w)). \end{array}$$

Moreover the generalised eigenspace corresponding to a subset A *has dimension* $\begin{pmatrix} n-1 \\ \#A \end{pmatrix}$.

The next two results are proved in [M2]. As this is not currently available, proofs repeated in appendix A.

Lemma 2.2.7 (Mann) *Suppose that*

$$\phi : \overline{k}[GL_n(\mathcal{O}_{F_w}) \backslash GL_n(F_w) / GL_n(\mathcal{O}_{F_w})] \longrightarrow \overline{k}$$

is a homomorphism. Then the map

$$\begin{array}{rcl} \overline{k}[U_1(w^n) \backslash GL_n(F_w) / GL_n(\mathcal{O}_{F_w})]_\phi & \longrightarrow & \mathcal{W}_n \\ T & \longmapsto & TW_\phi^0 \end{array}$$

is an injection.

Let η_w denote the diagonal matrix $\text{diag}(1, ..., 1, \varpi_w^n)$. Then there is a bijection $\widehat{\ }$:

$$\begin{array}{rcl} \mathbb{Z}[1/q_w][U_1(w^n) \backslash GL_n(F_w) / GL_n(\mathcal{O}_{F_w})] & \longrightarrow & \mathbb{Z}[1/q_w][GL_n(\mathcal{O}_{F_w}) \backslash GL_n(F_w) / U_1(w^n)] \\ {[U_1(w^n) g GL_n(\mathcal{O}_{F_w})]} & \longmapsto & [GL_n(\mathcal{O}_{F_w}) {}^t g \eta_w^{-1} U_1(w^n)]. \end{array}$$

(This is because $U_1(w^n) = \eta_w {}^t U_1(w^n) \eta_w^{-1}$.)

Proposition 2.2.8 (Mann) *There exists an element*

$$\theta_{n,w} \in \mathbb{Z}_l[U_1(w^n) \backslash GL_n(F_w) / GL_n(\mathcal{O}_{F,w})]$$

with the following properties.

1. *For* $i = 1, ..., n-1$ *we have* $U_w^{(i)} \theta_{n,w} = 0$.
2. *For any homomorphism* $\phi : \overline{k}[GL_n(\mathcal{O}_{F_w}) \backslash GL_n(F_w) / GL_n(\mathcal{O}_{F_w})] \to \overline{k}$ *we have* $\theta_{n,w} W_\phi^0 \neq 0$ *in* $\mathcal{W}_n$.

3. *If* $\chi_1, ..., \chi_n$ *are unramified characters* $F_w^\times \to K^\times$ *such that the induced representation* $\pi = \text{n-Ind}\,_{B_n(F_w)}^{GL_n(F_w)}(\chi_1, ..., \chi_n)$ *is irreducible, and if* $0 \neq v \in \pi^{GL_n(\mathcal{O}_{F_w})}$ *then* $\theta_{n,w}v$ *is nonzero and so a basis of* $\pi^{U_1(w^n), U_w^{(1)} = ... = U_w^{(n-1)} = 0}$.

4. *The composite*

$$\widehat{\theta}_{n,w}\theta_{n,w} \in \mathbb{Z}_l[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$$

has Satake transform

$$q_w^{n^2(n-1)/2}(X_1...X_n)^{-(n+1)} \prod_{i=1}^n \prod_{j=1}^n (q_w X_i - X_j).$$

Corollary 2.2.9 *Suppose that* π *is an irreducible unramified representation of* $GL_n(F_w)$ *over* $\overline{K}$ *such that* $r_l(\pi)^\vee(1-n)$ *is defined over* K. *If* $\widehat{\theta}_{n,w}\theta_{n,w}$ *acts on* $\pi^{GL_n(\mathcal{O}_{F,w})}$ *by* α *then* $\alpha \in \mathcal{O}$ *and*

$$\lg_{\mathcal{O}} \mathcal{O}/\alpha \geq \lg_{\mathcal{O}} H^0(\mathrm{Gal}\,(\overline{F}_w/F_w), (\mathrm{ad}\, r_l(\pi)^\vee(1-n)) \otimes_{\mathcal{O}} (K/\mathcal{O})(-1)).$$

Let M be an admissible $\overline{k}[GL_n(F_w)]$-module. We will say that M has the *Ihara property* if for every $v \in M^{GL_n(\mathcal{O}_{F_w})}$ which is an eigenvector of $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$, every irreducible submodule of the $\overline{k}[GL_n(F_w)]$-module generated by v is generic.

Lemma 2.2.10 *Suppose that* l *is quasi-banal for* $GL_n(F_w)$. *Suppose also that* M *is an admissible* $\overline{k}[GL_n(F_w)]$*-module with the Ihara property and that*

$$\ker(\theta_{n,w} : M^{GL_n(\mathcal{O}_{F_w})} \longrightarrow M)$$

is a $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$*-module. Then*

$$\theta_{n,w} : M^{GL_n(\mathcal{O}_{F_w})} \hookrightarrow M^{U_1(w^n), U_w^{(1)} = ... = U_w^{(n-1)} = 0}$$

is injective.

Proof: Suppose $\theta_{n,w}$ were not injective on $M^{GL_n(\mathcal{O}_{F,w})}$. We could choose a $\overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$-eigenvector $0 \neq v \in \ker \theta_{n,w}$, say

$$Tv = \phi(T)v$$

where

$$\phi : \overline{k}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \longrightarrow \overline{k}$$

is a $\overline{k}$-algebra homomorphism.

Let A denote the kernel of the map

$$\begin{array}{rcl} \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi & \longrightarrow & \mathcal{W}_n \\ T & \longmapsto & TW_\phi^0. \end{array}$$

Thus A has no generic subquotient and $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi/A$ has a unique irreducible submodule B/A. The module B/A is generic, but no subquotient of $\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi/B$ is generic.

Now consider the map

$$\begin{array}{rcl} \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi & \longrightarrow & M \\ T & \longmapsto & Tv. \end{array}$$

As M has the Ihara property, any irrreducible submodule of the image is generic. Thus A is contained in the kernel and moreover the induced map

$$\overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi/A \longrightarrow M$$

must be injective. Thus we have an injection

$$\begin{array}{rcl} \langle GL_n(F_w)W_\phi^0\rangle & \hookrightarrow & M \\ W_\phi^0 & \longmapsto & v. \end{array}$$

Proposition 2.2.8 then tells us that $\theta_{n,w}v \neq 0$, a contradiction. □

We would conjecture that the previous lemma remains true without the quasi-banal hypothesis. In fact, it is tempting to conjecture that the natural map

$$\begin{array}{rcl} \overline{k}[GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]_\phi & \longrightarrow & \mathcal{W}_n \\ [GL_n(\mathcal{O}_{F_w})] & \longmapsto & W_\phi^0 \end{array}$$

is in general injective.

2.3 Automorphic forms on unitray groups.

Fix a positive integer $n \geq 2$ and a prime $l > n$.

Fix an imaginary quadratic field E in which l splits and a totally real field F^+. Fix a finite non-empty set of places $S(B)$ of places of F^+ with the following properties:

- Every element of $S(B)$ splits in F.
- $S(B)$ contains no place above l.

- If n is even then

$$n[F^+ : \mathbb{Q}]/2 + \#S(B) \equiv 0 \bmod 2.$$

Choose a division algebra B with centre F with the following properties:

- $\dim_F B = n^2$.
- $B^{\mathrm{op}} \cong B \otimes_{E,c} E$.
- B splits outside $S(B)$.
- If $\widetilde{v}$ is a prime of F above an element of $S(B)$, then $B_{\widetilde{v}}$ is a division algebra.

If $\ddagger$ is an involution on B with $\ddagger|_F = c$ then we can define a reductive algebraic group $G_\ddagger/F^+$ by setting

$$G_\ddagger(R) = \{g \in B \otimes_{F^+} R : \ g^{\ddagger\otimes 1} g = 1\}$$

for any F^+-algebra R. Fix an involution $\ddagger$ on B such that

- $\ddagger|_F = c$,
- for a place $v|\infty$ of F^+ we have $G_\ddagger(F_v^+) \cong U(n)$, and
- for a finite place $v \notin S(B)$ of F^+ the group $G_\ddagger(F_v^+)$ is quasi-split.

This is always possible, by an argument exactly analogous to the proof of lemma 1.7.1 of [HT]. From now on we will write G for $G_\ddagger$.

We will also define an algebraic group G'/F^+ by setting

$$G'(R) = \{g \in B^{\mathrm{op}} \otimes_{F^+} R : \ g^{\ddagger\otimes 1} g = 1\}$$

for any F^+-algebra R. Note that there is an isomorphism

$$\begin{array}{rcl} I : G & \stackrel{\sim}{\longrightarrow} & G' \\ g & \longmapsto & g^{-1}. \end{array}$$

We can choose an order $\mathcal{O}_B$ in B such that $\mathcal{O}_B^\ddagger = \mathcal{O}_B$ and $\mathcal{O}_{B,w}$ is maximal for all primes w of F which are split over F^+. (Start with any order. Replacing it by its intersection with its image under $\ddagger$ gives an order $\mathcal{O}'_B$ with $(\mathcal{O}'_B)^\ddagger = \mathcal{O}'_B$. For all but finitely many primes v of F^+ the completion $\mathcal{O}'_{B,v}$ will be a maximal order in B_v. Let R denote the finite set of primes which split in F

and for which $\mathcal{O}'_{B,v}$ is not maximal. For $v \in R$ choose a maximal order $\mathcal{O}''_{B,v}$ of B_v with $(\mathcal{O}''_{B,v})^\ddagger = \mathcal{O}''_{B,v}$ (e.g. $\mathcal{O}_{B,w} \oplus \mathcal{O}^\ddagger_{B,w}$ where w is a prime of F above v and $\mathcal{O}_{B,w}$ is a maximal order in B_w). Let $\mathcal{O}_B$ be the unique order with $\mathcal{O}_{B,v} = \mathcal{O}''_{B,v}$ if $v \in R$ and $\mathcal{O}_{B,v} = \mathcal{O}'_{B,v}$ otherwise.) This choice gives models of G and G' over $\mathcal{O}_{F^+}$. (These models may be very bad at primes v which do not split in F, but this will not concern us.)

Let v be a place of F^+ which splits in F. If $v \notin S(B)$ choose an isomorphism $i_v : \mathcal{O}_{B,v} \xrightarrow{\sim} M_n(\mathcal{O}_{F_v})$ such that $i_v(x^\ddagger) = {}^t i_v(x)^c$. The choice of a prime w of F above v then gives us an identification

$$\begin{array}{rcl} i_w : G(F_v^+) & \xrightarrow{\sim} & GL_n(F_w) \\ i_v^{-1}(x, {}^t x^{-c}) & \longmapsto & x \end{array}$$

with $i_w G(\mathcal{O}_{F^+,v}) = GL_n(\mathcal{O}_{F,w})$ and $i_{w^c} = {}^t(c \circ i_w)^{-1}$. Using ${}^t i_v$ in place of i_v we also get

$$i_w^t : G'(F_v^+) \xrightarrow{\sim} GL_n(F_w)$$

with $i_w^t G'(\mathcal{O}_{F^+,v}) = GL_n(\mathcal{O}_{F,w})$ and $i_w^t \circ I = {}^t(i_w)^{-1} = c \circ i_{w^c}$. If $v \in S(B)$ and w is a prime of F above v we get an isomorphism

$$i_w : G(F_v^+) \xrightarrow{\sim} B_w^\times$$

with $i_w G(\mathcal{O}_{F^+,v}) = \mathcal{O}_{B,w}^\times$ and $i_{w^c} = i_w^{-\ddagger}$. We also get

$$i'_w : G'(F_v^+) \xrightarrow{\sim} (B_w^{\mathrm{op}})^\times$$

with $i'_w G'(\mathcal{O}_{F^+,v}) = \mathcal{O}^\times_{B^{\mathrm{op}},w}$.

Let S_l denote the primes of F^+ above l and let $T \supset S_l \cup S(B)$ denote a finite set of primes of F^+ which split in F. Fix a set $\widetilde{T}$ of primes of F such that $\widetilde{T} \coprod {}^c\widetilde{T}$ is the set of all primes of F above T. If $S \subset T$ write $\widetilde{S}$ for the preimage of S in $\widetilde{T}$. If $v \in T$ we will write $\widetilde{v}$ for the element of $\widetilde{T}$ above v. Write S_∞ for the set of infinite places of F^+.

Let k be an algebraic extension of $\mathbb{F}_l$ and K a finite, totally ramified extension of the fraction field of the Witt vectors of k such that K contains the image of every embedding $F \hookrightarrow \overline{K}$. Let $\mathcal{O}$ denote the ring of integers of K and let λ denote its maximal ideal. Let I_l denote the set of embeddings $F^+ \hookrightarrow K$, so that there is a natural surjection $I_l \twoheadrightarrow S_l$. Let $\widetilde{I}_l$ denote the set of embeddings $F \hookrightarrow K$ which give rise to a prime of $\widetilde{S}_l$. Thus there is a natural bijection $\widetilde{I}_l \xrightarrow{\sim} I_l$.

For an n-tuple of integers $a = (a_1, ..., a_n)$ with $a_1 \geq ... \geq a_n$ there is an irreducible representation defined over $\mathbb{Q}$:

$$\xi_a : GL_n \longrightarrow GL(W_a)$$

with highest weight

$$\mathrm{diag}(t_1, ..., t_n) \longmapsto \prod_{i=1}^{n} t_i^{a_i}.$$

(N.B. This is not the same convention used in [HT].) There is also a (unique up to scalar multiples) perfect pairing

$$\langle\ ,\ \rangle_a : W_a \times W_a \longrightarrow \mathbb{Q}$$

such that

$$\langle \xi_a(g)w, w' \rangle_a = \langle w, \xi_a({}^t g)w' \rangle_a$$

for all $w, w' \in W_a$ and $g \in GL_n(\mathbb{Q})$. We can choose a model

$$\xi_a : GL_n \longrightarrow GL(M_a)$$

of ξ_a over $\mathbb{Z}$. (So M_a is a $\mathbb{Z}$-lattice in W_a.) Let M_a' denote the $\langle\ ,\ \rangle_a$ dual of M_a and

$$\xi_a' : GL_n \longrightarrow GL(M_a')$$

the corresponding model over $\mathbb{Z}$ of ξ_a.

Let Wt_n denote the subset of $(\mathbb{Z}^n)^{\mathrm{Hom}\,(F,\overline{\mathbb{Q}}_l)}$ consisting of elements a which satisfy

- $a_{\tau c,i} = -a_{\tau,n+1-i}$ and
- $a_{\tau,1} \geq ... \geq a_{\tau,n}$.

If $a \in \mathrm{Wt}_n$ then we get a K-vector space W_a and irreducible representations

$$\begin{array}{rcl} \xi_a : G(F_l^+) & \longrightarrow & GL(W_a) \\ g & \longmapsto & \otimes_{\tau \in \widetilde{I}_l} \xi_{a_\tau}(\tau i_\tau g) \end{array}$$

and

$$\begin{array}{rcl} \xi_a' : G'(F_l^+) & \longrightarrow & GL(W_a) \\ g & \longmapsto & \prod_{\tau \in \widetilde{I}_l} \xi_{a_\tau}(\tau i_\tau^t g). \end{array}$$

The representation ξ_a contains a $G(\mathcal{O}_{F^+,l})$-invariant $\mathcal{O}$-lattice M_a and the representation ξ_a' contains a $G'(\mathcal{O}_{F^+,l})$-invariant $\mathcal{O}$-lattice M_a' such that there is a perfect pairing

$$\langle\ ,\ \rangle_a : M_a \times M_a' \longrightarrow \mathcal{O}$$

with

$$\langle \xi_a(g)x, \xi_a'(I(g))y \rangle_a = \langle x, y \rangle_a.$$

For $v \in S(B)$, let $\rho_v : G(F_v^+) \to GL(M_{\rho_v})$ denote a representation of $G(F_v^+)$ on a finite free $\mathcal{O}$-module such that ρ_v has open kernel and $M_{\rho_v} \otimes_{\mathcal{O}} \overline{K}$ is irreducible. Let $M'_{\rho_v} = \mathrm{Hom}\,(M_{\rho_v}, \mathcal{O})$ and define $\rho'_v : G(F_v^+) \to GL(M'_{\rho_v})$ by

$$\rho'_v(g)(x)(y) = x(\rho_v(I^{-1}(g))^{-1}y).$$

If we identify $G(F_v^+) \cong B_w^\times$ and $G'(F_v^+) \cong (B_w^{\mathrm{op}})^\times$ and if $g \in B_w^\times$ and $g' \in (B_w^{\mathrm{op}})^\times$ have the same characteristic polynomials then $\mathrm{tr}\,\rho_v(g) = \mathrm{tr}\,\rho_v(g')$. Let $e(\rho_v)$ denote the number of irreducible constituents of $\rho_v|_{G(\mathcal{O}_{F^+,v})} \otimes_K \overline{K}$.

If $\mathrm{JL}\,(\rho_v \circ i_w^{-1}) = \mathrm{Sp}\,_{m_v}(\pi_w)$ then set

$$\widetilde{r}_w = r_l(\pi_w|\ \ |^{(n/m-2)(1-m)/2}).$$

Note that we also have $\mathrm{JL}\,(\rho'_v \circ i_w^{-1}) = \mathrm{Sp}\,_{m_v}(\pi_w)$. We will *suppose that*

$$\widetilde{r}_w : \mathrm{Gal}\,(\overline{F}_w/F_w) \longrightarrow GL_{n/m}(\mathcal{O})$$

(as opposed to $GL_{n/m}(\overline{K})$) and that the reduction of $\widetilde{r}_w$ is absolutely irreducible. Thus $\widetilde{r}_w$ is well defined over $\mathcal{O}$.

We will call an open compact subgroup $U \subset G(\mathbb{A}_{F^+}^\infty)$ *sufficiently small* if for some place v its projection to $G(F_v^+)$ contains only one element of finite order, namely 1.

Suppose that U is an open compact subgroup of $G(\mathbb{A}_{F^+}^\infty)$, that $a \in \mathrm{Wt}_n$ and that for $v \in S(B)$, ρ_v is as in the last paragraph. Set

$$M_{a,\{\rho_v\}} = M_a \otimes \bigotimes_{v \in S(B)} M_{\rho_v}$$

and

$$M_{a,\{\rho_v\}} = M'_a \otimes \bigotimes_{v \in S(B)} M'_{\rho_v}.$$

Suppose that either R is a K-algebra or that the projection of U to $G(F_l^+)$ is contained in $G(\mathcal{O}_{F^+,l})$. Then we define a space of automorphic forms

$$S_{a,\{\rho_v\}}(U, R)$$

to be the space of functions

$$f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty) \longrightarrow R \otimes_{\mathcal{O}} M_{a,\{\rho_v\}}$$

such that

$$f(gu) = u_{S(B),l}^{-1} f(g)$$

for all $u \in U$ and $g \in G(\mathbb{A}^\infty_{F^+})$. Here $u_{S(B),l}$ denotes the projection of u to $G(F_l^+) \times \prod_{v \in S(B)} G(F_v^+)$. If V is any compact subgroup of $G(\mathbb{A}^\infty_{F^+})$ we define $S_{a,\{\rho_v\}}(V,R)$ to be the union of the $S_{a,\{\rho_v\}}(U,R)$ as U runs over open compact subgroups containing V. Similarly, if U' is an open compact subgroup of $G'(\mathbb{A}^\infty_{F^+})$ and either R is a K-algebra or the projection of U' to $G'(F_l^+)$ is contained in $G'(\mathcal{O}_{F^+,l})$ we define

$$S'_{a,\{\rho_v\}}(U',R)$$

to be the space of functions

$$f : G'(F^+)\backslash G'(\mathbb{A}^\infty_{F^+}) \longrightarrow R \otimes_{\mathcal{O}} M'_{a,\{\rho_v\}}$$

such that

$$f(gu) = u^{-1}_{S(B),l} f(g)$$

for all $u \in U'$ and $g \in G'(\mathbb{A}^\infty_{F^+})$. We make a corresponding definition of $S'_{a,\{\rho_v\}}(V',R)$ for V' any compact subgroup of $G'(\mathbb{A}^\infty_{F^+})$.

If $g \in G(\mathbb{A}^\infty_{F^+})$ (and either R is a K-algebra or $g_l \in G(\mathcal{O}_{F^+,l})$) and if $V \subset gUg^{-1}$ then there is a natural map

$$g : S_{a,\{\rho_v\}}(U,R) \longrightarrow S_{a,\{\rho_v\}}(V,R)$$

defined by

$$(gf)(h) = g_{l,S(B)} f(hg).$$

We see that if V is a normal subgroup of U then

$$S_{a,\{\rho_v\}}(U,R) = S_{a,\{\rho_v\}}(V,R)^U.$$

If U is open then the R-module $S_{a,\{\rho_v\}}(U,R)$ is finitely generated. If U is open and sufficiently small then it is free of rank $\#G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U$. If R is flat over $\mathcal{O}$ or if U is sufficiently small then

$$S_{a,\{\rho_v\}}(U,R) = S_{a,\{\rho_v\}}(U,\mathcal{O}) \otimes_{\mathcal{O}} R.$$

Suppose that U_1 and U_2 are compact subgroups and $g \in G(\mathbb{A}^\infty_{F^+})$. If R is not a K-algebra suppose that $g_l \in G(\mathcal{O}_{F^+,l})$ and that $u_l \in G(\mathcal{O}_{F^+,l})$ for all $u \in U_1 \cup U_2$. Suppose also that $\#U_1gU_2/U_2 < \infty$. (This will be automatic if U_1 and U_2 are open.) Then we define a linear map

$$[U_1gU_2] : S_{a,\{\rho_v\}}(U_2,R) \longrightarrow S_{a,\{\rho_v\}}(U_1,R)$$

by

$$([U_1gU_2]f)(h) = \sum_i (g_i)_{l,S(B)} f(hg_i)$$

if $U_1gU_2 = \coprod_i g_iU_2$. Exactly similar statements hold for G'.

Lemma 2.3.1 *Let $U \subset G(\mathbb{A}_{F^+}^\infty)$ be a sufficiently small open compact subgroup and let $V \subset U$ be a normal open subgroup. Let R be an $\mathcal{O}$-algebra. Suppose that either R is a K-algebra or the projection of U to $G(F_l^+)$ is contained in $G(\mathcal{O}_{F^+,l})$. Then $S_{a,\{\rho_v\}}(V,R)$ is a finite free $R[U/V]$-module and $\mathrm{tr}_{U/V}$ gives an isomorphism from the coinvariants $S_{a,\{\rho_v\}}(V,R)_{U/V}$ to $S_{a,\{\rho_v\}}(U,R)$.*

Proof: Suppose that
$$G(\mathbb{A}_{F^+}^\infty) = \coprod_{j\in J} G(F^+)g_jU.$$
Then
$$G(\mathbb{A}_{F^+}^\infty) = \coprod_{j\in J}\coprod_{u\in U/V} G(F^+)g_juV.$$
Moreover for all $j \in J$ we have $g_j^{-1}G(F^+)g_j \cap U = \{1\}$. (Because this intersection is finite and U is sufficiently small.) Thus
$$\begin{array}{rcl} S_{a,\{\rho_v\}}(U,R) & \stackrel{\sim}{\longrightarrow} & \bigoplus_{j\in J} M_{a,\{\rho_v\}} \otimes_{\mathcal{O}} R \\ f & \longmapsto & (f(g_j))_j. \end{array}$$
and
$$\begin{array}{rcl} S_{a,\{\rho_v\}}(V,R) & \stackrel{\sim}{\longrightarrow} & \bigoplus_{j\in J}\bigoplus_{u\in U/V} M_{a,\{\rho_v\}} \otimes_{\mathcal{O}} R \\ f & \longmapsto & (f(g_ju))_{j,u}. \end{array}$$
Alternatively we get an isomorphism of $R[U/V]$-modules
$$\begin{array}{rcl} S_{a,\{\rho_v\}}(V,R) & \stackrel{\sim}{\longrightarrow} & \bigoplus_{j\in J} M_{a,\{\rho_v\}} \otimes_{\mathcal{O}} R[U/V] \\ f & \longmapsto & (\sum_{u\in U/V} u_{S(B),l}f(g_ju)\otimes u^{-1})_j. \end{array}$$
Then
$$\begin{array}{rcl} S_{a,\{\rho_v\}}(V,R)_{U/V} & \stackrel{\sim}{\longrightarrow} & \bigoplus_{j\in J} M_{a,\{\rho_v\}} \otimes_{\mathcal{O}} R \\ f & \longmapsto & (\sum_{u\in U/V} u_{S(B),l}f(g_ju))_j. \end{array}$$
In fact we have a commutative diagram
$$\begin{array}{ccc} S_{a,\{\rho_v\}}(V,R)_{U/V} & \stackrel{\mathrm{tr}_{U/V}}{\longrightarrow} & S_{a,\{\rho_v\}}(U,R) \\ \downarrow & & \downarrow \\ \bigoplus_{j\in J} M_{a,\{\rho_v\}} \otimes_{\mathcal{O}} R & = & \bigoplus_{j\in J} M_{a,\{\rho_v\}} \otimes_{\mathcal{O}} R \end{array}$$
where the vertical maps are the above isomorphisms. The lemma follows. $\Box$

Suppose that U is an open compact subgroup of $G(\mathbb{A}_{F^+}^\infty)$ and that $\eta \in G'(\mathbb{A}_{F^+}^\infty)$. If R is not a K-algebra further assume that $\eta_l \in G'(\mathcal{O}_{F^+,l})$ and that for all $u \in U$ we also have $u_l \in G(\mathcal{O}_{F^+,l})$. Set $U' = \eta^{-1}I(U)\eta$. Define a pairing
$$\langle\ ,\ \rangle_{U,\eta} : S_{a,\{\rho_v\}}(U,R) \times S'_{a,\{\rho_v\}}(U',R) \longrightarrow R$$

by

$$\langle f, f' \rangle_{U,\eta} = \sum_{g \in G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U} \langle f(g), \eta_{l,S(B)} f'(I(g)\eta) \rangle_{a,\{\rho_v\}}.$$

If U is sufficiently small, or if R is a K-algebra, this is a perfect pairing. If we have two such pairs (U_1, η_1) and (U_2, η_2) with each U_i sufficiently small, if $U_i' = \eta_i^{-1} I(U_i) \eta_i$ and if $g \in G(\mathbb{A}^\infty_{F^+})$ (with $g_l \in G(\mathcal{O}_{F^+,l})$ if R is not a K-algebra) then

$$\langle [U_1 g U_2] f, f' \rangle_{U_1,\eta_1} = \langle f, [U_2' \eta_2^{-1} I(g)^{-1} \eta_1 U_1'] f' \rangle_{U_2,\eta_2}.$$

Proposition 2.3.2 *Fix $\iota : K \hookrightarrow \mathbb{C}$.*

1. *$S_{a,\{\rho_v\}}(\{1\}, \mathbb{C})$ is a semi-simple admissible $G(\mathbb{A}^\infty_{F^+})$-module.*

2. *If $S(B) \neq \emptyset$ and $\pi = \otimes_v \pi_v$ is an irreducible constituent of $S_{a,\{\rho_v\}}(\{1\}, \mathbb{C})$ then there is an automorphic representation $\mathrm{BC}_\iota(\pi)$ of $(B \otimes \mathbb{A})^\times$ with the following properties.*

   - $\mathrm{BC}_\iota(\pi) \circ (-\ddagger) = BC_\iota(\pi)$.
   - *If a prime v of F^+ splits as ww^c in F then $BC_\iota(\pi)_w \cong \pi_v \circ i_w^{-1}$.*
   - *If v is an infinite place of F^+ and $\tau : F \hookrightarrow \mathbb{C}$ lies above v then $\mathrm{BC}_\iota(\pi)_v$ is cohomological for $(\xi_{a_{\iota^{-1}\tau}} \circ \tau) \otimes (\xi_{a_{\iota^{-1}\tau c}} \circ \tau c)$.*
   - *If v is a prime of F^+ which is unramified, inert in F and if π_v has a fixed vector for a hyperspecial maximal compact subgroup of $G(F_v)$ then $\mathrm{BC}_\iota(\pi)_v$ has a $GL_n(\mathcal{O}_{F,v})$-fixed vector.*
   - *If $v \in S(B)$ and π_v has a $G(\mathcal{O}_{F,v})$ fixed vector and w is a prime of F above v then $\mathrm{BC}_\iota(\pi)_w$ is an unramified twist of $(\iota\rho_v^\vee) \circ i_w^{-1}$.*

3. *If $S(B) \neq \emptyset$ and $\pi = \otimes_v \pi_v$ is an irreducible constituent of $S_{a,\{\rho_v\}}(\{1\}, \mathbb{C})$ such that for $v \in S(B)$ the representation π_v has a $G(\mathcal{O}_{F^+,v})$-fixed vector, then one of the following two possibilities obtains. Either there is a cuspidal automorphic representation Π of $GL_n(\mathbb{A}_F)$ with the following properties.*

   - $\Pi \circ c = \Pi^\vee$.
   - *If a prime $v \notin S(B)$ of F^+ splits as ww^c in F then $\Pi_w \cong \pi_v \circ i_w^{-1}$.*
   - *If v is an infinite place of F^+ and $\tau : F \hookrightarrow \mathbb{C}$ lies above v then Π_v is cohomological for $(\xi_{a_{\iota^{-1}\tau}} \circ \tau) \otimes (\xi_{a_{\iota^{-1}\tau c}} \circ \tau c)$.*

- *If v is a prime of F^+ which is unramified, inert in F and if π_v has a fixed vector for a hyperspecial maximal compact subgroup of $G(F_v)$ then Π_v has a $GL_n(\mathcal{O}_{F,v})$-fixed vector.*
- *If $v \in S(B)$ and w is a prime of F above v then Π_w is an unramified twist of* $\mathrm{JL}\,((\iota\rho_v^\vee) \circ i_w^{-1})$.

Or there is an integer $m|n$ and a cuspidal automorphic representation Π of $GL_{n/m}(\mathbb{A}_F)$ with the following properties.

- $\Pi^\vee \circ c = \Pi|\ |^{m-1}$.
- *If a prime $v \notin S(B)$ of F^+ splits as ww^c in F then $\Pi_w \boxplus \Pi_w|\ | \boxplus \ldots \boxplus \Pi_w|\ |^{m-1} \cong \pi_v \circ i_w^{-1}$.*
- *If v is an infinite place of F^+ and $\tau : F \hookrightarrow \mathbb{C}$ lies above v then $\Pi_v|\ |^{n(m-1)/(2m)}$ is cohomological for $(\xi_{b_\tau} \circ \tau) \otimes (\xi_{b_{\tau c}} \circ \tau c)$ and $b_{\tau,i} = a_{\tau,m(i-1)+j} + (m-1)(i-1)$ for every $j = 1, \ldots, m$.*
- *If v is a prime of F^+ which is unramified, inert in F and if π_v has a fixed vector for a hyperspecial maximal compact subgroup of $G(F_v)$ then Π_v has a $GL_{n/m}(\mathcal{O}_{F,v})$-fixed vector.*
- *If $v \in S(B)$ and w is a prime of F above v then Π_w is cuspidal and* $\mathrm{JL}\,(\iota\rho_v \circ i_w^{-1})^\vee$ *is an unramified twist of* $\mathrm{Sp}_m(\Pi_w)$.

If for one place $v_0 \notin S(B)$ of F^+, which splits in F, the representation π_{v_0} is generic, then for all places $v \notin S(B)$ of F^+, which split in F, the representation π_v is generic.

4. *Suppose that Π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ with the following properties.*

   - $\Pi^\vee \circ c = \Pi$.
   - *If v is an infinite place of F^+ and $\tau : F \hookrightarrow \mathbb{C}$ lies above v then Π_v is cohomological for $(\xi_{a_{\iota^{-1}\tau}} \circ \tau) \otimes (\xi_{a_{\iota^{-1}\tau c}} \circ \tau c)$.*
   - *If $v \in S(B)$ and w is a prime of F above v then Π_w is an unramified twist of* $\mathrm{JL}\,((\iota\rho_v^\vee) \circ i_w^{-1})$.

   Then there is an irreducible constituent π of $S_{a,\{\rho_v\}}(\{1\}, \mathbb{C})$ with the following properties.

   - *For $v \in S(B)$ the representation π_v has a $G(\mathcal{O}_{F^+,v})$-fixed vector.*
   - *If a prime $v \notin S(B)$ of F^+ splits as ww^c in F then $\pi_v \cong \Pi_w \circ i_w$.*

- *If v is a prime of F^+ which is inert and unramified in F and if Π_w is unramified then π_v has a fixed vector for a hyperspecial maximal compact subgroup of $G(F_v)$.*

Exactly similar results hold for G' (with i_w^t replacing i_w).

Proof: If $\tau \in \widetilde{I}_l$ then $\iota\tau : F \to \mathbb{C}$ and hence $F_\infty \to \mathbb{C}$. Then $W_a \otimes_{K,\iota} \mathbb{C}$ is naturally a $G(\mathbb{R})$-module:

$$g \longmapsto \otimes_{\tau \in \widetilde{I}_l} \xi_{a_\tau}(\iota\tau g).$$

Denote this action by $\xi_{a,\iota}$. Let $\mathcal{A}$ denote the space of automorphic forms on $G(F^+)\backslash G(\mathbb{A}_{F^+})$. We have an isomorphism

$$i : S_{a,\{\rho_v\}}(U, \mathbb{C}) \xrightarrow{\sim} \mathrm{Hom}_{U \times G(F^+_\infty)}((M_{a,\{\rho_v\}} \otimes_{\mathcal{O},\iota} \mathbb{C})^\vee, \mathcal{A})$$

given by

$$i(f)(\alpha)(g) = \alpha(\xi_{a,\iota}(g_\infty)^{-1}(\xi_a(g_l)f(g^\infty))).$$

The first part now becomes a standard fact. The second part follows from theorem A.5.2 of [CL], except that theorem A.5.2 of [CL] only identifies $\mathrm{BC}_\iota(\pi)_v$ for all but finitely many v. We can easilly adapt the argument to identify $\mathrm{BC}_\iota(\pi)_v$ at all split places, as we described in the proof of theorem VI.2.1 of [HT] (page 202). It is equally easy to control $\mathrm{BC}_\iota(\pi)_v$ at places where π_v has a fixed vector for a hyperspecial maximal compact subgroup. One just chooses the set S in the proof of theorem A.5.2 of [CL] not containing v. The third part follows from the second, theorem VI.1.1 of [HT] and the main result of [MW]. As for the fourth part, the existence of some descent (controlled at all but finitely many places) follows from theorem VI.1.1 of [HT] and the argument for proposition 2.3 of [Cl] as completed by theorem A.3.1 of [CL]. That this descent has all the stated properties follows from the earlier parts of this proposition. □

Corollary 2.3.3 *$S_{a,\{\rho_v\}}(\{1\}, K)$ (resp. $S'_{a,\{\rho_v\}}(\{1\}, K)$) is a semi-simple admissible $G(\mathbb{A}^\infty_{F^+})$-module (resp. $G'(\mathbb{A}^\infty_{F^+})$-module).*

Combining the above proposition with theorem VII.1.9 of [HT] we obtain the following result.

Proposition 2.3.4 *Let $\overline{K}^0$ denote the algebraic closure of $\mathbb{Q}_l$ in $\overline{K}$. Suppose that $\pi = \otimes_v \pi_v$ is an irreducible constituent of $S_{a,\{\rho_v\}}(\{1\}, \overline{K})$ then there is a continuous semi-simple representation*

$$r_\pi : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{K}^0)$$

with the following properties.

1. *If $v \notin S(B) \cup S_l$ is a prime of F^+ which splits $v = ww^c$ in F, then*

$$r_\pi|_{G_{F_w}}^{\mathrm{ss}} = (r_l(\pi_w \circ i_w^{-1})^\vee(1-n))^{\mathrm{ss}}.$$

2. $r_\pi^c \cong r_\pi^\vee \epsilon^{1-n}$.

3. *If $v \in S(B)$ splits $v = ww^c$ in F then*

$$r_\pi|_{G_{F_w}}^{\mathrm{ss}} = (r_l(\mathrm{JL}\,(\pi_w \circ i_w^{-1}))^\vee(1-n))^{\mathrm{ss}}.$$

4. *If v is a prime of F^+ which is inert and unramified in F and if π_v has a fixed vector for a hyperspecial maximal compact subgroup of $G(F_v^+)$ then $r_\pi|_{W_{F_v}}$ is unramified.*

5. *If w is a prime of F above l then r_π is potentially semi-stable at w. If moreover $\pi_{w|_{F^+}}$ is unramified then r_π is crystalline at w.*

6. *If $\tau : F \hookrightarrow K$ gives rise to a prime w of F then*

$$\dim_{\overline{K}^0} \mathrm{gr}^{\,i}(r_\pi \otimes_{\tau,F_w} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_w/F_w)} = 0$$

unless $i = a_{\tau,j} + n - j$ for some $j = 1, ..., n$ in which case

$$\dim_{\overline{K}^0} \mathrm{gr}^{\,i}(r_\pi \otimes_{\tau,F_w} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_w/F_w)} = 1.$$

7. *If for some place $v \notin S(B)$ of F^+ which splits in F the representation π_v is not generic then r_π is reducible.*

Exactly similar results hold for G' (with i_w^t replacing i_w).

Proof: If the first possibility of part 3 of proposition 2.3.2 obtains then by theorem VII.1.9 of [HT] $r_\pi = R_l(\Pi)^\vee(1-n)$ will suffice. So suppose the second possibility obtains. Let $S' \supset S_l$ be any finite set of finite places of F^+ which are unramified in F. Choose a character $\psi : \mathbb{A}_F^\times \to \mathbb{C}^\times$ such that

- $\psi^{-1} = \psi^c$;
- ψ is unramified above S'; and
- if $\tau : F \hookrightarrow \mathbb{C}$ gives rise to an infinite place v of F then

$$\psi_v : z \longmapsto (\tau z/|\tau z|)^{\delta_\tau}$$

  where $|z|^2 = zz^c$ and $\delta_\tau = 0$ if either m or n/m is odd and $\delta_\tau = \pm 1$ otherwise.

The existence of such a character is proved as in the proof of lemma VII.2.8 of [HT]. Then

$$r_\pi = R_l(\Pi \otimes \psi| \ |^{(m-1)/2})^\vee(1-n) \otimes R_l(\psi^{-1}| \ |^{(n/m-1)(m-1)/2})^\vee \otimes (1 \oplus \epsilon^{-1} \oplus ... \oplus \epsilon^{1-m})$$

is independent of the choice of S' and ψ and satisfies the requirements of the proposition. (We use the freedom to vary S' to verify property 4.) $\square$

2.4 Unitary group Hecke algebras.

Keep the notation and assumptions of the last section. Further suppose that

$$U = \prod_v U_v \subset G(\mathbb{A}_{F^+}^\infty)$$

is a sufficiently small open compact subgroup and that, if $v \notin T$ splits in F, then $U_v = G(\mathcal{O}_{F^+,v})$. We will denote by

$$\mathbb{T}^T_{a,\{\rho_v\}}(U)$$

the $\mathcal{O}$-subalgebra of $\mathrm{End}\,(S_{a,\{\rho_v\}}(U, \mathcal{O}))$ generated by the Hecke operators $T_w^{(j)}$ (or strictly speaking $i_w^{-1}(T_w^{(j)}) \times U^v$) and $(T_w^{(n)})^{-1}$ for $j = 1, ..., n$ and for w a place of F which is split over a place $v \notin T$ of F^+. (Note that $T_{w^c}^{(j)} = (T_w^{(n)})^{-1} T_w^{(n-j)}$, so we need only consider one place w above a given place v of F^+.) If X is a $\mathbb{T}^T_{a,\{\rho_v\}}(U)$-stable subspace of $S_{a,\{\rho_v\}}(U, K)$ then we will write

$$\mathbb{T}^T(X)$$

for the image of $\mathbb{T}^T_{a,\{\rho_v\}}(U)$ in $\mathrm{End}_K(X)$.

Similarly suppose that

$$U' = \prod_v U'_v \subset G'(\mathbb{A}_{F^+}^\infty)$$

is a sufficiently small open compact subgroup and that, if $v \notin T$ splits in F, then $U'_v = G(\mathcal{O}_{F^+,v})$. We will denote by

$$\mathbb{T}^T_{a,\{\rho_v\}}(U')'$$

the $\mathcal{O}$-subalgebra of $\mathrm{End}\,(S'_{a,\{\rho_v\}}(U', \mathcal{O}))$ generated by the Hecke operators $T_w^{(j)}$ (or strictly speaking $(i_w^t)^{-1}(T_w^{(j)}) \times (U')^v$) and $(T_w^{(n)})^{-1}$ for $j = 1, ..., n$ and for w a place of F which is split over a place $v \notin T$ of F^+. (Again

$T^{(j)}_{w^c} = (T^{(n)}_w)^{-1} T^{(n-j)}_w$, so we need only consider one place w above a given place v of F^+.) If X' is a $\mathbb{T}^T_{a,\{\rho_v\}}(U')'$-stable subspace of $S'_{a,\{\rho_v\}}(U', K)$ then we will write

$$\mathbb{T}^T(X')'$$

for the image of $\mathbb{T}^T_{a,\{\rho_v\}}(U')'$ in $\mathrm{End}_K(X')$.

Note that $\mathbb{T}^T(X)$ and $\mathbb{T}^T(X')'$ are finite and free as $\mathcal{O}$-modules. Also by corollary 2.3.3 we see that they are reduced.

Proposition 2.4.1 *Suppose that* $\mathfrak{m}$ *is a maximal ideal of* $\mathbb{T}^T_{a,\{\rho_v\}}(U)$. *Then there is a unique continuous semisimple representation*

$$\overline{r}_\mathfrak{m} : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\mathbb{T}^T_{a,\{\rho_v\}}(U)/\mathfrak{m})$$

with the following properties. The first two of these properties already characterise $\overline{r}_\mathfrak{m}$ *uniquely.*

1. $\overline{r}_\mathfrak{m}$ *is unramified at all but finitely many places.*
2. *If a place* $v \notin T$ *of* F^+ *splits as* ww^c *in* F *then* $\overline{r}_\mathfrak{m}$ *is unramified at* w *and* $\overline{r}_\mathfrak{m}(\mathrm{Frob}_w)$ *has characteristic polynomial*
$$X^n - T^{(1)}_w X^{n-1} + ... + (-1)^j (\mathbf{N}w)^{j(j-1)/2} T^{(j)}_w X^{n-j} + \\ +... + (-1)^n (\mathbf{N}w)^{n(n-1)/2} T^{(n)}_w.$$
3. *If a place* v *of* F^+ *such that* V *is inert and unramified in* F *and such that* U_v *is a hyperspecial maximal compact, then* $\overline{r}_\mathfrak{m}$ *is unramified above* v.
4. $\overline{r}^c_\mathfrak{m} \cong \overline{r}^\vee_\mathfrak{m} \otimes \epsilon^{1-n}$.
5. *Suppose* $v \in S(B)$, $U_v = G(\mathcal{O}_{F^+,v})$ *and* w *is a prime of* F *above* v. *Then* $\overline{r}_\mathfrak{m}$ *has a* $\mathrm{Gal}\,(\overline{F}_w/F_w)$*-invariant filtration* $\overline{\mathrm{Fil}}^i_w$ *with*
$$\overline{\mathrm{gr}}^0_w \overline{r}_\mathfrak{m}|_{I_{F_w}} \cong \widetilde{r}_w|_{I_{F_w}} \otimes_\mathcal{O} k$$
and
$$\overline{\mathrm{gr}}^i_w \overline{r}_\mathfrak{m}|_{\mathrm{Gal}\,(\overline{F}_w/F_w)} \cong (\overline{\mathrm{gr}}^0_w \overline{r}_\mathfrak{m}|_{\mathrm{Gal}\,(\overline{F}_w/F_w)}(\epsilon^i)$$
for $i = 0, ..., m_v - 1$ *(and* $= (0)$ *otherwise). If (for instance) we further have*
$$\widetilde{r}_w \otimes_\mathcal{O} k(\epsilon^j) \not\cong \widetilde{r}_w \otimes_\mathcal{O} k$$
for $j = 1, ..., m_v$, *then this filtration is unique and*
$$\mathrm{Hom}_{\,\mathrm{Gal}\,(\overline{F}_w/F_w)}(\overline{\mathrm{Fil}}^j_w \overline{r}_\mathfrak{m}, \overline{\mathrm{gr}}^j_w \overline{r}_\mathfrak{m}) = k$$
for $j = 0, ..., m_v - 1$.

6. *Suppose that* $w \in \widetilde{S}_l$ *is unramified over* l*, that* $U_{w|_{F^+}} = G(\mathcal{O}_{F^+,w})$ *and that for each* $\tau \in \widetilde{I}_l$ *above* w *we have*

$$l - 1 - n \geq a_{\tau,1} \geq ... \geq a_{\tau,n} \geq 0.$$

Then

$$\overline{r}_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_w/F_w)} = \mathbb{G}_w(\overline{M}_{\mathfrak{m},w})$$

for some object $\overline{M}_{\mathfrak{m},w}$ *of* $\mathcal{MF}_{\mathbb{T}^T_{a,\{\rho_v\}}(U)/\mathfrak{m},w}$*. Moreover for all* $\tau \in \widetilde{I}_l$ *over* w *we have*

$$\dim_{\mathbb{T}^T_{a,\{\rho_v\}}(U)/\mathfrak{m}}(\mathrm{gr}^i \overline{M}_{\mathfrak{m},w}) \otimes_{\tau \otimes 1} \mathcal{O} = 1$$

if $i = a_{\tau,j} + n - j$ *for some* $j = 1, ..., n$ *and* $= 0$ *otherwise.*

Exactly similar statements are true for maximal ideals $\mathfrak{m}'$ *of* $\mathbb{T}^T_{a,\{\rho_v\}}(U')'$.

Proof: Choose a minimal prime ideal $\wp \subset \mathfrak{m}$ and an irreducible constituent π of $S_{a,\{\rho_v\}}(\{1\}, \overline{K})$ such that $\pi^U \neq (0)$ and $\mathbb{T}^T_{a,\{\rho_v\}}(U)$ acts on π^U via the quotient $\mathbb{T}^T_{a,\{\rho_v\}}(U)/\wp$. Choosing an invariant lattice in r_π, reducing and semisimplifying gives us the desired representation $\overline{r}_{\mathfrak{m}}$, except that it is defined over the algebraic closure of $\mathbb{T}^T_{a,\{\rho_v\}}(U)/\mathfrak{m}$. However, as the characteristic polynomial of every element of the image of $\overline{r}_{\mathfrak{m}}$ is rational over $\mathbb{T}^T_{a,\{\rho_v\}}(U)/\mathfrak{m}$ and as $\mathbb{T}^T_{a,\{\rho_v\}}(U)/\mathfrak{m}$ is a finite field we see that (after conjugation) we may assume that

$$\overline{r}_{\mathfrak{m}} : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\mathbb{T}^T_{a,\{\rho_v\}}(U)/\mathfrak{m}).$$

□

We will call $\mathfrak{m}$ (resp. $\mathfrak{m}'$) *Eisenstein* if $\overline{r}_{\mathfrak{m}}$ (resp. $\overline{r}_{\mathfrak{m}'}$) is absolutely reducible.

Proposition 2.4.2 *Suppose that* $\mathfrak{m}$ *is a non-Eisenstein maximal ideal of the Hecke algebra* $\mathbb{T}^T_{a,\{\rho_v\}}(U)$ *with residue field* k*. Then* $\overline{r}_{\mathfrak{m}}$ *has an extension to a continuous homomorphism*

$$\overline{r}_{\mathfrak{m}} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(k).$$

Pick such an extension. There is a unique continuous lifting

$$r_{\mathfrak{m}} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(\mathbb{T}^T_{a,\{\rho_v\}}(U)_{\mathfrak{m}})$$

of $\overline{r}_{\mathfrak{m}}$ *with the following properties. The first two of these properties already characterise the lifting* $r_{\mathfrak{m}}$ *uniquely.*

1. $r_{\mathfrak{m}}$ *is unramified at all but finitely many places.*

2. *If a place* $v \notin T$ *of* F^+ *splits as* ww^c *in* F *then* $r_{\mathfrak{m}}$ *is unramified at* w *and* $r_{\mathfrak{m}}(\mathrm{Frob}_w)$ *has characteristic polynomial*

$$X^n - T_w^{(1)} X^{n-1} + \ldots + (-1)^j (\mathbf{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j} + \ldots + (-1)^n (\mathbf{N}w)^{n(n-1)/2} T_w^{(n)}.$$

3. *If a place* v *of* F^+ *such that* v *is inert and unramified in* F *and if* U_v *is a hyperspecial maximal compact then* $r_{\mathfrak{m}}$ *is unramified at* v.

4. $\nu \circ r_{\mathfrak{m}} = \epsilon^{1-n} \delta_{F/F^+}^{\mu_{\mathfrak{m}}}$, *where* δ_{F/F^+} *is the nontrivial character of* $\mathrm{Gal}\,(F/F^+)$ *and where* $\mu_{\mathfrak{m}} \in \mathbb{Z}/2\mathbb{Z}$.

5. *Suppose that* $w \in \widetilde{S}_l$ *is unramified over* l, *that* $U_{w|_{F^+}} = G(\mathcal{O}_{F^+,w})$ *and that for each* $\tau \in \widetilde{I}_l$ *above* w *we have*

$$l - 1 - n \geq a_{\tau,1} \geq \ldots \geq a_{\tau,n} \geq 0.$$

   Then for each open ideal $I \subset \mathbb{T}^T_{a,\{\rho_v\}}(U)_{\mathfrak{m}}$

$$(r_{\mathfrak{m}} \otimes_{\mathbb{T}^T_{a,\{\rho_v\}}(U)_{\mathfrak{m}}} \mathbb{T}^T_{a,\{\rho_v\}}(U)_{\mathfrak{m}}/I)|_{\mathrm{Gal}\,(\overline{F}_w/F_w)} = \mathbb{G}_w(M_{\mathfrak{m},I,w})$$

   for some object $M_{\mathfrak{m},I,w}$ *of* $\mathcal{MF}_{\mathcal{O},w}$.

6. *Suppose* $v \in S(B)$, $U_v = G(\mathcal{O}_{F^+,v})$, *that* w *is a prime of* F *above* v *and that for* $j = 1, \ldots, m_v$

$$\widetilde{r}_w \otimes_{\mathcal{O}} k \not\cong \widetilde{r}_w \otimes_{\mathcal{O}} k(\epsilon^j).$$

   Then $r_{\mathfrak{m}}$ *has a* $\mathrm{Gal}\,(\overline{F}_w/F_w)$*-invariant filtration* $\mathrm{Fil}^{\,i}_w$ *such that*

$$\mathrm{gr}^{\,0}_w r_{\mathfrak{m}}|_{I_{F_w}} \cong \widetilde{r}_w|_{I_{F_w}} \otimes_{\mathcal{O}} \mathbb{T}^T_{a,\{\rho_v\}}(U)_{\mathfrak{m}}$$

   lifting any fixed isomorphism $\overline{\mathrm{gr}}^{\,0}_w \overline{r}_{\mathfrak{m}}|_{I_{F_w}} \cong \widetilde{r}_w|_{I_{F_w}} \otimes_{\mathcal{O}} k$, *and*

$$\mathrm{gr}^{\,i}_w r_{\mathfrak{m}} \cong (\mathrm{gr}^{\,0}_w r_{\mathfrak{m}})(\epsilon^i)$$

   if $i = 0, \ldots, m_v - 1$ *(and* $= (0)$ *otherwise).*

7. *Suppose that a place* $v \in T - (S_l \cup S(B))$ *splits as* ww^c *in* F *and that* $U_v = i_w U_1(w)$. *Let* ϕ_w *be a lift of* Frob_w *to* $\mathrm{Gal}\,(\overline{F}_w/F_w)$ *and let* ϖ_w *be an element of* $F_w^\times$ *such that* $\mathrm{Art}_{\,F_w} \varpi_w = \phi_w$ *on the maximal abelien extension of* F_w. *Suppose that* $a \in k$ *is a simple root of the characteristic*

polynomial of $\overline{r}_{\mathfrak{m}}(\phi_w)$. Then there is a unique root $A \in \mathbb{T}^T_{a,\{\rho_v\}}(U)_{\mathfrak{m}}$ of the characteristic polynomial of $r_{\mathfrak{m}}(\phi_w)$ which lifts a.

Suppose further that Y is a $\mathbb{T}^T_{a,\{\rho_v\}}(U)[V_{\varpi_w}]$-invariant subspace of $S_{a,\{\rho_v\}}(U,K)_{\mathfrak{m}}$ such that $V_{\varpi_w} - a$ is topologically nilpotent on Y. Then for each $\alpha \in F_w^\times$ with non-negative valuation the element V_α (in $\mathrm{End}_K(Y)$) lies in $\mathbb{T}^T(Y)$. Moreover $\alpha \mapsto V_\alpha$ extends to a continuous character $V : F_w^\times \to \mathbb{T}^T(Y)^\times$. Further $(X - V_{\varpi_w})$ divides the characteristic polynomial of $r_{\mathfrak{m}}(\phi_w)$ over $\mathbb{T}^T(Y)$.

If $\mathbf{N}w \equiv 1 \bmod l$ then

$$r_{\mathfrak{m}}|_{\mathrm{Gal}(\overline{F}_w/F_w)} = s \oplus (V \circ \mathrm{Art}_{F_w}^{-1}),$$

where s is unramified.

Exactly similar statements are true for non-Eisenstein maximal ideals $\mathfrak{m}'$ of $\mathbb{T}^T_{a,\{\rho_v\}}(U')'$.

Proof: By lemma 1.1.3 we can extend $\overline{r}_{\mathfrak{m}}$ to a homomorphism

$$\overline{r}_{\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(k)$$

with $\nu \circ \overline{r}_{\mathfrak{m}} = \epsilon^{n-1}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}$ and $\overline{r}_{\mathfrak{m}}(c_v) \notin GL_n(k)$ for any infinite place v of F^+. Moreover, up to $GL_n(k)$-conjugation, the choices of such extensions are parametrised by $k^\times/(k^\times)^2$.

Similarly, for any minimal primes $\wp \subset \mathfrak{m}$ we have a continuous homomorphism $r_\wp$ from $\mathrm{Gal}(\overline{F}/F^+)$ to the the points of $\mathcal{G}_n$ over the algebraic closure of $\mathbb{Q}_l$ in the algebraic closure of the field of fractions of $\mathbb{T}^T_{a,\{\rho_v\}}(U)/\wp$ such that

- $r_\wp$ is unramified almost everywhere;
- $r_\wp^{-1}GL_n = \mathrm{Gal}(\overline{F}/F)$; and
- for all places $v \notin T$ of F^+ which split $v = ww^c$ in F the characteristic polynomial of $r_{\mathfrak{m}}(\mathrm{Frob}_w)$ is

  $$X^n - T_w^{(1)}X^{n-1} + ... + (-1)^j(\mathbf{N}w)^{j(j-1)/2}T_w^{(j)}X^{n-j} + ... + (-1)^n(\mathbf{N}w)^{n(n-1)/2}T_w^{(n)}.$$

According to lemma 1.1.6 we may assume that $r_\wp$ is actually valued in $\mathcal{G}_n(\mathcal{O}_\wp)$ where $\mathcal{O}_\wp$ is the ring of integers of some finite extension of the field of fractions of $\mathbb{T}^T_{a,\{\rho_v\}}(U)/\wp$. Then by lemma 1.1.3 again we may assume that the reduction of $r_\wp$ modulo the maximal ideal of $\mathcal{O}_\wp$ *equals* $\overline{r}_{\mathfrak{m}}$. (Not simply conjugate to $\overline{r}_{\mathfrak{m}}$.) Let R denote the subring of $k \oplus \bigoplus_{\wp \subset \mathfrak{m}} \mathcal{O}_\wp$ consisting of elements $(a_{\mathfrak{m}}, a_\wp)$

such that for all $\wp$ the reduction of $a_\wp$ modulo the maximal ideal of $\mathcal{O}_\wp$ is $a_\mathfrak{m}$. Then

$$\oplus_\wp r_\wp : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(R).$$

Moreover the natural map

$$\mathbb{T}^T_{a,\{\rho_v\}}(U)_\mathfrak{m} \longrightarrow R$$

is an injection. (Because $\mathbb{T}^T_{a,\{\rho_v\}}(U)_\mathfrak{m}$ is reduced.) Thus by lemma 1.1.10 we see that $\oplus_\wp r_\wp$ is $GL_n(R)$ conjugate to a representation

$$r_\mathfrak{m} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(\mathbb{T}^T_{a,\{\rho_v\}}(U)_\mathfrak{m})$$

such that:

- If a place $v \notin T$ of F^+ splits as ww^c in F then $r_\mathfrak{m}$ is unramified at w and $r_\mathfrak{m}(\mathrm{Frob}_w)$ has characteristic polynomial

  $$X^n - T_w^{(1)}X^{n-1} + ... + (-1)^j(\mathbf{N}w)^{j(j-1)/2}T_w^{(j)}X^{n-j} + ... + (-1)^n(\mathbf{N}w)^{n(n-1)/2}T_w^{(n)}.$$

- If a place v of F^+ is inert in F then $r_\mathfrak{m}$ is unramified at v.

It is easy to verify that $r_\mathfrak{m}$ also satisfies properties 4 and 5 of the proposition.

We next turn to assertion 6. After base changing to an algebraicly closed field each $r_\wp|_{\mathrm{Gal}\,(\overline{F}_w/F_w)}$ has a unique filtration such that $\mathrm{gr}^0 r_\wp|_{I_{F_w}} \cong \tilde{r}_w|_{I_{F_w}}$, and

$$\mathrm{gr}^i r_\wp|_{\mathrm{Gal}\,(\overline{F}_w/F_w)} \cong (\mathrm{gr}^0 r_\wp|_{\mathrm{Gal}\,(\overline{F}_w/F_w)})(\epsilon^i)$$

for $i = 0, ..., m_v - 1$ (and $= (0)$ otherwise). Enlarging $\mathcal{O}_\wp$ if need be we may assume that this filtration is defined over the field of fractions of $\mathcal{O}_\wp$. As $\tilde{r}_w \otimes_\mathcal{O} k$ is irreducible, such a filtration also exists over $\mathcal{O}_\wp$. Because of the uniqueness of the filtration $\overline{\mathrm{Fil}}^i_w$ on $\overline{r}_\mathfrak{m}$ we see that these filtrations piece together to give a filtration of $\oplus_\wp r_\wp$ over R. As the isomorphisms $\overline{\mathrm{gr}}^i_w \overline{r}_\mathfrak{m} \cong (\overline{\mathrm{gr}}^0_w \overline{r}_\mathfrak{m})(\epsilon^i)$ are unique up to scalar multiples we get a isomorphisms

$$\mathrm{gr}^i_w(\oplus_\wp r_\wp) \cong (\mathrm{gr}^0_w(\oplus_\wp r_\wp))(\epsilon^i)$$

over $R[\mathrm{Gal}\,(\overline{F}_w/F_w)]$ which are compatible with the chosen isomorphism between $\overline{\mathrm{gr}}^i_w \overline{r}_\mathfrak{m}$ and $(\overline{\mathrm{gr}}^0_w \overline{r}_\mathfrak{m})(\epsilon^i)$. As

$$Z_{GL_{n/m_v}(\mathcal{O}_\wp)}(\mathrm{gr}^0 r_\wp(I_{F_w})) \twoheadrightarrow Z_{GL_{n/m_v}(\mathcal{O}_\wp/\mathfrak{m}_{\mathcal{O}_\wp})}(\mathrm{gr}^0 r_\wp(I_{F_w}))$$

(see lemma 1.3.14), we see that we get an isomorphism

$$\mathrm{gr}^0_w(\oplus_\wp r_\wp) \cong \widetilde{r}_w \otimes_\mathcal{O} R$$

over $R[I_{F_w}]$ compatible with the chosen isomorphism $\overline{\mathrm{gr}}^0_w\overline{r}_\mathfrak{m} \cong \widetilde{r}_w \otimes_\mathcal{O} k$. Then using lemmas 1.1.8 and 1.3.14 we see that these isomorphism persist over $\mathbb{T}^T_{a,\{\rho_v\}}(U)_\mathfrak{m}$.

Finally we turn to part 7 of the proposition. The existence of A follows at once from Hensel's lemma. Let $P(X) \in \mathbb{T}^T_{a,\{\rho_v\}}(U)_\mathfrak{m}[X]$ denote the characteristic polynomial of $r_\mathfrak{m}(\phi_w)$. Thus $P(X) = (X - A)Q(X)$ where $Q(A) \in \mathbb{T}^T_{a,\{\rho_v\}}(U)^\times_\mathfrak{m}$.

Write $Y \otimes_K \overline{K} = \bigoplus((Y \otimes \overline{K}) \cap \pi)$ as π runs over irreducible smooth representations of $G(\mathbb{A}^\infty_{F^+})$. From lemmas 2.1.3 and 2.1.5 and the fact that $V_{\varpi_w} - a$ is topologically nilpotent we see that $\dim((Y \otimes \overline{K}) \cap \pi) \leq 1$ for all π. Let ϕ'_w be any lift of Frob_w to $\mathrm{Gal}\,(\overline{F}_w/F_w)$ and let $\mathrm{Art}_{F_w}\varpi'_w = \phi'_w$. Let P' denote the characteristic polynomial of $r_\mathfrak{m}(\phi'_w)$ and let A' be its unique root in $\mathbb{T}^T(Y)$ over a. As V_{ϖ_w} and $V_{\varpi'_w}$ commute, each $(Y \otimes \overline{K}) \cap \pi$ is invariant under $V_{\varpi'_w}$. By lemma 2.1.5 $V_{\varpi'_w}V^{-1}_{\varpi_w}$ is topologically unipotent on $(Y \otimes \overline{K}) \cap \pi$. Lemmas 2.1.3 and 2.1.5 imply that $P'(V_{\varpi'_w}) = 0$ on $(Y \otimes \overline{K}) \cap \pi$. Thus $V_{\varpi'_w} = A'$ on $(Y \otimes \overline{K}) \cap \pi$. Hence $V_{\varpi'_w} = A' \in \mathbb{T}^T(Y) \subset \mathrm{End}_K(Y)$. It follows that $V_\alpha \in \mathbb{T}^T(X)$ for all $\alpha \in F_w^\times$ with non-negative valuation and that $\alpha \mapsto V_\alpha$ extends to a continuous character $V : F_w^\times \to \mathbb{T}^T(Y)^\times$.

Now suppose that $\mathbf{N}w \equiv 1 \bmod l$. From lemma 2.1.5 we see that if $(Y \otimes \overline{K}) \cap \pi \neq (0)$ then either π is unramified or $\pi^{U_0(w)} = (0)$. Thus $(r_\mathfrak{m} \otimes \mathbb{T}^T(Y))(\mathrm{Gal}\,(\overline{F}_w/F_w))$ is abelian. We have a decomposition

$$\mathbb{T}^T(Y)^n = Q(\phi_w)\mathbb{T}^T(Y)^n \oplus (\phi_w - A)\mathbb{T}^T(Y)^n.$$

As $(r_\mathfrak{m} \otimes \mathbb{T}^T(Y))(\mathrm{Gal}\,(\overline{F}_w/F_w))$ is abelian we see that this decomposition is preserved by $\mathrm{Gal}\,(\overline{F}_w/F_w)$. By lemma 2.1.5 we see that after projection to any $\pi \cap (Y \otimes \overline{K})$, $\mathrm{Gal}\,(\overline{F}_w/F_w)$ acts on $Q(\phi_w)\mathbb{T}^T(Y)^n$ by $V_\pi \circ \mathrm{Art}^{-1}_{F_w}$ and its action on $(\phi_w - A)\mathbb{T}^T(Y)^n$ is unramified. We conclude that $\mathrm{Gal}\,(\overline{F}_w/F_w)$ acts on $Q(\phi_w)\mathbb{T}^T(Y)^n$ by V and that its action on $(\phi_w - A)\mathbb{T}^T(Y)^n$ is unramified. This completes the proof of part 7 of the proposition. □

Corollary 2.4.3 *Suppose that* $\mathfrak{m}$ *is a non-Eistenstein maximal ideal of the Hecke algebra* $\mathbb{T}^T_{a,\{\rho_v\}}(U)$. *Suppose also that* $v \in T - (S(B) \cup S_l)$ *and that* $U_v = G(\mathcal{O}_{F^+,v})$. *If* w *is a prime of* F *above* v *then for* $j = 1, ..., n$ *we have*

$$T^{(j)}_w \in \mathbb{T}^T_{a,\{\rho_v\}}(U)_\mathfrak{m} \subset \mathrm{End}\,(S_{a,\{\rho_v\}}(U,K)_\mathfrak{m}).$$

An exactly similar statement is true for a non-Eistenstein maximal ideal $\mathfrak{m}'$ *of* $\mathbb{T}^T_{a,\{\rho_v\}}(U')'$.

Proof: One need only remark that

$$T_w^{(j)} = (\mathbf{N}w)^{j(1-j)/2} \mathrm{tr} \wedge^j r_{\mathfrak{m}}(\mathrm{Frob}_w).$$

□

Lemma 2.4.4 *Let $R \subset T - (S(B) \cup S_l)$ and let $\widetilde{R} \coprod \widetilde{R}^c$ be a partition of the primes of F above R. For $v \in R$ let $\widetilde{v}$ denote the prime of $\widetilde{R}$ above v. Suppose that for $v \in R$ the group U_v is $i_w^{-1}\mathrm{Iw}_1(\widetilde{v})$. Suppose also that $V = U^R \times \prod_{v \in R} i_{\widetilde{v}}^{-1} \mathrm{Iw}(\widetilde{v})$ is sufficiently small. Then $V/U = \prod_{\widetilde{v} \in \widetilde{R}} (k(\widetilde{v})^\times)^n$ acts on $S_{a,\{\rho_v\}}(U,K)$. Suppose that χ and χ' are two characters $V/U \to \mathcal{O}^\times$ with $\chi \bmod \lambda = \chi' \bmod \lambda$. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^T_{a,\{\rho_v\}}(U)$. Then*

$$S_{a,\{\rho_v\}}(U,K)^{\chi}_{\mathfrak{m}} \neq (0)$$

if and only if

$$S_{a,\{\rho_v\}}(U,K)^{\chi'}_{\mathfrak{m}} \neq (0).$$

Proof: If R is an $\mathcal{O}$-algebra and $\psi : V/U \to \mathcal{O}^\times$ let $S_{a,\{\rho_v\},\psi}(V,R)$ denote the set of functions

$$f : G(F^+) \backslash G(\mathbb{A}^\infty_{F^+}) \longrightarrow M_{a,\{\rho_v\}} \otimes_{\mathcal{O}} R$$

such that

$$f(gu) = \psi(u_R)^{-1} u_{l,S(B)}^{-1} f(g)$$

for all $u \in V$ and $g \in G(\mathbb{A}^\infty_{F^+})$. As V is sufficiently small we see that $S_{a,\{\rho_v\},\psi}(V,\mathcal{O})$ is finite and free over $\mathcal{O}$ and that

$$S_{a,\{\rho_v\},\psi}(V,R) = S_{a,\{\rho_v\},\psi}(V,\mathcal{O}) \otimes_{\mathcal{O}} R.$$

The spaces $S_{a,\{\rho_v\},\psi}(V,R)$ have a natural action of $\mathbb{T}^T_{a,\{\rho_v\}}(U)$. We have

$$S_{a,\{\rho_v\}}(U,K)^{\chi}_{\mathfrak{m}} = S_{a,\{\rho_v\},\chi^{-1}}(U,K)_{\mathfrak{m}}$$

and

$$S_{a,\{\rho_v\}}(U,K)^{\chi'}_{\mathfrak{m}} = S_{a,\{\rho_v\},(\chi')^{-1}}(U,K)_{\mathfrak{m}}.$$

Moreover $S_{a,\{\rho_v\},\chi^{-1}}(U,K)_{\mathfrak{m}} = (0)$ (resp. $S_{a,\{\rho_v\},(\chi')^{-1}}(U,K)_{\mathfrak{m}} = (0)$) if and only if $S_{a,\{\rho_v\},\chi^{-1}}(U,\mathcal{O})_{\mathfrak{m}} = (0)$ (resp. $S_{a,\{\rho_v\},(\chi')^{-1}}(U,\mathcal{O})_{\mathfrak{m}} = (0)$) if and only if $S_{a,\{\rho_v\},\chi^{-1}}(U,k)_{\mathfrak{m}} = (0)$ (resp. $S_{a,\{\rho_v\},(\chi')^{-1}}(U,k)_{\mathfrak{m}} = (0)$). However

$$S_{a,\{\rho_v\},\chi^{-1}}(U,k)_{\mathfrak{m}} = S_{a,\{\rho_v\},(\chi')^{-1}}(U,k)_{\mathfrak{m}}$$

and the lemma follows. □

2.5 Ihara's lemma and raising the level.

In this section we will discuss congruences between modular forms of different levels. Unfortunately we can not prove anything. Rather we will explain how the congruence results we expect would follow from an analogue of Ihara's lemma for elliptic modular forms (see [I], [Ri]). Let us first describe this conjecture more precisely.

Conjecture I *Let G, l, T and U be as in the last section with U sufficiently small. Suppose that $v \in T - (S(B) \cup S_l)$ with $U_v = G(\mathcal{O}_{F^+,v})$ and that $\mathfrak{m}$ is a non-Eisenstein maximal ideal of $\mathbb{T}^T_{0,\{1\}}(U)$. If $f \in S_{0,\{1\}}(U, \overline{k})[\mathfrak{m}]$ and if π is an irreducible $G(F_v^+)$-submodule of*

$$\langle G(F_v^+) f \rangle \subset S_{0,\{1\}}(U^v, \overline{k})$$

then π is generic.

In fact we suspect something stronger is true. Although we will not need this stronger form we state it here. We will call an irreducible $G(F_v^+)$-submodule π of $S_{a,\{\rho_v\}}(\{1\}, \overline{k})$ *Eisenstein* if for some (and hence all) open compact subgroups $U = \prod_x U_x$ with $\pi^U \neq (0)$ there is a finite set T (containing v) of split primes and an Eisenstein maximal ideal $\mathfrak{m}$ of $\mathbb{T}^T_{a,\{\rho_v\}}(\{1\}, \overline{k})$ with $\pi_{\mathfrak{m}} \neq (0)$.

Conjecture II *Let G and l be as in the last section. Suppose that $v \notin S(B) \cup S_l$ is a prime of F^+ which splits in F. Let π be a non-Eisenstein irreducible $G(F_v^+)$-submodule of $S_{0,\{1\}}(\{1\}, \overline{k})$. Then π is generic.*

We should point out that these conjectures are certainly false if we replace 'submodule' by 'subquotient'. If we replace $\overline{k}$ by $\overline{K}$ and $\mathbb{T}^T_{0,\{1\}}(U)$ by $\mathbb{T}^T_{0,\{1\}}(U) \otimes_{\mathcal{O}} K$, then the conjectures would be true by part 7 of proposition 2.3.4. In the case $n = 2$ the conjecture is an easy consequence of the strong approximation theorem for G. We believe that we can prove many cases of conjecture I in the case $n = 3$. We hope to return to this in another paper.

Lemma 2.5.1 *Let G be as in the last section. Suppose conjecture I holds for all T and U with U sufficiently small. Let T, U, a and $\{\rho_v\}$ be as in the last section. Let $v \in T - (S(B) \cup S_l)$ with $U_v = G(\mathcal{O}_{F^+,v})$ and let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}^T_{a,\{\rho_v\}}(U)$. If $f \in S_{a,\{\rho_v\}}(U, \overline{k})[\mathfrak{m}]$ and if π is an irreducible $G(F_v^+)$-submodule of*

$$\langle G(F_v^+) f \rangle \subset S_{a,\{\rho_v\}}(U^v, \overline{k})$$

then π is generic.

Proof: We need only prove the lemma for U small, because its truth for some U implies its truth for all $U' \supset U$. But for U small enough we have

$$S_{a,\{\rho_v\}}(U,\overline{k}) = S_{0,\{1\}}(U,\overline{k})^r$$

for some r. □

Lemma 2.5.2 *Conjecture II (and hence conjecture I) is true if $n=2$.*

Proof: Let G_1 denote the derived subgroup of G. Then we have exact sequences

$$(0) \longrightarrow G_1(F^+) \longrightarrow G(F^+) \stackrel{\det}{\longrightarrow} F^{\mathbf{N}_{F/F^+}=1}$$

and

$$(0) \longrightarrow G_1(\mathbb{A}_{F^+}^{\infty}) \longrightarrow G(\mathbb{A}_{F^+}^{\infty}) \stackrel{\det}{\longrightarrow} \mathbb{A}_F^{\mathbf{N}_{F/F^+}=1}.$$

Suppose π is as in the statement of conjecture II, but π is not generic. Then π is one dimensional and trivial on $G_1(F_v^+)$. Let $0 \neq f \in \pi$ be invarinat by an open compact U. Then for all $g \in G(\mathbb{A}_{F^+}^{\infty})$, the function f is constant on

$$G(F^+)gUG_1(F_v^+) = G(F^+)G_1(\mathbb{A}_{F^+}^{\infty})gU$$

(by the strong approximation theorem). Thus f factors through

$$\det : G(F^+)\backslash G(\mathbb{A}_{F^+}^{\infty})/U \longrightarrow \det G(F^+)\backslash (\mathbb{A}_F^{\infty})^{\mathbf{N}=1}/\det U.$$

Thus we can find a character

$$\chi : \det G(F^+)\backslash (\mathbb{A}_F^{\infty})^{\mathbf{N}=1}/\det U \longrightarrow \overline{k}^{\times}$$

such that

$$\sum_{g \in (\det G(F^+))\backslash(\det G(\mathbb{A}_{F^+}^{\infty}))/(\det U)} \chi(g)^{-1} f(g) \neq 0.$$

It follows that, for all but finitely many places w of F which are split over F^+, $\overline{r}_{\mathfrak{m}}(\mathrm{Frob}_w)$ has characteristic polynomial

$$(X - \chi(\varpi_w/\varpi_w^c))(X - q_w\chi(\varpi_w/\varpi_w^c))...(X - q_w^{n-1}\chi(\varpi_w/\varpi_w^c)).$$

We deduce that

$$(\mathrm{ad}\, \overline{r}_{\mathfrak{m}})^{\mathrm{ss}} = \bigoplus_{i=1-n}^{n-1} (\epsilon^i)^{\oplus(n-|i|)}.$$

Thus $\overline{r}_{\mathfrak{m}}$ is reducible and $\mathfrak{m}$ is Eisenstein. □

We now turn to 'raising the level' congruences. For the rest of this section we keep the notation and assumptions of the last two sections.

Let $R \subset T - (S(B) \cup S_l)$ and assume that $U_v = G(\mathcal{O}_{F^+,v})$ for all $v \in R$. For $v \in R$ choose a prime $\widetilde{v}$ of F above v. Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}^T_{a,\{\rho_v\}}(U)$ and let

$$\phi : \mathbb{T}^T_{a,\{\rho_v\}}(U)_{\mathfrak{m}} \longrightarrow \mathcal{O}.$$

If $S \subset R$ set

$$U(S) = U^S \prod_{v \in S} i_{\widetilde{v}}^{-1} U_1(\widetilde{v}^n).$$

Also set

$$X_S = S_{a,\{\rho_v\}}(U(S), \mathcal{O})_{\mathfrak{m},\mathfrak{n}}$$

where $\mathfrak{n}$ denotes the maximal ideal

$$(\lambda, U^{(1)}_{\widetilde{v}}, ..., U^{(n-1)}_{\widetilde{v}} : \ v \in S)$$

of $\mathcal{O}[U^{(1)}_{\widetilde{v}}, ..., U^{(n-1)}_{\widetilde{v}} : \ v \in S]$. Further set

$$\mathbb{T}_S = \mathbb{T}^T(X_S),$$

so that $\mathbb{T}_\emptyset = \mathbb{T}^T_{a,\{\rho_v\}}(U)_{\mathfrak{m}}$.

If $S \subset R$ let

$$\theta_S = \prod_{v \in R} i_{\widetilde{v}}^{-1} \theta_{n,\widetilde{v}}.$$

If $S_1 \subset S_2 \subset R$ then we get an injection

$$\theta_{S_2 - S_1} : X_{S_1} \hookrightarrow X_{S_2}.$$

(To see that this map is an injection we may suppose that $S_2 = S_1 \cup \{v\}$. Let π be an irreducible constituent of $S_{a,\{\rho_v\}}(\{1\}, \overline{K})$ with $\pi \cap X_{S_1} \neq (0)$. Because $\mathfrak{m}$ is not Eisenstein we see that π_v is generic (see part 7 of proposition 2.3.4). Thus by proposition 2.2.8

$$i_{\widetilde{v}}^{-1} \theta_{n,\widetilde{v}} : \pi \cap X_{S_1} \hookrightarrow \pi \cap X_{S_2}.)$$

Thus we also have a surjection

$$\mathbb{T}_{S_2} \twoheadrightarrow \mathbb{T}_{S_1}$$

which takes $T^{(j)}_w$ to $T^{(j)}_w$ for all w (a prime of F which is split over a prime of F^+ not in T) and j $(= 1, ..., n)$. Let ϕ_S denote the composite

$$\phi_S : \mathbb{T}_S \twoheadrightarrow \mathbb{T}_\emptyset \xrightarrow{\phi} \mathcal{O}.$$

We will be interested in congruences between ϕ and other homomorphisms $\mathbb{T}_S \to \overline{K}$. In particular we will be interested in how these congruences vary with S. A useful measure of these congruences is provided by the ideal $\mathfrak{c}_S(\phi)$, defined by

$$\phi_S : \mathbb{T}_S/(\ker \phi_S + \operatorname{Ann}_{\mathbb{T}_S} \ker \phi_S) \xrightarrow{\sim} \mathcal{O}/\mathfrak{c}_S(\phi).$$

If $S \subset R$ let $X_S[\phi]$ denote the subspace of X_S where $\mathbb{T}_S$ acts via ϕ_S. Let $i_S : X_S[\phi] \hookrightarrow X_S$ denote the canonical inclusion and let $\pi_S : X_S \twoheadrightarrow X_S[\phi]$ denote the $\mathbb{T}_S$-equivariant projection. (This exists because $\mathbb{T}_S$ is reduced.) The next lemma is now clear.

Lemma 2.5.3 *Keep the above notation. The module $X_S[\phi]/\pi_S i_S X_S[\phi]$ is an $\mathcal{O}/\mathfrak{c}_S(\phi)$-module. If X_S is free over $\mathbb{T}_S$ then $X_S[\phi]/\pi_S i_S X_S[\phi]$ is free over $\mathcal{O}/\mathfrak{c}_S(\phi)$.*

Lemma 2.5.4 *Keep the above notation. Then*

$$\theta_S : X_\emptyset[\phi] \otimes_{\mathcal{O}} K \xrightarrow{\sim} X_S[\phi] \otimes_{\mathcal{O}} K.$$

Proof: It suffices to prove that if π is an irreducible constituent of the space $S_{a,\{\rho_v\}}(\{1\}, \overline{K})$ then

$$\theta_S : (X_\emptyset[\phi] \otimes_{\mathcal{O}} \overline{K}) \cap \pi \xrightarrow{\sim} (X_S[\phi] \otimes_{\mathcal{O}} \overline{K}) \cap \pi.$$

As $\phi r_{\mathfrak{m}}$ is unramified at $v \in S$, proposition 2.3.4 tells us that if $(X_S[\phi] \otimes_{\mathcal{O}} \overline{K}) \cap \pi \neq (0)$ then π_v is unramified. In particular $(X_\emptyset[\phi] \otimes_{\mathcal{O}} \overline{K}) \cap \pi \neq (0)$. If $(X_\emptyset[\phi] \otimes_{\mathcal{O}} \overline{K}) \cap \pi \neq (0)$ then for $v \in S$ the representation π_v is unramified and, by part 7 of proposition 2.3.4, generic. Write

$$\pi_v \circ i_{\widetilde{v}}^{-1} = \text{n-Ind}\,_{B_n(F_{\widetilde{v}})}^{GL_n(F_{\widetilde{v}})}(\chi_{v,1}, ..., \chi_{v,n})$$

with each $\chi_{v,i}$ unramified. Again by proposition 2.3.4 we see that for $v \in S$, each $\chi_{v,i}(\varpi_{\widetilde{v}}) \in \mathcal{O}_{\overline{K}}^\times$. From lemma 2.2.6 we deduce that

$$\pi_{\mathfrak{n}}^{U(S)}$$

is the subspace of $\pi^{U(S)}$ on which $i_{\widetilde{v}}^{-1} U_{\widetilde{v}}^{(j)} = 0$ for each $v \in S$ and each $j = 1, ..., n-1$. Proposition 2.2.8 then tells us that

$$\theta_S : \pi^{U(\emptyset)} \xrightarrow{\sim} \pi_{\mathfrak{n}}^{U(S)}$$

as desired. □

Proposition 2.5.5 *Keep the above notation and assumptions. In particular assume that U is sufficiently small. Suppose that conjecture I is true for the groups G and G', for l, for T, for $v \in R$ and for the various open compact subgroups U_S (with $S \subset R$). Also suppose that $X_\emptyset$ is free over $\mathbb{T}_\emptyset$. Finally suppose that for each $v \in R$, l is quasi-banal for $G(F_v^+)$. Then*

$$\lg_{\mathcal{O}} \mathcal{O}/\mathfrak{c}_R(\phi) \geq \lg_{\mathcal{O}} \mathcal{O}/\mathfrak{c}_\emptyset(\phi) + \sum_{v \in R} \lg_{\mathcal{O}} H^0(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}), (\mathrm{ad}\, r_{\mathfrak{m}}) \otimes_{\mathbb{T}_\emptyset, \phi} K/\mathcal{O}(\epsilon^{-1})).$$

Proof: Let $\eta_\emptyset \in G'(\mathbb{A}_{F^+}^\infty)$ equal 1 away from $T - (R \cup S(B) \cup S_l)$. If $S \subset R$ set

$$\eta_S = \eta_\emptyset \prod_{v \in S} (i_{\widetilde{v}}^t)^{-1} \begin{pmatrix} 1_{n-1} & 0 \\ 0 & \varpi_{\widetilde{v}}^n \end{pmatrix}$$

and

$$U(S)' = \eta_S^{-1} U(S) \eta_S = (U(\emptyset)')^S \times \prod_{v \in S} (i_{\widetilde{v}}^t)^{-1} U_1(\widetilde{v}^n).$$

Let $\mathfrak{m}'$ denote the ideal of $\mathbb{T}^T_{a,\{\rho_v\}}(U(S)')'$ generated by λ and $T_w^{(j)} - a$ whenever $a \in \mathcal{O}$, w is a prime of F split above a prime of F^+ not in T and $T_w^{(j)} - a \in \mathfrak{m}$. Then $\mathfrak{m}'$ is either maximal or the whole Hecke algebra. Set

$$X'_S = S'_{a,\{\rho_v\}}(U(S)', \mathcal{O})_{\mathfrak{m}',\mathfrak{n}}$$

where $\mathfrak{n}$ denotes the maximal ideal

$$(\lambda, U_{\widetilde{v}}^{(1)}, ..., U_{\widetilde{v}}^{(n-1)})$$

of $\mathcal{O}[U_{\widetilde{v}}^{(1)}, ..., U_{\widetilde{v}}^{(n-1)}]$, and

$$\mathbb{T}'_S = \mathbb{T}^T(X_S)'.$$

Also set

$$\theta'_S = \prod_{v \in R} (i_{\widetilde{v}}^t)^{-1} \theta_{n,\widetilde{v}}$$

and

$$\widehat{\theta}'_S = \prod_{v \in S} (i_{\widetilde{v}}^t)^{-1} (\widehat{\theta}_{n,\widetilde{v}}).$$

If $S_1 \subset S_2 \subset R$ then we get an injection

$$\theta'_{S_2 - S_1} : X'_{S_1} \hookrightarrow X'_{S_2}$$

and exactly as in the proof of lemma 2.5.4 we see that

$$\theta'_{S_2 - S_1} : X'_{S_1} \otimes_{\mathcal{O}} K \xrightarrow{\sim} X'_{S_2} \otimes_{\mathcal{O}} K.$$

Also by corollary 2.4.3

$$\widehat{\theta}'_S \theta_S = \prod_{v \in S} i_{\widetilde{v}}^{-1}(\widehat{\theta}_{n,\widetilde{v}} \theta_{n,\widetilde{v}})$$

acts on $X_\emptyset$ by an element of $\mathbb{T}_\emptyset$.

Under the perfect pairing

$$\langle \ , \ \rangle_{U(S),\eta_S} : S_{a,\{\rho_v\}}(U(S), \mathcal{O}) \times S'_{a,\{\rho_v\}}(U(S)', \mathcal{O}) \longrightarrow \mathcal{O}$$

we have that:

- for $v \in S$ the adjoint of $i_{\widetilde{v}}^{-1} U_{\widetilde{v}}^{(j)}$ is $(i_{\widetilde{v}}^t)^{-1} U_{\widetilde{v}}^{(j)}$, and
- for w a prime of F split over a prime of F^+ not in T, the adjoint of $T_w^{(j)}$ is $T_w^{(j)}$.

Thus $\mathbb{T}_S \cong \mathbb{T}'_S$ (with $T_w^{(j)}$ matching $T_w^{(j)}$ for w a prime of F split over a prime of F^+ not in T), and $\langle \ , \ \rangle_{U(S),\eta_S}$ induces a perfect pairing

$$\langle \ , \ \rangle_S : X_S \times X'_S \longrightarrow \mathcal{O}$$

under which the actions of $\mathbb{T}_S \cong \mathbb{T}'_S$ are self-adjoint. If $S_1 \subset S_2 \subset R$, then

$$\widehat{\theta}'_{S_2 - S_1} : X_{S_2} \longrightarrow X_{S_1}$$

is the adjoint of $\theta'_{S_2-S_1}$.

It follows from conjecture I and lemma 2.2.10 that

$$\theta_{\{v\}} : X_S \longrightarrow X_{S \cup \{v\}}$$

has torsion free cokernel, and that

$$\widehat{\theta}'_{\{v\}} : X_{S \cup \{v\}} \longrightarrow X_S$$

is surjective. Thus

$$\theta_R : X_\emptyset \longrightarrow X_R$$

has torsion free cokernel, and

$$\widehat{\theta}'_R : X_R \longrightarrow X_\emptyset$$

is surjective. In particular

$$\theta_R : X_\emptyset[\phi] \xrightarrow{\sim} X_R[\phi],$$

and we may take

$$i_R = \theta_R \circ i_\emptyset \circ \theta_R|_{X_\emptyset[\phi]}^{-1}$$

and

$$\pi_R = \theta_R|_{X_\emptyset[\phi]} \circ \pi_\emptyset \circ \widehat{\theta}'_R.$$

Thus

$$\begin{array}{rcl} X_R[\phi]/\pi_R i_R X_R[\phi] & \cong & X_\emptyset[\phi]/\phi(\widehat{\theta}'_R \theta_R)\pi_\emptyset i_\emptyset X_\emptyset[\phi] \\ & \cong & X_\emptyset[\phi]/(\prod_{v \in R} \phi i_{\widetilde{v}}^{-1}(\widehat{\theta}_{n,\widetilde{v}} \theta_{n,\widetilde{v}}))\pi_\emptyset i_\emptyset X_\emptyset[\phi]. \end{array}$$

The proposition follows from corollary 2.2.9. □

3 $R = T$ theorems.

Fix a positive integer $n \geq 2$ and a prime $l > n$.

Fix an imaginary quadratic field E in which l splits and a totally real field F^+ such that

- $F = F^+E/F^+$ is unramified at all finite primes, and
- $F^+/\mathbb{Q}$ is unramified at l.

Fix a finite non-empty set of places $S(B)$ of places of F^+ with the following properties:

- Every element of $S(B)$ splits in F.
- $S(B)$ contains no place above l.
- If n is even then

$$n[F^+ : \mathbb{Q}]/2 + \#S(B) \equiv 0 \bmod 2.$$

Choose a division algebra B with centre F with the following properties:

- $\dim_F B = n^2$.
- $B^{\mathrm{op}} \cong B \otimes_{E,c} E$.
- B splits outside $S(B)$.
- If $\widetilde{v}$ is a prime of F above an element of $S(B)$, then $B_{\widetilde{v}}$ is a division algebra.

Fix an involution $\ddagger$ on B such that

- $\ddagger|_F = c$,
- for a place $v|\infty$ of F^+ we have $G_\ddagger(F_v^+) \cong U(n)$, and
- for a finite place $v \notin S(B)$ of F^+ the group $G_\ddagger(F_v^+)$ is quasi-split.

Also define an algebraic group G'/F^+ by setting

$$G'(R) = \{g \in B^{\mathrm{op}} \otimes_{F^+} R : \ g^{\ddagger \otimes 1} g = 1\}$$

for any F^+-algebra R.

Choose an order $\mathcal{O}_B$ in B such that $\mathcal{O}_B^\ddagger = \mathcal{O}_B$ and $\mathcal{O}_{B,w}$ is maximal for all primes w of F which are split over F^+. This gives a model of G over $\mathcal{O}_{F^+}$. If $v \notin S(B)$ is a prime of F^+ which splits in F choose an isomorphism $i_v : \mathcal{O}_{B,v} \xrightarrow{\sim} M_n(\mathcal{O}_{F,v})$ such that $i_v(x^\ddagger) = {}^t i_v(x)^c$. If w is a prime of F above v this gives rise to an isomorphism $i_w : G(F_v^+) \xrightarrow{\sim} GL_n(F_w)$ as in section 2.3. If $v \in S(B)$ and w is a prime of F above v choose isomorphisms $i_w : G(F_v^+) \xrightarrow{\sim} B_w^\times$ such that $i_{w^c} = i_w^{-\ddagger}$ and $i_w G(\mathcal{O}_{F^+,v}) = \mathcal{O}_{B,w}^\times$.

Let S_l denote the set of primes of F^+ above l. Let S_1 denote a non-empty set, disjoint from $S_l \cup S(B)$, of primes of F^+ such that

- if $v \in S_1$ then v splits in F, and
- if $v \in S_1$ lies above a rational prime p then $[F(\zeta_p) : F] > n$.

Let R denote a set, disjoint from $S_l \cup S(B) \cup S_1$, of primes of F^+ such that

- if $v \in R$ then v splits in F, and
- if $v \in R$ then either $\mathbf{N}v \equiv 1 \bmod l$ or $l \not| \# GL_n(k(v))$.

Let $T = R \cup S(B) \cup S_l \cup S_1$. Let $\widetilde{T}$ denote a set of primes of F above T such that $\widetilde{T} \coprod \widetilde{T}^c$ is the set of all primes of F above T. If $v \in T$ we will let $\widetilde{v}$ denote the prime of $\widetilde{T}$ above v. If $S \subset T$ we will let $\widetilde{S}$ denote the set of $\widetilde{v}$ for $v \in S$.

If $S \subset R$ let $U(S) = \prod_v U(S)_v$ denote an open compact subgroup of $G(\mathbb{A}_{F^+}^\infty)$ such that

- if v is not split in F then U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$,
- if $v \notin S_1 \cup S$ splits in F then $U_v = G(\mathcal{O}_{F^+,v})$,
- if $v \in S$ then $U_v = i_{\widetilde{v}}^{-1} U_1(\widetilde{v}^n)$, and
- if $v \in S_1$ then $U_v = i_{\widetilde{v}}^{-1} \ker(GL_n(\mathcal{O}_{F,\widetilde{v}}) \to GL_n(\mathcal{O}_{F,\widetilde{v}}/(\varpi_{\widetilde{v}}^{m_v})))$ for some $m_v \geq 1$.

Then $U(S)$ is sufficiently small. If $S = \emptyset$ we will drop it from the notation, i.e. we will write $U = \prod_v U_v$ for $U(\emptyset)$.

Let $K/\mathbb{Q}_l$ be a finite extension which contains the image of every embedding $F^+ \hookrightarrow \overline{K}$. Let $\mathcal{O}$ denote its ring of integers, λ the maximal ideal of $\mathcal{O}$ and k the residue field $\mathcal{O}/\lambda$.

For each $\tau : F \hookrightarrow K$ choose integers $a_{\tau,1}, ..., a_{\tau,n}$ such that

- $a_{\tau c,i} = -a_{\tau,n+1-i}$, and

- if τ gives rise to a place in $\widetilde{S}_l$ then

$$l-1-n \geq a_{\tau,1} \geq \ldots \geq a_{\tau,n} \geq 0.$$

For each $v \in S(B)$ let $\rho_v : G(F_v^+) \longrightarrow GL(M_{\rho_v})$ denote a representation of $G(F_v^+)$ on a finite free $\mathcal{O}$-module such that ρ_v has open kernel and $M_{\rho_v} \otimes_{\mathcal{O}} \overline{K}$ is irreducible. For $v \in S(B)$, define m_v, $\pi_{\widetilde{v}}$ and $\widetilde{r}_{\widetilde{v}}$ by

$$\mathrm{JL}\,(\rho_v \circ i_{\widetilde{v}}^{-1}) = \mathrm{Sp}_{\,m_v}(\pi_{\widetilde{v}})$$

and

$$\widetilde{r}_{\widetilde{v}} = r_l(\pi_{\widetilde{v}}|\ \ |^{(n/m_{\widetilde{v}}-1)(1-m_{\widetilde{v}})/2}).$$

We will suppose that

$$\widetilde{r}_{\widetilde{v}} : \mathrm{Gal}\,(\overline{F}_w/F_w) \longrightarrow GL_{n/m_{\widetilde{v}}}(\mathcal{O})$$

(as opposed to $GL_{n/m_{\widetilde{v}}}(\overline{K})$), that the reduction of $\widetilde{r}_{\widetilde{v}}$ mod λ is absolutely irreducible and that for $i = 1, \ldots, m_v$ we have

$$\widetilde{r}_{\widetilde{v}} \otimes_{\mathcal{O}} k \not\cong \widetilde{r}_{\widetilde{v}} \otimes_{\mathcal{O}} k(\epsilon^i).$$

Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}^T_{a,\{\rho_v\}}(U)$ with residue field k and let

$$\overline{r}_{\mathfrak{m}} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(k)$$

be a continuous homomorphism associated to $\mathfrak{m}$ as in propositions 2.4.1 and 2.4.2. Note that

$$\nu \circ \overline{r}_{\mathfrak{m}} = \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}$$

where δ_{F/F^+} is the non-trivial character of $\mathrm{Gal}\,(F/F^+)$ and where $\mu_{\mathfrak{m}} \in \mathbb{Z}/2\mathbb{Z}$. We will *assume* that $\overline{r}_{\mathfrak{m}}$ has the following properties.

- $\overline{r}_{\mathfrak{m}}(\mathrm{Gal}\,(\overline{F}/F^+(\zeta_l)))$ is big in the sense of section 1.4.
- If $v \in S_1$ then $\overline{r}_{\mathfrak{m}}$ is unramified at v and

$$H^0(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}), (\mathrm{ad}\,\overline{r}_{\mathfrak{m}})(1)) = (0).$$

We will also assume that $\mathbb{T}^T_{a,\{\rho_v\}}(U)$ admits a section $\mathbb{T}^T_{a,\{\rho_v\}}(U) \to \mathcal{O}$.

Recall that if $v \in S(B)$ then by proposition 2.4.1 there is a unique filtration $\overline{\mathrm{Fil}}^i_{\widetilde{v}}$ of $\overline{r}_{\mathfrak{m}}$ invariant by $\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})$ and such that

$$\overline{\mathrm{gr}}^0_{\widetilde{v}}\overline{r}_{\mathfrak{m}}|_{I_{F_{\widetilde{v}}}} \cong \widetilde{r}_{\widetilde{v}}|_{I_{F_{\widetilde{v}}}} \otimes_{\mathcal{O}} k$$

and

$$\overline{\mathrm{gr}}^i_{\widetilde{v}}\overline{r}_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})} \cong (\overline{\mathrm{gr}}^0_{\widetilde{v}}\overline{r}_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})})(\epsilon^i)$$

for $i = 1, ..., m_v - 1$ and $= (0)$ otherwise. Moreover

$$\mathrm{Hom}_{\,\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}(\overline{\mathrm{Fil}}^i_{\widetilde{v}}\overline{r}_{\mathfrak{m}}, \overline{\mathrm{gr}}^i_{\widetilde{v}}\overline{r}_{\mathfrak{m}}) = k$$

for $i = 0, ..., m_v - 1$.

For $S \subset R$ write $X_{\mathfrak{m},S}$ for the space

$$S_{a,\{\rho_v\}}(U(S), \mathcal{O})_{\mathfrak{m},\mathfrak{n}}$$

where $\mathfrak{n}$ is the maximal ideal

$$(\lambda, U^{(1)}_{\widetilde{v}}, ..., U^{(n-1)}_{\widetilde{v}} :\ v \in S)$$

of $\mathcal{O}[U^{(1)}_{\widetilde{v}}, ..., U^{(n-1)}_{\widetilde{v}} :\ v \in S]$. Also write $\mathbb{T}_{\mathfrak{m},S}$ for the algebra $\mathbb{T}^T(X_{\mathfrak{m},S})$. Thus $\mathbb{T}_{\mathfrak{m},S}$ is a quotient of $\mathbb{T}^T_{a,\{\rho_v\}}(U(S))_{\mathfrak{m}}$, and these two algebras are equal if $S = \emptyset$. The algebra $\mathbb{T}_{\mathfrak{m},S}$ is local and reduced. It is finite and free as a $\mathcal{O}$-module. Let

$$r_{\mathfrak{m},S} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(\mathbb{T}_{\mathfrak{m},S})$$

denote the continuous lifting of $\overline{r}_{\mathfrak{m}}$ provided by proposition 2.4.2. Then $\mathbb{T}_{\mathfrak{m},S}$ is generated as a $\mathcal{O}$-algebra by the coefficients of the characteristic polynimials of $r_{\mathfrak{m},S}(\sigma)$ for $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$.

For $S \subset R$, consider the deformation problem $\mathcal{S}_S$ given by

$$(G_{F^+,T} \supset G_{F,T}, T \supset S(B), \{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})\}_{v\in T}, \mathcal{O}, \overline{r}_{\mathfrak{m}}, \epsilon^{1-n}\delta^{\mu_{\mathfrak{m}}}_{F/F^+}, \{\overline{\mathrm{Fil}}^i_{\widetilde{v}}\}_{\widetilde{v}\in S(B)}, \{\mathcal{D}_{\widetilde{v}}\}_{v\in T}, \{L_{\widetilde{v}}\}_{v\in T})$$

where:

- For $v \in S_1$, $\mathcal{D}_{\widetilde{v}}$ will consist of all lifts of $\overline{r}_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}$ and

$$L_{\widetilde{v}} = H^1(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}), \mathrm{ad}\,\overline{r}_{\mathfrak{m}}) = H^1(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}_{\mathfrak{m}}).$$

- For $v \in S_l$, $\mathcal{D}_{\widetilde{v}}$ and $L_{\widetilde{v}}$ are as described in section 1.3.1 or 1.3.2.
- For $v \in S(B)$, $\mathcal{D}_{\widetilde{v}}$ and $L_{\widetilde{v}}$ are as described in section 1.3.5.
- For $v \in R - S$, $\mathcal{D}_{\widetilde{v}}$ will consist of all unramified lifts of $\overline{r}_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}$ and

$$L_{\widetilde{v}} = H^1(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})/I_{F_{\widetilde{v}}}, \mathrm{ad}\,\overline{r}_{\mathfrak{m}}).$$

- For $v \in S$, $\mathcal{D}_{\widetilde{v}}$ will consist of all lifts of $\overline{r}_{\mathfrak{m}}|_{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})}$ and

$$L_{\widetilde{v}} = H^1(\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}), \mathrm{ad}\,\overline{r}_{\mathfrak{m}}).$$

Also let

$$r_{\mathfrak{m},S}^{\mathrm{univ}} : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(R_{\mathfrak{m},S}^{\mathrm{univ}})$$

denote the universal deformation of $\overline{r}_{\mathfrak{m}}$ of type $\mathcal{S}_S$. By proposition 2.4.2 there is a natural surjection

$$R_S^{\mathrm{univ}} \twoheadrightarrow \mathbb{T}_{\mathfrak{m},S}$$

such that $r_{\mathfrak{m},S}^{\mathrm{univ}}$ pushes forward to $r_{\mathfrak{m}}$.

We can now state and prove our main results.

Theorem 3.1.1 *Keep the notation and assumptions of the start of this section. Then*

$$R_{\mathfrak{m},\emptyset}^{\mathrm{univ}} \xrightarrow{\sim} \mathbb{T}_{\mathfrak{m},\emptyset}$$

is an isomorphism of complete intersections and $X_{\mathfrak{m},\emptyset}$ is free over $\mathbb{T}_{\mathfrak{m},\emptyset}$. Moreover $\mu_{\mathfrak{m}} \equiv n \bmod 2$.

Proof: To prove this we will appeal to Diamond's and Fujiwara's improvement to Faltings' understanding of the method of [TW]. More precisely we will appeal to theorem 2.1 of [Dia]. We remark that one may easilly weaken the hypotheses of this theorem in the following minor ways. The theorem with the weaker hypotheses is easilly deduced from the theorem as it is stated in [Dia]. In the notation of [Dia] one can take $B = k[[X_1, ..., X_{r'}]]$ with $r' \leq r$. Also in place of his assumption (c) one need only assume that H_n is free over $A/\mathfrak{n}_n$, where $\mathfrak{n}_n$ is an open ideal contained in $\mathfrak{n}$ with the property that $\bigcap_n \mathfrak{n}_n = (0)$. We also remark with these weakened hypotheses one may also deduce from the proof of theorem 2.1 of [Dia] that in fact $r = r'$.

Choose an integer r as in proposition 1.4.5. Set

$$r' = r - n[F^+ : \mathbb{Q}](1 + (-1)^{n-1+\mu_{\mathfrak{m}}})/2.$$

For each $N \in \mathbb{Z}_{\geq 1}$ choose a set of primes Q_N of F^+ as in proposition 1.4.5, and, for each $v \in Q_N$, choose a prime $\widetilde{v}$ of F above v and an eigenvalue $a_{\widetilde{v}}$ of $\overline{r}_{\mathfrak{m}}(\mathrm{Frob}_{\widetilde{v}})$ as in example 1.3.6. (In the notation of example 1.3.6, $a_{\widetilde{v}} = \overline{\chi}(\mathrm{Frob}_{\widetilde{v}})$.) Let $\mathcal{S}_{\emptyset,Q_N}$ denote the deformation problem

$$(G_{F^+,T\cup Q_N} \supset G_{F,T\cup Q_N}, T \cup Q_N \supset S(B), \{\mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}}\}_{v\in T\cup Q_N}, \mathcal{O}, \overline{r}_{\mathfrak{m}},$$
$$\epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}, \{\overline{\mathrm{Fil}}_{\widetilde{v}}^i\}_{\widetilde{v}\in S(B)}, \{\mathcal{D}_{\widetilde{v}}\}_{v\in T\cup Q_N}, \{L_{\widetilde{v}}\}_{v\in T\cup Q_N}),$$

where for $v \in T$, $\mathcal{D}_{\widetilde{v}}$ and $L_{\widetilde{v}}$ are as in $\mathcal{S}_\emptyset$, and for $v \in Q_N$ they are as in section 1.3.6. (Thus in the notation of theorem 1.4.5 $\mathcal{S}_{\emptyset,Q_N} = \mathcal{S}'$.) Let $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset,Q_N}$ denote the universal deformation ring $R^{\mathrm{univ}}_{\mathcal{S}_{\emptyset,\mathbb{Q}_N}}$. By proposition 1.4.5 there is a surjection

$$\mathcal{O}[[X_1, ..., X_{r'}]] \twoheadrightarrow R^{\mathrm{univ}}_{\mathfrak{m},\emptyset,Q_N}.$$

Let ψ_N denote the composite

$$\psi_N : \mathcal{O}[[X_1, ..., X_{r'}]] \twoheadrightarrow R^{\mathrm{univ}}_{\mathfrak{m},\emptyset,Q_N} \twoheadrightarrow R^{\mathrm{univ}}_{\mathfrak{m},\emptyset}.$$

For $v \in Q_N$ let $\Delta_{\widetilde{v}}$ denote the maximal l-power quotient of $\mathcal{O}^\times_{F,\widetilde{v}}$. Let $\Delta_{Q_N} = \prod_{v\in Q_N} \Delta_{\widetilde{v}}$. As explained in example 1.3.6, $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset,Q_N}$ is naturally a $\mathcal{O}[\Delta_{Q_N}]$-module and $(R^{\mathrm{univ}}_{\mathfrak{m},\emptyset,Q_N})_{\Delta_{Q_N}} = R^{\mathrm{univ}}_{\mathfrak{m},\emptyset}$. There is a surjection

$$\mathcal{O}[[S_1, ..., S_r]] \twoheadrightarrow \mathcal{O}[\Delta_{Q_N}]$$

such that, if $\mathfrak{n}_N$ denotes the kernel, then $\bigcap_N \mathfrak{n}_N = (0)$. We can lift the map

$$\mathcal{O}[[S_1, ..., S_r]] \twoheadrightarrow \mathcal{O}[\Delta_{Q_N}] \longrightarrow R^{\mathrm{univ}}_{\mathfrak{m},\emptyset,Q_N}$$

to a map

$$\phi_N : \mathcal{O}[[S_1, ..., S_r]] \longrightarrow \mathcal{O}[[X_1, ..., X_{r'}]].$$

Then the composite

$$\mathcal{O}[[S_1, ..., S_r]] \stackrel{\psi_N\circ\phi_N}{\longrightarrow} R^{\mathrm{univ}}_{\mathfrak{m},\emptyset}/\lambda$$

has kernel $(\lambda, S_1, ..., S_r)$.

Note that $X_{\mathfrak{m},\emptyset}$ is a $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset}$-module via $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset} \twoheadrightarrow \mathbb{T}_{\mathfrak{m},\emptyset}$.

Define open compact subgroups $U_1(Q_N) = \prod_v U_1(Q_N)_v$ and $U_0(Q_N) = \prod_v U_0(Q_N)_v$ of $G(\mathbb{A}^\infty_{F^+})$ by

- $U_1(Q_N)_v = U_0(Q_N)_v = U_v$ if $v \notin Q_N$,
- $U_1(Q_N)_v = i^{-1}_{\widetilde{v}} U_1(\widetilde{v})$ if $v \in Q_N$, and
- $U_0(Q_N)_v = i^{-1}_{\widetilde{v}} U_0(\widetilde{v})$ if $v \in Q_N$.

By corollary 2.4.3 we see that we have

$$\mathbb{T}^{T\cup Q_N}_{a,\{\rho_v\}}(U_1(Q_N))_{\mathfrak{m}} \twoheadrightarrow \mathbb{T}^{T\cup Q_N}_{a,\{\rho_v\}}(U_0(Q_N))_{\mathfrak{m}} \twoheadrightarrow \mathbb{T}^{T\cup Q_N}_{a,\{\rho_v\}}(U)_{\mathfrak{m}} = \mathbb{T}^{T}_{a,\{\rho_v\}}(U)_{\mathfrak{m}}.$$

For $v \in Q_N$ choose $\phi_{\widetilde{v}} \in \mathrm{Gal}\,(\overline{F}_{\widetilde{v}}/F_{\widetilde{v}})$ lifting $\mathrm{Frob}_{\widetilde{v}}$ and $\varpi_{\widetilde{v}} \in F^\times_{\widetilde{v}}$ with $\phi_{\widetilde{v}} = \mathrm{Art}_{F_{\widetilde{v}}} \varpi_{\widetilde{v}}$ on the maximal abelian extension of $F_{\widetilde{v}}$. Let

$$P_{\widetilde{v}} \in \mathbb{T}^{T\cup Q_N}_{a,\{\rho_v\}}(U_1(Q_N))_{\mathfrak{m}}[X]$$

denote the characteristic polynomial of $r_{\mathfrak{m}}(\phi_{\tilde{v}})$. By Hensel's lemma we have a unique factorisation

$$P_{\tilde{v}}(X) = (X - A_{\tilde{v}})Q_{\tilde{v}}(X)$$

over $\mathbb{T}^{T\cup Q_N}_{a,\{\rho_v\}}(U_1(Q_N))_{\mathfrak{m}}$, where $A_{\tilde{v}}$ lifts $a_{\tilde{v}}$ and $Q_{\tilde{v}}(A_{\tilde{v}}) \in \mathbb{T}^{T\cup Q_N}_{a,\{\rho_v\}}(U_1(Q_N))^{\times}_{\mathfrak{m}}$. By lemmas 2.1.3 and 2.1.5 we see that $P_{\tilde{v}}(V_{\varpi_{\tilde{v}}}) = 0$ on $S_{a,\{\rho_v\}}(U_1(Q_N), \mathcal{O})_{\mathfrak{m}}$. Set

$$H_{1,Q_N} = (\prod_{v\in Q_N} Q_{\tilde{v}}(V_{\varpi_{\tilde{v}}}))S_{a,\{\rho_v\}}(U_1(Q_N), \mathcal{O})_{\mathfrak{m}}$$

and

$$H_{0,Q_N} = (\prod_{v\in Q_N} Q_{\tilde{v}}(V_{\varpi_{\tilde{v}}}))S_{a,\{\rho_v\}}(U_0(Q_N), \mathcal{O})_{\mathfrak{m}}.$$

We see that H_{1,Q_N} is a $\mathbb{T}^{T\cup Q_N}_{a,\{\rho_v\}}(U_1(Q_N))$-direct summand of $S_{a,\{\rho_v\}}(U_1(Q_N), \mathcal{O})$, and hence by lemma 2.3.1

$$\mathrm{tr}_{\,U_0(Q_N)/U_1(Q_N)} : (H_{1,Q_N})_{U_0(Q_N)/U_1(Q_N)} \xrightarrow{\sim} H_{0,Q_N}.$$

Moreover for all $v \in Q_N$, $V_{\varpi_{\tilde{v}}} = A_{\tilde{v}}$ on H_{1,Q_N}. By part 7 of proposition 2.4.2 we see that for each $v \in Q_N$ there is a character

$$V_{\tilde{v}} : F^{\times}_{\tilde{v}} \longrightarrow \mathbb{T}^{T\cup Q_N}(H_{1,Q_N})^{\times}$$

such that

- if $\alpha \in F^{\times}_{\tilde{v}} \cap \mathcal{O}_{F,\tilde{v}}$ then $V_{\tilde{v}}(\alpha) = V_{\alpha}$ on H_{1,Q_N}, and
- $r_{\mathfrak{m}}|_{W_{F_{\tilde{v}}}} = s \oplus (V_{\tilde{v}} \circ \mathrm{Art}^{-1}_{F_{\tilde{v}}})$ where s is unramified.

Thus $r_{\mathfrak{m}}$ gives rise to a surjection

$$R^{\mathrm{univ}}_{\mathfrak{m},\emptyset,Q_N} \twoheadrightarrow \mathbb{T}^{T\cup Q_N}(H_{Q_N}).$$

The composite

$$\prod_{v\in Q_N} \mathcal{O}^{\times}_{F,\tilde{v}} \twoheadrightarrow \Delta_{Q_N} \longrightarrow (R^{\mathrm{univ}}_{\mathfrak{m},\emptyset,Q_N})^{\times} \longrightarrow \mathbb{T}^{T\cup Q_N}(H_{Q_N})^{\times}$$

is just $\prod_v V_{\tilde{v}}$. As H_{1,Q_N} is a direct summand of $S_{a,\{\rho_v\}}(U_1(Q_N), \mathcal{O})$ over $\mathbb{T}^{T\cup Q_N}_{a,\{\rho_v\}}(U_1(Q_N))$, lemma 2.3.1 now tells us that H_{1,Q_N} is a free $\mathcal{O}[\Delta_{Q_N}]$-module and that

$$(H_{1,Q_N})_{\Delta_{Q_N}} \xrightarrow{\sim} H_{0,Q_N}.$$

Also lemma 2.2.2, combined with lemma 2.1.5, tells us that

$$(\prod_{v \in Q_N} Q_{\tilde{v}}(V_{\varpi_t v})) : X_{\mathfrak{m},\emptyset} \stackrel{\sim}{\longrightarrow} H_{0,Q_N}.$$

Now we apply theorem 2.1 of [Dia] (as reformulated in the first paragraph of this proof) to $A = k[[S_1, ..., S_r]]$, $B = k[[X_1, ..., X_{r'}]]$, $R = R^{\mathrm{univ}}_{\mathfrak{m},\emptyset}/\lambda$, $H = X_{\mathfrak{m},\emptyset}/\lambda$ and $H_N = H_{1,Q_N}/\lambda$. We deduce that $r = r'$, that $X_{\mathfrak{m},\emptyset}/\lambda$ is free over $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset}/\lambda$ via $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset}/\lambda \twoheadrightarrow \mathbb{T}_{\mathfrak{m},\emptyset}/\lambda$ and that $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset}/\lambda$ is a complete intersection. As $X_{\mathfrak{m},\emptyset}$ is free over $\mathcal{O}$ we see that $X_{\mathfrak{m},\emptyset}$ is also free over $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset}$ via $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset} \twoheadrightarrow \mathbb{T}_{\mathfrak{m},\emptyset}$. Thus $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset} \stackrel{\sim}{\to} \mathbb{T}_{\mathfrak{m},\emptyset}$ is free over $\mathcal{O}$ and hence a complete intersection. The equality $r = r'$ tells us that $\mu_{\mathfrak{m}} \equiv n \bmod 2$. □

Theorem 3.1.2 *Keep the notation and assumptions of the start of this section. Assume also that conjecture I is true for G and G'. Then*

$$R^{\mathrm{univ}}_{\mathfrak{m},R} \stackrel{\sim}{\longrightarrow} \mathbb{T}_{\mathfrak{m},R}$$

is an isomorphism of complete intersections.

Proof: As in section 2.5 we see that we have a commutative diagram

$$\begin{array}{ccccc} R^{\mathrm{univ}}_{\mathfrak{m},R} & \twoheadrightarrow & \mathbb{T}_{\mathfrak{m},R} & & \\ \downarrow & & \downarrow & & \\ R^{\mathrm{univ}}_{\mathfrak{m},\emptyset} & \stackrel{\sim}{\longrightarrow} & \mathbb{T}_{\mathfrak{m},\emptyset} & \stackrel{\phi}{\longrightarrow} & \mathcal{O}. \end{array}$$

Let ϕ_R denote the composite $\mathbb{T}_{\mathfrak{m},R} \to \mathbb{T}_{\mathfrak{m},\emptyset} \stackrel{\phi}{\longrightarrow} \mathcal{O}$. Let $\mathfrak{c}_\emptyset(\phi)$ (resp. $\mathfrak{c}_R(\phi)$) be the ideals $\phi(\mathrm{Ann}_{\mathbb{T}_{\mathfrak{m},\emptyset}} \ker \phi)$ (resp. $\phi_R(\mathrm{Ann}_{\mathbb{T}_{\mathfrak{m},R}} \ker \phi_R)$). Also let $\wp_\emptyset$ (resp. $\wp_R$) denote the kernel of the composite $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset} \twoheadrightarrow \mathbb{T}_{\mathfrak{m},\emptyset} \stackrel{\phi}{\longrightarrow} \mathcal{O}$ (resp. $R^{\mathrm{univ}}_{\mathfrak{m},R} \twoheadrightarrow \mathbb{T}_{\mathfrak{m},R} \stackrel{\phi_R}{\longrightarrow} \mathcal{O}$).

Because $R^{\mathrm{univ}}_{\mathfrak{m},\emptyset} \stackrel{\sim}{\to} \mathbb{T}_{\mathfrak{m},\emptyset}$ is an isomorphism of complete intersections we see from the main theorem of [Le] that

$$\lg_{\mathcal{O}} \wp_\emptyset/\wp_\emptyset^2 = \lg_{\mathcal{O}} \mathcal{O}/\mathfrak{c}_\emptyset(\phi).$$

Hence by lemma 1.3.17 and proposition 2.5.5 we see that

$$\begin{array}{cl} & \lg_{\mathcal{O}} \wp_R/\wp_R^2 \\ \leq & \lg_{\mathcal{O}} \wp_\emptyset/\wp_\emptyset^2 + \sum_{v \in R} \lg_{\mathcal{O}} H^0(\mathrm{Gal}\,(\overline{F}_{\tilde{v}}/F_{\tilde{v}}), (\mathrm{ad}\, r_{\mathfrak{m}}) \otimes_{\mathbb{T}_{\mathfrak{m},\emptyset},\phi} K/\mathcal{O}(\epsilon^{-1})) \\ \leq & \lg_{\mathcal{O}} \mathcal{O}/\mathfrak{c}_R(\phi). \end{array}$$

Another application of the main theorem of [Le] tells us that $R^{\mathrm{univ}}_{\mathfrak{m},R} \to \mathbb{T}_{\mathfrak{m},R}$ is an isomorphism of complete intersections. □

4 Applications.

4.1 Some algebraic number theory.

We start with some elementary algebraic number theory. The first three lemmas are well known.

Lemma 4.1.1 *Suppose that F is a number field and that S is a finite set of places of F. Suppose also that*

$$\chi_S : \prod_{v \in S} F_v^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

is a continuous character of finite order. Then there is a continuous character

$$\chi : F^\times \backslash \mathbb{A}_F^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

such that $\chi|_{\prod_{v \in S} F_v^\times} = \chi_S$.

Proof: One may suppose that S contains all infinite places. Then we choose an open subgroup $U \subset (\mathbb{A}_F^S)^\times$ such that χ_S is trivial on $U \cap F^\times$. Then we can extend χ_S to $U \prod_{v \in S} F_v^\times/(U \cap F^\times)$ by setting it to one on U. Finally we can extend this character to $\mathbb{A}_F^\times/F^\times$ (which contains $U \prod_{v \in S} F_v^\times/(U \cap F^\times)$ as an open subgroup). $\Box$

Lemma 4.1.2 *Suppose that F is a number field, D/F is a finite Galois extension and S is a finite set of places of F. For $v \in S$ let E_v'/F_v be a finite Galois extension. Then we can find a finite, soluble Galois extension E/F linearly disjoint from D such that for each $v \in S$ and each prime w of E above v, the extension E_w/F_v is isomorphic to E_v'/F_v.*

Proof: For each $D \supset D_i \supset F$ with D_i/F Galois with a simple Galois group, choose a prime $v_i \notin S$ of F which does not split completely in F. Add the v_i to S along with $E_{v_i}' = F_{v_i}$. Then we can drop the condition that E/F is disjoint from D/F - it will be automatically satisfied.

Using induction on the maximum of the degrees $[E_v' : F_v]$ we may reduce to the case that each E_v'/F_v is cyclic. Then we can choose a continuous finite order character

$$\chi_S : \prod_{v \in S} F_v^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

such that $\ker \chi_S|_{F_v^\times}$ corresponds (under local class field theory) to E'_v/F_v for all $v \in S$. According to the previous lemma we can extend χ to a continuous character

$$\chi : F^\times \backslash \mathbb{A}_F^\times \longrightarrow \overline{\mathbb{Q}}^\times.$$

Let E/F correspond, under global class field theory, to $\ker \chi$. $\Box$

Let F be a number field. A character

$$\chi : \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times$$

is called *algebraic* if for $\tau \in \mathrm{Hom}\,(F, \mathbb{C})$ there exist $m_\tau \in \mathbb{Z}$ such that

$$\chi|_{(F_\infty^\times)^0}(x) = \prod_{\tau \in \mathrm{Hom}\,(F,\mathbb{C})} \tau(x)^{-m_\tau}.$$

A set of integers $\{m_\tau\}$ arises from some algebraic character if and only if there is an integer d and a CM subfield $E \subset F$ such that if $\tau_1|_E = (\tau_2|_E) \circ c$ then $d = m_{\tau_1} + m_{\tau_2}$. For this and the proof of the next lemma see [Se].

We will call a continuous character

$$\chi : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow \overline{\mathbb{Q}}_l^\times$$

algebraic if it is de Rham at all places above l.

Lemma 4.1.3 *Let $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let F be a number field. Let*

$$\chi : \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times$$

be an algebraic character and for $\tau \in \mathrm{Hom}\,(F, \mathbb{C})$ let $m_\tau \in \mathbb{Z}$ satisfy

$$\chi|_{(F_\infty^\times)^0}(x) = \prod_{\tau \in \mathrm{Hom}\,(F,\mathbb{C})} \tau(x)^{-m_\tau}.$$

Then there is a continuous character

$$r_{l,\imath}(\chi) : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow \overline{\mathbb{Q}}_l^\times$$

with the following properties.

1. *For every prime $v \not| l$ of F we have*

$$r_{l,\imath}(\chi)|_{\mathrm{Gal}\,(\overline{F}_v/F_v)} = \chi_v \circ \mathrm{Art}_{F_v}^{-1}.$$

2. *If $v|l$ is a prime of F then $r_{l,\imath}(\chi)|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ is potentially semistable, and if χ_v is unramified then it is crystalline.*

3. *If $v|l$ is a prime of F and if $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ lies above v then*

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i (r_{l,\imath}(\chi) \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 0$$

unless $i = m_{\imath\tau}$ in which case

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i (r_{l,\imath}(\chi) \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1.$$

Any continuous algebraic character $\psi : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow \overline{\mathbb{Q}}_l^\times$ arises in this way.

The character $r_{l,\imath}(\chi)$ is explicitly $\chi_{(l)} \circ \mathrm{Art}_F^{-1}$ where $\chi_{(l)} : \mathbb{A}_F^\times / \overline{F^\times (F_\infty^\times)^0} \rightarrow \overline{\mathbb{Q}}_l^\times$ is given by

$$\chi_{(l)}(x) = \left(\prod_{\tau \in \mathrm{Hom}\,(F,\mathbb{C})} (\imath^{-1}\tau)(x_l)^{-m_\tau} \right) \imath^{-1} \left(\left(\prod_{\tau \in \mathrm{Hom}\,(F,\mathbb{C})} \tau(x_\infty)^{m_\tau} \right) \chi(x) \right).$$

Lemma 4.1.4 *Let F be an imaginary CM field with maximal totally real subfield F^+. Let S be a finite set of primes of F^+ which split in F. Let I be a set of embeddings $F \hookrightarrow \mathbb{C}$ such that $I \coprod Ic$ is the set of all embeddings $F \hookrightarrow \mathbb{C}$. For $\tau \in I$ let m_τ be an integer. Suppose that*

$$\chi : \mathbb{A}_{F^+}^\times / (F^+)^\times \longrightarrow \mathbb{C}^\times$$

is algebraic, unramified at S and such that $\chi_v(-1)$ is independent of $v|\infty$. Then there is an algebraic character

$$\psi : \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times$$

which is unramified above S and satisfies

$$\psi \circ \mathbf{N}_{F/F^+} = \chi \circ \mathbf{N}_{F/F^+}$$

and

$$\psi|_{F_\infty^\times} = \prod_{\tau \in I} \tau^{m_\tau} (c\tau)^{w - m_\tau}$$

for some w.

Proof: From the discussion before lemma 4.1.3 we have that

$$\chi|_{((F_\infty^+)^\times)^0} = \prod_{\tau\in I} \tau^w$$

for some integer w. Choose an algebraic character

$$\phi : \mathbb{A}_F^\times/F^\times \longrightarrow \mathbb{C}^\times$$

which is unramified above S and such that

$$\phi|_{F_\infty^\times} = \prod_{\tau\in I} \tau^{m_\tau}(c\tau)^{w-m_\tau}.$$

Replacing χ by $\chi\phi|^\times_{\mathbb{A}_F^+}$ we may suppose that χ has finite order and that $m_\tau = 0$ for all $\tau \in I$.

Let $U_S = \prod_{v\in S} \mathcal{O}_{F,v}^\times$ and $U_S^+ = \prod_{v\in S} \mathcal{O}_{F^+,v}^\times$. It suffices to prove that

$$\chi|_{(\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)\cap U_S\overline{F^\times F_\infty^\times}} = 1.$$

If $\gamma_i \in F^\times$ and $x_i \in F_\infty^\times$ and $\gamma_i x_i$ tends to an element of $\mathbb{A}_{F^+}^\times U_S$, then $\gamma_i^c/\gamma_i \in F^{\mathbf{N}_{F/F^+}=1}$ is a unit at all primes above S and tends to 1 in $(\mathbb{A}_F^{S,\infty})^\times$. As $\mathcal{O}_F^{N_{F/F^+}=1}$ is the group of roots of unity in F and hence is finite, we conclude that for i sufficiently large $\gamma_i^c/\gamma_i = 1$, i.e. $\gamma_i \in F^+$. Thus

$$(\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)\cap U_S\overline{F^\times F_\infty^\times} = (\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)\cap U_S^+\overline{(F^+)^\times(F_\infty^+)^\times}.$$

We know that χ is trivial on $(\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)\cap U_S^+\overline{(F^+)^\times((F_\infty^+)^\times)^0}$.

Note that $\mathbb{A}_{F^+}^\times/(\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)(F^+)^\times(F_\infty^+)^\times$ corresponds to the maximal quotient of $\mathrm{Gal}\,(F/F^+)$ in which all complex conjugations are trivial. Hence $\mathbb{A}_{F^+}^\times = (\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)(F^+)^\times(F_\infty^+)^\times$ and we have an exact sequence

$$\begin{array}{l}(0) \to ((\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)\cap U_S^+\overline{(F^+)^\times(F_\infty^+)^\times})/((\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)\cap U_S^+\overline{(F^+)^\times((F_\infty^+)^\times)^0}) \\ \to (F^+)^\times(\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)/U_S^+\overline{(F^+)^\times((F_\infty^+)^\times)^0} \to \mathbb{A}_{F^+}^\times/U_S^+\overline{(F^+)^\times(F_\infty^+)^\times}) \to (0).\end{array}$$

If M/F^+ denotes the maximal abelian extension unramified in S and if L/F^+ denotes the maximal totally real abelian extension unramified in S, then by class field theory this exact sequence corresponds to the exact sequence

$$(0) \to \mathrm{Gal}\,(M/LF) \to \mathrm{Gal}\,(M/F) \to \mathrm{Gal}\,(L/F^+) \to (0).$$

If $v|\infty$ write c_v for a complex conjugation at v. As $\mathrm{Gal}\,(M/LF)$ is generated by elements $c_{v_1}c_{v_2}$ where v_1 and v_2 are infinite places we see that the image of $((\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)\cap U_S^+\overline{(F^+)^\times(F_\infty^+)^\times})/((\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)\cap U_S^+\overline{(F^+)^\times((F_\infty^+)^\times)^0})$ in

$(F^+)^\times(\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)/U_S^+\overline{(F^+)^\times((F_\infty^+)^\times)^0}$ is generated by elements $(-1)_{v_1}(-1)_{v_2}$, where v_1 and v_2 are two infinite places. Thus χ will be trivial on $(\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)\cap U_S^+\overline{(F^+)^\times(F_\infty^+)^\times}$ if and only if $\chi_{v_1}(-1)\chi_{v_2}(-1)=1$ for all infinite places v_1 and v_2. The lemma follows. □

Lemma 4.1.5 *Let F be an imaginary CM field with maximal totally real subfield F^+. Let I be a set of embeddings $F\hookrightarrow\overline{\mathbb{Q}}_l$ such that $I\coprod I^c$ is the set of all such embeddings. Choose an integer m_τ for all $\tau\in I$. Choose a finite set S of primes of F^+ which split in F and do not lie above l. Suppose that*

$$\chi:\mathrm{Gal}\,(\overline{F}/F^+)\longrightarrow\overline{\mathbb{Q}}_l^\times$$

is a continuous algebraic character which is unramified above S, crystalline at all primes above l and for which $\chi(c_v)$ is independent of the infinite place v of F^+. (Here c_v denotes complex conjugation at v.) Then there is a continuous algebraic character

$$\psi:\mathrm{Gal}\,(\overline{F}/F)\longrightarrow\overline{\mathbb{Q}}_l^\times$$

which is unramified above S and crystalline above l, such that

$$\psi\psi^c=\chi|_{\mathrm{Gal}\,(\overline{F}/F)},$$

and

$$\mathrm{gr}^{\,m_\tau}(\overline{\mathbb{Q}}_l(\psi)\otimes_{\tau,F_{v(\tau)}}B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_{v(\tau)}/F_{v(\tau)})}\neq(0)$$

for all $\tau\in I$. (Here $v(\tau)$ is the place above l induced by τ.)

Proof: This is the Galois theoretic analogue of the previous lemma. It follows from lemmas 4.1.3 and 4.1.4. □

A slight variant on these lemmas is the following.

Lemma 4.1.6 *Suppose that $l>2$ is a rational prime. Let F be an imaginary CM field with maximal totally real subfield F^+. Let S be a finite set of finite places of F containing all primes above l and satisfying $S^c=S$. Let*

$$\chi:\mathrm{Gal}\,(\overline{F}/F^+)\longrightarrow\mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$$

and

$$\overline{\theta}:\mathrm{Gal}\,(\overline{F}/F)\longrightarrow\overline{\mathbb{F}}_l^\times$$

be continuous characters with $\overline{\theta\theta}^c$ equal to the reduction of $\chi|_{\mathrm{Gal}\,(\overline{F}/F)}$. For $v\in S$, let

$$\psi_v:\mathrm{Gal}\,(\overline{F}_v/F_v)\longrightarrow\mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$$

be a continuous character lifting $\overline{\theta}|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ such that

$$(\psi_v\psi_{v^c}^c)|_{I_{F_v}} = \chi|_{I_{F_v}}.$$

Suppose also that if $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ lies above $v \in S$ then

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^{m_\tau}(\psi_v \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1,$$

and that $m_\tau + m_{\tau\circ c}$ is independent of τ.

Then there is a continuous character

$$\theta : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow \mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$$

lifting $\overline{\theta}$ and such that

$$\theta\theta^c = \chi|_{\mathrm{Gal}\,(\overline{F}/F)}$$

and, for all $v \in S$,

$$\theta|_{I_{F_v}} = \psi|_{I_{F_v}}.$$

In particular θ is algebraic.

Proof: Choose an algebraic character ϕ of $\mathrm{Gal}\,(\overline{F}/F)$ such that if $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ lies above $v \in S$ then

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^{m_\tau}(\phi \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1.$$

Replace ψ_v by $\psi_v\phi|^{-1}_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$; $\overline{\theta}$ by $\overline{\theta}\phi^{-1}$; and χ by $\chi\phi_0^{-1}$, where ϕ_0 denotes ϕ composed with the transfer $\mathrm{Gal}\,(\overline{F}/F^+)^{\mathrm{ab}} \to \mathrm{Gal}\,(\overline{F}/F)^{\mathrm{ab}}$. Then we see that we may suppose that χ has finite image and each $\psi_v|_{I_{F_v}}$ has finite image.

Using class field theory, think of χ as a character of $\mathbb{A}_{F^+}^\times/\overline{(F^+)^\times((F_\infty^+)^\times)^0}$; $\overline{\theta}$ as a character of $\mathbb{A}_F^\times/\overline{F^\times F_\infty^\times}$; and ψ_v as a character of $\mathcal{O}_{F,v}^\times$. Let $U_S = \prod_{v\in S}\mathcal{O}_{F,v}^\times$, $U_S^+ = \prod_{v\in S}\mathcal{O}_{F^+,v}^\times$ and $\psi = \prod_{v\in S}\psi_v : U_S \to \overline{\mathbb{Q}}_l^\times$. Note that $\psi|_{U_S^+} = \chi|_{U_S^+}$, that the reduction of χ equals $\overline{\theta}$ on $\mathbf{N}_{F/F^+}\mathbb{A}_F^\times$ and that the reduction of ψ equals $\overline{\theta}$ on U_S.

We get a character

$$\chi' = \chi\psi : U_S\mathbf{N}_{F/F^+}\mathbb{A}_F^\times/((U_S^+\mathbf{N}_{F/F^+}\mathbb{A}_F^\times) \cap \overline{(F^+)^\times((F_\infty^+)^\times)^0}) \longrightarrow \mathcal{O}_{\overline{\mathbb{Q}}_l}^\times.$$

The reduction of χ' equals $\overline{\theta}$. As in the proof of lemma 4.1.4 we see that

$$U_S(\mathbf{N}_{F/F^+}\mathbb{A}_F^\times) \cap \overline{F^\times F_\infty^\times} = U_S^+(\mathbf{N}_{F/F^+}\mathbb{A}_F^\times) \cap \overline{(F^+)^\times(F_\infty^+)^\times}.$$

However

$$(U_S^+(\mathbf{N}_{F/F^+}\mathbb{A}_F^\times) \cap \overline{(F^+)^\times(F_\infty^+)^\times})/((U_S^+\mathbf{N}_{F/F^+}\mathbb{A}_F^\times) \cap \overline{(F^+)^\times((F_\infty^+)^\times)^0})$$

is a 2-group on which $\overline{\theta}$ vanishes. As $l > 2$ we see that χ' also vanishes on this group.

Extend χ' to a continuous character

$$\chi' : \mathbb{A}_F^\times/\overline{F^\times F_\infty^\times} \longrightarrow \overline{\mathbb{Q}}_l^\times$$

and let $\overline{\chi}'$ denote its reduction. Then $\overline{\theta}(\overline{\chi}')^{-1}$ is a continuous character

$$\mathbb{A}_F^\times/(U_S(\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)\overline{F^\times F_\infty^\times} \longrightarrow \overline{\mathbb{F}}_l^\times.$$

Lift it to a continuous character

$$\chi'' : \mathbb{A}_F^\times/(U_S(\mathbf{N}_{F/F^+}\mathbb{A}_F^\times)\overline{F^\times F_\infty^\times} \longrightarrow \overline{\mathbb{Q}}_l^\times.$$

Then $\theta = \chi'\chi''$ will suffice. $\square$

4.2 Some determinants.

Lemma 4.2.1 *We have the following evaluations of determinants.*

1. *For an $n \times n$ determinant:*

$$\det \begin{pmatrix} 1 & b & 0 & 0 & & 0 & 0 \\ 1 & c & b & 0 & \dots & 0 & 0 \\ 1 & c & c & b & & 0 & 0 \\ & & \vdots & & \ddots & & \vdots \\ 1 & c & c & c & & c & b \\ 1 & c & c & c & \dots & c & c \end{pmatrix} = (c-b)^{n-1}.$$

2. *For an $n \times n$ determinant:*

$$\det \begin{pmatrix} a & b & b & b & & b & b \\ c & a & b & b & \dots & b & b \\ c & c & a & b & & b & b \\ & & \vdots & & \ddots & & \vdots \\ c & c & c & c & & a & b \\ c & c & c & c & \dots & c & a \end{pmatrix} = (c(a-b)^n - b(a-c)^n)/(c-b).$$

3. For an $(n+1)\times(n+1)$ determinant:

$$\det\begin{pmatrix} 0 & 1 & 2 & 3 & & n-2 & n-1 & 2n-1 \\ n & 0 & 1 & 2 & \dots & n-3 & n-2 & 2n-1 \\ n+1 & n & 0 & 1 & & n-4 & n-3 & 2n-1 \\ & & \vdots & & \ddots & & \vdots & \\ 2n-3 & 2n-4 & 2n-5 & 2n-6 & & 0 & 1 & 2n-1 \\ 2n-2 & 2n-3 & 2n-4 & 2n-5 & \dots & n & 0 & 2n-1 \\ 2n-1 & 2n-1 & 2n-1 & 2n-1 & & 2n-1 & 2n-1 & 2(2n-1) \end{pmatrix}$$

$$= (-1)^n(2n-1)((n+1)^n+(n-1)^n)/2.$$

Proof: For the first part subtract the penultimate row from the last row, then the three from last row from the penultimate row and so on finally subtracting the first row from the second. One ends up with an upper triangular matrix.

For the second matrix let Δ_n denote the determinant. Subtract the first row from each of the others and expand down the last column. Using the first part, we obtain

$$\begin{aligned} \Delta_n = \; & b(a-c)^{n-1} + (a-b)\det\begin{pmatrix} a & b & b & b & & b \\ c-a & a-b & 0 & 0 & \dots & 0 \\ c-a & c-b & a-b & 0 & & 0 \\ & & \vdots & & \ddots & \vdots \\ c-a & c-b & c-b & c-b & \dots & a-b \end{pmatrix} \\ = \; & b(a-c)^{n-1} + (a-b)\Delta_{n-1}. \end{aligned}$$

The second assertion follows easilly by induction.

For the third matrix subtract the second row from the first, the third from the second and so on, finally subtracting the penultimate row from the two from last row. One obtains

$$\det\begin{pmatrix} -n & 1 & 1 & 1 & & 1 & 1 & 0 \\ -1 & -n & 1 & 1 & \dots & 1 & 1 & 0 \\ -1 & -1 & -n & 1 & & 1 & 1 & 0 \\ & & \vdots & & \ddots & & \vdots & \\ -1 & -1 & -1 & -1 & & -n & 1 & 0 \\ 2n-2 & 2n-3 & 2n-4 & 2n-5 & \dots & n & 0 & 2n-1 \\ 2n-1 & 2n-1 & 2n-1 & 2n-1 & & 2n-1 & 2n-1 & 2(2n-1) \end{pmatrix}.$$

Then add half the sum of the first $n-1$ rows to the penultimate row making it

$$n-1 \quad n-1 \quad n-1 \quad n-1 \quad \dots \quad n-1 \quad (n-1)/2 \quad 2n-1.$$

Now subtract 1/2 of the last column from each of the first n columns. This leaves the first $n-1$ rows unchanged and the last two rows become

$$\begin{array}{cccccccc} -1/2 & -1/2 & -1/2 & -1/2 & \dots & -1/2 & -n/2 & 2n-2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2(2n-1). \end{array}$$

Thus the determinant becomes

$$(2n-1)\det\begin{pmatrix} -n & 1 & 1 & 1 & & 1 & 1 \\ -1 & -n & 1 & 1 & \dots & 1 & 1 \\ -1 & -1 & -n & 1 & & 1 & 1 \\ & & \vdots & & \ddots & & \vdots \\ -1 & -1 & -1 & -1 & & -n & 1 \\ -1 & -1 & -1 & -1 & \dots & -1 & -n \end{pmatrix}.$$

The result follows on applying the second part. □

4.3 CM fields I.

Let F be a CM field. By a *RACSDC* (regular, algebraic, conjugate self dual, cuspidal) automorphic representation π of $GL_n(\mathbb{A}_F)$ we mean a cuspidal automorphic representation such that

- $\pi^\vee \cong \pi^c$, and
- π_∞ has the same infinitessimal character as some irreducible algebraic representation of the restriction of scalars from F to $\mathbb{Q}$ of GL_n.

Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(F,\mathbb{C})}$ satisfy

- $a_{\tau,1} \geq ... \geq a_{\tau,n}$, and
- $a_{\tau c,i} = -a_{\tau,n+1-i}$.

Let Ξ_a denote the irreducible algebraic representation of $GL_n^{\mathrm{Hom}\,(F,\mathbb{C})}$ which is the tensor product over τ of the irreducible representations of GL_n with highest weights a_τ. We will say that a RACSDC automorphic representation π of $GL_n(\mathbb{A}_F)$ has *weight* a if π_∞ has the same infinitessimal character as $\Xi_a^\vee$.

Let S be a finite set of finite places of F. For $v \in S$ let ρ_v be an irreducible square integrable representation of $GL_n(F_v)$. We will say that a RACSDC automorphic representation π of $GL_n(\mathbb{A}_F)$ has *type* $\{\rho_v\}_{v\in S}$ if for each $v \in S$, π_v is an unramified twist of $\rho_v^\vee$.

The following is a restatement of theorem VII.1.9 of [HT].

Proposition 4.3.1 *Let $\imath : \overline{\mathbb{Q}}_l \stackrel{\sim}{\to} \mathbb{C}$. Let F be an imaginary CM field, S a finite non-empty set of finite places of F and, for $v \in S$, ρ_v a square integrable representation of $GL_n(F_v)$. Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(F,\mathbb{C})}$ be as above. Suppose that π is a RACSDC automorphic representation of $GL_n(\mathbb{A}_F)$ of weight a and type $\{\rho_v\}_{v\in S}$. Then there is a continuous semisimple representation*

$$r_{l,\imath}(\pi) : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

with the following properties.

1. *For every prime $v \not| l$ of F we have*

$$r_{l,\imath}(\pi)|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_v/F_v)} = r_l(\imath^{-1}\pi_v)^\vee(1-n)^{\mathrm{ss}}.$$

2. *$r_{l,\imath}(\pi)^c = r_{l,\imath}(\pi)^\vee \epsilon^{1-n}$.*

3. *If $v|l$ is a prime of F then $r_{l,\imath}(\pi)|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ is potentially semistable, and if π_v is unramified then it is crystalline.*

4. *If $v|l$ is a prime of F and if $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ lies above v then*

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^{\,i}(r_{l,\imath}(\pi) \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 0$$

   unless $i = a_{\imath\tau,j} + n - j$ for some $j = 1, ..., n$ in which case

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^{\,i}(r_{l,\imath}(\pi) \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1.$$

Proof: We can take $r_{l,\imath}(\pi) = R_l(\pi^\vee)(1-n)$ in the notation of [HT]. Note that the definition of highest weight we use here differs from that in [HT].
$\Box$

The representation $r_{l,\imath}(\pi)$ can be taken to be valued in $GL_n(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of some finite extension of $\mathbb{Q}_l$. Thus we can reduce it modulo the maximal ideal of $\mathcal{O}$ and semisimplify to obtain a continuous semisimple representation

$$\overline{r}_{l,\imath}(\pi) : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{F}}_l)$$

which is independent of the choices made. Note that if $r_{l,\imath}(\pi)$ (resp. $\overline{r}_{l,\imath}(\pi)$) is irreducible it extends to a continuous homomorphism

$$r_{l,\imath}(\pi)' : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(\overline{\mathbb{Q}}_l)$$

(resp.

$$\overline{r}_{l,\imath}(\pi)' : \mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(\overline{\mathbb{F}}_l)).$$

We will call a continuous semisimple representation

$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

(resp.

$$\overline{r} : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{F}}_l))$$

automorphic of weight a and type $\{\rho_v\}_{v\in S}$ if it equals $r_{l,\imath}(\pi)$ (resp. $\overline{r}_{l,\imath}(\pi)$) for some $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ and some RACSDC automorphic form π of weight a and type $\{\rho_v\}_{v\in S}$ (resp. and with π_l unramified). We will say that r is automorphic of weight a and type $\{\rho_v\}_{v\in S}$ and *level prime to l* if it equals $r_{l,\imath}(\pi)$ for some $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ and some RACSDC automorphic form π of weight a and type $\{\rho_v\}_{v\in S}$ with π_l unramified.

The following lemma is well known.

Lemma 4.3.2 *Suppose that E/F is a soluble Galois extension of CM fields. Suppose that*

$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

is a continuous semisimple representation and that $r|_{\mathrm{Gal}\,(\overline{F}/E)}$ is irreducible and automorphic of weight a and type $\{\rho_v\}_{v\in S}$. Let S_F denote the set of places of F under an element of S. Then we have the following.

1. *$a_\tau = a_{\tau'}$ if $\tau|_F = \tau'|_F$ so we can define a_F by $a_{F,\sigma} = a_{\widetilde{\sigma}}$ for any extension $\widetilde{\sigma}$ of σ to E.*

2. *r is automorphic over F of weight a_F and type $\{\rho'_v\}_{v\in S_F}$ for some square integrable representations ρ'_v.*

Proof: Inductively we may reduce to the case that E/F is cyclic of prime order. Suppose that $\mathrm{Gal}\,(E/F) = \langle\sigma\rangle$ and that $r = r_{l,\imath}(\pi)$, for π a RACSDC automorphic representation of $GL_n(\mathbb{A}_E)$ of weight a and level $\{\rho_v\}_{v\in S}$. Then $r|^{\sigma}_{\mathrm{Gal}\,(\overline{F}/E)} \cong r|_{\mathrm{Gal}\,(\overline{F}/E)}$ so that $\pi^\sigma = \pi$. By theorem 4.2 of [AC] π descends to a RACSDC automorphic representation π_F of $GL_n(\mathbb{A}_F)$. As r and $r_{l,\imath}(\pi_F)$ are irreducible and have the same restriction to $\mathrm{Gal}\,(\overline{F}/E)$ we see that $r = r_{l,\imath}(\pi_F) \otimes \chi = r_{l,\imath}(\pi_F \otimes (\chi \circ \mathrm{Art}_F))$ for some character χ of $\mathrm{Gal}\,(E/F)$. The lemma follows. $\Box$

Lemma 4.3.3 *Let F^+ be a totally real field of even degree and E an imaginary quadratic field such that $F = F^+E/F^+$ is unramified at all finite primes. Let $n \in \mathbb{Z}_{\geq 2}$ and let $l > n$ be a prime which splits in E. Let $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ and let S_l denote the set of primes of F above l. Let π be a RACSDC automorphic representation of $GL_n(\mathbb{A}_F)$ of weight a and type $\{\rho_v\}_{v \in S}$ where S is a finite non-empty set of primes split over F^+. Assume that $4 | \#(S \cup S^c)$. Suppose that π_v is unramified if v is not split over F^+ or if $v | l$. Let R be a finite set of primes of F such that if $v \in R$ then*

- *$v \notin S \cup S^c \cup S_l$,*
- *v is split over F^+,*
- *$\mathbf{N}v \equiv 1 \bmod l$,*
- *$\pi_v^{\mathrm{Iw}(v)} \neq (0)$.*

Let S_1 be a non-empty finite set of primes of F such that $S_1 = S_1^c$ and $S_1 \cap (R \cup S \cup S_l) = \emptyset$.

Then there is a RACSDC automorphic representation π' of $GL_n(\mathbb{A}_F)$ of weight a and type $\{\rho_v\}_{v \in S}$ with the following properties:

- *$\overline{r}_{l,\imath}(\pi) \cong \overline{r}_{l,\imath}(\pi')$;*
- *if $v \notin S_1$ and π_v is unramified then π'_v unramified;*
- *if v in R then $r_l(\pi'_v)^\vee(1-n)(I_{F_v})$ is finite.*

Proof: Let $S(B)$ denote the set of primes of F^+ below an element of S. Choose B and $\ddagger$ as at the start of section 2.3. These define an algebraic group G. Consider open campact subgroups $U = \prod_v U_v$ of $G(\mathbb{A}_{F^+}^\infty)$ where

- if v is inert in F, then U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$;
- if v is split in F and v lies below S then $U_v = G(\mathcal{O}_{F^+,v})$;
- if v does not lie below $R \cup S_1$, if v is split in F and if π_v is unramified then $U_v = G(\mathcal{O}_{F^+,v})$;
- if v lies below R and if w is a prime of F above v then $U_v = i_w^{-1}\mathrm{Iw}(w)$;
- if v lies below S_1 then U_v conatins only one element of finite order, namely 1.

The lemma now follows from lemma 2.1.6, proposition 2.3.2 and lemma 2.4.4.
□

Theorem 4.3.4 *Let F be an imaginary CM field and let F^+ denote its maximal totally real subfield. Let $n \in \mathbb{Z}_{\geq 1}$ and let $l > n$ be a prime which is unramified in F. Let*

$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

be a continuous irreducible representation with the following properties. Let $\overline{r}$ denote the semisimplification of the reduction of r and let r' denote the extension of r to a continuous homomorphism $\mathrm{Gal}\,(\overline{F}/F^+) \longrightarrow \mathcal{G}_n(\overline{\mathbb{Q}}_l)$.

1. *$r^c \cong r^\vee \epsilon^{1-n}$.*
2. *r is unramified at all but finitely many primes.*
3. *For all places $v|l$ of F, $r|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ is crystalline.*
4. *There is an element $a \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(F,\overline{\mathbb{Q}}_l)}$ such that*
   - *for all $\tau \in \mathrm{Hom}\,(F, \overline{\mathbb{Q}}_l)$ we have*
   $$l - 1 - n \geq a_{\tau,1} \geq ... \geq a_{\tau,n} \geq 0$$
   or
   $$l - 1 - n \geq a_{\tau c,1} \geq ... \geq a_{\tau c,n} \geq 0;$$
   - *for all $\tau \in \mathrm{Hom}\,(F, \overline{\mathbb{Q}}_l)$ and all $i = 1, ..., n$*
   $$a_{\tau c,i} = -a_{\tau,n+1-i};$$
   - *for all $\tau \in \mathrm{Hom}\,(F, \overline{\mathbb{Q}}_l)$ above a prime $v|l$ of F,*
   $$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i(r \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 0$$
   unless $i = a_{\tau,j} + n - j$ for some $j = 1, ..., n$ in which case
   $$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i(r \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1.$$

5. *There is a non-empty finite set S of places of F not dividing l and for each $v \in S$ a square integrable representation ρ_v of $GL_n(F_v)$ over $\overline{\mathbb{Q}}_l$ such that*
$$r|^{\rm ss}_{{\rm Gal}\,(\overline{F}_v/F_v)} = r_l(\rho_v)^\vee(1-n)^{\rm ss}.$$
If $\rho_v = {\rm Sp}_{m_v}(\rho'_v)$ then set
$$\widetilde{r}_v = r_l((\rho'_v)^\vee|\ \ |^{(n/m_v-1)(1-m_v)/2}).$$
Note that $r|_{{\rm Gal}\,(\overline{F}_v/F_v)}$ has a unique filtration ${\rm Fil}^j_v$ such that
$${\rm gr}^j_v r|_{{\rm Gal}\,(\overline{F}_v/F_v)} \cong \widetilde{r}_v\epsilon^j$$
for $j = 0, ..., m_v - 1$ and equals (0) otherwise. We assume that $\widetilde{r}_v$ has irreducible reduction $\overline{r}_v$. Then $\overline{r}|_{{\rm Gal}\,(\overline{F}_v/F_v)}$ inherits a filtration $\overline{\rm Fil}^j_v$ with
$$\overline{\rm gr}^j_v\overline{r}|_{{\rm Gal}\,(\overline{F}_v/F_v)} \cong \overline{r}_v\epsilon^j$$
for $j = 0, ..., m_v - 1$. Finally we suppose that for $j = 1, ..., m_v$ we have
$$\overline{r}_v \not\cong \overline{r}_v\epsilon^i.$$
6. *Assume that $\overline{F}^{\ker {\rm ad}\, \overline{r}}$ does not contain $F(\zeta_l)$.*
7. *Assume that ${\rm ad}\,\overline{r}'{\rm Gal}\,(\overline{F}/F^+(\zeta_l))$ is big in the sense of section 1.4.*
8. *Assume that the representation $\overline{r}$ is irreducible and automorphic of weight a and type $\{\rho_v\}_{v\in S}$ with $S \neq \emptyset$.*

Assume further that conjecture I is valid (for all unitary groups of the type considered there over any totally real field.)

Then r is automorphic of weight a and type $\{\rho_v\}_{v\in S}$ and level prime to l.

Proof: Suppose that $\overline{r} = \overline{r}_{l,\imath}(\pi)$, where π is a RACSDC automorphic representation of $GL_n(\mathbb{A}_F)$ of weight a and type $\{\rho_v\}_{v\in S}$. Let S_l denote the primes of F above l. Let R denote the primes of F outside $S^c \cup S \cup S_l$ at which r or π is ramified. Because $\overline{F}^{\ker {\rm ad}\, \overline{r}}$ does not contain $F(\zeta_l)$, we can choose a prime v_1 of F with the following properties

- $v_1 \notin R \cup S_l \cup S \cup S^c$,
- v_1 is unramified over a rational prime p for which $[F(\zeta_p) : F] > n$,

- v_1 does not split completely in $F(\zeta_l)$,
- $\mathrm{ad}\,\overline{r}(\mathrm{Frob}_{v_1}) = 1$.

Choose a CM field L/F with the following properties

- $L = L^+E$ with E an imaginary quadratic field and L^+ totally real.
- $[L^+ : F^+]$ is even.
- L/F is Galois and soluble.
- L is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F.
- L/L^+ is everywhere unramified.
- l splits in E and is unramified in L.
- v_1 splits completely in L/F and in L/L^+.
- All primes in S split completely in L/F and in L/L^+.
- Let π_L denote the base change of π to L. If v is a prime of L not lying above $S \cup S^c$ then $\pi_v^{\mathrm{Iw}(v)} \neq (0)$.
- If v is a place of L above R then $\mathbf{N}v \equiv 1 \bmod l$.

Let $S(L)$ (resp. $S_l(L)$) denote the set of places of L above S (resp. l). Let $a_L \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(L,\overline{\mathbb{Q}}_l)}$ be defined by $a_{L,\tau} = a_{\tau|_F}$. By theorem 4.2 of [AC] we know that $\overline{r}|_{\mathrm{Gal}\,(\overline{F}/L)}$ is automorphic of weight a_L and type $\{\rho_{v|_F}\}_{v\in S(L)}$. (The base change must be cuspidal as it is square integrable at finite places in S.) By lemma 4.3.3 there is a RACSDC automorphic representation π' of $GL_n(\mathbb{A}_L)$ of weight a_L and type $\{\rho_{v|_F}\}_{v\in S(L)}$ such that

- $\overline{r}|_{\mathrm{Gal}\,(\overline{F}/L)} = \overline{r}_{l,\imath}(\pi')$, and
- $r_{l,\imath}(\pi')$ is finitely ramified at all primes outside $S(L) \cup S(L)^c \cup S_l(L)$.

Choose a CM field M/L with the following properties.

- M/L is Galois and soluble.
- M is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over L.
- l is unramified in M.

- v_1 splits completely in M/F.
- All primes in S split completely in M/L.
- Let π'_M denote the base change of π' to M. If v is a prime of M not lying above $S \cup S^c$ then $\pi'_{M,v}$ is unramified.

Let $S(M)$ denote the set of places of M above S. Let $a_M \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(M,\overline{\mathbb{Q}}_l)}$ be defined by $a_{M,\tau} = a_{\tau|_F}$. Let $S(M^+)$ denote the set of places of M^+ below an element of $S(M)$. Then $\#S(M^+)$ is even and every element of $S(M^+)$ splits in M. Choose a division algebra B/M and an involution $\ddagger$ of B as at the start of section 2.3, with $S(B) = S(M^+)$. Let $R(M^+)$ denote the set of primes of M^+ above the restriction to F^+ of a prime of R. Let $S_l(M^+)$ denote the primes of M^+ above l and let $S_1(M^+)$ denote the primes of M^+ above $v_1|_{F^+}$. Let $T(M^+) = S(M^+) \cup S_l(M^+) \cup R(M^+) \cup S_1(M^+)$. It follows from proposition 2.3.2 and theorem 3.1.2 that $r|_{\mathrm{Gal}\,(\overline{F}/M)}$ is automorphic of weight a_M and type $\{\rho_{v|F}\}_{v \in S(M)}$. The theorem now follows from lemma 4.3.2. $\Box$

4.4 CM Fields II

In this section we will consider the following situation.

- $M/\mathbb{Q}$ is a Galois imaginary CM field of degree n with $\mathrm{Gal}\,(M/\mathbb{Q})$ cyclic generated by an element τ.
- $l > 1 + (n-1)((n+2)^{n/2} - (n-2)^{n/2})/2^{n-1}$ (e.g. $l > 8((n+2)/4)^{1+n/2}$) is a prime which splits completely in M and is $\equiv 1 \bmod n$.
- p is a rational prime which is inert and unramified in M.
- $Q \not\ni l$ is a finite set of rational primes, such that if $q \in Q$ then q splits completely in M and $q^i \not\equiv 1 \bmod l$ for $i = 1, ..., n-1$.
- $\overline{\theta} : \mathrm{Gal}\,(\overline{\mathbb{Q}}/M) \longrightarrow \overline{\mathbb{F}}_l^\times$ is a continuous character such that
    - $\theta\theta^c = \epsilon^{1-n}$;
    - there exists a prime $w|l$ of M such that for $i = 0, ..., n/2 - 1$ we have $\overline{\theta}|_{I_{\tau^i w}} = \epsilon^{-i}$;
    - if $v_1, ..., v_n$ are the primes of M above $q \in Q$ then $\{\overline{\theta}(\mathrm{Frob}_{v_i})\} = \{\alpha_q q^{-j} : j = 0, ..., n-1\}$ for some $\alpha_q \in \overline{\mathbb{F}}_l^\times$;
    - $\overline{\theta}|_{\mathrm{Gal}\,(\overline{M}_p/M_p)} \neq \overline{\theta}^{\tau^j}|_{\mathrm{Gal}\,(\overline{M}_p/M_p)}$ for $j = 1, ..., n-1$.

Let $S(\overline{\theta})$ denote the set of rational primes above which M or $\overline{\theta}$ is ramified.

- $E/\mathbb{Q}$ is an imaginary quadratic field linearly disjoint from the Galois closure of $\overline{M}^{\ker\overline{\theta}}(\zeta_l)/\mathbb{Q}$ in which every element of $S(\overline{\theta}) \cup Q \cup \{l, p\}$ splits; and whose class number is not divisible by l.

Set $L/\mathbb{Q}$ equal to the composite of E with the Galois closure of $\overline{M}^{\ker\overline{\theta}}(\zeta_l)/\mathbb{Q}$. Also let $(EM)^+$ denote the maximal totally real subfield of EM. Then $\overline{\theta}$ extends to a homomorphism, which we will also denote $\overline{\theta}$,

$$\overline{\theta} : \mathrm{Gal}\,(L/(EM)^+) \longrightarrow \mathcal{G}_1(\overline{\mathbb{F}}_l)$$

such that $\overline{\theta}(c) = (1, 1, \jmath)$ and $\nu \circ \overline{\theta} = \epsilon^{1-n}$. Let $\overline{r} : \mathrm{Gal}\,(L/\mathbb{Q}) \to \mathcal{G}_n(\overline{\mathbb{F}}_l)$ denote the induction with multiplier ϵ^{1-n} from $(\mathrm{Gal}\,(L/(EM)^+), \mathrm{Gal}\,(L/EM))$ to $(\mathrm{Gal}\,(L/\mathbb{Q}), \mathrm{Gal}\,(L/E))$ of $\overline{\theta}$.

We have an embedding

$$\begin{array}{rcl} \mathrm{Gal}\,(L/EM) & \hookrightarrow & (\overline{\mathbb{F}}_l^{\times})^{n/2} \times \mathbb{F}_l^{\times} \\ \alpha & \longmapsto & (\overline{\theta}(\alpha), \overline{\theta}^{\tau}(\alpha), ..., \overline{\theta}^{\tau^{n/2-1}}(\alpha); \epsilon(\alpha)). \end{array}$$

fix a primitive n^{th} root of unity $\zeta_n \in \mathbb{F}_l$. Suppose $\alpha = (\alpha_0, ..., \alpha_{n/2-1}) \in (\overline{\mathbb{F}}_l^{\times})^{n/2}$, $\beta^2 = \alpha_0 ... \alpha_{n/2-1}$. If $n/2 \leq i \leq n-1$ set $\alpha_i = \alpha_{i-n/2}^{-1}$. Let $\Gamma_{\alpha,\beta} = \Gamma$ denote the group generated by $(\overline{\mathbb{F}}_l^{\times})^{n/2} \times \mathbb{F}_l^{\times}$ and two elements C and T satisfying

- $C^2 = 1$ and $T^n = 1$;
- $CTCT^{-1} = (\alpha_0, ..., \alpha_{n/2-1}; 1)$;
- $T(a_0, ..., a_{n/2-1}; b)T^{-1} = (a_1, ..., a_{n/2-1}, b^{1-n}a_0^{-1}; b)$;
- and $C(a_0, ..., a_{n/2-1}; b)C = (b^{1-n}a_0^{-1}, ..., b^{1-n}a_{n/2-1}^{-1}; b)$.

Define characters $\Xi : \Gamma \to \mathbb{F}_l^{\times}$ by

- $\Xi(T) = \zeta_n$,
- $\Xi(C) = -1$,
- and $\Xi(a_0, ..., a_{n/2-1}; b) = b$;

and $\Theta : \langle (\overline{\mathbb{F}}_l^{\times} \times \mathbb{F}_l^{\times}, CT^{n/2} \rangle \to \overline{\mathbb{F}}_l^{\times}$ such that

- $\Theta(a_0, ..., a_{n/2-1}; b) = a_0$,
- and $\Theta(CT^{n/2}) = \beta$.

Note that

- $\Theta(CT^iCT^{-i}) = \alpha_0...\alpha_{i-1}$ (because $(CTCT^{-1})T(CT^iCT^{-i})T^{-1} = CT^{i+1}CT^{-(i+1)}$), and
- $\Theta(T^iCT^{n/2}T^{-i}) = \beta(\alpha_0...\alpha_{i-1})^{-1}$ (because $(CT^iCT^{-i})T^i(CT^{n/2})T^{-i} = CT^{n/2}$).

Let $\Gamma_0 = \Gamma_{\alpha,\beta,0}$ denote the subgroup generated by $((\mathbb{F}_l^\times)^{\kappa_n})^{\oplus n/2+1}$ and by C and T, where $\kappa_n = (n-1)((n+2)^{n/2} + (n-2)^{n/2})/2^{n+1}$.

Lemma 4.4.1 *There exist α and β such that the embedding*

$$\mathrm{Gal}\,(L/EM) \hookrightarrow (\overline{\mathbb{F}}_l^\times)^{n/2} \times \mathbb{F}_l^\times$$

extends to an embedding

$$j : \mathrm{Gal}\,(L/\mathbb{Q}) \hookrightarrow \Gamma$$

satisfying

- $\Xi \circ j = \epsilon$*;*
- $\Theta \circ j = \overline{\theta}$*;*
- *the image of j contains Γ_0;*
- *some complex conjugation maps to C;*
- *and some lifting $\widetilde{\tau} \in \mathrm{Gal}\,(L/E)$ of the generator τ of $\mathrm{Gal}\,(EM/E) \xrightarrow{\sim} \mathrm{Gal}\,(M/\mathbb{Q})$ maps to T.*

If such an embedding exists for some α it also exists for any element of $\alpha((\mathbb{F}_l^\times)^{2\kappa_n})^{\oplus n/2}$.

Proof: Note that EM and $\mathbb{Q}(\zeta_l)$ are linearly disjoint over $\mathbb{Q}$. Thus we may choose a lifting $\widetilde{\tau} \in \mathrm{Gal}\,(L/E)$ of the generator τ of $\mathrm{Gal}\,(EM/E) \xrightarrow{\sim} \mathrm{Gal}\,(M/\mathbb{Q})$ with $\epsilon(\widetilde{\tau}) = \zeta_n$. Also choose a complex conjugation $c \in \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$. Then $\epsilon(c\widetilde{\tau}^{n/2}) = 1$ and so

$$\begin{aligned} \overline{\theta}(\widetilde{\tau}^n) &= \overline{\theta}(c(c\widetilde{\tau}^{n/2})c(c\widetilde{\tau}^{n/2})) \\ &= (\overline{\theta}\overline{\theta}^c)(c\widetilde{\tau}^{n/2}) \\ &= \epsilon(c\widetilde{\tau}^{n/2})^{1-n} \\ &= 1. \end{aligned}$$

Also note that $\epsilon(c\widetilde{\tau}c\widetilde{\tau}^{-1}) = 1$. Setting $\alpha_i = \overline{\theta}^{\tau^i}(c\widetilde{\tau}c\widetilde{\tau}^{-1})$ we get a homomorphism

$$j : \mathrm{Gal}\,(L/\mathbb{Q}) \hookrightarrow \Gamma$$

extending the embedding $\mathrm{Gal}\,(L/EM) \hookrightarrow (\overline{\mathbb{F}}_l^\times)^{n/2} \times \mathbb{F}_l^\times$ and which sends $\widetilde{\tau}$ to T and c to C. We have $\Xi \circ j = \epsilon$. Note that

$$\overline{\theta}(c\widetilde{\tau}^{n/2})^2 = \overline{\theta}(c\widetilde{\tau}^{n/2}c\widetilde{\tau}^{-n/2}) = \overline{\theta}(c\widetilde{\tau}c\widetilde{\tau}^{-1})\overline{\theta}^{\tau}(c\widetilde{\tau}c\widetilde{\tau}^{-1})...\overline{\theta}^{\tau^{n/2-1}}(c\widetilde{\tau}c\widetilde{\tau}^{-1}),$$

and so for some choice of β we have $\Theta \circ j = \overline{\theta}$.

Choose a place u of E above l. Let A denote the subgroup of the image of $\mathrm{Ind}_{\mathrm{Gal}\,(\overline{E}/EM)}^{\mathrm{Gal}\,(\overline{E}/E)}\overline{\theta}$ generated by the decomposition groups above u. Let w be a place of EM above u. For any integer i define β_i to be

- $-i_0$ if $i \equiv i_0 \bmod n$ and $0 \leq i_0 \leq n/2 - 1$, and
- $i_0 + 1 - 3n/2$ if $i \equiv i_0 \bmod n$ and $n/2 \leq i_0 \leq n - 1$.

Note that $\beta_i + \beta_{i+n/2} = 1 - n$. We have

$$\prod_{i=0}^{n-1} I_{M_{\sigma^i w}} \twoheadrightarrow \prod_{i=0}^{n-1} \mathbb{F}_l^\times \twoheadrightarrow A \hookrightarrow (\overline{\mathbb{F}}_l^\times)^{n/2+1}.$$

The composite map sends

$$(a_i)_i \longmapsto (\prod_{i=0}^{n-1} a_i^{\beta_i}, \prod_{i=0}^{n-1} a_i^{\beta_{i-1}}, ..., \prod_{i=0}^{n-1} a_i^{\beta_{i+1-n/2}}, (\prod_{i=0}^{n-1} a_i)^{1-n}).$$

Moreover by lemma 4.2.1 we see that the image has index dividing κ_n. Thus the image of j contains Γ_0.

Finally note that

$$((a_0, ..., a_{n/2-1}; 1)T)^n = 1$$

and

$$C(a_0, ..., a_{n/2-1}; 1)TC((a_0, ..., a_{n/2-1}; 1)T)^{-1} = (\alpha_0 a_0^{-2}, ..., \alpha_{n/2-1} a_{n/2-1}^{-2}; 1).$$

□

There is a homomorphism

$$\widetilde{\Theta} : \langle (\overline{\mathbb{F}}_l^\times)^{n/2} \times \mathbb{F}_l^\times, C\rangle \longrightarrow \mathcal{G}_1(\overline{\mathbb{F}}_l^\times)$$

extending $\Theta|_{(\overline{\mathbb{F}}_l^\times)^{n/2}\times\mathbb{F}_l^\times}$ and with $\nu\circ\widetilde{\Theta}=\Xi^{1-n}$. It takes C to $(1,1,\jmath)$. Consider I, the induction of $\widetilde{\Theta}$ from $(\langle(\overline{\mathbb{F}}_l^\times)^{n/2}\times\mathbb{F}_l^\times, C\rangle, (\overline{\mathbb{F}}_l^\times)^{n/2}\times\mathbb{F}_l^\times)$ to $(\Gamma, \langle(\overline{\mathbb{F}}_l^\times)^{n/2}\times\mathbb{F}_l^\times, T\rangle)$ with multiplier Ξ^{1-n}. Then I has a basis consisting of functions e_i for $i=0,...,n-1$ with $e_i(T^j)=\delta_{ij}$ for $j=0,...,n-1$. Let $f_0,...,f_{n-1}$ be the dual basis of $I^\vee$. If $(a_0,...,a_{n/2-1};b)\in(\overline{\mathbb{F}}_l^\times)^{n/2}\times\mathbb{F}_l^\times$ set $a_i=b^{1-n}a_{i-n/2}^{-1}$ for $i=n/2,...,n-1$. Then we have

- $Te_i=e_{i-1}$ (with $e_{-1}=e_{n-1}$);
- $(a_0,...,a_{n/2-1};b)e_i=a_ie_i$ for $i=0,...,n-1$;
- $Tf_i=f_{i-1}$;
- and $(a_0,...,a_{n/2-1};b)f_i=a_i^{-1}f_i$ for $i=0,...,n-1$.

Moreover
$$\langle e_i,e_j\rangle=\zeta_n^i\alpha_0...\alpha_{i-1}\delta_{ij}.$$
We have $\overline{r}=I\circ j$.

Then Γ acts on $\operatorname{ad} I$ via

- $Te_i\otimes f_j=e_{i-1}\otimes f_{j-1}$;
- $(a_0,...,a_{n/2-1};b)e_i\otimes f_j=a_i/a_je_i\otimes f_j$;
- $Ce_i\otimes f_j=-\zeta_n^{i-j}\alpha_j...\alpha_{i-1}e_j\otimes f_i$ if $0\le j\le i\le n-1$;
- and $Ce_i\otimes f_j=-\zeta_n^{i-j}(\alpha_i...\alpha_{j-1})^{-1}e_j\otimes f_i$ if $0\le i\le j\le n-1$.

Hence if $0\le i\le j\le n/2-1$ then

- $CT^{n/2}e_i\otimes f_j=-\zeta_n^{i-j}\alpha_i...\alpha_{j-1}e_{j+n/2}\otimes f_{i+n/2}$;
- $CT^{n/2}e_{j+n/2}\otimes f_{i+n/2}=-\zeta_n^{j-i}\alpha_i...\alpha_{j-1}e_i\otimes f_j$;
- $CT^{n/2}e_j\otimes f_i=-\zeta_n^{j-i}\alpha_i^{-1}...\alpha_{j-1}^{-1}e_{i+n/2}\otimes f_{j+n/2}$;
- $CT^{n/2}e_{i+n/2}\otimes f_{j+n/2}=-\zeta_n^{i-j}\alpha_i^{-1}...\alpha_{j-1}^{-1}e_j\otimes f_i$;
- $CT^{n/2}e_i\otimes f_{j+n/2}=\zeta_n^{i-j}\alpha_0^{-1}...\alpha_{i-1}^{-1}\alpha_j...\alpha_{n/2-1}e_j\otimes f_{i+n/2}$;
- $CT^{n/2}e_j\otimes f_{i+n/2}=\zeta_n^{j-i}\alpha_0^{-1}...\alpha_{i-1}^{-1}\alpha_j...\alpha_{n/2-1}e_i\otimes f_{j+n/2}$;
- $CT^{n/2}e_{i+n/2}\otimes f_j=\zeta_n^{i-j}\alpha_0...\alpha_{i-1}\alpha_j^{-1}...\alpha_{n/2-1}^{-1}e_{j+n/2}\otimes f_i$;

- and $CT^{n/2}e_{j+n/2}\otimes f_i = \zeta_n^{j-i}\alpha_0...\alpha_{i-1}\alpha_j^{-1}...\alpha_{n/2-1}^{-1}e_{i+n/2}\otimes f_j$.

For $j = 1, ..., n/2-1$ let $W_j^{\pm}$ denote the span of the vectors

$$e_i \otimes f_{i+j} \mp \zeta_n^{-j} e_{n/2+i+j} \otimes f_{n/2+i}$$

for $i = 0, ..., n-1$ (and where we consider the subscripts modulo n). Then $W_j^{\pm}$ is a Γ-invariant subspace of $\mathrm{ad}\, I$. The space W_j^+ is isomorphic to the induction from $\langle (\overline{\mathbb{F}}_l^{\times})^{n/2} \times \mathbb{F}_l^{\times}, CT^{n/2}\rangle$ to Γ of Θ/Θ^{T^j}. The space W_j^- is isomorphic to the induction from $\langle (\overline{\mathbb{F}}_l^{\times})^{n/2} \times \mathbb{F}_l^{\times}, CT^{n/2}\rangle$ to Γ of Θ/Θ^{T^j} times the order two character with kernel $(\overline{\mathbb{F}}_l^{\times})^{n/2} \times \mathbb{F}_l^{\times}$.

If χ is a character of $\Gamma/((\overline{\mathbb{F}}_l^{\times})^{n/2} \times \mathbb{F}_l^{\times})$ with $\chi(C) = -1$ let W_χ denote the span of

$$e_0 \otimes f_0 + \chi(T)e_1 \otimes f_1 + ... + \chi(T)^{n-1}e_{n-1} \otimes f_{n-1}.$$

Then W_χ is an Γ invariant subspace of $\mathrm{ad}\, I$ on which Γ acts via χ.

Let $W_{n/2}$ denote the span of the vectors $e_i \otimes f_{i+n/2}$ for $i = 0, ..., n-1$ (with the subscripts taken modulo n). Then $W_{n/2}$ is a Γ-invariant subspace of $\mathrm{ad}\, I$ isomorphic to the induction from $\langle (\overline{\mathbb{F}}_l^{\times})^{n/2} \times \mathbb{F}_l^{\times}, CT^{n/2}\rangle$ to Γ of $\Theta/\Theta^{T^{n/2}}$. We have

$$\mathrm{ad}\, I = W_{n/2} \oplus (\bigoplus_\chi W_\chi) \oplus (\bigoplus_{j=1}^{n/2-1} W_j^+) \oplus (\bigoplus_{j=1}^{n/2-1} W_j^-).$$

Lemma 4.4.2 *The restrictions to $\Gamma_0^{\Xi=1}$ of the $2n-1$ representations $W_{n/2}$, $W_j^{\pm}$ (for $j = 1, ..., n/2-1$) and W_χ are all irreducible, non-trivial and pairwise non-isomorphic.*

Proof: It suffices to show the following:

- If $1 \leq j \leq n/2$ then $\Theta \neq \Theta^{T^j}$ on $((\overline{\mathbb{F}}_l^{\times})^{\kappa_n})^{\oplus n/2} \times \{1\}$.
- If $1 \leq j, j' \leq n/2$ and $0 \leq k \leq n-1$ then

$$\Theta/\Theta^{T^j} \neq \Theta^{T^k}/\Theta^{T^{j'+k}}$$

  on $((\overline{\mathbb{F}}_l^{\times})^{\kappa_n})^{\oplus n/2} \times \{1\}$ unless $j = j'$ and $k = 0$.

These facts are easily checked because $(l-1)/\kappa_n > 4$. $\Box$

Lemma 4.4.3 *Keep the notation and assumptions listed at the start of this section. There is a continuous homomorphism*

$$r : G_{\mathbb{Q}} \longrightarrow \mathcal{G}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

such that

- *r lifts $\overline{r}$;*
- *$\nu \circ r = \epsilon^{1-n}$;*
- *r is ramified at only finitely many primes, all of which split in E;*
- *for all places $v|l$ of E, $r|_{\mathrm{Gal}\,(\overline{E}_v/E_v)}$ is crystalline;*
- *for all $\tau \in \mathrm{Hom}\,(E, \overline{\mathbb{Q}}_l)$ above a prime $v|l$ of E;*

  $$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau,E_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{E}_v/E_v)} = 1$$

  for $i = 0, ..., n-1$ and $= 0$ otherwise;
- *for any place v of E above a rational prime $q \in Q$, the restriction $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{E}_v/E_v)}$ is unramified and $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{E}_v/E_v)}(\mathrm{Frob}_v)$ has eigenvalues $\{\alpha q^{-j} : j = 0, ..., n-1\}$ for some $\alpha \in \overline{\mathbb{Q}}_l^{\times}$.*

Proof: Consider the following deformation problem $\mathcal{S}_1$ for $\overline{r}$. We take $S_{1,0} = Q$ and $S_1 = Q \cup S(\overline{\theta}) \cup \{l\}$. Let $\widetilde{S}_1$ denote a choice of one prime of E above each prime of S_1. For $v \in \widetilde{S}_1$ we define $\mathcal{D}_v$ and L_v as follows.

- If $v|l$ the choice of $\mathcal{D}_v$ and L_v is described in subsection 1.3.2.
- If $v|q \in Q$ then $(\mathcal{D}_v, L_v)$ is as in example 1.3.5 with $m = n$ and $\widetilde{r}_{\widetilde{v}} = 1$.
- If $v|r \in S(\overline{\theta})$ then $(\mathcal{D}_v, L_v)$ is as in example 1.3.4.

Also set $W_0 = \bigoplus_\chi W_\chi \subset \mathrm{ad}\,\overline{r}$ and $\delta_{E/\mathbb{Q}} : G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}\,(E/\mathbb{Q}) \cong \{\pm 1\}$.

Then $H^1_{\mathcal{L}_1}(G_{\mathbb{Q},S_1}, W_0)$ is the kernel of

$$H^1(G_{\mathbb{Q}}, W_0) \longrightarrow \bigoplus_{v \notin Q} H^1(I_{\mathbb{Q}_v}, W_0) \oplus \bigoplus_{v \in Q} (H^1(I_{\mathbb{Q}_v}, W_{\delta_{E/\mathbb{Q}}}) \oplus \bigoplus_{\chi \neq \delta_{E/\mathbb{Q}}} H^1(G_{\mathbb{Q}_v}, W_\chi)).$$

(See the definition of L_v for $v|q \in Q$ given in sectiuon 1.3.5 at the start of the second paragraph after lemma 1.3.14.) Because l does not divide the order of the class group of E we see that

$$\ker\left(H^1(G_{\mathbb{Q}}, W_{\delta_{E/\mathbb{Q}}}) \longrightarrow \bigoplus_v H^1(I_{\mathbb{Q}_v}, W_{\delta_{E/\mathbb{Q}}})\right) = (0).$$

On the other hand if $\chi \neq \delta_{E/\mathbb{Q}}$ then

$$\ker\left(H^1(G_{\mathbb{Q}}, W_\chi) \longrightarrow \bigoplus_{v \notin Q} H^1(I_{\mathbb{Q}_v}, W_\chi) \oplus \bigoplus_{v \in Q} H^1(G_{\mathbb{Q}_v}, W_\chi)\right)$$

is contained in $\mathrm{Hom}\,(\mathrm{Cl}_Q(EM), k)$, where $\mathrm{Cl}_Q(EM)$ denotes the quotient of the class group of EM by the subgroup generated by the classes of primes above elements of Q. Because the maximal elementary l extension of EM unramified everywhere is linearly disjoint from L over EM, the Cebotarev density theorem implies that we can enlarge Q so that $\mathrm{Hom}\,(\mathrm{Cl}_Q(EM), k) = (0)$. Make such an enlargement. Then $H^1_{\mathcal{L}_1}(G_{\mathbb{Q},S_1}, W_0) = (0)$.

Moreover $H^1_{\mathcal{L}_1^\perp}(G_{\mathbb{Q}}, W_{\delta_{E/\mathbb{Q}}}(1))$ is the kernel of

$$H^1(G_{\mathbb{Q}}, W_{\delta_{E/\mathbb{Q}}}(1)) \to H^1(G_{\mathbb{Q}_l}, W_{\delta_{E/\mathbb{Q}}}(1))/H^1(G_{\mathbb{Q}_l}/I_{\mathbb{Q}_l}, W_{\delta_{E/\mathbb{Q}}})^\perp \oplus \bigoplus_{v \neq l} H^1(I_{\mathbb{Q}_v}, W_0).$$

From theorem 2.19 of [DDT] we deduce that

$$\# H^1_{\mathcal{L}_1^\perp}(G_{\mathbb{Q},S_1}, W_{\delta_{E/\mathbb{Q}}}(1)) = \# H^1_{\mathcal{L}_1}(G_{\mathbb{Q},S_1}, W_{\delta_{E/\mathbb{Q}}}) = 1,$$

i.e. $H^1_{\mathcal{L}_1^\perp}(G_{\mathbb{Q},S_1}, W_{\delta_{E/\mathbb{Q}}}(1)) = (0)$.

Now consider a second deformation problem $\mathcal{S}_2$ for $\overline{r}$. We take $S_{2,0} = Q$ and $S_2 = Q \cup S(\overline{\theta}) \cup \{l\} \cup Q'$, where Q' will be a set of primes disjoint from S_1 such that if $q' \in Q'$ then

$$j(\mathrm{Frob}_{q'}) = T(a_0(q'), ..., a_{n/2-1}(q'); b(q'))$$

with $b(q')^n = 1$ and $\zeta_n b(q') \neq 1$. Thus the eigenvalues of $\overline{r}(\mathrm{Frob}_{q'})$ are the n^{th} roots of $b(q')^{n/2}$ each with multiplicity 1, and $\overline{\epsilon}(\mathrm{Frob}_{q'}) \neq 1$. Set $a_{i+n/2}(q') = b(q')^{1-n} a_i(q')^{-1}$ for $i = 0, ..., n/2-1$. Let $\widetilde{S}_2 \supset \widetilde{S}_1$ denote a choice of one prime of E above each prime of S_2. For $v \in \widetilde{S}_1$ we define $\mathcal{D}_v$ and L_v as before. For $v \in \widetilde{S}_2$ above Q' choose an unramified character $\overline{\chi}_v$ of G_{E_v} with $\overline{\chi}_v(\mathrm{Frob}_v)^n = b(q')^{n/2}$, and let $\mathcal{D}_v$ and L_v be as in example 1.3.7 with $\overline{\chi} = \overline{\chi}_v$. Let π_v (resp. i_v, resp. π'_v, resp. i'_v) denote the projection onto the $\overline{\chi}_v(\mathrm{Frob}_v)$

(resp. inclusion of the $\overline{\chi}_v(\mathrm{Frob}_v)$, resp. projection onto the $b(q')\zeta_n\overline{\chi}_v(\mathrm{Frob}_v)$, resp. inclusion of the $b(q')\zeta_n\overline{\chi}_v(\mathrm{Frob}_v)$) eigenspace of Frob_v in $\overline{r}$. Then $i'_v\pi_v$ is in the k-span of

$$\sum_{i,j=0}^{n-1} b(q')^i\zeta_n^i\overline{\chi}_v(\mathrm{Frob}_v)^{i-j}(a_1(q')...a_i(q'))^{-1}a_1(q')...a_j(q')e_i\otimes f_j.$$

Thus $i'_v\pi_v \notin W_0$ and so $H^1_{\mathcal{L}_2}(G_{\mathbb{Q},S_2},W_0)\subset H^1_{\mathcal{L}_1}(G_{\mathbb{Q},S_1},W_0)=(0)$.

On the other hand $i'_v\pi'_v - i_v\pi_v$ is in the k-linear span of

$$\sum_{i,j=0}^{n-1}((b(q')\zeta_n)^{i-j}-1)\overline{\chi}_v(\mathrm{Frob}_v)^{i-j}(a_1(q')...a_i(q'))^{-1}a_1(q')...a_j(q')e_i\otimes f_j$$

and so $i'_v\pi'_v - i_v\pi_v \notin W_0$ (because $b(q')\zeta_n \neq 1$). Thus

$$H^1_{\mathcal{L}_2^\perp}(G_{\mathbb{Q},S_2},W_0(1)) = \ker\left(H^1_{\mathcal{L}_1^\perp}(G_{\mathbb{Q},S_1},W_0(1)) \longrightarrow \bigoplus_{q'\in Q'} H^1(G_{\mathbb{Q}_{q'}}/I_{\mathbb{Q}_{q'}},k)\right),$$

where the map onto the factor $H^1(G_{\mathbb{Q}_{q'}}/I_{\mathbb{Q}_{q'}},k)$ is induced by $A\longmapsto \pi_vAi'_v$ for $v\in\widetilde{S}_2$ with $v|q'$, i.e. by

$$\sum_{i=0}^{n-1}x_ie_i\otimes f_i \longmapsto \sum_{i=0}^{n-1}x_i(b(q')\zeta_n)^i.$$

If $[\phi]\in H^1_{\mathcal{L}_1^\perp}(G_{\mathbb{Q},S_1},W_0(1))$ then the extension P_ϕ of EM cut out by ϕ is nontrivial and l-power order and hence linearly disjoint from L over EM. Because $H^1_{\mathcal{L}_1^\perp}(G_{\mathbb{Q},S_1},W_{\delta_{E/\mathbb{Q}}}(1))=(0)$ we see that $\phi(\mathrm{Gal}\,(P_\phi/EM))\not\subset W_{\delta_{E/\mathbb{Q}}}(1)$. Thus we can choose $b\neq\zeta_n^{-1}$ so that

$$\sum_{i=0}^{n-1}x_ie_i\otimes f_i \longmapsto \sum_{i=0}^{n-1}x_i(b\zeta_n)^i$$

is not identically zero on $\phi(\mathrm{Gal}\,(P_\phi/EM))$. Then choose $a_0,...,a_{n/2-1}\in\overline{\mathbb{F}}_l^\times$ and $\sigma\in\mathrm{Gal}\,(LP_\phi/\mathbb{Q})$ such that $j(\sigma)=T(a_0,...,a_{n/2-1};b)$ and, if

$$\phi(\sigma)=\sum_{i=0}^{n-1}\phi_i(\sigma)e_i\otimes f_i$$

then

$$\sum_{i=0}^{n-1}(b\zeta_n)^i\phi_i(\sigma)\neq 0.$$

Let $q'\notin S_1$ be a rational prime unramified in LP_ϕ with $\mathrm{Frob}_{q'}=\sigma\in\mathrm{Gal}\,(LP_\phi/\mathbb{Q})$. Then if $q'\in Q'$ and $b(q')=b$ then $[\phi]\notin H^1_{\mathcal{L}_2^\perp}(G_{\mathbb{Q},S_2},W_0(1))$. Thus we can choose Q' and the $b(q')$ for $q'\in Q'$ such that

$$H^1_{\mathcal{L}_2^\perp}(G_{\mathbb{Q},S_2},W_0(1))=(0).$$

Make such a choice.

Finally we will apply theorem 1.4.6 with $W_1=W_0$ to complete the proof of the lemma. In the notation of theorem 1.4.6, given W and W', each equal to $W_{n/2}$ or some $W_j^\pm$, we will show that the conditions of theorem 1.4.6 can be verified with σ a lift of $T(a_0,...,a_{n/2-1};b)\in\Gamma_0$ for a suitable $a_0,...,a_{n/2-1},b$. We shall suppose that $b^n=1$ but that $b\neq\zeta_n^{-1}$, so that $\epsilon(\sigma)^n=1$ but $\epsilon(\sigma)\neq 1$. For $i=0,...,n/2-1$ write $a_{i+n/2}=b^{1-n}a_i^{-1}$. There is a decomposition

$$\overline{r}=\bigoplus_{\mu^n=b^{n/2}}V_\mu$$

into σ-eigenspaces, where σ acts on V_μ as μ and where V_μ is the span of

$$e_0+\mu a_1^{-1}e_1+...+\mu^{n-1}a_1^{-1}...a_{n-1}^{-1}e_{n-1}.$$

Let i_μ denote the inclusion $V_\mu\hookrightarrow\overline{r}$ and let π_μ denote the σ-equivariant projection $\overline{r}\twoheadrightarrow V_\mu$, so that $\pi_\mu i_\mu=\mathrm{Id}_{V_\mu}$. Note that

- $i_{\mu\epsilon(\sigma)}\pi_\mu=\sum_{i,j=0}^{n-1}a_1...a_j(a_1...a_i)^{-1}\mu^{i-j}\epsilon(\sigma)^ie_i\otimes f_j\notin W_0$
- and $i_{\mu\epsilon(\sigma)}\pi_{\mu\epsilon(\sigma)}-i_\mu\pi_\mu=\sum_{i,j=0}^{n-1}a_1...a_j(a_1...a_i)^{-1}\mu^{i-j}(\epsilon(\sigma)^{i-j}-1)\notin W_0$.

Moreover

- $\pi_\mu(e_i\otimes f_{i+n/2})i_{\mu\epsilon(\sigma)}=\epsilon(\sigma)^{i+n/2}\mu^{n/2}(a_{i+1}...a_{i+n/2})^{-1}$;
- $\pi_\mu(e_i\otimes f_{i+j}\mp\zeta_n^{-j}e_{n/2+i+j}\otimes f_{n/2+i})i_{\mu\epsilon(\sigma)}=(a_{i+1}...a_{i+j})^{-1}\mu^j\epsilon(\sigma)^{i+j}(1\pm b^{n/2}(\mu\zeta_n)^{-2j})$;
- $\pi_{\mu\epsilon(\sigma)}(e_i\otimes f_{i+n/2})i_{\mu\epsilon(\sigma)}-\pi_\mu(e_i\otimes f_{i+n/2})i_\mu=(\epsilon(\sigma)^{n/2}-1)\mu^{n/2}(a_{i+1}...a_{i+n/2})^{-1}$;
- and $\pi_{\mu\epsilon(\sigma)}(e_i\otimes f_{i+j}\mp\zeta_n^{-j}e_{n/2+i+j}\otimes f_{n/2+i})i_{\mu\epsilon(\sigma)}-\pi_\mu(e_i\otimes f_{i+j}\mp\zeta_n^{-j}e_{n/2+i+j}\otimes f_{n/2+i})i_\mu=(1\pm(\zeta_n\mu)^{-2j})(\epsilon(\sigma)^j-1)\mu^j(a_{i+1}...a_{i+j})^{-1}$.

Let β (resp. γ) denote a primitive $(n/2)^{th}$ (resp. $(2n)^{th}$) root of 1. Then we have:

- In the cases $W, W' \in \{W_{n/2}, W_1^-, ..., W_{n/2-1}^-\}$ taking $b = \mu = 1$ will satisfy the conditions of theorem 1.4.6.
- In the cases $W, W' \in \{W_{n/2}, W_1^+, ..., W_{n/2-1}^+\}$ taking $b = 1$ and $\mu = \zeta_n^{-1}$ will satisfy the conditions of theorem 1.4.6.
- If $W, W' \in \{W_1^{\pm}, ..., W_{n/2-1}^{\pm}\}$ taking $b = \zeta_n^{-1}\beta$ and $\mu = \zeta_n^{-1}\gamma$ will satisfy the conditions of theorem 1.4.6.

□

Theorem 4.4.4 *Keep the notation and assumptions listed at the start of this section. Let F/F_0 be a Galois extension of imaginary CM fields with F linearly disjoint from the normal closure of $\overline{M}^{\ker \overline{\theta}}(\zeta_l)$ over $\mathbb{Q}$. Assume that l is unramified in F and that there is a prime $v_{p,0}$ of F_0 split above p. Let*

$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

be a continuous irreducible representation with the following properties. Let $\overline{r}$ denote the semisimplification of the reduction of r.

1. $\overline{r} \cong \mathrm{Ind}\,_{\mathrm{Gal}\,(\overline{F}/FM)}^{\mathrm{Gal}\,(\overline{F}/F)} \overline{\theta}|_{\mathrm{Gal}\,(\overline{F}/FM)}$.
2. $r^c \cong r^\vee \epsilon^{1-n}$.
3. *r ramifies at only finitely many primes.*
4. *For all places $v|l$ of F, $r|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ is crystalline.*
5. *For all $\tau \in \mathrm{Hom}\,(F, \overline{\mathbb{Q}}_l)$ above a prime $v|l$ of F,*

   $$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau, F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1$$

   for $i = 0, ..., n-1$ and $= 0$ otherwise.
6. *There is a place v_q of F above a rational prime $q \in Q$ such that $(\#k(v_q))^j \not\equiv 1 \bmod l$ for $j = 1, ..., n$, and such that $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}/F_{v_q})}$ is unramified, and such that $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}/F_{v_q})}(\mathrm{Frob}_{v_q})$ has eigenvalues $\{\alpha(\#k(v_q))^j : \ j = 0, ..., n-1\}$ for some $\alpha \in \overline{\mathbb{Q}}_l^\times$.*

Assume further that conjecture I is valid (for all unitary groups of the type considered there over any totally real field.)

Then r is automorphic over F of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{v_q\}}$ and level prime to l.

Proof: Replacing F by EF if necessary we may suppose that $F \supset E$ (see lemma 4.3.2).

Choose a continuous character

$$\theta : \mathrm{Gal}(\overline{M}/M) \longrightarrow \mathcal{O}_{\overline{\mathbb{Q}}_l}^{\times}$$

such that

- θ lifts $\overline{\theta}$;
- $\theta^{-1} = \epsilon^{n-1}\theta^c$;
- for $i = 0, ..., n/2-1$ we have $\theta|_{I_{M_{\sigma^i w}}} = \epsilon^{-i}$; and
- $l \not| \#\theta(I_v)$ for all places $v|p$ of M.

(See lemma 4.1.6.) We can extend $\theta|_{\mathrm{Gal}(\overline{E}/EM)}$ to a continuous homomorphism

$$\theta : \mathrm{Gal}(\overline{E}/(EM)^+) \longrightarrow \mathcal{G}_1(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

with $\nu \circ \theta = \epsilon^{1-n}$. We will let $\overline{\theta}$ also denote the reduction

$$\overline{\theta} : \mathrm{Gal}(\overline{E}/(EM)^+) \longrightarrow \mathcal{G}_1(\overline{\mathbb{F}}_l)$$

of θ. Consider the pairs $\mathrm{Gal}(\overline{E}/(EM)^+) \supset \mathrm{Gal}(\overline{E}/(EM))$ and $\mathrm{Gal}(\overline{E}/\mathbb{Q}) \supset \mathrm{Gal}(\overline{E}/E)$. Set

$$r_0 = \mathrm{Ind}_{\mathrm{Gal}(\overline{E}/(EM)^+)}^{\mathrm{Gal}(\overline{E}/\mathbb{Q}),\epsilon^{1-n}} \theta : \mathrm{Gal}(\overline{E}/\mathbb{Q}) \longrightarrow \mathcal{G}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l}).$$

Note also that

$$r_0|_{\mathrm{Gal}(\overline{E}/E)} = ((\mathrm{Ind}_{\mathrm{Gal}(\overline{E}/M)}^{\mathrm{Gal}(\overline{E}/\mathbb{Q})} \theta)|_{\mathrm{Gal}(\overline{E}/E)}, \epsilon^{1-n}).$$

By lemma 4.4.3 there is a continuous homomorphism

$$r_1 : \mathrm{Gal}(\overline{E}/\mathbb{Q}) \longrightarrow \mathcal{G}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

with the following properties.

- r_1 lifts $\mathrm{Ind}\,^{\mathrm{Gal}\,(\overline{E}/\mathbb{Q}),\epsilon^{1-n}}_{\mathrm{Gal}\,(\overline{E}/(EM)^+)}\overline{\theta}$.
- $\nu \circ r_1 = \epsilon^{1-n}$.
- For all places $w|l$ of E, $r_1|_{\mathrm{Gal}\,(\overline{E}_w/E_w)}$ is crystalline.
- For all $\tau \in \mathrm{Hom}\,(E, \overline{\mathbb{Q}}_l)$ corresponding to prime $w|l$,

  $$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^{\,i}(r_1 \otimes_{\tau,E_w} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{E}_w/E_w)} = 1$$

  for $i = 0, ..., n-1$ and $= 0$ otherwise.
- $r_1|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{E}_{v_q}/E_{v_q})}$ is unramified and $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{E}_{v_q}/E_{v_q})}(\mathrm{Frob}_{v_q|E})$ has eigenvalues $\{\alpha q^{-j} : \ j = 0, ..., n-1\}$ for some $\alpha \in \overline{\mathbb{Q}}_l^{\times}$.
- $r_1|_{\mathrm{Gal}\,(\overline{E}_{v_p}/E_{v_p})}$ is an unramified twist of $\mathrm{Ind}\,^{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/M_p)}\theta|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/M_p)}$.

Let v_p be a prime of F above $v_{p,0}$ and let $F_1 \subset F$ denote the fixed field of the decomposition group of v_p in $\mathrm{Gal}\,(F/F_0)$. Thus $v_p|_{F_1}$ is split over p and F/F_1 is soluble.

The restriction $r_0|_{\mathrm{Gal}\,(\overline{E}/F_1)}$ is automorphic of weight 0 and type $\{\rho_p\}_{\{v_p|_{F_1}\}}$ and level prime to l, for a suitable cuspidal representation ρ_p (by theorem 4.2 of [AC]). We will apply theorem 4.3.4 to deduce that $r_1|_{\mathrm{Gal}\,(\overline{F}/F_1)}$ is automorphic of weight 0 and type $\{\rho_p\}_{\{v_p|_{F_1}\}}$ and level prime to l. We need only check that $\overline{r}_1(G_{F^+(\zeta_l)})$ is big (see section 1.4). This follows from lemmas 4.4.1 and 4.4.2, the fact that $l \not| \#\overline{r}_0(G_{\mathbb{Q}})$ and the following calculations.

- Take $a_0 \in (\mathbb{F}_l^{\times})^{\kappa_n}$ with $a_0^2 \neq 1$ and take $\sigma \in G_{F(\zeta_l)}$ with $j(\sigma) = (a_0, 1, ..., 1; 1) \in \Delta_0$. Then

  $$\pi_{\sigma,a_0} W_{\chi} i_{\sigma,a_0} \neq (0).$$

- Take $(a_0, ..., a_{n/2-1}) \in (\mathbb{F}_l^{\times})^{\oplus n/2}$ and $\sigma \in G_{F(\zeta_l)}$ with $j(\sigma) = T(a_0, ..., a_{n/2-1}; \zeta_n^{-1})$. Also take μ to be the product of ζ_n^{-1} with a primitive $(2n)^{th}$ root of 1. Set $a_{i+n/2} = \zeta_n^{-1} a_i$ for $i = 0, ..., n/2-1$. Then

  $$\pi_{\sigma,\mu} e_i \otimes f_{i+n/2} i_{\sigma,\mu} = \mu^{n/2}(a_{i+1}...a_{i+n/2})^{-1}$$

  and

  $$\pi_{\sigma,\mu}(e_i \otimes f_{i+j} \mp \zeta_n^{-j} e_{n/2+i+j} \otimes f_{n/2+i}) i_{\sigma,\mu} = (1 \mp (\mu\zeta_n)^{-2j})\mu^j (a_{i+1}...a_{i+j})^{-1}.$$

  Thus $\pi_{\sigma,\mu} W_{n/2} i_{\sigma,\mu} \neq (0)$ and $\pi_{\sigma,\mu} W_j^{\pm} i_{\sigma,\mu} \neq (0)$.

It follows from corollary VII.1.11 of [HT] that $r_1|_{\mathrm{Gal}\,(\overline{F}/F_1)}$ is also automorphic of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{v_q|_{F_1}\}}$ and level prime to l. (The only tempered representations π of $GL_n(F_{1,v_q|_{F_1}})$ for which $r_l(\pi)^\vee(1-n)^{\mathrm{ss}}$ unramified and $r_l(\pi)^\vee(1-n)^{\mathrm{ss}}(\mathrm{Frob}_{v_q|_{F_1}})$ has eigenvalues of the form $\{\alpha q^{-j}\,:\, j = 0,...,n-1\}$ are unramified twists of $\mathrm{Sp}_n(1)$.) From theorem 4.2 of [AC] we deduce that $r_1|_{\mathrm{Gal}\,(\overline{F}/F)}$ is automorphic of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{v_q\}}$ and level prime to l. (The base change must be cuspidal as it is square integrable at one place.)

Finally we again apply theorem 4.3.4 to deduce that r is automorphic of weight 0 and type $\{\rho_p\}_{\{v_p\}}$ and level prime to l. The verification that $\overline{r}(G_{F^+(\zeta_l)})$ is big is exactly as above. $\Box$

4.5 Totally real fields.

Let F^+ be a totally real field. By a *RAESDC* (regular, algebraic, essentially self dual, cuspidal) automorphic representation π of $GL_n(\mathbb{A}_{F^+})$ we mean a cuspidal automorphic representation such that

- $\pi^\vee \cong \chi\pi$ for some character $\chi : (F^+)^\times \backslash \mathbb{A}_{F^+}^\times \to \mathbb{C}^\times$ with $\chi_v(-1)$ independent of $v|\infty$, and
- π_∞ has the same infinitessimal character as some irreducible algebraic representation of the restriction of scalars from F^+ to $\mathbb{Q}$ of GL_n.

One can ask whether if these conditions are met for some $\chi : (F^+)^\times \backslash \mathbb{A}_{F^+}^\times \to \mathbb{C}^\times$, they will automatically be met for some such χ' with $\chi'_v(-1)$ independent of $v|\infty$. This is certainly true if n is odd. (As then χ^n is a square, so that $\chi_v(-1) = 1$ for all $v|\infty$.) It is also true if $n = 2$ (As in this case we can take χ to be the inverse of the central character of π and the parity condition is equivalent to the fact that if a holomorphic Hilbert modular form has weight $(k_\tau)_{\tau\in \mathrm{Hom}\,(F^+,\mathbb{R})}$ then k_τ mod 2 is independent of τ.)

Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(F^+,\mathbb{C})}$ satisfy

$$a_{\tau,1} \geq ... \geq a_{\tau,n}$$

Let Ξ_a denote the irreducible algebraic representation of $GL_n^{\mathrm{Hom}\,(F^+,\mathbb{C})}$ which is the tensor product over τ of the irreducible representations of GL_n with highest weights a_τ. We will say that a RAESDC automorphic representation π of $GL_n(\mathbb{A}_F)$ has *weight* a if π_∞ has the same infinitessimal character as $\Xi_a^\vee$. In that case there is an integer w_a such that

$$a_{\tau,i} + a_{\tau,n+1-i} = w_a$$

for all $\tau \in \mathrm{Hom}(F^+, \mathbb{C})$ and all $i = 1, ..., n$.

Let S be a finite set of finite places of F^+. For $v \in S$ let ρ_v be an irreducible square integrable representation of $GL_n(F_v^+)$. We will say that a RAESDC automorphic representation π of $GL_n(\mathbb{A}_{F^+})$ has *type* $\{\rho_v\}_{v \in S}$ if for each $v \in S$, π_v is an unramified twist of $\rho_v^\vee$.

Proposition 4.5.1 *Let $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let F^+ be a totally real field, S a finite non-empty set of finite places of F^+ and, for $v \in S$, ρ_v a square integrable representation of $GL_n(F_v^+)$. Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F^+,\mathbb{C})}$ be as above. Suppose that π is a RAESDC automorphic representation of $GL_n(\mathbb{A}_{F^+})$ of weight a and type $\{\rho_v\}_{v \in S}$. Specifically suppose that $\pi^\vee \cong \pi\chi$ where $\chi : \mathbb{A}_{F^+}^\times/(F^+)^\times \to \mathbb{C}^\times$. Then there is a continuous semisimple representation*

$$r_{l,\imath}(\pi) : \mathrm{Gal}(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

with the following properties.

1. *For every prime $v \not| l$ of F^+ we have*
$$r_{l,\imath}(\pi)|^{\mathrm{ss}}_{\mathrm{Gal}(\overline{F}_v^+/F_v^+)} = r_l(\imath^{-1}\pi_v)^\vee(1-n)^{\mathrm{ss}}.$$

2. *$r_{l,\imath}(\pi)^\vee = r_{l,\imath}(\pi)\epsilon^{n-1}r_{l,\imath}(\chi)$.*

3. *If $v|l$ is a prime of F^+ then $r_{l,\imath}(\pi)|_{\mathrm{Gal}(\overline{F}_v^+/F_v^+)}$ is potentially semistable, and if π_v is unramified then it is crystalline.*

4. *If $v|l$ is a prime of F^+ and if $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_l$ lies above v then*
$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i(r_{l,\imath}(\pi) \otimes_{\tau,F_v^+} B_{\mathrm{DR}})^{\mathrm{Gal}(\overline{F}_v^+/F_v^+)} = 0$$
unless $i = a_{\imath\tau,j} + n - j$ for some $j = 1, ..., n$ in which case
$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i(r_{l,\imath}(\pi) \otimes_{\tau,F_v^+} B_{\mathrm{DR}})^{\mathrm{Gal}(\overline{F}_v^+/F_v^+)} = 1.$$

Proof: Let F be an imaginary CM field with maximal totally real subfield F^+, such that all primes above l and all primes in S split in F/F^+. Choose an algebraic character $\psi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ such that $\chi \circ \mathbf{N}_{F/F^+} = \psi \circ \mathbf{N}_{F/F^+}$. (See lemma 4.1.4.) Let π_F denote the base change of π to F. Applying proposition 4.3.1 to $\pi_F\psi$, we obtain a continuous semi-simple representation

$$r_F : \mathrm{Gal}(\overline{F}^+/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

such that for every prime $v \not| l$ of F we have

$$r_F|^{\rm ss}_{{\rm Gal}\,(\overline{F}_v^+/F_v)} = r_l(\imath^{-1}\pi_{v|_{F^+}})^\vee(1-n)|^{\rm ss}_{{\rm Gal}\,(\overline{F}_v^+/F_v)}.$$

Letting the field F vary we can piece together the representations r_F to obtain r. (See the argument of the second half of the proof of theorem VII.1.9 of [HT].) $\Box$

The representation $r_{l,\imath}(\pi)$ can be taken to be valued in $GL_n(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of some finite extension of $\mathbb{Q}_l$. Thus we can reduce it modulo the maximal ideal of $\mathcal{O}$ and semisimplify to obtain a continuous semisimple representation

$$\overline{r}_{l,\imath}(\pi) : {\rm Gal}\,(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbb{F}}_l)$$

which is independent of the choices made.

We will call a continuous semisimple representation

$$r : {\rm Gal}\,(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

(resp.

$$\overline{r} : {\rm Gal}\,(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbb{F}}_l))$$

automorphic of weight a and level $\{\rho_v\}_{v\in S}$ if it equals $r_{l,\imath}(\pi)$ (resp. $\overline{r}_{l,\imath}(\pi)$) for some $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ and some RAESDC automorphic form π of weight a and type $\{\rho_v\}_{v\in S}$ (resp. and with π_l unramified). We will say that r is automorphic of weight a and type $\{\rho_v\}_{v\in S}$ and *level prime to l* if it equals $r_{l,\imath}(\pi)$ for some $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ and some RAESDC automorphic form π of weight a and type $\{\rho_v\}_{v\in S}$ with π_l unramified.

The following lemma is proved just as lemma 4.3.2.

Lemma 4.5.2 *Suppose that E^+/F^+ is a soluble Galois extension of CM fields. Suppose that*

$$r : {\rm Gal}\,(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

is a continuous semisimple representation and that $r|_{{\rm Gal}\,(\overline{F}^+/E^+)}$ is irreducible and automorphic of weight a and type $\{\rho_v\}_{v\in S}$. Let S_{F^+} denote the set of places of F^+ under an element of S. Then we have the following.

1. *$a_\tau = a_{\tau'}$ if $\tau|_{F^+} = \tau'|_{F^+}$ so we can define a_{F^+} by $a_{F^+,\sigma} = a_{\widetilde{\sigma}}$ for any extension $\widetilde{\sigma}$ of σ to E^+.*

2. *r is automorphic over F^+ of weight a_{F^+} and type $\{\rho'_v\}_{v\in S_{F^+}}$ for some square integrable representations ρ'_v.*

Theorem 4.5.3 *Let F^+ be a totally real field. Let $n \in \mathbb{Z}_{\geq 1}$ and let $l > n$ be a prime which is unramified in F^+. Let*

$$r : \mathrm{Gal}\,(\overline{F}^+/F^+) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

be a continuous irreducible representation with the following properties. Let $\overline{r}$ denote the semisimplification of the reduction of r.

1. *$r^\vee \cong r\epsilon^{n-1}\chi$ for some character $\chi : \mathrm{Gal}\,(\overline{F}^+/F^+) \longrightarrow \overline{\mathbb{Q}}_l^\times$ with $\chi(c_v)$ independent of $v|\infty$. (Here c_v denotes a complex conjugation at v.)*

2. *r ramifies at only finitely many primes.*

3. *For all places $v|l$ of F^+, $r|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)}$ is crystalline.*

4. *There is an element $a \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(F^+,\overline{\mathbb{Q}}_l)}$ such that*

   - *for all $\tau \in \mathrm{Hom}\,(F^+,\overline{\mathbb{Q}}_l)$ we have*

   $$l - 1 - n + a_{\tau,n} \geq a_{\tau,1} \geq ... \geq a_{\tau,n};$$

   - *for all $\tau \in \mathrm{Hom}\,(F^+,\overline{\mathbb{Q}}_l)$ above a prime $v|l$ of F^+,*

   $$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau,F_v^+} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = 0$$

   unless $i = a_{\tau,j} + n - j$ for some $j = 1, ..., n$ in which case

   $$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau,F_v^+} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = 1.$$

5. *There is a finite non-empty set S of places of F^+ not dividing l and for each $v \in S$ a square integrable representation ρ_v of $GL_n(F_v^+)$ over $\overline{\mathbb{Q}}_l$ such that*

   $$r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = r_l(\rho_v)^\vee(1-n)^{\mathrm{ss}}.$$

   If $\rho_v = \mathrm{Sp}_{\,m_v}(\rho_v')$ then set

   $$\widetilde{r}_v = r_l((\rho_v')^\vee|\ \ |^{(n/m_v-1)(1-m_v)/2}).$$

   Note that $r|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)}$ has a unique filtration $\mathrm{Fil}_v^{\,j}$ such that

   $$\mathrm{gr}_v^{\,j} r|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} \cong \widetilde{r}_v\epsilon^j$$

for $j = 0, ..., m_v - 1$ and equals (0) otherwise. We assume that $\widetilde{r}_v$ has irreducible reduction $\overline{r}_v$ such that

$$\overline{r}_v \not\cong \overline{r}_v \epsilon^j$$

for $j = 1, ..., m_v$. Then $\overline{r}|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)}$ inherits a unique filtration $\overline{\mathrm{Fil}}_v^j$ with

$$\overline{\mathrm{gr}}_v^j \overline{r}|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} \cong \overline{r}_v \epsilon^j$$

for $j = 0, ..., m_v - 1$.

6. *$(\overline{F}^+)^{\ker \mathrm{ad}\,\overline{r}}$ does not contain $F^+(\zeta_l)$.*

7. *$H^i(\mathrm{ad}\,\overline{r}\mathrm{Gal}\,(\overline{F}^+/F^+(\zeta_l)), \mathrm{ad}^0\overline{r}) = (0)$ for $i = 0$ and 1.*

8. *For all irreducible $k[\mathrm{ad}\,\overline{r}\mathrm{Gal}\,(\overline{F}^+/F^+(\zeta_l))]$-submodules W of $\mathrm{ad}\,\overline{r}$ we can find $h \in \mathrm{ad}\,\overline{r}\mathrm{Gal}\,(\overline{F}^+/F^+(\zeta_l))$ and $\alpha \in k$ with the following properties. The α generalised eigenspace $V_{h,\alpha}$ of h in $\overline{r}$ is one dimensional. Let $\pi_{h,\alpha} : \overline{r} \to V_{h,\alpha}$ (resp. $i_{h,\alpha}$) denote the h-equivariant projection of $\overline{r}$ to $V_{h,\alpha}$ (resp. h-equivariant injection of $V_{h,\alpha}$ into $\overline{r}$). Then $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq (0)$.*

9. *$\overline{r}$ is irreducible and automorphic of weight a and type $\{\rho_v\}_{v\in S}$ with $S \neq \emptyset$.*

Assume further that conjecture I is valid (for all unitary groups of the type considered there over any totally real field.)

Then r is automorphic of weight a and type $\{\rho_v\}_{v\in S}$ and level prime to l.

Proof: Choose an imaginary CM field F with maximal totally real subfield F^+ such that

- all primes above l split in F/F^+,
- all primes in S split in F/F^+, and
- F is linearly disjoint from $(\overline{F}^+)^{\ker \overline{r}}(\zeta_l)$ over F^+.

Choose an algebraic character

$$\psi : \mathrm{Gal}\,(\overline{F}^+/F) \longrightarrow \overline{\mathbb{Q}}_l^\times$$

such that

- $\chi|_{\mathrm{Gal}\,(\overline{F}^+/F)} = \psi\psi^c$,

- ψ is unramified above S,
- ψ is crystalline above l, and
- for each $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_l$ there exists an extension $\widetilde{\tau} : F \hookrightarrow \overline{\mathbb{Q}}_l$ such that

$$\mathrm{gr}^{-a_{\tau,n}}(\overline{\mathbb{Q}}_l(\psi) \otimes_{\widetilde{\tau},F_{v(\widetilde{\tau})}} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_{v(\widetilde{\tau})}/F_{v(\widetilde{\tau})})} \neq (0),$$

  where $v(\widetilde{\tau})$ is the place of F above l determined by $\widetilde{\tau}$.

(This is possible by lemma 4.1.5.) Now apply theorem 4.3.4 to $r|_{\mathrm{Gal}\,(\overline{F}^+/F)}\psi$ and this theorem follows easilly by the argument for lemma 4.3.2. $\Box$

As the conditions of this theorem are a bit complicated we give a special case as a corollary.

Corollary 4.5.4 *Let $n \in \mathbb{Z}_{\geq 1}$ be even and let $l > \max\{3, n\}$ be a prime. Let*

$$r : \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GSp_n(\mathbb{Z}_l)$$

be a continuous irreducible representation with the following properties.

1. *r ramifies at only finitely many primes.*
2. *$r|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)}$ is crystalline.*
3. *$\dim_{\mathbb{Q}_l} \mathrm{gr}^i(r \otimes_{\mathbb{Q}_l} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)} = 0$ unless $i \in \{0, 1, ..., n-1\}$ in which case it has dimension 1.*
4. *There is a prime $q \neq l$ such that $q^i \not\equiv 1 \bmod l$ for $i = 1, ..., n$ and $r|^{\mathrm{ss}}_{G_{\mathbb{Q}_q}}$ is unramified and $r|^{\mathrm{ss}}_{G_{\mathbb{Q}_q}}(\mathrm{Frob}_q)$ has eigenvalues $\{\alpha q^i : \ i = 0, 1, ..., n-1\}$ for some α.*
5. *The image of $r \bmod l$ contains $Sp_n(\mathbb{F}_l)$.*
6. *$r \bmod l$ is automorphic of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{q\}}$.*

Assume further that conjecture I is valid (for all unitary groups of the type considered there over any totally real field.)

Then r is automorphic of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{q\}}$ and level prime to l.

Proof: Let $\overline{r} = r \bmod l$. As $PSp_n(\mathbb{F}_l)$ is simple, the maximal abelian quotient of $\mathrm{ad}\,\overline{r}(G_\mathbb{Q})$ is

$$\overline{r}(G_\mathbb{Q})/(\overline{r}(G_\mathbb{Q}) \cap \mathbb{F}_l^\times)Sp_n(\mathbb{F}_l) \subset PGSp_n(\mathbb{F}_l)/PSp_n(\mathbb{F}_l) \xrightarrow{\sim} (\mathbb{F}_l^\times)/(\mathbb{F}_l^\times)^2.$$

Thus $\overline{\mathbb{Q}}^{\ker \mathrm{ad}\,\overline{r}}$ does not contain $\mathbb{Q}(\zeta_l)$.

Suppose that $Sp_n = \{g \in GL_n : \; gJ^tg = J\}$ where

$$J = \begin{pmatrix} 0 & 1_{n/2} \\ -1_{n/2} & 0 \end{pmatrix}.$$

Define submodules R_0, R_1 and R_2 of $\mathrm{ad}\,\overline{r}$ as follows. R_0 consists of scalar matrices. R_1 consists of matrices A such that $AJ + J^tA = 0$. Finally R_2 consists of matrices A such that $\mathrm{tr}\,A = 0$ and $AJ - J^tA = 0$. Each is preserved by $GSp_n(\mathbb{F}_l)$. As $l > n$ we see that

$$\mathrm{ad}\,\overline{r} = R_0 \oplus R_1 \oplus R_2$$

and each R_i is an irreducible $Sp_n(\mathbb{F}_l)$-module. (The latter fact is because each R_i is a Weyl module with l-restricted highest weight.) Thus $H^0(G_{\mathbb{Q}(\zeta_l)}, \mathrm{ad}^0\overline{r}) = (0)$. Moreover condition 8 of the theorem is verified by choosing $\alpha \in \mathbb{F}_l^\times$ with $\alpha^2 \neq 1$ and taking h to be the diagonal matrix

$$\mathrm{diag}(\alpha, 1, ..., 1, \alpha^{-1}, 1, ..., 1)$$

in $Sp_n(\mathbb{F}_l)$.

Finally let B_n denote the Borel subgroup of elements of Sp_n of the form

$$\begin{pmatrix} a & b \\ 0 & {}^ta^{-1} \end{pmatrix}$$

with a upper triangular. Then $(\mathrm{ad}\,\overline{r})^{B_n(\mathbb{F}_l)} = R_0$. Also let T_n denote the subgroup of Sp_n consisting of diagonal elements. Associate the character group $X^*(T_n)$ with $\mathbb{Z}^{n/2}$ by

$$(a_1, ..., a_{n/2})\mathrm{diag}(t_1, ..., t_{n/2}, t_1^{-1}, ..., t_{n/2}^{-1}) = t_1^{a_1} ... t_{n/2}^{a_{n/2}}.$$

Corollary 2.9 of [CPS] tells us that $H^1(Sp_n(\mathbb{F}_l), \mathrm{ad}^0\overline{r}) = (0)$. (According to footnote (23) on page 182 of [CPS], because $l > 3$, we may take ψ of corollary 2.9 of [CPS] to consist of $(1, -1, 0, ..., 0)$, $(0, 1, -1, ..., 0)$, ..., $(0, 0, ..., 1, -1)$, and $(0, 0, ..., 0, 2)$. Then that corollary tells us that

$$\dim H^1(Sp_n(\mathbb{F}_l), \mathrm{ad}^0\overline{r}) = 2(n/2 - 1) + 1 - (n - 1) = 0.)$$

It follows that $H^1(G_{\mathbb{Q}(\zeta_l)}, \mathrm{ad}^0\overline{r}) = (0)$.

The corollary now follows from the theorem. □

Theorem 4.5.5 *Let $n > 1$ be an even integer. Let $M/\mathbb{Q}$ be an imaginary CM field which is cyclic Galois of degree n. Let τ generate $\mathrm{Gal}\,(M/\mathbb{Q})$. Let l be a rational prime such that $l > 8((n+2)/4)^{n/2+1}$, $l \equiv 1 \bmod n$, and l splits completely in M. Let p be a prime which is inert and unramified in M. Let $q \neq l$ be a prime which splits completely in M and satisfies $q^i \not\equiv 1 \bmod l$ for $i = 1, ..., n$.*

Let

$$\overline{\theta} : \mathrm{Gal}\,(\overline{\mathbb{Q}}/M) \longrightarrow \overline{\mathbb{F}}_l^\times$$

be a continuous character such that

- $\overline{\theta\theta}^c = \epsilon^{1-n}$;
- *there exists a prime $w|l$ of M such that for $i = 0, ..., n/2-1$ we have $\overline{\theta}|_{I_{M_{\tau^i w}}} = \epsilon^{-i}$;*
- *$\overline{\theta}$ is unramified above q;*
- *and for $j = 1, ..., n-1$ we have $\overline{\theta}|_{\mathrm{Gal}\,(\overline{M}_p/M_p)} \neq \overline{\theta}^{\tau^j}|_{\mathrm{Gal}\,(\overline{M}_p/M_p)}$.*

Suppose that there exists an imaginary quadratic field $E/\mathbb{Q}$ linearly disjoint from $\overline{M}^{\ker \overline{\theta}}(\zeta_l)$ in which l, p, q and all primes above which $\overline{\theta}$ or M is ramified split, and such that l does not divide the class number of E.

Let F^+/F_0^+ be a Galois extension of totally real fields with F^+ linearly disjoint from the Galois closure of $E(\zeta_l)\overline{M}^{\ker \overline{\theta}}$ over $\mathbb{Q}$. Suppose that that l is unramified in F^+ and that there is a prime $v_{p,0}$ of F_0^+ split over p. Let

$$r : \mathrm{Gal}\,(\overline{F^+}/F^+) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

be a continuous representation such that

- $\overline{r} \cong (\mathrm{Ind}\,_{\mathrm{Gal}\,(\overline{\mathbb{Q}}/M)}^{\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})} \overline{\theta})|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}/F^+)}$;
- $r^\vee \cong r\epsilon^{n-1}$;
- *r is unramified at all but finitely many primes;*
- *For all places $v|l$ of F^+, $r|_{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)}$ is crystalline.*
- *For all $\tau \in \mathrm{Hom}\,(F^+, \overline{\mathbb{Q}}_l)$ above a prime $v|l$ of F^+,*

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^{\,i}(r \otimes_{\tau, F_v^+} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v^+/F_v^+)} = 1$$

for $i = 0, ..., n-1$ and $= 0$ otherwise.

- *There is a place* $v_q|q$ *of* F^+ *such that* $\#k(v_q)^j \not\equiv 1 \bmod l$ *for* $j = 1, ..., n-1$*, and such that* $r|^{\rm ss}_{{\rm Gal}\,(\overline{F}^+_{v_q}/F^+_{v_q})}$ *is unramified and finally such that* $r|^{\rm ss}_{{\rm Gal}\,(\overline{F}^+_{v_q}/F^+_{v_q})}({\rm Frob}_{v_q})$ *has eigenvalues* $\{\alpha(\#k(v_q))^j : \; j = 0, ..., n-1\}$ *for some* $\alpha \in \overline{\mathbb{Q}}_l^\times$.

Assume further that conjecture I is valid (for all unitary groups of the type considered there over any totally real field.)

Then r *is automorphic over* F^+ *of weight* 0 *and type* $\{{\rm Sp}\,_n(1)\}_{\{v_q\}}$ *and level prime to* l.

Proof: Apply theorem 4.4.4 to $F = F^+E$ and use the argument of lemma 4.3.2. □

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APPENDIX A: The level raising operator after Russ Mann.

In this appendix we will explain Russ Mann's proof of lemma 2.2.7 and proposition 2.2.8. A preliminary write-up of most of the arguments can be found in [M2], but as Russ has left academia it seems increasingly unlikely that he will finish [M2]. Hence this appendix. Russ actually found more general results concerning level raising for forms of level greater than 1, which we do not report on here. We stress that the arguments of this appendix are entirely due to Russ Mann, though we of course take responsibility for any errors in their presentation.

Write B_n for the Borel subgroup of GL_n consisting of upper triangular matrices and write N_n for its unipotent radical. Also write T_n for the maximal torus in GL_n consisting of diagonal matrices and write P_n for the subgroup of GL_n consisting of matrices with last row $(0, ..., 0, 1)$.

Let F_w be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_{F_w}$. Let $w : F_w^\times \twoheadrightarrow \mathbb{Z}$ denote the valuation, let ϖ_w denote a uniformiser of $\mathcal{O}_{F_w}$ and let $q_w = \#\mathcal{O}_{F_w}/(\varpi_w)$. Also let $\mathcal{O}$ denote the subring of $\mathbb{C}$ generated by $q_w^{-1/2}$ and all p-power roots of 1. Let S_n denote the symmetric group on n letters and set

$$R_n^+ = \mathcal{O}[X_1, ..., X_n]^{S_n} \subset R_n = \mathcal{O}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^{S_n},$$

where S_n permutes the variables X_i. Sometimes we will want to consider R_n and R_{n-1} at the same time. To make the notation clearer we will write $R_{n-1} = \mathcal{O}[Y_1^{\pm 1}, ..., Y_{n-1}^{\pm 1}]^{S_{n-1}}$ and $R_{n-1}^+ = \mathcal{O}[Y_1, ..., Y_{n-1}]^{S_{n-1}}$. We will also set

$$R_{n-1}^\wedge = \mathcal{O}[[Y_1, ..., Y_{n-1}]]^{S_{n-1}}$$

and $R_{n-1}^{\leq m}$ to equal to the $\mathcal{O}$-submodule of R_{n-1}^+ consisting of polynomials of degree $\leq m$ in each variable separately.

Let $\alpha_j = \varpi_w 1_j \oplus 1_{n-j}$ and let $T^{(j)}$ denote the double coset

$$T^{(j)} = GL_n(\mathcal{O}_{F_w})\alpha_j GL_n(\mathcal{O}_{F_w}).$$

Let $GL_n(\mathcal{O}_{F_w})^+$ denote the sub-semigroup of $GL_n(F_w)$ consisting of matrices with entries in $\mathcal{O}_{F_w}$. Then

$$\mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)^+/GL_n(\mathcal{O}_{F_w})] = \mathcal{O}[T^{(1)}, T^{(2)}, ..., T^{(n)}]$$

and

$$\mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] = \mathcal{O}[T^{(1)}, T^{(2)}, ..., T^{(n)}, (T^{(n)})^{-1}].$$

Define $\sim$ from $\mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ to itself by

$$[GL_n(\mathcal{O}_{F_w})gGL_n(\mathcal{O}_{F_w})]^\sim = [GL_n(\mathcal{O}_{F_w})g^{-1}GL_n(\mathcal{O}_{F_w})].$$

Then $(T^{(j)})^\sim = (T^{(n)})^{-1}T^{(n-j)}$.

There is an isomorphism (a certain normalisation of the the Satake isomorphism)

$$S : \mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \xrightarrow{\sim} R_n$$

which sends $T^{(j)}$ to $q_w^{j(1-j)/2}$ times the j^{th} elementary symmetric function in the X_i's (i.e. to the sum of all products of j distinct X_i's). We have

$$S(\mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)^+/GL_n(\mathcal{O}_{F_w})]) = R_n^+$$

and

$$S(T^\sim)(X_1, ..., X_n) = S(T)(q_w^{n-1}X_1^{-1}, ..., q_w^{n-1}X_n^{-1}).$$

If we write

$$\mathcal{O}[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m}$$

for the submodule of $\mathcal{O}[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]$ spanned by the double cosets

$$GL_{n-1}(\mathcal{O}_{F_w})\mathrm{diag}(t_1, ..., t_{n-1})GL_{n-1}(\mathcal{O}_{F_w}),$$

where $m \geq w(t_1) \geq ... \geq w(t_{n-1}) \geq 0$, then

$$S(\mathcal{O}[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m}) = (\mathcal{O}[Y_1, ..., Y_{n-1}]^{S_{n-1}})^{\leq m}.$$

Let $U_1(w^m)$ denote the subgroup of $GL_n(\mathcal{O}_{F_w})$ consisting of elements which reduce modulo ϖ_w^m to an element of $P_n(\mathcal{O}_{F_w}/(\varpi_w^m))$. For $j = 1, ..., n-1$ let

$$U^{(j)} = P_n(\mathcal{O}_{F_w})\alpha_j P_n(\mathcal{O}_{F_w}).$$

Note that $U^{(j)}/P_n(\mathcal{O}_{F_w})$ has finite cardinality. If π is a smooth representation of $GL_n(F_w)$ and if $m \in \mathbb{Z}_{\geq 1}$ then

- the operators $U^{(j)}$ on $\pi^{P_n(\mathcal{O}_{F_w})}$ commute, and
- the action of $U^{(j)}$ preserves $\pi^{U_1(w^m)}$ and in fact acts the same way as

  $$U_1(w^m)\alpha_j U_1(w^m)$$

  on this space.

(This is proved by writing down explicit coset decompositions, see for instance proposition 4.1 of [M1] .)

Let A be an $\mathcal{O}$-module and suppose that

$$T = \sum_i a_i GL_{n-1}(\mathcal{O}_{F_w}) g_i GL_{n-1}(\mathcal{O}_{F_w})$$

is in $A[GL_{n-1}(\mathcal{O}_{F_w}) \backslash GL_{n-1}(F_w)^+ / GL_{n-1}(\mathcal{O}_{F_w})]$. Define

$$V(T) = \sum_i a_i |\det g_i|^{n-1/2} P_n(\mathcal{O}_{F_w}) \begin{pmatrix} g_i^{-1} & 0 \\ 0 & 1 \end{pmatrix} GL_n(\mathcal{O}_{F_w}).$$

Note that if $h \in GL_{n-1}(F_w)^+$ and

$$GL_{n-1}(\mathcal{O}_{F_w}) h^{-1} GL_{n-1}(\mathcal{O}_{F_w}) = \coprod_j h_j GL_{n-1}(\mathcal{O}_{F_w})$$

then

$$P_n(\mathcal{O}_{F_w}) \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} GL_n(\mathcal{O}_{F_w}) = \coprod_j \begin{pmatrix} h_j & 0 \\ 0 & 1 \end{pmatrix} GL_n(\mathcal{O}_{F_w}).$$

Similarly if $m \in \mathbb{Z}_{\geq 1}$ and if

$$T = \sum_i a_i GL_{n-1}(\mathcal{O}_{F_w}) g_i GL_{n-1}(\mathcal{O}_{F_w})$$

is in $A[GL_{n-1}(\mathcal{O}_{F_w}) \backslash GL_{n-1}(F_w)^+ / GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m}$ define

$$V_m(T) = \sum_i a_i |\det g_i|^{n-1/2} U_1(w^m) \begin{pmatrix} g_i^{-1} & 0 \\ 0 & 1 \end{pmatrix} GL_n(\mathcal{O}_{F_w}).$$

Note that if $h \in GL_{n-1}(F_w)^+$ is such that $GL_{n-1}(\mathcal{O}_{F_w}) h GL_{n-1}(\mathcal{O}_{F_w})$ lies in $A[GL_{n-1}(\mathcal{O}_{F_w}) \backslash GL_{n-1}(F_w)^+ / GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m}$, and if

$$GL_{n-1}(\mathcal{O}_{F_w}) h^{-1} GL_{n-1}(\mathcal{O}_{F_w}) = \coprod_j h_j GL_{n-1}(\mathcal{O}_{F_w})$$

then

$$U_1(w^m) \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} GL_n(\mathcal{O}_{F_w}) = \coprod_j \begin{pmatrix} h_j & 0 \\ 0 & 1 \end{pmatrix} GL_n(\mathcal{O}_{F_w}).$$

We deduce that if π is any smooth representation of $GL_n(F_w)$ and if $T \in A[GL_{n-1}(\mathcal{O}_{F_w}) \backslash GL_{n-1}(F_w)^+ / GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m}$ then $V(T)$ preserves the space

$\pi^{U_1(w^m)}$ and acts on it via $V_m(T)$. In the case $A = R_n$ the map V_m induces a map, which we will also denote V_m:

$$R_n[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m} \to \mathcal{O}[U_1(w^m)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$$

by the formula

$$\begin{array}{rl} & V_m(\sum_i a_i[GL_{n-1}(\mathcal{O}_{F_w})g_i GL_{n-1}(\mathcal{O}_{F_w})]) \\ = & \sum_i |\det g_i|^{n-1/2} \left[U_1(w^m)\begin{pmatrix} g_i^{-1} & 0 \\ 0 & 1 \end{pmatrix} GL_n(\mathcal{O}_{F_w})\right] \circ S^{-1}(a_i). \end{array}$$

Proposition 5.2 of [M1] says that the set of

$$V_m(GL_{n-1}(\mathcal{O}_{F_w})\mathrm{diag}(t_1, ..., t_{n-1})GL_{n-1}(\mathcal{O}_{F_w})),$$

where $t \in T_{n-1}(F_w)/T_{n-1}(\mathcal{O}_{F_w})$ with $m \geq w(t_1) \geq ... \geq w(t_{n-1}) \geq 0$ is a basis of $\mathcal{O}[U_1(w^m)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ as a right R_n-module. Hence

$$V_m : R_n[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m} \to \mathcal{O}[U_1(w^m)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$$

is an isomorphism of free R_n-modules.

Let

$$\psi : F_w \longrightarrow \mathcal{O}^\times$$

be a continuous character with kernel $\mathcal{O}_{F_w}$. We will also think of ψ as a character of $N_n(F_w)$ by setting

$$\psi(u) = \psi(u_{1,2} + u_{2,3} + ... + u_{n-1,n}).$$

If A is a $\mathcal{O}$-algebra we will write $\mathcal{W}_n(A, \psi)$ for the set of functions

$$W : GL_n(F_w) \longrightarrow A$$

such that

- $W(ug) = \psi(u)W(g)$ for all $g \in GL_n(F_w)$ and $u \in N_n(F_w)$,
- and W is invariant under right translation by some open subgroup of $GL_n(F_w)$.

Thus $\mathcal{W}_n(A, \psi)$ is a smooth representation of $GL_n(F_w)$ (acting by right translation).

There is a unique element $W_n^0(\psi) \in \mathcal{W}_n(R_n, \psi)^{GL_n(\mathcal{O}_{F_w})}$ such that

- $W_n^0(\psi)(1_n) = 1$ and

- $TW_n^0(\psi) = S(T)W_n^0(\psi)$ for all $T \in \mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$.

Moreover if the last row of g is integral then $W_n^0(\psi)(g) \in R_n^+$. (These facts are proved exactly as in [Sh].)

Suppose again that A is a $\mathcal{O}$-algebra. If $W \in \mathcal{W}_n(A,\psi)^{P_n(\mathcal{O}_{F_w})}$ we heuristically define $\Phi(W) \in A \otimes_{\mathcal{O}} R_{n-1}^\wedge = A[[Y_1, ..., Y_{n-1}]]^{S_{n-1}}$ by

$$\Phi(W) = \int_{N_{n-1}(F_w)\backslash GL_{n-1}(F_w)} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W_{n-1}^0(\psi^{-1})(g)|\det g|^{s-n+1/2}dg\Big|_{s=0}$$

where the implies Haar measures give $GL_{n-1}(\mathcal{O}_{F_w})$ and $N_{n-1}(\mathcal{O}_{F_w})$ volume 1. Rigorously one can for instance set

$$\Phi(W) = \sum_t W\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} W_{n-1}^0(\psi^{-1})(t)|\det t|^{s-n+1/2}|t_1|^{2-n}|t_2|^{4-n}...|t_{n-1}|^{n-2}$$

where $t = \mathrm{diag}(t_1, ..., t_{n-1})$ runs over elements of $T_{n-1}(F_w)/T_{n-1}(\mathcal{O}_{F_w})$ with

$$w(t_1) \geq w(t_2) \geq ... \geq w(t_{n-1}) \geq 0.$$

For such t the value $W_{n-1}^0(\psi^{-1})(t)$ is a homogeneous polynomial in the Y_i's of degree $w(\det t)$ and these polynomials are linearly independent over A for $t \in T_{n-1}(F_w)/T_{n-1}(\mathcal{O}_{F_w})$ with $w(t_1) \geq w(t_2) \geq ... \geq w(t_{n-1}) \geq 0$. (As in [Sh].) In particular if $W \in \mathcal{W}_n(A,\psi)^{P_n(\mathcal{O}_{F_w})}$ then $\Phi(W)$ determines $W|_{P_n(F_w)}$. As in section (1.4) of [JS2] we see that

$$\Phi(W_n^0(\psi)) = \prod_{i,j}(1 - X_iY_j)^{-1}.$$

Fix an embedding $\imath : R_n \hookrightarrow \mathbb{C}$. There is a unique irreducible smooth representation π of $GL_n(F_w)$ such that $\mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ acts on $\pi^{GL_n(\mathcal{O}_{F_w})}$ via $\imath \circ S$. Moreover there is an embedding $\pi \hookrightarrow \mathcal{W}_n(\mathbb{C},\psi)$ which is unique up to $\mathbb{C}^\times$-multiples. It follows from [Sh] that $\imath W_n^0(\psi)$ is in the image of π. It follows from sections (3.5) and (4.2) of [JPSS] that

$$\Phi : (R_n[GL_n(F_w)]W_n^0(\psi))^{P_n(\mathcal{O}_{F_w})} \hookrightarrow \prod_{i,j}(1 - X_iY_j)^{-1}R_n[Y_1, ..., Y_{n-1}]^{S_{n-1}}.$$

From corollary 3.5 of [M1] we see also see that

$$\dim_{\mathbb{C}}(R_n[GL_n(F_w)]W_n^0(\psi))^{U_1(w^m)}) \otimes_{R_n,\imath} \mathbb{C} \leq \dim_{\mathbb{C}} \pi^{U_1(w^m)} = \begin{pmatrix} m+n-1 \\ n-1 \end{pmatrix}.$$

If $W \in (R_n[GL_n(F_w)]W_n^0(\psi))^{P_n(\mathcal{O}_{F_w})}$ and $\Phi(W) = 1$ then we see that $W|_{P_n(F_w)}$ is supported on $N_n(F_w)P_n(\mathcal{O}_{F_w})$ and that $W(1_n) = 1$. Thus we have $(U^{(j)}W)|_{P_n(F_w)} = 0$. (Recall that we only have to check this at elements $\mathrm{diag}(t_1, ..., t_{n-1}, 1)$ and that any element of $\mathcal{W}_n(R_n, \psi)$ will vanish at $\mathrm{diag}(t_1, ..., t_{n-1}, 1)$ unless $w(t_i) \geq 0$ for all i. To check at the remaining diagonal matrices one uses the explicit single coset decomposition in proposition 4.1 of [M1].) Hence $\Phi(U^{(j)}W) = 0$ and so $U^{(j)}W = 0$.

Recall that if $h \in GL_{n-1}(F_w)^+$ and

$$GL_{n-1}(\mathcal{O}_{F_w})h^{-1}GL_{n-1}(\mathcal{O}_{F_w}) = \coprod_j h_j GL_{n-1}(\mathcal{O}_{F_w})$$

then

$$P_n(\mathcal{O}_{F_w}) \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} GL_n(\mathcal{O}_{F_w}) = \coprod_j \begin{pmatrix} h_j & 0 \\ 0 & 1 \end{pmatrix} GL_n(\mathcal{O}_{F_w}).$$

From this and a simple change of variable in the integral defining Φ we see that if $T \in A[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]$ and $f \in \mathcal{W}_n(A, \psi)^{GL_n(\mathcal{O}_{F_w})}$ then

$$\Phi(V(T)f) = S(T)\Phi(f).$$

Thus we have

$$\begin{array}{ccc} R_n[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m} & T & \\ \downarrow & \downarrow & \\ (R_n[GL_n(F_w)]W_n^0(\psi))^{U_1(w^m)} & V_m(T)W_n^0(\psi) & W \\ \downarrow & & \downarrow \\ \prod_{i,j}(1 - X_iY_j)^{-1}R_n[Y_1, ..., Y_{n-1}]^{S_{n-1}} & & \Phi(W). \end{array}$$

The composite sends

$$T \longmapsto S(T)\prod_{i,j}(1 - X_iY_j)^{-1}.$$

The composite is an isomorphism to its image:

$$\prod_{i,j}(1 - X_iY_j)^{-1}(R_n[Y_1, ..., Y_{n-1}]^{S_{n-1}})^{\leq m},$$

which is a direct summand of $\prod_{i,j}(1 - X_iY_j)^{-1}R_n[Y_1, ..., Y_{n-1}]^{S_{n-1}}$ and which is free over R_n of rank

$$\begin{pmatrix} m+n-1 \\ n-1 \end{pmatrix}.$$

As

$$\dim_{\mathbb{C}}(R_n[GL_n(F_w)]W_n^0(\psi))^{U_1(w^m)}) \otimes_{R_n,\imath} \mathbb{C} \leq \left(\begin{array}{c} m+n-1 \\ n-1 \end{array} \right),$$

we deduce that

$$\begin{array}{rl} & R_n[GL_{n-1}(\mathcal{O}_{F_w})\backslash GL_{n-1}(F_w)^+/GL_{n-1}(\mathcal{O}_{F_w})]_{\leq m} \\ \stackrel{\sim}{\longrightarrow} & \mathcal{O}[U_1(w^m)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})] \\ \stackrel{\sim}{\longrightarrow} & (R_n[GL_n(F_w)]W_n^0(\psi))^{U_1(w^m)} \\ \stackrel{\sim}{\longrightarrow} & \prod_{i,j}(1-X_iY_j)^{-1}(R_n[Y_1,...,Y_{n-1}]^{S_{n-1}})^{\leq m}. \end{array}$$

Lemma 2.2.7 follows immediately from this.

Let θ denote the element of

$$\mathcal{O}[U_1(w^m)\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$$

which is $V_n(\prod_{i,j}(1-X_iY_j))$. Then

$$\Phi(\theta W_n^0(\psi)) = 1.$$

Moreover $U^{(j)}\theta W_n^0(\psi) = 0$ and so $U^{(j)}\theta = 0$ for $j = 1,...,n-1$. Thus θ satisfies the first three parts of proposition 2.2.8.

We now turn to the proof the final part of proposition 2.2.8. Write

$$\theta = \sum_{\underline{a}} [U_1(w^n)\mathrm{diag}(\varpi_w^{-a_1},...,\varpi_w^{-a_{n-1}},1)GL_n(\mathcal{O}_{F_w})]T_{\underline{a}}$$

where $T_{\underline{a}} \in \mathcal{O}[GL_n(\mathcal{O}_{F_w})\backslash GL_n(F_w)/GL_n(\mathcal{O}_{F_w})]$ and where $\underline{a} = (a_1,...,a_{n-1})$ runs over elements of $\mathbb{Z}^{n-1}$ with

$$n \geq a_1 \geq ... \geq a_{n-1} \geq 0.$$

As

$$\sum_{\underline{a}} S(T_{\underline{a}})S(GL_{n-1}(\mathcal{O}_{F_w})\mathrm{diag}(\varpi_w^{a_1},...,\varpi_w^{a_{n-1}})GL_{n-1}(\mathcal{O}_{F_w})) = \prod_{i,j}(1-X_iY_j)$$

we see that

$$S(T_{(n,...,n)}) = (X_1...X_n)^{n-1},$$

i.e. $T_{(n,...,n)} = q_w^{n(n-1)^2/2}(T^{(n)})^{n-1}$. Let $\eta = 1_{n-1} \oplus \varpi_w^n$ and define $\widehat{\theta}$ as we did just before proposition 2.2.8. Thus we have

$$\widehat{\theta} = \sum_{\underline{a}} (T^{(n)})^{-n}T_{\underline{a}}[GL_n(\mathcal{O}_{F_w})\mathrm{diag}(\varpi_w^{n-a_1},...,\varpi_w^{n-a_{n-1}},1)U_1(w^n)].$$

Again π denote the $GL_n(F_w)$-subrepresentation of $\mathcal{W}_n(\mathbb{C},\psi)$ generated by $\imath W_n^0(\psi)$. Define $\widetilde{\imath}: R_n \hookrightarrow \mathbb{C}$ to be the $\mathcal{O}$-linear map sending X_i to $q_w^{n-1}\imath(X_i)^{-1}$. Let $\widetilde{\pi}$ denote the $GL_n(F_w)$-subrepresenetation of $\mathcal{W}_n(\mathbb{C},\psi^{-1})$ generated by $\widetilde{\imath}(W_n^0(\psi^{-1}))$. Then $\widetilde{\pi}$ is the contragredient of π. Write gen_n for the compact induction c-Ind$_{N_n(F_w)}^{P_n(F_w)}\mathbb{C}(\psi)$. It follows from proposition 3.2 and lemma 4.5 of [BZ] that gen embeds in $\pi|_{P_n(F_w)}$ and in $\widetilde{\pi}|_{P_n(F_w)}$. Moreover it follows from proposition 3.8 and lemma 4.5 of [BZ] that any $P_n(F_w)$ bilinear form

$$\langle\ ,\ \rangle : \pi \times \widetilde{\pi} \longrightarrow \mathbb{C}$$

restricts non-trivially to $\mathrm{gen}_n \times \mathrm{gen}_n$. Hence there is a unique such bilinear form up to scalar multiples and so any $P_n(F_w)$-bilinear pairing $\pi \times \widetilde{\pi} \to \mathbb{C}$ is also $GL_n(F_w)$-bilinear. Such a pairing is given by

$$\langle W, \widetilde{W}\rangle = \int_{N_n(F_w)\backslash P_n(F_w)} W(g)\widetilde{W}(g)|\det g|^s dg\bigg|_{s=0}.$$

Here we use a Haar measure on $N_n(F_w)$ giving $N_n(\mathcal{O}_{F_w})$ volume 1 and a right Haar measure on $P_n(F_w)$ giving $P_n(\mathcal{O}_{F_w})$ volume 1. The integral may not converge for $s=0$, but in its domain of convergence it is a rational function of q_w^s and so has meromorphic continuation to the whole complex plane.

We will complete the proof of proposition 2.2.8 by evaluating

$$\langle \imath\widehat{\theta}\theta W_n^0(\psi), \widetilde{\imath}W_n^0(\psi^{-1})\rangle$$

in two ways. Firstly moving the $\widehat{\theta}$ to the other side of the pairing we obtain

$$\begin{array}{l}[GL_n(\mathcal{O}_{F_w}) : U_1(w^n)] \sum_{\underline{a}} \widetilde{\imath} \circ S(\widetilde{T}_{\underline{a}}(T^{(n)})^n) \\ \langle \imath\theta W_n^0(\psi), \widetilde{\imath}[U_1(w^n)\mathrm{diag}(\varpi_w^{a_1-n},\ldots,\varpi_w^{a_{n-1}-n},1)GL_n(\mathcal{O}_{F_w})]W_n^0(\psi^{-1})\rangle.\end{array}$$

But $(\theta W_n^0(\psi))|_{P_n(F_w)}$ is supported on $N_n(F_w)P_n(\mathcal{O}_{F_w})$ and equals 1 on $P_n(\mathcal{O}_{F_w})$. Thus $\langle \imath\theta W_n^0(\psi), \widetilde{W}\rangle$ simply equals $\widetilde{W}(1_n)$. We deduce that

$$\begin{array}{c}\langle \imath\widehat{\theta}\theta W_n^0(\psi), \widetilde{\imath}W_n^0(\psi^{-1})\rangle = (q_w^n-1)q_w^{n(n-1)} \sum_{\underline{a}} \widetilde{\imath} \circ S(\widetilde{T}_{\underline{a}}(T^{(n)})^n) \\ \widetilde{\imath}([U_1(w^n)\mathrm{diag}(\varpi_w^{a_1-n},\ldots,\varpi_w^{a_{n-1}-n},1)GL_n(\mathcal{O}_{F_w})]W_n^0(\psi^{-1}))(1_n).\end{array}$$

The terms of this sum are zero except for the term $a_1 = \ldots = a_{n-1} = n$ which gives

$$(q_w^n-1)q_w^{n(n-1)}\widetilde{\imath}S(q_w^{n(n-1)^2/2}T^{(n)}),$$

i.e.

$$(q_w^n-1)q_w^{(n+2)n(n-1)/2}\imath(X_1\ldots X_n)^{-1}.$$

On the other hand

$$\langle \imath\widehat{\theta}\theta W_n^0(\psi), \widetilde{\imath} W_n^0(\psi^{-1})\rangle$$

equals

$$\imath(S(\widehat{\theta})\theta)\langle \imath W_n^0(\psi), \widetilde{\imath} W_n^0(\psi^{-1})\rangle.$$

We consider the integral

$$\int_{N_n(F_w)\backslash P_n(F_w)} W(g)\widetilde{W}(g)|\det g|^s dg$$

with the Haar measures described above. It equals

$$\sum_t \imath(W_n^0(\psi)(t))\widetilde{\imath}(W_n^0(\psi^{-1})(t))|t_1|^{2-n+s}|t_2|^{4-n+s}...|t_n|^{n+s},$$

where the sum runs over $t = \mathrm{diag}(t_1, ..., t_n) \in T_n(F_w)/T_n(\mathcal{O}_{F_w})$ with

$$w(t_1) \geq w(t_2) \geq ... \geq w(t_n) = 0.$$

Because $\imath(W_n^0(\psi)(t))\widetilde{\imath}(W_n^0(\psi^{-1})(t))$ is invariant under the multiplication of t by an element of $F_w^\times$ this in turn equals

$$(1 - q_w^{-n(s+1)})\sum_t \imath(W_n^0(\psi)(t))\widetilde{\imath}(W_n^0(\psi^{-1})(t))|t_1|^{2-n+s}|t_2|^{4-n+s}...|t_n|^{n+s},$$

where now the sum runs over $t = \mathrm{diag}(t_1, ..., t_n) \in T_n(F_w)/T_n(\mathcal{O}_{F_w})$ with

$$w(t_1) \geq w(t_2) \geq ... \geq w(t_n) \geq 0.$$

This in turn equals $(1 - q_w^{-n(s+1)})$ times

$$\int_{N_n(F_w)\backslash GL_n(F_w)} \imath(W_n^0(\psi)(g))\widetilde{\imath}(W_n^0(\psi^{-1})(g))\varphi((0, ..., 0, 1)g)|\det g|^{1+s}dg,$$

where φ is the characteristic function of $\mathcal{O}_{F_w}^n$ and where we use the Haar measures on $N_n(F_w)$ (resp. $GL_n(F_w)$) which give $N_n(\mathcal{O}_{F_w})$ (resp. $GL_n(\mathcal{O}_{F_w})$) volume 1. As in proposition 2 of [JS1] this becomes

$$(1 - q_w^{-n(s+1)})\prod_{i=1}^n\prod_{j=1}^n(1 - \imath(X_i/X_j)q_w^{-(1+s)})^{-1}.$$

Thus

$$\langle \imath\widehat{\theta}\theta W_n^0(\psi), \widetilde{\imath} W_n^0(\psi^{-1})\rangle = \imath(S(\widehat{\theta})\theta)(1 - q_w^{-n})\prod_{i=1}^n\prod_{j=1}^n(1 - \imath(X_i/X_j)q_w^{-1})^{-1}.$$

Thus we conclude that

$$S(\widehat{\theta}\theta) = q_w^{n^2(n-1)/2}(X_1 ... X_n)^{-(n+1)} \prod_{i=1}^{n} \prod_{j=1}^{n} (q_w X_i - X_j),$$

and we have completed the proof of proposition 2.2.8.

APPENDIX B: Unipotent representations of $GL(n,F)$ in the quasi-banal case.

By M.-F.Vigneras

Let F be a local non archimedean field of residual characteristic p and let R be an algebraically closed field of characteristic 0 or $\ell > 0$ different from p. Let $G = GL(n,F)$. The category $\mathrm{Mod}_R G$ of (smooth) R-representations of G is equivalent to the category of *right* modules $\mathcal{H}_R(G)$ for the global Hecke algebra (the convolution algebra of locally constant functions $f : G \to R$ with compact support, isomorphic to the opposite algebra by $f(g) \to f(g^{-1})$.)

$$\mathrm{Mod}_R G \simeq \mathrm{Mod}\mathcal{H}_R(G).$$

Definitions. We are in the *quasi-banal case* when the order of the maximal compact subgroup of G is invertible in R (the *banal case*), or when $q = 1$ in R and the characteristic of R is $\ell > n$ (the *limit case*).

A *block* of $\mathrm{Mod}_R G$ is an abelian subcategory of $\mathrm{Mod}_R G$ which is a direct factor of $\mathrm{Mod}_R G$ and is minimal for this property. One proves that $\mathrm{Mod}_R G$ is a product of blocks [V2, III.6]. The *unipotent block* $\mathcal{B}_{R,1}(G)$ is the block containing the trivial representation. An R-representation of G is *unipotent* if it belongs to the unipotent block.

Notations. Let $I, B = TU$ be a standard Iwahori, Borel, diagonal, stritly upper triangular subgroup of G, T_o the maximal compact subgroup of T, I_p the pro-p-radical of I. The functor $\mathrm{Ind}_B^G : \mathrm{Mod}_R B \to \mathrm{Mod}_R G$ is the normalized induction. The group I has a normal subgroup I^ℓ of pro-order prime to ℓ and a finite ℓ subgroup I_ℓ such that $I = I^\ell I_\ell$. To get a uniform notation, we set $I^\ell = I, I_\ell = \{1\}$ when the characteristic of R is 0. We have $I = I^\ell, I_\ell = \{1\}$ in the banal case and $I \neq I^\ell, I_\ell \neq \{1\}$ in the limit case. Let $\mathrm{Mod} H_R(G,I)$ be the category of *right* modules for the Iwahori Hecke algebra (isomorphic to its opposite)

$$H_R(G,I) := \mathrm{End}_{RG} R[I\backslash G] \simeq_R R[I\backslash G/I].$$

Let $\mathrm{Mod}_R(G,I)$ be the category of R-representations of G generated by their I-invariant vectors.

1 Theorem *In the quasi-banal case,*

1) The category $\mathrm{Mod}_R(G,I)$ *is stable by subquotients.*

2) For any $V \in \mathrm{Mod}_R(G,I)$*, one has* $V^{I_p} = V^I$*, in particular* $R[I\backslash G]$ *is projective in* $\mathrm{Mod}_R(G,I)$*.*

3) The I*-invariant functor*

$$V \to V^I : \mathrm{Mod}_R(G,I) \to \mathrm{Mod} H_R(G,I)$$

is an equivalence of categories.

4) The I^ℓ*-invariant functor on the unipotent block* $\mathcal{B}_{R,1}(G)$

$$V \to V^{I^\ell} : \mathcal{B}_{R,1}(G) \to \mathrm{Mod} H_R(G,I^\ell)$$

is an equivalence of categories.

5) In the banal case, $\mathrm{Mod}_R(G,I)$ *is the unipotent block.*

6) In the limit case, $\mathrm{Mod}_R(G,I)$ *is not the unipotent block.*

7) The parabolically induced representation $\mathrm{Ind}_B^G 1$ *is semi-simple (hence also* $\mathrm{Ind}_P^G 1$ *for all parabolic subgroups* P *of* G*). In the limit case,* $\mathrm{Ind}_B^G \mathbb{X}$ *is semi-simple for any unramified* R*-character* $\mathbb{X} : T/T_o \to R^*$ *of* T*.*

8) In the limit case, $H_R(G,I^\ell) \simeq H_R(G,I) \otimes_R R[I^\ell]$*.*

The proof of the theorem uses some general results (A), ... , (H), valid in the non quasi-banal case (except (E) and (G)) and for most of them when G is a general reductive connected p-adic group. We recall them first.

(A) The algebra $R[T/T_o]$ is identified to its image in $H_R(G,I)$ by the *Bernstein embedding*

$$t_B : R[T/T_o] \to H_R(G,I) \tag{1}$$

such that the U-coinvariants induces a $R[T/T_o]$-isomorphism

$$V^I \simeq (V_U)^{T_o} \tag{2}$$

for any $V \in \mathrm{Mod}_R G$ [V2, II.10.2].

(B) By [Dat], we have a $(G, R[T/T_o])$-isomorphism

$$R[I\backslash G] \simeq \mathrm{Ind}_B^G R[T/T_o] \tag{3}$$

when $R[T/T_o]$ is embedded in $H_R(G,I)$ by the Bernstein embedding $t_{\overline{B}}$: $R[T/T_o] \to H_R(G,I)$, defined by the opposite (lower triangular) $\overline{B}$ of B as in (A), where $R[T/T_o]$ is the universal representation of T inflated to B.

Hence for any character $\mathbb{X} : T/T_o \to R^*$ i.e. an algebra homomorphism $R[T/T_o] \to R$

$$R \otimes_{\mathbb{X}, R[T/T_o], t_{\overline{B}}} R[I\backslash G] \simeq \mathrm{Ind}_B^G \mathbb{X} \tag{4}$$

$$R \otimes_{\mathbb{X}, R[T/T_o], t_{\overline{B}}} H_R(G, I) \simeq (\mathrm{Ind}_B^G \mathbb{X})^I. \tag{5}$$

(C) The compact induction from an open compact subgroup K of G to G has a *right* adjoint the restriction from G to K [V1, I.5.7]. In particular, a representation generated by its I-invariant vectors is a quotient of a direct sum of $R[I\backslash G]$ (denoted $\oplus R[I\backslash G]$).

(D) The double cosets of G modulo (I_p, I) are in bijection with the double cosets of G modulo (I, I). This is clear by the Bruhat decomposition. In particular, the I_p-invariants of $R[I\backslash G]$ is equal to the I-invariants.

(E) In the quasi-banal case, every cuspidal irreducible representation of every Levi subgroup of G is supercuspidal [V1, III.5.14].

(F) The irreducible unipotent representations are the irreducible subquotients of $R[I\backslash G]$ by [V2, IV.6.2].

(G) When $q = 1$ in R, the Iwahori-Hecke algebra is the group algebra of the affine symmetric group

$$N/T_o \simeq W.(T/T_o) \simeq S_n \mathbb{Z}^n$$

(semi-direct product) where N is the normalizer of T in G and $W := N/T$ with its natural action on T/T_o. Naturally $T/T_o \simeq \mathbb{Z}^n$ by choice of a uniformizing parameter p_F of F and $W \simeq S_n$ the symmetric group on n letters with its natural action on $\mathbb{Z}^n$. The natural embedding

$$R[T/T_o] \to H_R(G, I) \simeq R[W.(T/T_o)] \tag{6}$$

is equal to $t_B = t_{\overline{B}}$. These properties are deduced without difficulty from [V1, I.3.14], [V2, II.8].

(H) When $q = 1$ in R, let π_i be an irreducible R-representation of the group $GL(n_i d_i, F)$ wich cuspidal support $\otimes^{n_i} \sigma_i$, for an irreducible cuspidal R-representation σ_i of $GL(d_i, F)$ for all $1 \leq i \leq k$. Suppose that σ_i is not equivalent to σ_j if $i \neq j$. Then the representation of $GL(\sum_i n_i d_i, F)$ parabolically induced from $\pi_1 \otimes \ldots \otimes \pi_k$ is irreducible by [V2, V.3].

Proof of the theorem 1 We suppose that we are in the quasi-banal case.

a) We prove that any irreducible subquotient V of $R[I\backslash G]$ has a non zero I-invariant vector. The U-coinvariants V_U of any irreducible subquotient V of the representation (3) have a non zero vector invariant by T_o, by (E). By (2), V_U has a non zero I-invariant vector.

b) We prove that if $W \subset V$ are subrepresentations of $\oplus R[I\backslash G]$, then $W^I = V^I$ implies $W = V$, and $V^I = V^{I_p}$. The geometric property (D) implies that the I_p-invariants of any subrepresentation of $\oplus R[I\backslash G]$ is equal to its I-invariants. Hence $W^I = W^{I_p}, V^I = V^{I_p}$. The functor of I_p-invariants is exact and any irreducible subquotient of $R[I\backslash G]$ has a non zero I_p-invariant vector by a). Hence $W^{I_p} = V^{I_p}$ implies $W = V$.

c) We prove the property 1) of the theorem. The property is trivial with quotient instead of subquotient. Let $Y \subset X$ and $p : \oplus R[I\backslash G] \to X$ a surjective G-homomorphism. Let us denote by V the inverse image of Y by p, and by W the subrepresentation of V generated by V^I. We have $W^I = V^I$ by construction, hence $W = V$ by b). Hence V is generated by its I-invariant vectors. The same is true for its quotient Y.

d) We prove the property 2) of the theorem. In c) V is a subrepresentation of $\oplus R[I\backslash G]$ hence we have $V^I = V^{I_p}$ by b). The functor of I_p-invariants is exact hence $p(V^{I_p}) = Y^{I_p}$. As $Y^I \subset Y^{I_p}$ and $p(V^I) \subset Y^I$ we have $Y^I = Y^{I_p} = p(V^I)$. This is valid for any Y hence for any representation of $\mathrm{Mod}_R(G, I)$.

e) We prove the property 3) of the theorem. All the conditions of the theorem of Arabia [A, th.4 2) (b-2)] are satisfied.

f) We prove the property 4) of the theorem. Let V be a unipotent representation. Then V is generated by V^{I^ℓ} by (F). The irreducible subquotients of the action of I on V^{I^ℓ} are trivial, because I/I^ℓ is an ℓ-group. Conversely let V be a representation generated by V^{I^ℓ}. Then the irreducible subquotients of V are unipotent, and a representation such that all its irreducible subquotients are unipotent is unipotent. As the pro-order of I^ℓ is invertible in R, and the unipotent block is generated by $\mathrm{Ind}_{I^\ell}^G 1_R = R[I^\ell\backslash G]$, the I^ℓ-invariant functor is an equivalence of category with the Hecke algebra $H_R(G, I^\ell)$.

g) We prove the property 5) of the theorem. In the banal case $I = I^\ell$ and compare the properties 3) and 4) of the theorem.

h) We prove the property 6) of the theorem. In the limit case, $I \neq I^\ell$. The I-invariants of $\mathrm{Ind}_{I^\ell}^G 1$ can be computed using the decomposition of the parahoric restriction-induction functor [V3, C.1.4] and the simple property

$$\dim(\mathrm{Ind}_{I^\ell}^I 1)^I = 1.$$

One finds that the I-invariants of $\mathrm{Ind}_{I^\ell}^G 1$ are the I-invariants of its proper subrepresentation $\mathrm{Ind}_I^G 1 = R[I\backslash G]$. Hence the unipotent representation $\mathrm{Ind}_{I^\ell}^G 1$ is not generated by its I-invariant vectors.

i) We prove the property 7) of the theorem. In the banal case $\mathrm{Ind}_B^G 1$ is irreducible. We suppose that we are in the limit case. By (4), $\mathrm{Ind}_B^G 1$ is generated by its I-invariant vectors. Hence by the property 3) of the theorem, $\mathrm{Ind}_B^G 1$ is semi-simple if $(\mathrm{Ind}_B^G 1)^I$ is a semi-simple right $H_R(G, I)$-module. By (5) for the trivial character of T, we have

$$(\mathrm{Ind}_B^G 1)^I \simeq R \otimes_{1, R[T/T_o], t_{\overline{B}}} H_R(G, I).$$

By (6), the action of $H_R(G, I) \simeq R[W.(T/T_o)]$ on $(\mathrm{Ind}_B^G 1)^I$ restricted to $R[T/T_o]$ is trivial. As $R[W]$ is semi-simple, $(\mathrm{Ind}_B^G 1)^I$ is a semi-simple right $H_R(G, I)$-module.

Every parabolic subgroup of G is conjugate to a parabolic group P which contains B, and the isomorphism class of $\mathrm{Ind}_P^G 1$ does not change when P is replaced by a conjugate in G. We have an inclusion $\mathrm{Ind}_P^G 1 \subset \mathrm{Ind}_B^G 1$ in $\mathrm{Mod}_R G$. As $\mathrm{Ind}_B^G 1$ is semi-simple, the same is true for $\mathrm{Ind}_P^G 1$.

Let $\mathbb{X}$ be an unramified R-character of T. Modulo conjugaison $\mathbb{X} = \otimes_i \mathbb{X}_i$ is the external product of characters $\mathbb{X}_i := x_i 1$ of the diagonal subgroups T_i of $G_i := GL(n_i, F)$, which are different multiples of the identity character, $x_i \neq x_j \in R^*$ if $i \neq j$ and $\sum_i n_i = n$. The parabolic induction $\mathrm{Mod}_R \prod_i G_i \rightarrow \mathrm{Mod}_R G$ sends any irreducible subquotient of $\otimes_i \mathrm{Ind}_{B_i}^G x_i 1$ to an irreducible representation of G by (H). This implies the semi-simplicity of $\mathrm{Ind}_B^G \mathbb{X}$.

j) The property 8) of the theorem results from the (known) formula (8) and (10) below, applied to $V = R[I/I^\ell]$.

Let R be any commutative ring. An R-representation $\sigma : I/I_p \rightarrow GL_R V$ of I/I_p identifies to an R-representation of I trivial on I_p. We have $I = T_o I_p$. The Weyl group $W \simeq S_n$ acts on $T_o/T_o \cap I_p \simeq I/I_p$ by conjugation, and by inflation the affine group $W.(T/T_o)$ acts on I/I_p. One denotes by $\mathrm{Int}w.V$ the representation of I/I_p deduced from V by conjugation by $w \in W.(T/T_o)$. The endomorphism algebra $\mathrm{End}_{RG} \mathrm{Ind}_I^G V$ is isomorphic as an R-module to ([V2, II.2 page 562] and [V3,C.1.5]):

$$\text{(8)} \qquad \mathrm{End}_{RG} \mathrm{Ind}_I^G V \simeq \oplus_{w \in W.(T/T_o)} \mathrm{Hom}_{RI}(V, \mathrm{Int}w.V),$$

where $\mathrm{Hom}_{RI}(V, \mathrm{Int}w.V)$ is the space of $A \in \mathrm{End}_R V$ such that $A \circ \sigma(k) = \sigma(wkw^{-1}) \circ A$, $\forall k \in I/I_p$. A function in $\mathrm{Ind}_I^G V$ with support Ig and value v at g is denoted by $[Ig, v]$ for all $g \in G, v \in V$. The representation $\mathrm{Ind}_I^G V$ is generated by $[I,v]$ for all $v \in V$. The endomorphism $T_{w,A}$ in $\mathrm{End}_{RG}\mathrm{Ind}_I^G V$ corresponding to (w,A) by (8) sends $[I,v]$ to [V2, II.2 page 562]:

$$[I,v]T_{w,A} = \sum_x [Ix, A_x(v)] \tag{9}$$

where $IwI = \cup_x Ix$ is a disjoint decomposition and $A_x = A \circ \sigma(w^{-1}x)$. Let $w, w' \in W.(T/T_o)$ and $A \in \mathrm{Hom}_{RI}(V, \mathrm{Int}w'.V), B \in \mathrm{Hom}_{RI}(V, \mathrm{Int}w'.V)$. We use that the image of $[I,v]$ by g^{-1} is $g^{-1}[I,v] = [Ig, v]$ for $g \in G, v \in V$, (9) and the G-equivariance of $T_{w',B}$, to see that the product $T_{w,A}T_{w',B}$ sends $[I,v]$ to

$$[I,v]T_{w,A}T_{w',B} = \sum_x [Ix, A_x(v)]T_{w',B} = \sum_{x,y} [Iyx, (B_y \circ A_x)(v)], \tag{10}$$

where $IwI = \cup_x Ix,\ Iw'I = \cup_y Iy$. One can choose $x, y \in I^\ell$ because $I_\ell \subset T_o \subset I \cap wIw^{-1}$ for any $w \in W.(T/T_o)$.

We prove the property 8. The Iwahori-Hecke algebra $H_R(G,I)$ is the algebra of RG endomorphisms of $\mathrm{Ind}_I^G 1_R$. The canonical basis $(T_w)_{w \in W.(T/T_o)}$ corresponds to $A = \mathrm{Id}$ for all w. When $V = R[I/I^\ell]$, $\mathrm{Hom}_{RI}(V, \mathrm{Int}w.V) = \mathrm{End}_{RI}V \simeq R[I_\ell]$. The group I_ℓ is commutative, and $A_x = A$ commute with $B_y = B$ for any x, y in (10). ⋄

We suppose again that R is an algebraically closed field of characteristic 0 or $\ell > 0$ different from p. Let $\mathcal{J}_R$ *be the annihilator of* $R[G/I]$. The Schur R-algebra of G is Morita equivalent to $\mathcal{H}_R(G)/\mathcal{J}_R$ [V3, 2]. It is clear that the abelian category $\mathrm{Mod}_R(G,I)$ is annihilated by $\mathcal{J}_R$.

2 Theorem *In the quasi-banal case, the category* $\mathrm{Mod}_R(G,I)$ *is the category of representations of* G *which are annihilated by* $\mathcal{J}_R$. *In other terms, the Schur* R*-algebra of* G *is Morita equivalent to the Iwahori-Hecke* R*-algebra of* G.

This is already known in the banal case. The proof of the theorem results from properties of the Gelfand-Graev representation Γ_R and of the Steinberg representation St_R of $GL(n, \mathbb{F}_q)$.

We need more notation.

a) The subcategory $\mathrm{Mod}_{R,1}GL(n,\mathbb{F}_q)$ of $\mathrm{Mod}_R GL(n,\mathbb{F}_q)$ generated by (the irreducible subquotients of) $R[GL(n,\mathbb{F}_q)/B(\mathbb{F}_q)]$ is a sum of blocks by a theorem of Broué-Malle. Representations in $\mathrm{Mod}_{R,1}GL(n,\mathbb{F}_q)$ are called *unipotent*.

The annihilator $\mathcal{J}_R(q)$ of $R[GL(n,\mathbb{F}_q)/B(\mathbb{F}_q)]$ in $R[GL(n,\mathbb{F}_q)]$ is the Jacobson radical of the unipotent part of the group algebra $R[GL(n,\mathbb{F}_q)]$, because the representation $R[GL(n,\mathbb{F}_q)/B(\mathbb{F}_q)]$ is semi-simple.

b) Let $\psi : \mathbb{F}_q \to R^*$ be a non trivial character. We extend ψ to a character $(u_{i,j}) \to \psi(\sum u_{i,i+1})$ of the strictly upper triangular subgroup $U(\mathbb{F}_q)$ of $GL(n,\mathbb{F}_q)$, still denoted by ψ. The representation of $GL(n,\mathbb{F}_q)$ induced by the character ψ of $U(\mathbb{F}_q)$ is the *Gelfand-Graev representation* Γ_R. Its isomorphism class does not depend on ψ. We denote by $\Gamma_{R,1}$ the unipotent part of Γ_R.

c) The Steinberg representation St_R of $GL(n,\mathbb{F}_q)$ is the unique irreducible R-representation such that its $B(\mathbb{F}_q)$-invariants is isomorphic to the sign representation as a right module for the Hecke algebra $H_R(GL(n,\mathbb{F}_q), B(\mathbb{F}_q))$.

d) The inflation followed by the compact induction is an exact functor

$$i^G : \mathrm{Mod}_R GL(n,\mathbb{F}_q) \to \mathrm{Mod}_R GL(n,O_F) \to \mathrm{Mod}_R G$$

e) The global Hecke algebra $\mathcal{H}_R(G)$ contains the Hecke algebra

$$\mathcal{H}_R^o := H_R(GL(n,O_F), 1 + p_F M(n,O_F))$$

isomorphic via inflation to the group algebra $R[GL(n,\mathbb{F}_q)]$. The Jacobson radical $\mathcal{J}_R(q)$ of the unipotent part of the group algebra $R[GL(n,\mathbb{F}_q)]$ identifies with a two-sided ideal of $\mathcal{H}_R^o$.

We recall [V3, theorem 4.1.4]:

(I) The representation of $GL(n,\mathbb{F}_q)$ on the $1 + p_F M(n,O_F)$-invariants of $R[G/I]$ is isomorphic to a direct sum $\oplus R[GL(n,\mathbb{F}_q)/B(\mathbb{F}_q)]$.

(J) $i^G V$ is generated by its I-invariant vectors if $V \in \mathrm{Mod}_R GL(n,\mathbb{F}_q)$ is generated by its $B(\mathbb{F}_q)$-invariant vectors.

4 Lemma *Suppose that we are in the quasi-banal case. Then*

1) $\mathcal{J}_R$ is the Jacobson radical of the unipotent bloc of $\mathrm{Mod}_R G$ (same for $\mathcal{J}_R(q)$ and $GL(n,\mathbb{F}_q)$).

2) The unipotent part $\Gamma_{R,1}$ of the Gelfand-Graev R-representation of the group $GL(n,\mathbb{F}_q)$ is the projective cover of the Steinberg R-representation St_R of $GL(n,\mathbb{F}_q)$.

3) $\Gamma_{R,1}\mathcal{J}_R(q)$ is the kernel of the map $\Gamma_{R,1} \to \mathrm{St}_R$.

4) $\mathcal{J}_R(q) \subset \mathcal{J}_R$.

5) $i^G\Gamma_{R,1}/(i^G\Gamma_{R,1})\mathcal{J}_R$ is a quotient of $i^G\mathrm{St}_R$ and is generated by its I-invariant vectors.

Proof of the lemma This is known in the banal case, hence we suppose that we are in the limit case.

We prove the property 1). The semi-simplicity of $\mathrm{Ind}_B^G \mathbb{X}$ for all unramified characters (theorem 1 7)) implies with (3) that $\mathcal{J}_R$ *is the Jacobson radical of the unipotent bloc*. This means that $\mathcal{J}_R$ is the intersection of the annihilators in the global Hecke algebra $\mathcal{H}_R(G)$ of the irreducible unipotent R-representations of G.

We prove the property 2). The induced representation $\mathrm{Ind}_{B(\mathbb{F}_q)}^{GL(n,\mathbb{F}_q)} 1_R$ is semi-simple, and St_R is the *unique* subquotient which is isomorphic to a quotient of the Gelfand-Graev representation Γ_R. By the *uniqueness* theorem,

$$\dim_R \mathrm{Hom}_{RG}(\Gamma_R, \mathrm{St}_R) = 1.$$

The unipotent part $\Gamma_{R,1}$ of the Gelfand-Graev representation Γ_R is projective (because the characteristic of R is different from p) and is a direct sum of indecompable projective representations of $GL(n, \mathbb{F}_q)$. In the quasi-banal case, the two properties of uniqueness imply that $\Gamma_{R,1}$ is projective cover of St_R.

The property 3) results from 1) and 2) by general results [CRI 18.1].

The property 4) results from e) and (I).

We prove the property 5). By definition $(i^G\Gamma_R)\mathcal{J}_R = \Gamma_R \otimes_{\mathcal{H}_R^\circ} \mathcal{J}_R$.

By 4) $\Gamma_R \otimes_{\mathcal{H}_R^\circ} \mathcal{J}_R(q)\mathcal{H}_R(G) \subset \Gamma_R \otimes_{\mathcal{H}_R^\circ} \mathcal{J}_R$.

We have [V1 I.5.2.c)] $\Gamma_R \otimes_{\mathcal{H}_R^\circ} \mathcal{J}_R(q)\mathcal{H}_R(G) = \Gamma_R\mathcal{J}_R(q) \otimes_{\mathcal{H}_R^\circ} \mathcal{H}_R(G) = i^G W$ where $W = \Gamma_R\mathcal{J}_R(q)$. Clearly $i^G\Gamma_R/(i^G\Gamma_R)\mathcal{J}_R$ is a quotient of $i^G\Gamma_R/i^G W$.

The functor i^G is exact hence $i^G\Gamma_R/i^G W \simeq i^G(\Gamma_R/W)$. By 3) $\Gamma_R/W \simeq \mathrm{St}_R$. Hence $i^G\Gamma_R/(i^G\Gamma_R)\mathcal{J}_R$ is a quotient of $i^G\mathrm{St}_R$. By c), St_R is irreducible and has a non zero vector invariant by $B(\mathbb{F}_q)$. By (J), $i^G\mathrm{St}_R$ is generated by its I-invariant vectors. $\diamond$

The lemma 4 extends to the standard Levi subgroups $M_\lambda(\mathbb{F}_q)$ of $GL(n, \mathbb{F}_q)$, quotients of the parahoric subgroup $P_\lambda(O_F)$. These groups are parametrized by the partitions λ of n. The group $GL(n, \mathbb{F}_q)$ corresponds to the partition (n). One denotes by an index λ the objects relative to λ.

We recall:

(K) $\mathbf{Q_R} := \Gamma_{\mathbf{R}}/\Gamma_{\mathbf{R}}\mathcal{J}_R$ is a projective generator of $\mathrm{Mod}\mathcal{H}_R(G)/\mathcal{J}_R$ where $\Gamma_{\mathbf{R}} := \oplus_\lambda i_\lambda^G \Gamma_{R,\lambda}$ [V3, theorem 5.13].

Proof of the theorem 3 By lemma 4 for the group $M_\lambda(\mathbb{F}_q)$, the quotient $i_\lambda^G\Gamma_{R,\lambda}/i_\lambda^G\Gamma_{R,\lambda}\mathcal{J}_R$ of $i_\lambda^G\mathrm{St}_{R,\lambda}$ is generated by its I-invariant vectors. Hence the progenerator $\mathbf{Q_R}$ of $\mathrm{Mod}\mathcal{H}_R(G)/\mathcal{J}_R$ is generated by its I-invariant vectors. $\diamond$

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